Chapter 13

ALTERNATING CURRENT THEORY

INTRODUCTION

On the last afternoon of Chapter 12, with the steady current of batteries and bulbs still warm in everyone's memory, Mr. Akilikubwa made a tactical error. He asked the class an open question. He wanted to know which they thought was the finer kind of electricity: the loyal, one-way current they had just spent a whole chapter studying, or the restless current that comes out of the wall. He really should have known better than to hand Kipanga an opening for a speech.

Kipanga rose to his feet, holding up his pocket torch like a small trophy.

Kipanga: *Sir, there is no contest. In Chapter 12 I met a current with* ***character.*** *It chooses a direction and it commits. The electrons in this torch set out from the battery, crawl to the bulb and back, slower than an ant carrying a leaf, the very same way every second of every day. That is a current a man can trust. But the current from the wall? It sets off, loses its nerve, turns around, comes back, and changes its mind again, fifty times a second! That is not a current, sir. That is a* ***quarrel.*** *Why would TANESCO build an entire country on a current that cannot decide whether it is coming or going?*

Kipute did not even look up from her notebook.

Kipute: *Because the indecision is the entire point, Kipanga. Your faithful torch current can never be lifted to a high voltage and carried across the country. The quarrelsome one can. That is the only reason the wires above your house are humming while your torch sits quiet in your hand.*

Mr. Akilikubwa: *Kipute has gone straight to the heart of it. In Chapter 12 we told you, and asked you simply to trust us, that TANESCO carries its power across Tanzania at hundreds of thousands of volts, then steps it gently back down to the 240 at your socket. We never showed you the machine that does the stepping. That machine is the* ***transformer****, and it keeps one stubborn condition: it works only on a current that will not hold still. Offer it the dependable, committed current of Kipanga's torch and it does precisely nothing: no stepping up, no stepping down, only a coil of wire sitting there uselessly warm. A hundred years ago the engineers of the world fought bitterly over this very question, the one Kipanga has just settled so confidently in his own favour. The restless current won. It won for a single reason, the transformer, and that reason is the quiet hero of this chapter.*

Kipanga, unwilling to surrender either his torch or his thesis, tried one last line of defence.

Kipanga: *But sir, in Chapter 11 the capacitor was perfectly content to sit still and hold its charge for a week, long enough, as my thumb has not forgotten, to bite me through the back of a television that was not even plugged in. Surely the components themselves prefer a current that behaves.*

Mr. Akilikubwa (with the faint smile of a man about to spring a trap): *That same capacitor, Kipanga, fed this restless current, will never sit still again. It will fill and empty fifty times a second, and in doing so it will pass a current it could never have passed from your torch. Fully half of what the last two chapters trained you to expect is about to misbehave. I suspect you are going to enjoy the misbehaviour a great deal more than you currently intend to.*

Kipanga sat down, still holding the torch, but eyeing the wall socket with the wary respect of a man who has just been told that the quiet one in the room is the dangerous one.

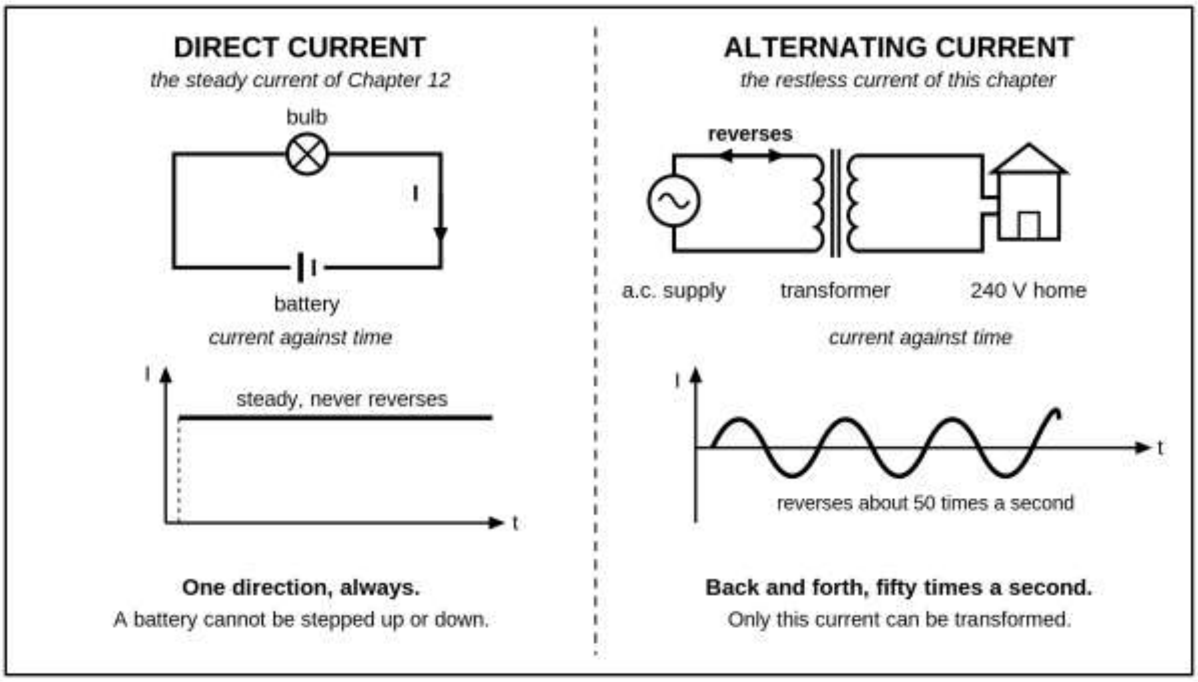


Figure: ***Two currents, side by side.*** *On the left, the steady one-way current of a torch: a battery drives a fixed current through the bulb in one direction only, and the current against time is a flat line that never reverses. On the right, the supply from the wall reverses about fifty times each second, and it feeds a transformer that steps the voltage gently down to the 240V of a home. Only the restless current on the right can be stepped up or down, which is the whole reason the grid is built from it.*

Two chapters stand directly behind this one, and alternating current unsettles them both. In **Chapter 11** a capacitor charged once, through its resistor, in a handful of time constants, and then held its charge in perfect stillness, long enough to sting a careless hand reaching into an unplugged set. In **Chapter 12** the current settled into one patient direction and kept it forever, from the torch in a student's pocket to the immersion heater in a Nyamagana kitchen.

Alternating current keeps neither promise. Give a capacitor an alternating voltage and it never finishes charging. Drive a wire from an alternating source and the patient drift of Chapter 12 never commits to a direction at all; it sloshes back and forth and arrives nowhere. Almost every steady habit those two chapters taught us must now be re-examined for a current that simply will not stand still. And, as Mr. Akilikubwa promised, most of it becomes more interesting for the unrest, not less.

This chapter builds that new physics in the order it is best learned. We begin where the alternating current is born, in a coil turned inside a magnetic field, and show why the EMF it delivers is bound to be a sine wave. We learn to measure it honestly, through the root-mean-square value hiding behind every "240 V" label. We meet the phasor, a spinning arrow that turns the clumsy trigonometry of alternating quantities into pictures the eye can trust. We then ask how the three building blocks of any circuit answer an alternating voltage: the resistor of O-level, the capacitor of Chapter 11, and the inductor we are meeting properly for the first time. We combine them in series and in parallel, and we arrive at resonance, the knife-edge that lets a small radio pick one station out of the crowded air. We build the transformer in full and discover why the grid simply had to be alternating. We settle the accounts of AC power, and find out why a Dar es Salaam workshop can be charged for power it never used. And we close by taming the restless current back into the steady one of Chapter 12, with the humble rectifier hidden inside every phone charger, radio, and laptop in the country.

Welcome to Chapter 13. Kipanga is about to spend an entire chapter discovering that the most useful current in the world is the one that cannot make up its mind. And he, of all people, should have been the first to see the appeal.

ALTERNATING EMF AND CURRENT

The introduction left us with a current that changes its mind fifty times a second. That is a fine thing to say in words, but physics asks two harder questions, and this section answers both. *First, what does that restless current actually look like when we write it down honestly? Second, where does it come from in the first* *place?* By the end of this section you will be able to draw the wave, name every feature on it, and build its shape from nothing grander than a coil turning in a magnetic field.

Picturing a current that will not sit still

Imagine watching the voltage at a wall socket on a screen that plots its value against time. It refuses to hold at one level the way the torch of Chapter 12 did. Instead it climbs smoothly from zero up to a greatest value in one direction, eases back down through zero, sinks to an equally great value in the opposite direction, and climbs back to zero again. Then it repeats, with the patience of a tide. That smooth, symmetric, endlessly repeating swing is a **sine wave**, and it is the gentlest back-and-forth that nature allows.

This behaviour is what we now call alternating current. An **alternating current (a.c.)** *is an electric current that periodically reverses its direction:* it grows to a maximum one way, falls back to zero, reverses to an equal maximum the other way, and returns, over and over in a regular cycle.

The source that drives it, an **alternating EMF**, *is one whose magnitude and polarity change periodically with time.* When that change follows a sine curve, as it does for the supply from a rotating generator, the EMF and current are said to be **sinusoidal**, and sinusoidal a.c. is the kind this chapter studies. (A steady torch current, by contrast, is **direct current**: one fixed direction, never reversing.)

Two quantities follow this same swing: the alternating EMF of the generator, and the alternating current it drives. Writing **E** for the **instantaneous EMF**, the value at one chosen instant, and $\mathbf{E_0}$ for the **peak value**, the height of the swing and the exact counterpart of the amplitude from Chapter 6, the whole wave is captured by $\boldsymbol{E = E_0 sin\omega t}$. The current keeps the same law, $\boldsymbol{I = I_0 sin\omega t}$, where I_0 is the peak current, and I is the **instantaneous current** (to distinguish it from I, which is commonly used for a steady or constant current, the lowercase letter $\boldsymbol{i}$ is used to denote instantaneous current).

The sine, the peak, and the rhythm of the swing are all we need. The voltage at a household socket swings between $+339\text{V}$ and -339V, fifty times every second. Why we politely call this 240V rather than 339V is the business of the next section.

Period, frequency and angular frequency

Three numbers describe the rhythm, and they are really three ways of saying one thing.

The **period** T is the time for one complete swing, measured in seconds. The **frequency** f is the number of complete swings in one second, measured in hertz. They are reciprocals of one another, $\text{T} = \frac{1}{\text{f}}$. Tanzania's mains supply runs at $\text{f} = 50\text{Hz}$, so one full swing takes $\text{T} = \frac{1}{50\text{Hz}} = 0.02\text{s}$, a fiftieth of a second. A single blink of your eye contains about five complete cycles of the current lighting the room.

The third number, the **angular frequency** ω, is the one students meet last and fear most, quite unnecessarily. The sine function does not count swings; it counts radians, and one complete swing carries the wave through 2π radians. So the angular frequency is simply how many radians of the cycle go by each second, $\omega = 2\pi\text{f}$, measured in radians per second. For our mains, $\omega = 2\pi \times 50\text{Hz} = 314\text{rad s}^{-1}$. You can also read the period straight back from it, since $\text{T} = \frac{2\pi}{\omega}$.

Keep all three in view: f counts cycles for the engineer, T counts seconds for the clock, and ω counts radians for the sine. They never disagree with one another.

If all of this feels familiar, it should. It is the very mathematics of Chapter 6, where a mass on a spring and Kipute on her swing rose and fell as $\text{x} = \text{A}\sin\omega\text{t}$. Everything carries straight across. The peak value E_0 here plays exactly the part the amplitude A played there; the period T, the frequency f, and the angular frequency $\omega = 2\pi\text{f}$ are defined identically and obey the same relations; and the sine is the same sine. Only the oscillating quantity has changed. In Chapter 6 it was a displacement in space, while here it is a voltage and a current in a wire. The travelling waves of Chapter 7 told the same story with $\text{y} = \text{A}\sin\omega\text{t}$. Learn the alternating-current sinusoid once and you have, honestly, already learned it three times.

One thread is worth tying now. In Chapter 6, pushing the swing at exactly its natural rhythm made its amplitude grow dramatically, and we called that resonance. That same idea returns near the end of this chapter, where tuning a circuit to the right rhythm is precisely how a radio picks one station out of the crowded air.

Where the sine comes from: a coil turning in a field

We promised to earn the sine wave, not merely assert it. The machine that produces alternating current is the alternating-current generator, and Mtera's turbines turn day and night to keep ours running. Before writing a single symbol, let us first picture the arrangement shown in the next figure.

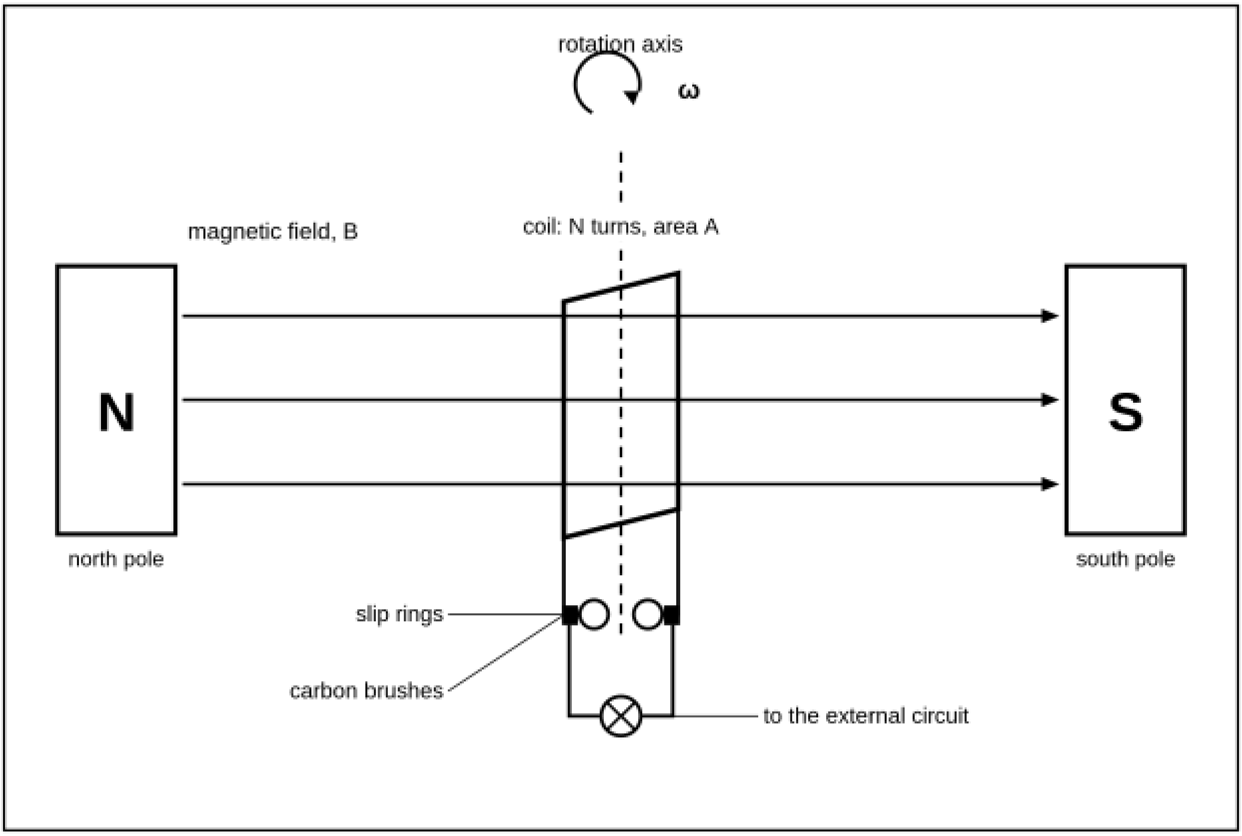


Figure: ***An alternating-current generator***. *A coil of N turns and area A is spun at a steady angular velocity ω about an axis lying across a uniform magnetic field B. The turning coil feeds the outside world through two slip rings pressed by carbon brushes.*

A flat coil of N turns, each enclosing an area A, is mounted so that it can spin about an axis lying across a uniform magnetic field B. The turbine spins it at a steady angular velocity ω. As the coil turns, the amount of field threading through it, the **magnetic flux**, rises and falls. *A changing magnetic flux linked with a coil produces an induced EMF.* The goal of the derivation that follows is to convert this physical statement into a quantitative formula.

Start with the flux.

When the face of the coil looks straight along the field, the field pours through it squarely and the flux is greatest. A quarter-turn later the coil's plane lies along the field, no lines thread it, and the flux is zero. In between, the flux follows the cosine of the angle the coil has turned, and because the coil turns at a steady rate that angle is just ωt. So the flux through one turn is $\Phi = BA\cos\omega t$, and for all N turns together the **flux linkage** is:

$$\Lambda = N\Phi = NBA\cos\omega t$$

Now apply Faraday's law.

The induced EMF is the rate at which the flux linkage changes, carrying a minus sign for Lenz's law:

$$E = -\frac{d\Lambda}{dt} = -\frac{d(NBA\cos\omega t)}{dt} = NBA\omega\sin\omega t$$

Set this beside the wave we first drew, $E = E_0\sin\omega t$, and the peak EMF names itself:

$$E_0 = NBA\omega$$

Look at what just happened, drawn out in the following figure.

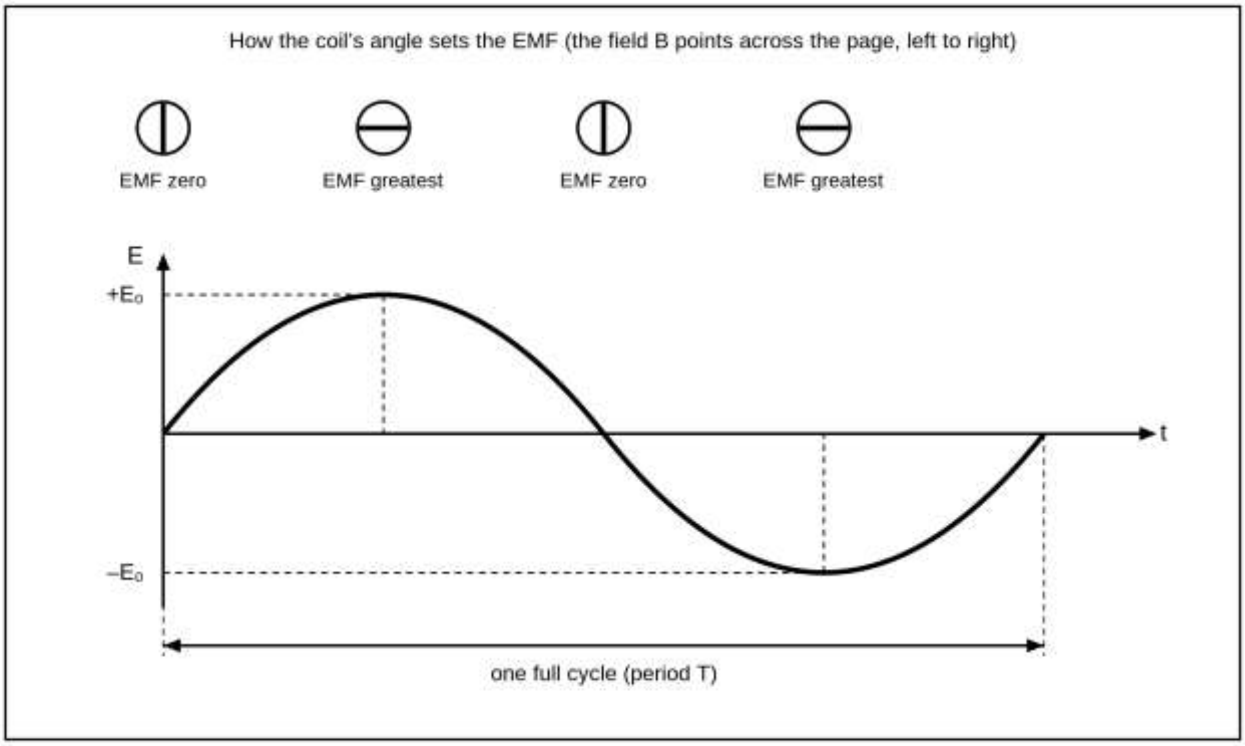


Figure: ***The EMF traces one full sine wave as the coil makes one full turn.*** *The EMF is greatest when the coil sweeps edge-on through the field (flux changing fastest) and zero when the coil faces the field squarely (flux greatest, but momentarily not changing).*

The EMF is greatest at exactly the moments the flux is changing fastest, which is when the coil sweeps edge-on through the field and its wires cut the field lines at full speed. The EMF falls to zero at the moments the flux is greatest, when the coil faces the field squarely and, just for that instant, is not changing at all. This is the part students reliably get backwards, so hold tight to the picture: *greatest EMF where the coil moves **along** the field, zero EMF where it **faces** the field*. The sine wave is not a decoration we chose. It is the unavoidable fingerprint of steady rotation, because the flux varies as a cosine and the rate of change of a cosine is a sine.

The result $E_0 = NBA\omega$ hands you four honest ways to raise the peak voltage:

- wind on more turns N,
- use a stronger magnet B,
- enlarge the coil's area A, or
- spin it faster ω.

Notice that ω quietly does two jobs at once: it sets both the size of the peak and the frequency of the swing. Keep that in mind, because it is the whole engine of the Mtera example coming up. (At Mtera each generator actually carries three coils set a third of a turn apart, which is why the national supply is three-phase; but one coil is all we need for the physics, and all the examination will ask of you.)

One promise for later: this very idea, a changing flux inducing an EMF in a coil, returns near the end of the chapter as the secret of the transformer. Keep it warm.

The theory is on its feet; time to make it work for a living. Five examples follow. Kipanga lobbied hard for two, on the grounds that he had already met enough Greek letters for one afternoon. He was outvoted by the arithmetic.

BINDER Example 1

Tanzania's mains supply has a frequency of 50Hz and a peak voltage of 339V. (a) Find its period. (b) Find its angular frequency. (c) Write the equation for the instantaneous voltage, and use it to find the voltage 0.002s after the voltage last passed through zero on its way up.

Solution

(a) The period is the reciprocal of the frequency:

$$T = \frac{1}{f} = \frac{1}{50\text{Hz}} = 0.02\text{s}$$

(b) The angular frequency follows from the frequency:

$$\omega = 2\pi f = 2\pi \times 50\text{Hz} = 314\text{rad s}^{-1}$$

(c) The instantaneous voltage obeys $E = E_0 \sin\omega t$, so:

$$E = 339\sin 314t$$

Putting $t = 0.002\text{s}$ (so that $\omega t = 314 \times 0.002 = 0.628\text{rad}$):

$$E = 339\sin(0.628)\text{V} = 339 \times 0.588\text{V} = 199\text{V}$$

Making Sense of the Answer: *The period came out as a fiftieth of a second, which is exactly what* $50Hz$ *promises, so the first answer checks itself. The instant we asked about is only a tenth of the way into the cycle, so the voltage has not yet climbed to its crest;* $199V$ *sits comfortably below the* $339V$ *peak, just where it ought to.*

Think Like a Physicist: *Whenever a frequency appears, expect* $2\pi f$ *to walk into every equation behind it, because the sine is fed radians, not cycles. Convert to* ω *once at the very start and the rest of the problem flows downhill.*

BINDER Example 2

An alternating current in a lamp has a maximum value of 4.0A and a frequency of 50Hz. (a) Write the equation for its instantaneous value. (b) Find the current 0.0025s after it last passed through zero on its way up. (c) Find the time it first takes to reach its peak.

Solution

(a) The instantaneous current obeys $I = I_0 \sin\omega t$, and the angular frequency is :

$$\omega = 2\pi f = 2\pi \times 50\text{Hz} = 314\text{rad s}^{-1}$$

With $I_0 = I_{max} = 4.0\text{A}$:

$$I = 4.0\sin 314t$$

(b) Putting $t = 0.0025\text{s}$, the angle reached is $\omega t = 314 \times 0.0025 = 0.785\text{rad}$, hence:

$$I = 4.0\sin(0.785)\text{A} = 4.0 \times 0.707\text{A} = 2.83\text{A}$$

(c) The current first reaches its peak when $\sin\omega t = 1$, that is when $\omega t = \frac{\pi}{2}$. Solving for the time gives

$$t = \frac{\pi}{2\omega} = \frac{\pi}{2 \times 314\text{rad s}^{-1}} = 0.005\text{s}$$

Making Sense of the Answer: *At one eighth of a cycle the current has climbed to* 0.707 *of its peak, about* $2.83A$*, sitting comfortably between zero and the* $4.0A$ *crest. The first peak then arrives a quarter of a period after a zero, at* $0.005s$*, exactly where the sine first reaches one.*

Think Like a Physicist: *The current law* $I = I_0 sin\omega t$ *is the exact twin of the voltage law* $E = E_0 sin\omega t$*: the same* ω*, the same sine, only the peak symbol changes from* E_0 *to* I_0*. Read a time-to-peak straight from* $\omega t = \frac{\pi}{2}$*, never by guessing.*

BINDER Example 3

A rectangular coil of 200 turns, each of area 0.020m^2, spins at an angular velocity of 314rad s^{-1} in a uniform magnetic field of 0.15T. Calculate the peak EMF it generates.

Solution

The peak EMF is already the subject of $E_0 = NBA\omega$, so we substitute directly, keeping every unit in place:

$$E_0 = NBA\omega = 200 \times 0.15\text{T} \times 0.020\text{m}^2 \times 314\text{rad s}^{-1} = 188.4\text{V}$$

Making Sense of the Answer: *A peak of roughly* $188V$ *is the right size for a modest generator, a little below the household crest we have been quoting, which is reassuring rather than surprising.*

Think Like a Physicist: *The formula $E_0 = NBA\omega$ has four handles and no hidden traps. If a question changes just one of N, B, A or ω and asks what happens to the peak, you rarely need to compute at all; you reason in proportions.*

REAL Example 4

At the Mtera hydroelectric station the engineers can open the gates and spin the generator coils faster. Kipanga, freshly armed with $E_0 = NBA\omega$, announces that spinning faster simply makes every light in the country brighter, and proposes that TANESCO should therefore spin the turbines as fast as the river allows. Explain (a) what really happens to the peak EMF when the coil spins faster, (b) what happens at the same time to the frequency, and (c) why TANESCO cannot take his advice.

Solution

(a) Kipanga is right about the voltage. In $E_0 = NBA\omega$, the turns N, the field B and the area A are all fixed once the machine is built, so the peak EMF is directly proportional to the spin rate ω. Spin the coil faster and the crest of the wave rises in exact step. So far, so bright.

(b) But ω is not free to change on its own. The frequency rides on the very same ω, through $f = \frac{\omega}{2\pi}$. Speeding the coil up does not only lift the voltage; in the same breath it raises the frequency of the whole supply. Faster spin means a higher pitch to the national hum.

(c) And there is the trap. Every mains clock, every motor, and every timing circuit in the country is built to expect exactly 50Hz. Wander away from 50Hz and clocks run fast, motors overheat, and equipment misbehaves from Mtera to Mwanza. So the grid is held at 50Hz with real care: the turbines are governed to a fixed ω, and when the country needs more power the engineers bring more generators online or push more current, never more frequency. Kipanga's lights would indeed glow brighter, for the few minutes before half the appliances in Tanzania filed a complaint.

Making Sense of the Answer: *The mistake was an honest one, because ω really does raise the voltage. The catch is that ω cannot be changed in private; it drags the frequency along with it, and the frequency is the one number the grid is least allowed to touch.*

Think Like a Physicist: *When one symbol plays two roles, changing it has two consequences, not one. Here ω sets both the peak, through $E_0 = NBA\omega$, and the rhythm, through $f = \frac{\omega}{2\pi}$. A careful physicist always asks what else a quantity controls before reaching for its dial.*

HOT Example 5

A coil of 120 turns, each of area $2.5 \times 10^{-2}\text{m}^2$, rotates at 1500 revolutions per minute in a uniform field of 0.20T. (a) Find its angular velocity. (b) Find its period. (c) Find the peak EMF. (d) Write the EMF as a function of time. (e) Find the EMF at $t = 0.01\text{s}$ and comment on the result.

Solution

(a) The coil completes 1500 cycles in every 60s, so the number of cycles each second is:

$$f = \frac{1500}{60\text{s}} = 25\text{Hz}$$

$$\omega = 2\pi f = 2\pi \times 25\text{Hz} = 157\text{rad s}^{-1}$$

The angular velocity is therefore 157rad s^{-1}.

(b) The period is the reciprocal of frequency:

$$T = \frac{1}{f} = \frac{1}{25\text{Hz}} = 0.04\text{s}$$

(c) Now the peak EMF:

$$E_0 = NBA\omega = 120 \times 0.20\text{T} \times 2.5 \times 10^{-2}\text{m}^2 \times 157\text{rad s}^{-1} = 94.2\text{V}$$

(d) The instantaneous EMF then reads

$$E = E_0 \sin\omega t = 94.2\sin 157t$$

(e) At $t = 0.01\text{s}$, the angle is $\omega t = 157 \times 0.01 = 1.57\text{rad}$, which is a right angle, so:

$$E = 94.2\sin(1.57)V = 94.2V$$

Comment: the EMF has reached its full peak value, and this is no accident. The time $t = 0.01s$ is exactly one quarter of the period $T = 0.04s$, and a quarter of a cycle after a zero is precisely where the sine first equals one. A quarter period after passing through zero, an alternating quantity always sits at its crest.

Making Sense of the Answer: *The answer lands squarely on the peak, and the figure drawn earlier shows why: the quarter-period mark is the very top of the sine. Had we chosen a time that was not a neat fraction of the period, the EMF would have fallen somewhere between zero and* $94.2V$*, and only the formula, not the eye, could have told us where.*

Think Like a Physicist: *Tame the units before you tame the physics. Revolutions per minute become radians per second by multiplying by* 2π *and dividing by* 60*; once ω is in proper units, every other quantity drops out in a single line.*

Five examples, and not a single bite. Kipanga has stopped flinching at the sight of ω, which we will record as a modest triumph for education and a quiet disappointment for anyone who came hoping for drama.

MEAN AND RMS VALUES

We can now draw the alternating wave and name its peak. But here is an awkward question. Clamp an a.c. ammeter onto the wire feeding Kipanga's lamp and it reads neither the peak nor zero, but some third number that nobody has yet explained. The supply is stamped $240V$, and yet we have just watched the voltage shoot up to $339V$ and down to $-339V$. *So which number, honestly, is the supply?* This section settles the matter, and the answer turns out to be one of the most useful ideas in all of a.c. theory.

Why the ordinary average is useless

The first instinct is to take the **mean value**, the way we average a set of test scores: add everything up and divide by how much there was. Try it on the current. Over one complete cycle the current is positive for exactly half the time and negative, by an equal amount, for the other half. Every positive instant is matched by a negative one, the two cancel perfectly, and the ***mean value*** *of an alternating current over a complete cycle is exactly zero.*

That is honest arithmetic, and it is also useless. A current whose average is zero still boils Kipanga's water and still turns the meter that bills his father. Plainly, zero is not what the ammeter is reading. The ordinary average has thrown away the very thing we care about, because it lets the negative half-cycle erase the positive one. We need a cleverer kind of average, one the minus signs cannot cancel.

The value that matters: the root-mean-square

Here is the escape. The reason an alternating current still heats a wire is that *heating does not care about direction.* A current flowing one way heats a resistor exactly as much as the same current flowing the other way, because the power turned into heat is $P = i^2R$, and squaring a negative number makes it positive. Heat is blind to the sign of the current, and that single observation hands us the right kind of average.

Follow the power. At any instant the power delivered to a resistance R is $P = i^2R = I_0{}^2R\sin^2\omega t$. The graphs below make the whole idea visible before we touch the algebra.

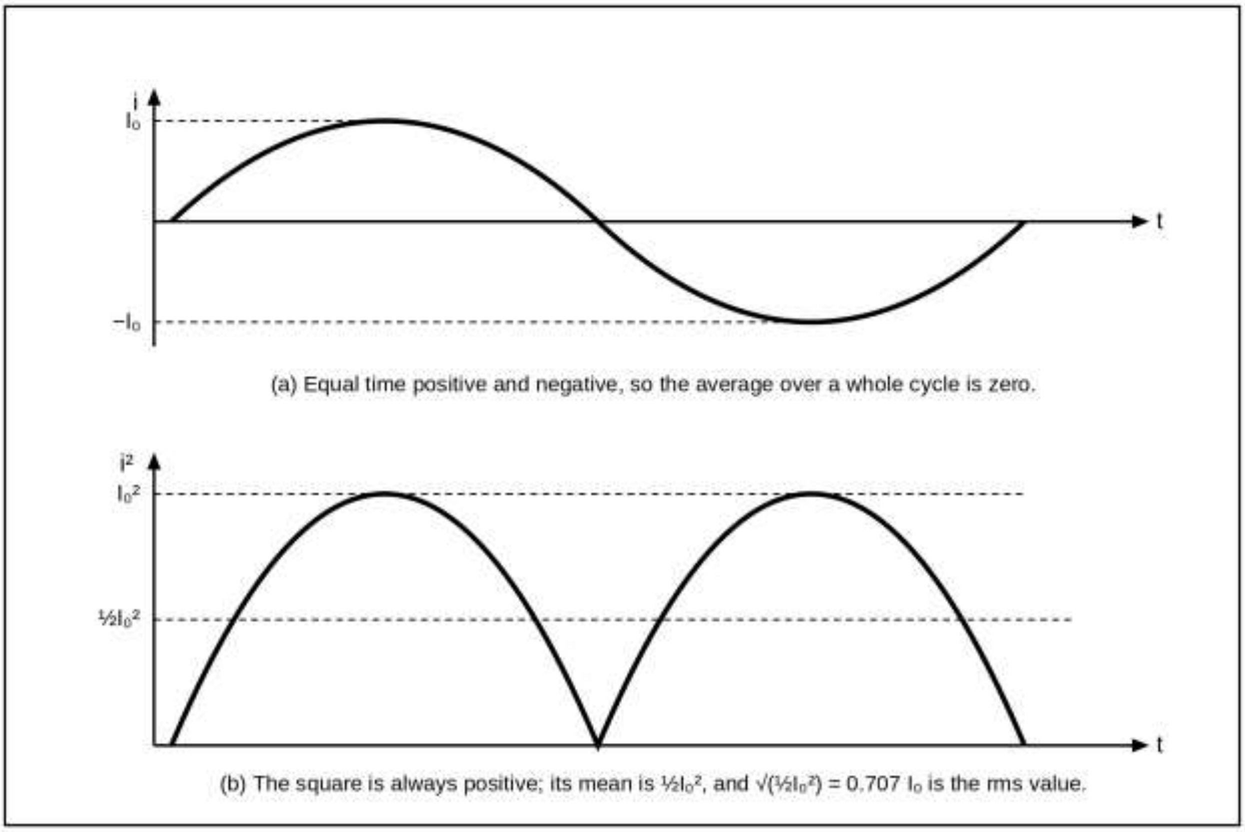


Figure: ***The two averages of an alternating current.*** *The plain average over a full cycle is zero, but the average of the squared current is one half of the peak squared, and the square root of that mean is the root-mean-square value, the steady current that would heat a resistor at the same rate.*

To get the average power we need the average of $\sin^2\omega t$ over a cycle. Here is a clean way to see it: $\sin^2\omega t$ and $\cos^2\omega t$ must average to the same value, since one is only the other shifted along, and at every instant they add to one because $\sin^2\omega t + \cos^2\omega t = 1$. Two equal things that add to one must each be one half. So the average of $\sin^2\omega t$ is ½, and the average power is:

$$P_{av} = ½{I_0}^2R$$

Now the key question: *what steady, direct current would heat the same resistor at this same rate?* Call it I_{rms}. A steady current delivers power ${I_{rms}}^2R$, so we set the two equal,

$${I_{rms}}^2R = ½{I_0}^2R$$

The resistance cancels, and the square root gives the headline result,

$$\mathbf{I_{rms} = \frac{I_0}{\sqrt{2}} = 0.707I_0},$$

and in exactly the same way for the voltage:

$$\mathbf{E_{rms} = \frac{E_0}{\sqrt{2}} = 0.707E_0}$$

So generally:

$$\textbf{rms value} = \frac{\textbf{peak value}}{\sqrt{\mathbf{2}}}$$

This number wears a name that describes precisely how it was built. The **root-mean-square value (rms)** of an alternating quantity is found in three steps, in order: **square** the instantaneous value, take the **mean** of the square over a cycle, then take the **square root**. Root, of mean, of square, read backwards. And it carries a physical meaning worth remembering: **The rms value of an a.c.** *is the steady d.c. value that would deliver the same average power, the same heating, to a given resistor.*

If that three-step recipe feels familiar, it should. We used exactly it in Chapter 4, the kinetic theory of gases, where the root-mean-square speed of the molecules stood in for their wildly differing individual speeds. Square, average, take the root. There it tamed a crowd of molecules; here it tames a current that will not sit still. The procedure is identical; only the quantity being averaged has changed.

The mean over half a cycle

There is a second average that occasionally earns its keep, so we name it and move on. If we average the current over just one half-cycle, the positive hump alone, the negatives never get their turn to cancel and the result is not zero. The average height of a single sine arch works out to $\frac{2}{\pi}$ of the peak, so the **mean value over a half cycle** is:

$$\mathbf{I_m} = \frac{I_0}{\frac{2}{\pi}} = \frac{\mathbf{2I_0}}{\boldsymbol{\pi}} = \mathbf{0.637I_0},$$

and likewise:

$$\mathbf{E_m} = \mathbf{0.637E_0}$$

This is a marginal quantity for now, because almost every a.c. calculation reaches for the rms value instead. Its one real moment of usefulness comes later in this chapter, when we pass a.c. through a diode to make d.c.: the steady output of such a rectifier is governed by exactly this half-cycle mean. Until then, keep it in your back pocket.

Form factor

One small ratio (called form factor) ties the two useful averages together. The **form factor** of a waveform *is its rms value divided by its mean value over a half cycle,*

$$\text{form factor} = \frac{I_{rms}}{I_m} = \frac{0.707I_0}{0.637I_0} = 1.11$$

For a pure sine wave this is always 1.11, whatever the peak. It is the number an rms-reading meter must apply internally if what it actually measures is the half-cycle mean, a point we return to when we open up an a.c. meter later in the chapter.

Every rating you read is an rms value

Now let us clear up the confusion that trips nearly every student. *Every a.c. voltage quoted* ***anywhere****, the* $240V$ *at a Tanzanian socket, the* $11kV$ *and* $400kV$ *on TANESCO's transmission lines, is an* ***rms value****, never the peak*. Now from:

$$\text{rms value} = \frac{\text{peak value}}{\sqrt{2}};\ \text{peak value} = \sqrt{2} \times \text{rms value}$$

The peak is larger by a factor of $\boldsymbol{\sqrt{2}}$. A 240V socket therefore rises to a peak of $E_0 = \sqrt{2} \times 240\text{V} = 339\text{V}$, which is why 339V kept appearing in the last section. The distinction is not pedantry. The insulation around a wire and the gap inside a switch must survive the full peak of 339V, so engineers size them on the peak; but the heat in the kettle, the reading on the meter, and the figure on the TANESCO bill all follow the rms value, so those use 240V. *Peak for safety, rms for power.* Keep the two apart and a great deal of a.c. falls into place.

Three averages, one ammeter reading, and a kettle that refuses to care about signs. Four worked examples turn all of it into numbers. Kipanga asked that none of them contain a square root; we have politely ignored him.

BINDER Example 6

(a) An a.c. supply has a peak voltage of $E_0 = 325\text{V}$; find its rms value. (b) A Tanzanian socket has an rms voltage of $E_{rms} = 240\text{V}$; find its peak voltage.

Solution

(a) The rms value is the peak divided by $\sqrt{2}$:

$$E_{rms} = \frac{E_0}{\sqrt{2}} = \frac{325V}{\sqrt{2}} = 230V$$

(b) Going the other way, the peak is $\sqrt{2}$ times the rms:

$$E_0 = \sqrt{2} \times E_{rms} = \sqrt{2} \times 240V = 339V$$

Making Sense of the Answer: *The rms is always about* 0.707 *of the peak, and the peak always about* 1.41 *times the rms. Our Tanzanian socket duly returns its familiar* 339*V peak.*

Think Like a Physicist: *There are only two conversions here and they are reciprocals: divide by* $\sqrt{2}$ *to go from peak to rms, multiply by* $\sqrt{2}$ *to come back. Decide which way you are travelling before you reach for the calculator.*

REAL Example 7

Kipanga's mother owns two identical 240V, 2.0kW kettles. One is plugged into the mains. The other, during a power cut, runs from his uncle's solar battery bank, which holds a steady 240V. Kipanga bets that the mains kettle, whose voltage is "really 339V at the peak", must boil faster than the battery kettle stuck at a flat 240V. Explain who loses the bet, and why.

Solution

Kipanga loses, and the reason is the whole point of the rms value. The mains voltage does peak at 339V, but it also passes through zero, dwells at every value in between, and spends half of each cycle negative. The single number that captures its heating effect is its rms value, and that rms value is exactly 240V, defined precisely so that the a.c. heats a resistor at the same average rate a steady 240V would. The battery sits at a flat 240V, which is its own rms value, because a steady value is its own root-mean-square.

Both kettles therefore deliver the same average power $P_{av} = \frac{{E_{rms}}^2}{R}$, draw the same rms current, and boil the same litre of water in the same time. The 339V peak is real, but it does its extra work for only an instant and is paid back by the moments near zero. Averaged over a cycle, the two kettles are twins, and Kipanga owes his mother the wager.

Making Sense of the Answer: *The rms value was defined to make exactly this true. If the mains boiled faster than a battery of equal rms voltage, we would have defined the rms value wrongly.*

Think Like a Physicist: *Whenever a problem compares a.c. with d.c. heating, convert the a.c. to its rms value and then treat it as though it were a steady d.c. The rms value is the bridge between the two worlds.*

HOT Example 8

A 2.0kW heating element is connected to the 240V, 50Hz mains, where 240V is the rms voltage. Find (a) the peak voltage, (b) the resistance of the element, (c) the rms current, (d) the peak current, and (e) the average power, confirming it matches the rating.

Solution

(a) The peak is $\sqrt{2}$ times the rms:

$$E_0 = \sqrt{2} \times 240V = 339V$$

(b) The element's resistance follows from the average power $P_{av} = \frac{{E_{rms}}^2}{R}$, rearranged for R:

$$R = \frac{{E_{rms}}^2}{P_{av}} = \frac{(240V)^2}{2000W} = 28.8\Omega$$

(c) The rms current is the rms voltage over the resistance:

$$I_{rms} = \frac{E_{rms}}{R} = \frac{240V}{28.8\Omega} = 8.33A$$

(d) The peak current is $\sqrt{2}$ times the rms current:

$$I_0 = \sqrt{2} \times 8.33A = 11.8A$$

(e) The average power is the product of the rms voltage and rms current:

$$P_{av} = E_{rms} \times I_{rms} = 240V \times 8.33A = 2000W = 2.0kW,$$

which matches the element's rating, as it must.

Making Sense of the Answer: *The rating "$2.0kW$ at $240V$" is an rms statement, and everything, the resistance, the currents, and the power, flowed from it. The peak current of $11.8A$ is just $\sqrt{2}$ times the rms and plays no part in the power.*

Think Like a Physicist: *Appliance ratings are rms. Begin every appliance problem by reading the plate as rms values, then lean on $P_{av} = E_{rms}I_{rms} = \frac{E_{rms}{}^2}{R}$.*

HOT Example 9

An alternating current is described by $i = 6.0\sin(314t)A$, with t in seconds. Find: (a) its peak value, (b) its frequency, (c) its rms value, (d) its mean value over a half cycle, (e) its form factor, and (f) the average power it dissipates in a 12Ω resistor.

Solution

(a) Comparing with $i = I_0\sin\omega t$, the peak current is $I_0 = 6.0A$.

(b) The angular frequency is $\omega = 314\text{rad s}^{-1}$, and since $\omega = 2\pi f$,

$$f = \frac{\omega}{2\pi} = \frac{314\text{rad s}^{-1}}{2\pi} = 50\text{Hz}$$

(c) The rms value is

$$I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{6.0A}{\sqrt{2}} = 4.24A$$

(d) The mean over a half cycle is

$$I_m = \frac{2I_0}{\pi} = \frac{2 \times 6.0A}{\pi} = 3.82A$$

(e) The form factor is the ratio of the two,

$$\text{form factor} = \frac{I_{rms}}{I_m} = \frac{4.24A}{3.82A} = 1.11$$

(f) The average power uses the rms current, never the peak:

$$P_{av} = I_{rms}{}^2R = (4.24A)^2 \times 12\Omega = 216W$$

Making Sense of the Answer: *The form factor came out* 1.11, *exactly as it must for any pure sine, which is a free check on the rms and half-cycle figures. And the power leans on the rms current squared, not the peak.*

Think Like a Physicist: *Read I_0 and ω straight off the equation $i = I_0 sin\omega t$; the frequency, the rms, the mean, the form factor, and the power are each a one-line consequence of those two numbers.*

Four examples, three averages, and one ammeter reading finally explained. Kipanga has made his peace with the square root, mostly because it kept turning the alarming 339V back into the friendly 240V he had trusted all along.

PHASORS: A ROTATING-VECTOR METHOD

So far each of our alternating quantities has lived alone. But the moment we wire two components together we shall have to add their voltages, and here a nasty surprise waits. The voltage across one component may be in step with the current while the voltage across another runs a quarter-cycle ahead, and adding two sine waves that are out of step, by trigonometry, is genuinely unpleasant. Physicists almost never do it. Instead they reach for a picture so simple a child could draw it and so powerful that it turns the whole of a.c. circuit theory into geometry. The picture is the **phasor**, and this short section is given over entirely to it. Master it now, on easy ground, and every circuit in the rest of the chapter becomes a matter of drawing arrows.

A sine wave is the shadow of a spinning arrow

As always, let us begin with an imagination. Imagine an arrow of fixed length, pinned at one end and spinning steadily anticlockwise, like the hand of a clock running backwards at a constant rate. Fix your eye on the height of its tip above the centre line. As the arrow turns, that height climbs from zero up to the arrow's full length, falls back to zero, drops to the full length below the line, and returns. Plot the height against time and you trace a perfect sine wave. A sine wave, in other words, is nothing more exotic than the moving shadow of a spinning arrow, as the following figure shows.

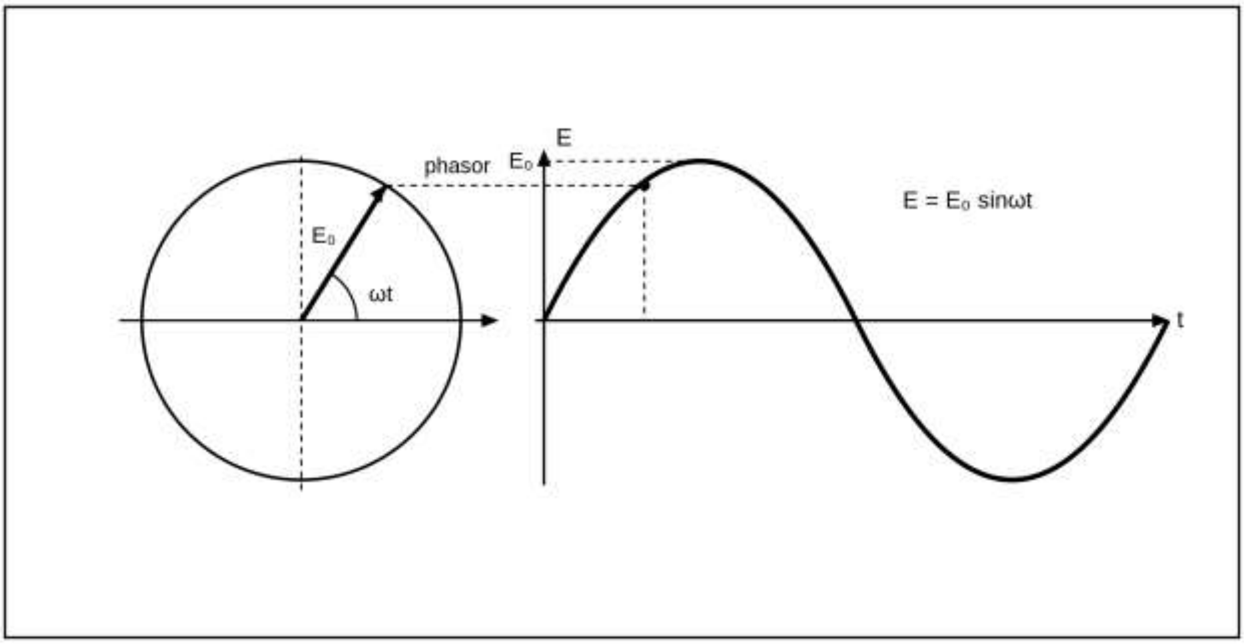


Figure: *A sine wave is the moving shadow of a spinning arrow.* *As the phasor of length E_0 turns anticlockwise through the angle ωt, the height of its tip traces out the curve $E = E_0 \sin\omega t$. This is the reference circle of simple harmonic motion, now standing for a voltage.*

You have met this picture before. In Chapter 6 we built simple harmonic motion as the shadow of uniform circular motion: a point ran steadily around a circle and its projection onto a diameter traced out $x = A\sin\omega t$. The **reference circle**, we called it. A **phasor** is that very same rotating radius, borrowed now to stand for a voltage or a current in place of a displacement. The geometry of Chapter 6 has quietly become the bookkeeping of this Chapter.

What a phasor is

Let us pin the idea down. A **phasor** *is a rotating vector that represents a sinusoidal quantity*, and it carries three pieces of information at once:

- its **length** equals the peak value (the amplitude) of the quantity;
- it **rotates anticlockwise** at the angular frequency ω of the quantity; and
- its **projection** onto the vertical at any instant gives the instantaneous value.

A phasor of length E_0, turned through an angle ωt from the horizontal, therefore has a vertical projection $E_0\sin\omega t$, which is exactly the instantaneous EMF we started from.

We fix a **convention** so that every diagram agrees: *phasors are drawn rotating anticlockwise, with angles measured anticlockwise from a horizontal reference line pointing to the right.* Now comes the liberating fact. *Because every phasor in a given circuit spins at the same frequency ω, the angles between them never change.* We may therefore freeze the rotation at a single instant and draw a still photograph, a **phasor diagram**, in which only two things matter: the length of each arrow, which is its peak value, and the angle between the arrows, which is their phase relationship. The spinning is understood; we never need to draw it twice.

The fixed angle between two phasors is their **phase difference**. *If one phasor sits further round anticlockwise than another, its quantity reaches each stage of the cycle earlier,* and we say it **leads**; *the one behind* **lags**. A phasor a quarter-turn (that is 90°, or $\frac{\pi}{2}$ radians) ahead of another describes a quantity that peaks a quarter-cycle sooner. These two words, leads and lags, will be the entire vocabulary of the circuits to come, where we shall find the voltage across an inductor leading the current and the voltage across a capacitor lagging behind it.

Why the trouble is worth it: adding sinusoids becomes adding arrows

Now the payoff that justifies the whole construction. Two sinusoids of the same frequency, however far out of step, always add to give a third sinusoid of the same frequency. Proving that by trigonometry is a chore; finding it with phasors is a single drawing. Because each sinusoid is the vertical shadow of its phasor, the shadow of the sum is just the sum of the shadows, so we simply add the phasors as vectors, tip to tail, exactly as we added forces and displacements back in mechanics.

Here is the case we shall meet over and over. A resistor and an inductor carry the same current, so we lay the current along the horizontal reference. The voltage across the resistor, V_R, is in step with the current, so its phasor lies along the reference. The voltage across the inductor, V_L, leads the current by a quarter-cycle, so its phasor stands straight up at right angles. The supply voltage is the sum of the two, found by completing the right-angled triangle in the figure below.

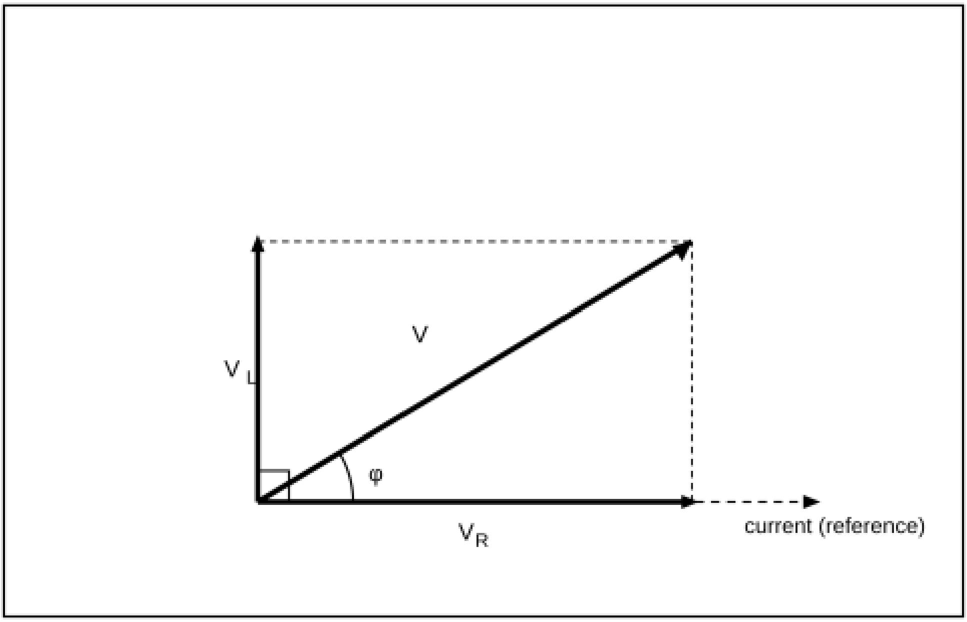


Figure: ***Adding two voltage phasors.*** *The resistor voltage lies along the current; the inductor voltage leads it by a quarter-cycle and points straight up. Their vector sum is the supply voltage V, the hypotenuse, which leads the current by the angle φ.*

Reading the triangle straight off, the peak supply voltage is the hypotenuse,

$$V = \sqrt{{V_R}^2 + {V_L}^2}$$

and it leads the current by the angle φ, given by:

$$\tan\varphi = \frac{V_L}{V_R}$$

Two arrows, one triangle, and Pythagoras: that is the entire method, and it will not change however many components we add.

Four examples to make the arrows behave. Kipanga, who once mistrusted ω, has decided he rather likes phasors, on the grounds that they are merely arrows and he has been drawing arrows since primary school. Let us not disabuse him.

BINDER Example 10

A voltage is represented by a phasor of length 50V rotating anticlockwise at angular frequency ω. Find the instantaneous voltage at the moment the phasor has turned through an angle of (a) 30°, (b) 90°, and (c) 210°.

Solution

The instantaneous value is the vertical projection of the phasor, $v = 50\sin\theta$ V, where θ is the angle turned. So:

(a) $v = 50\sin 30°V = 50 \times 0.5V = 25V$

(b) $v = 50\sin 90°V = 50 \times 1V = 50V$

(c) $v = 50\sin 210°V = 50 \times (-0.5)V = -25V$

Making Sense of the Answer*: At* $90°$ *the phasor points straight up, so its projection is the full* $50V$ *peak. At* $210°$ *it has swung below the centre line, so the value has gone negative. Both readings match the picture exactly.*

Think Like a Physicist: *The projection is the instantaneous value. Reading a phasor is nothing more than dropping a perpendicular and measuring a height; no extra formula is needed.*

REAL Example 11

During a particularly slow afternoon lesson, Kipanga finds himself staring at the second hand of the classroom clock. Spotting an opportunity, Mr. Akilikubwa asks him to explain in what sense the second hand is a phasor. Help Kipanga answer by identifying what plays the role of (a) the phasor's length, (b) its angular frequency, and (c) its projection, and by stating the period and angular frequency of this particular phasor.

Solution

(a) The second hand is an arrow of fixed length, pinned at one end and turning steadily, which is precisely a phasor. Its length stands for the amplitude of the quantity it represents, here the greatest height its tip reaches above the centre of the clock.

(b) Its angular frequency is its steady rate of turning. The hand completes one full turn in 60s, so its period is $T = 60s$ and its angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{60s} = 0.105 \text{rad s}^{-1}$$

(It happens to turn clockwise rather than anticlockwise, but that is merely a choice of direction; the physics of the projection is identical.)

(c) Its projection, the height of the tip above the centre line, rises and falls as a sine wave with that same period of 60s. If you filmed only the tip's height and plotted it against time, you would draw a perfect sinusoid, one full cycle every minute.

Making Sense of the Answer: *Every steadily rotating arrow is a phasor. The only thing special about a.c. is that the arrow stands for a voltage or current and spins far faster, fifty turns a second rather than one a minute.*

Think Like a Physicist: *When a new idea feels strange, find it inside something you already know. A phasor is just a clock hand whose shadow we happen to care about.*

HOT Example 12

In a series circuit the peak voltage across the resistor is 8.0V, in phase with the current, while the peak voltage across the inductor is 6.0V, leading the current by a quarter-cycle. Using a phasor diagram, find (a) the peak supply voltage and (b) the angle by which it leads the current.

Solution

(a) The two voltage phasors ($V_R = 8.0V, V_L = 6.0V$) meet at right angles, so the supply voltage is the hypotenuse of their triangle:

$$V = \sqrt{V_R^2 + V_L^2} = \sqrt{(8.0V)^2 + (6.0V)^2} = 10V$$

(b) The supply voltage leads the current by the angle φ, where:

$$\tan\varphi = \frac{V_L}{V_R} = \frac{6.0V}{8.0V} = 0.75, \qquad \text{so } \varphi = \tan^{-1}0.75 = 36.9°$$

Making Sense of the Answer: *The sides* 8, 6, 10 *are the familiar* 3-4-5 *triangle scaled up, a reassuring check. And a positive φ means the supply leads the current, exactly as it must when an inductor is in the circuit.*

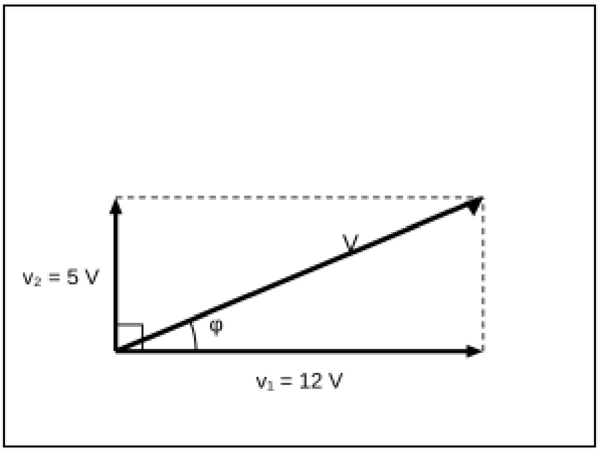
Think Like a Physicist: *Whenever two phasors meet at a right angle, the resultant is Pythagoras and the angle is an arctangent. You will lean on this one triangle more than any other relation in the chapter.*

HOT Example 13

Two alternating voltages of the same frequency act in series, $v_1 = 12\sin\omega t\,V$ and $v_2 = 5\sin(\omega t + 90°)V$. Find the single sinusoid that is their sum.

Solution

Draw each as a phasor, measuring angles from the horizontal. The first, v_1, has length 12V along the reference. The second, v_2, leads it by 90°, so its phasor of length 5V points straight up, as the following sketch shows.



The two phasors meet at a right angle, so the resultant is the hypotenuse,

$$V = \sqrt{(12V)^2 + (5V)^2} = 13V$$

at an angle ahead of the first phasor given by:

$$\tan\varphi = \frac{5V}{12V} = 0.417, \qquad \text{so } \varphi = \tan^{-1}0.417 = 22.6°$$

Since both phasors spin together at ω, the resultant spins with them, and its shadow is the sum we were after:

$$v = v_1 + v_2 = 13\sin(\omega t + 22.6°)\ V$$

Making Sense of the Answer: *The sides* 5, 12, 13 *form another whole-number right triangle, a clean check. The sum leads v_1 by only* 22.6°*, less than v_2's full* 90° *lead, because v_1 is the larger arrow and pulls the resultant towards itself.*

Think Like a Physicist: *To add any number of same-frequency sinusoids, drop them all onto one phasor diagram, add the arrows as vectors, and read off the single resultant. Trigonometric identities are for those who have forgotten that they can draw.*

Four examples, and not one sine-addition formula in sight. Kipanga has noticed, with some satisfaction, that the frightening trigonometry of adding waves has quietly been replaced by the drawing of triangles, which he was good at all along.

PURE AC CIRCUITS: RESISTOR (R), INDUCTOR (L), CAPACITOR (C)

We now have the wave, a way to measure it, and the phasor to picture it. Time for the central question of the whole chapter. *When an alternating voltage is applied across each of the three basic components on its own, a resistor, an inductor, and a capacitor, how does the current answer?* Each answers differently, and those three answers are the alphabet from which every a.c. circuit is spelled. We take them one at a time, and for each we ask the same five questions: *is the current in step with the voltage, ahead of it, or behind? How* *strongly does the component oppose the current? Does it consume power? How does its behaviour change with frequency? And what happens at zero frequency, that is, on plain d.c.?* Master these three letters (R, L, C) and you can read any word.

First, the three circuits themselves. Each is the simplest thing imaginable, a single component connected straight across an alternating source:

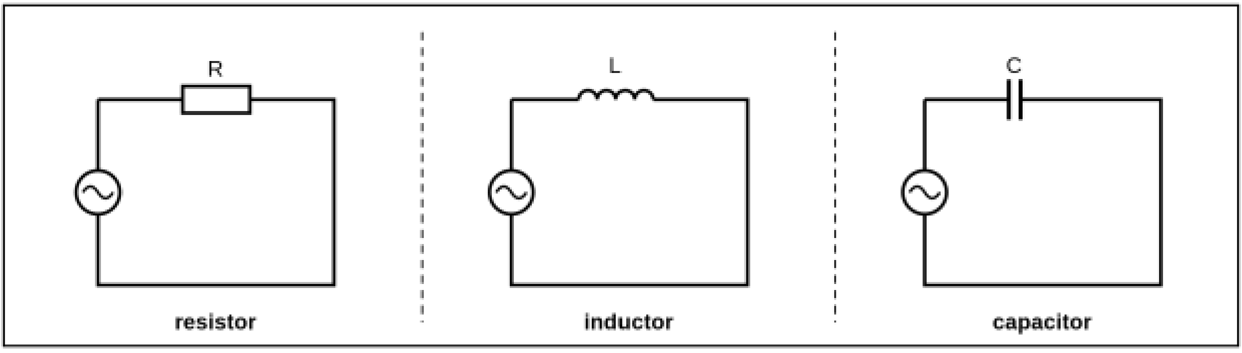


Figure: ***The three pure circuits of this section:*** *a resistor, an inductor, and a capacitor, each connected on its own across an alternating source. We ask the same five questions of each.*

And here, in a single glance, is the answer the section will earn, how the current sits relative to the voltage in each case:

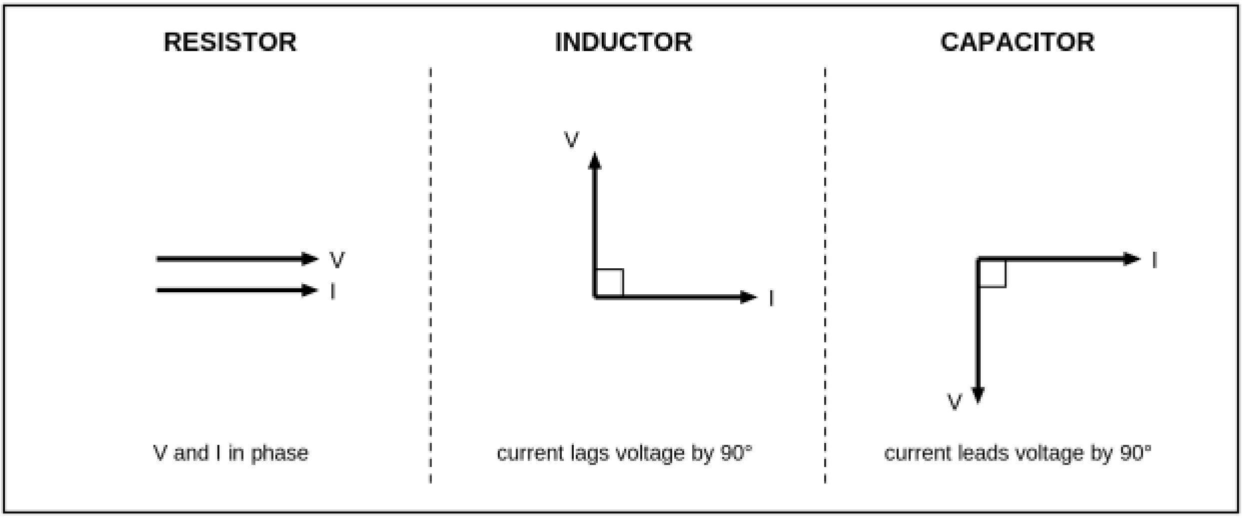


Figure: *The phase relationship between voltage and current in each of the three pure components, drawn as phasor diagrams with the current along the reference. In a resistor the two are in step; in an inductor the current lags the voltage by a quarter-cycle; in a capacitor the current leads the voltage by a quarter-cycle.*

The resistor: the honest baseline

Put an alternating voltage $v = V_0 \sin\omega t$ across a pure resistor and nothing surprising happens. Ohm's law from Chapter 12 still holds, instant by instant, $v = iR$, so the current is $i = \frac{v}{R} = \frac{V_0}{R}\sin\omega t = I_0 \sin\omega t$ with peak $I_0 = \frac{V_0}{R}$.

Run through the five questions.

- The current is exactly **in phase** with the voltage: they rise, peak, and cross zero together.
- The opposition is simply the **resistance R**, in ohms, the very same number as on d.c.; nothing new is needed.
- The resistor does **dissipate power**, and its average value is the one we built in the last section,
$$P_{av} = V_{rms} I_{rms} = I_{rms}^2 R$$
- Frequency makes no difference: R is R at any frequency, and the same at **zero frequency** too.

The resistor is the honest baseline against which the other two will look strange.

The inductor: the current lags behind

A pure inductor is a coil of wire with negligible resistance, and it makes its first appearance here, as a circuit component in its own right. A resistor is defined by Ohm's law, $v = iR$, and a capacitor by $i = C\frac{dv}{dt}$; an inductor is defined, in the same spirit, by its **inductance L**, measured in henries. Its defining rule is the exact mirror of the capacitor's: the voltage across the coil equals L times the rate at which the current through it changes:

$$\mathbf{v = L\frac{di}{dt}}$$

In plain words, *a coil does not care about the current itself, only about how fast that current is changing,* and it sets itself against that change. (Why a coil behaves this way is a matter for the deeper physics of magnetism, which we need not open up here.)

Watch what that rule forces. Suppose the current is $i = I_0 \sin\omega t$. Its rate of change is $\frac{di}{dt} = \omega I_0 \cos\omega t$ (differentiating a sine pulls down a factor of ω and turns it into a cosine, exactly as we saw for the spinning coil). The voltage across the inductor is therefore:

$$\mathbf{v} = L\frac{di}{dt} = \mathbf{\omega L I_0 \cos\omega t}$$

Two results fall out of that one line. First, the voltage follows a cosine while the current follows a sine, and a cosine peaks a quarter-cycle before a sine, so **the current lags the voltage by 90°** (equivalently, the voltage leads the current). Second, the peak voltage is $V_0 = \omega L I_0$, so the ratio of peak voltage to peak current is:

$$\frac{V_0}{I_0} = \frac{\omega L I_0}{I_0} = \omega L$$

That ratio is the heart of the matter. For a resistor the ratio of voltage to current is the resistance R; for a coil it is ωL, and we give it its own name, the **inductive reactance $\mathbf{X_L}$**, measured in ohms exactly like a resistance:

$$X_L = \omega L = 2\pi f L$$

So **reactance is simply** *the a.c. opposition a component offers, the ratio of voltage to current*, and the current obeys an Ohm's-law lookalike,

$\mathbf{I_{rms}} = \frac{\mathbf{V_{rms}}}{\mathbf{X_L}}$ (the ratio is the same whether you use peak or rms values).

Two things, though, make reactance a genuinely new idea rather than a dressed-up resistance: it depends on frequency, and, as we are about to see, it costs no power. The frequency dependence is already in plain view, since ω sits right inside $X_L = \omega L$. Physically, the coil fights the current in proportion to how fast it is changing, and a higher frequency means a faster change at every instant, so the *reactance rises with frequency*. Double the frequency and you double the opposition.

Does it consume power? Not on average. During one quarter-cycle the source pushes the current up and pours energy into the inductor's magnetic field; during the next quarter-cycle the field collapses and hands every joule back to the source. Over a full cycle the **net power is zero**. This is the same bookkeeping as a swing in Chapter 6, where kinetic and potential energy traded back and forth without loss; here the trade is between the source and the magnetic field. Only a resistor actually spends energy.

Finally, zero frequency. On steady d.c. there is no change in the current for the inductor to fight, so $X_L = \omega L = 0$: the inductor becomes an ordinary piece of wire. An inductor passes d.c. freely and chokes a.c., the more harshly the higher the frequency, a fact we shall put to work when we meet the choke in a fluorescent lamp. A memory aid, offered only after the reasoning above: in an inductor the **E**MF leads the current (**I**), **ELI**.

The capacitor: the current leads

Now a pure capacitor. From Chapter 11 the charge it holds is $Q = CV$, and the current is the rate at which that charge flows on and off the plates, $i = C\frac{dv}{dt}$. Read that equation carefully: *the current depends not on the voltage itself but on how fast the voltage is changing.*

When the alternating voltage is changing fastest, as it sweeps through zero, charge floods on and off the plates and the current is greatest. When the voltage sits momentarily still at its peak, no charge moves and the current is zero. So the current reaches its maximum a quarter-cycle **before** the voltage does: **the current leads the voltage by a quarter-cycle, by 90°.** The capacitor is the mirror image of the inductor.

Put it in symbols, exactly as we did for the coil.

Let $v = V_0 \sin\omega t$.

Then the current is $i = C\frac{dv}{dt} = \omega C V_0 \cos\omega t$.

The current is a cosine to the voltage's sine, which confirms that **the current leads the voltage by 90°**; and the peak current is $I_0 = \omega C V_0$, so the ratio of peak voltage to peak current is:

$$\frac{V_0}{I_0} = \frac{V_0}{\omega C V_0} = \frac{1}{\omega C}$$

That ratio is the **capacitive reactance**, the a.c. opposition of a capacitor in ohms:

$$\mathbf{X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}}$$

with $I_{rms} = \frac{V_{rms}}{X_C}$ as before.

Notice the reciprocal, and with it the physical reason *the reactance falls with frequency*: the faster the voltage alternates, the less time the capacitor has to charge up before the polarity flips, so the less it manages to oppose, and the more current it passes. A capacitor finds high frequencies easy and low ones hard.

Power, again, averages to zero: energy stored in the electric field while the capacitor charges is returned in full while it discharges, so over a cycle the capacitor spends nothing. And at zero frequency the reciprocal bites hard: $X_C = \frac{1}{\omega C} = \infty$ when $\omega = 0$. A capacitor **blocks d.c. completely**: once charged, no further steady current flows. This is the very fact that bit Kipanga through the back of an unplugged television in Chapter 11, and the reason the capacitor of that chapter, fed a.c. instead, never settles but passes current cycle after cycle.

The matching memory aid: in a capacitor the current leads the EMF, **ICE**. Put the two together and generations of students have remembered the lot as ***ELI the ICE man***.

Reactance, frequency, and the d.c. limits

Step back and compare. Both X_L and X_C are measured in ohms and both play, for their component, the role resistance plays for a resistor. But unlike resistance they depend on frequency, and in opposite ways, as the graph below shows.

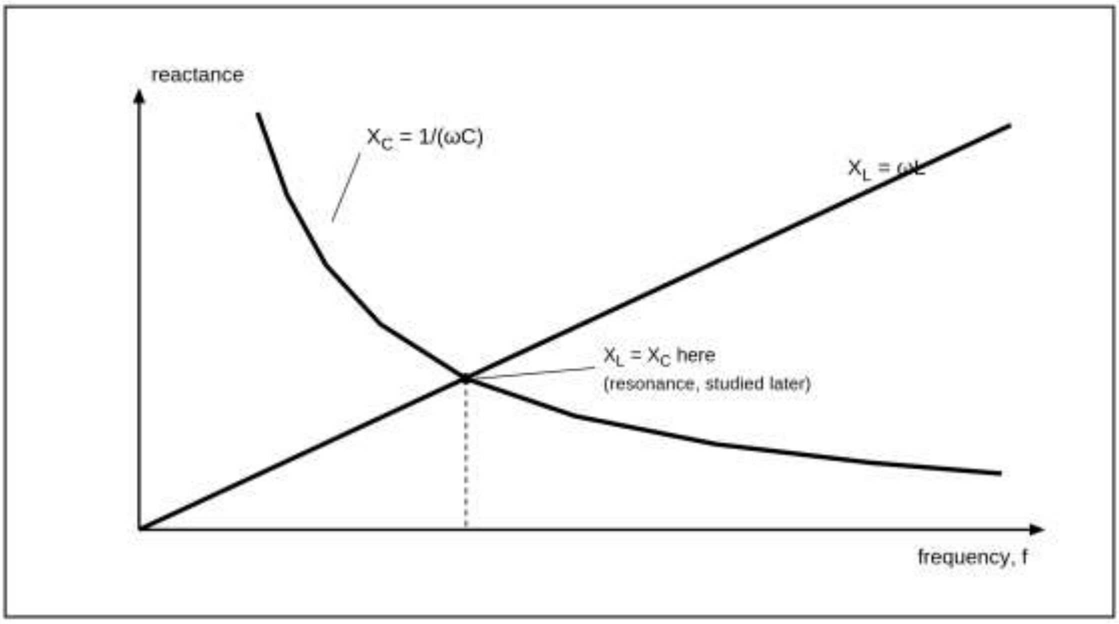


Figure: ***How the two reactances depend on frequency.*** *The inductive reactance grows in proportion to frequency, while the capacitive reactance falls as its reciprocal. At zero frequency (d.c.) the inductor offers no opposition while the capacitor offers infinite opposition; at very high frequency the roles reverse. The one frequency at which the two are equal is the seed of resonance, which we study later.*

Read the two limits straight off. At $f = 0$, the inductive reactance $X_L = \omega L = 0$ (a wire) while the capacitive reactance $X_C = \infty$ (a block); at very high frequency the inductor blocks while the capacitor waves the current through. An inductor and a capacitor are mirror images, and the single frequency where their reactances cross will turn out, in a later section, to be the most useful frequency in the whole circuit.

One last idea to plant. The amount of power a component draws depends on the angle φ between its voltage and its current, through $P_{av} = V_{rms} I_{rms} \cos\varphi$, a formula we develop properly in the power section. For a resistor $\varphi = 0$ and $\cos\varphi = \cos 0° = 1$: all the power is taken. For a pure inductor or capacitor $\varphi = 90°$ and $\cos\varphi = \cos 90° = 0$: none is. The quantity $\cos\varphi$, the **power factor**, will measure exactly how resistor-like or how reactance-like a real circuit behaves.

Five examples to drill the three letters. Kipanga, fresh from his victory over phasors, has promised to object the instant a current does something a current has no business doing. We are counting on him.

BINDER Example 14

A coil of inductance 0.50H and negligible resistance is connected to the 240V, 50Hz mains. Find (a) its inductive reactance and (b) the rms current it draws.

Solution

(a) The inductive reactance is:

$$X_L = \omega L = 2\pi f L = 2\pi \times 50\text{Hz} \times 0.50\text{H} = 157\Omega$$

(b) The rms current follows from the a.c. Ohm's law,

$$I_{rms} = \frac{V_{rms}}{X_L} = \frac{240\text{V}}{157\Omega} = 1.53\text{A}$$

Making Sense of the Answer: *A coil that would be almost a dead short on d.c. now opposes the current with 157Ω. That opposition is purely an a.c. effect, conjured entirely by the changing current.*

Think Like a Physicist: *For an inductor it is the reactance $X_L = \omega L$, not the resistance, that limits the a.c. current. Compute X_L first, then divide.*

BINDER Example 15

A $4.0\mu\text{F}$ capacitor is connected to the same 240V, 50Hz supply. Find (a) its capacitive reactance and (b) the rms current it draws.

Solution

(a) The capacitive reactance is:

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 50\text{Hz} \times 4.0 \times 10^{-6}\text{F}} = 796\Omega$$

(b) The rms current is:

$$I_{rms} = \frac{V_{rms}}{X_C} = \frac{240\text{V}}{796\Omega} = 0.302\text{A}$$

Making Sense of the Answer: *At $50Hz$ this small capacitor opposes the current more stiffly (796Ω) than the coil did (157Ω). Raise the frequency and that opposition would shrink.*

Think Like a Physicist: *Capacitive reactance is $X_C = \frac{1}{\omega C}$; mind the reciprocal. A smaller capacitor, or a lower frequency, means a larger reactance and a smaller current.*

REAL Example 16

Kipanga has been brooding over the last result, and finally he cannot hold it in.

Kipanga: *Sir, this one I refuse to accept. You say that in a capacitor the current leads the voltage, that the current reaches its peak a quarter-cycle before the voltage reaches its own. But the voltage is the cause and* *the current is the effect.* ***How can an effect arrive before its cause? Has the capacitor learned to see into the future?***

Mr. Akilikubwa: *A fair objection, and an honest one. But it hides a hidden assumption: that there was a first moment, a cause that started it all. In steady a.c. there is no such moment. The supply has been swinging back and forth for many cycles, and "leads" is not a story of cause and effect at all. It is a fixed relationship between two quantities that have been dancing together for a long time.*

The key is the capacitor's own law, $i = C\frac{dv}{dt}$*. The current follows not the voltage but the slope of the voltage. The slope of a sine is steepest as the sine crosses zero and vanishes at the sine's peak. So the current is largest exactly when the voltage is changing fastest, and zero when the voltage is momentarily still. Trace those points and the current's peak falls a quarter-cycle ahead of the voltage's peak, not because it foresaw anything, but because the rate of change of a sine is itself a cosine, which simply peaks earlier. The lead is calculus, not prophecy.*

Making Sense of the Answer: *Nothing is caused before its cause. The current at any instant is set by how fast the voltage is changing at that same instant, and that rate happens to peak a quarter-cycle before the voltage does.*

Think Like a Physicist: *When a "lead" or a "lag" feels paranormal, remember that a steady oscillation has no beginning. Phase is a standing relationship between quantities, never a claim that one event triggered another.*

HOT Example 17

A pure inductor of $L = 0.20\text{H}$ is connected across a 12V, 400Hz a.c. supply. Find (a) its reactance, (b) the rms current, and (c) the average power it consumes.

Solution

(a) The inductive reactance at this frequency is:

$$X_L = 2\pi fL = 2\pi \times 400\text{Hz} \times 0.20\text{H} = 503\Omega$$

(b) The rms current is:

$$I_{rms} = \frac{V_{rms}}{X_L} = \frac{12\text{V}}{503\Omega} = 0.0239\text{A} = 23.9\text{mA}$$

(c) The average power, with the phase angle $\varphi = 90°$ for a pure inductor, is:

$$P_{av} = V_{rms}I_{rms}\cos 90° = 0$$

Making Sense of the Answer: *A real current of* $23.9mA$ *flows, yet the inductor consumes no net power. The energy shuttles into the magnetic field and back out again each cycle; the source does work for half of every cycle and is repaid the other half.*

Think Like a Physicist: *Current flowing does not by itself mean power consumed. Only the part of the current in phase with the voltage carries power; a purely reactive current carries none.*

HOT Example 18

A $2.0\mu\text{F}$ capacitor is connected across a 12V supply. Find (a) the rms current when the supply is a.c. at 400Hz, (b) the average power it then consumes, and (c) the current that flows if the 12V supply is instead steady d.c.

Solution

(a) The capacitive reactance and current at 400Hz are:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 400\text{Hz} \times 2.0 \times 10^{-6}\text{F}} = 199\Omega$$

$$I_{rms} = \frac{V_{rms}}{X_C} = \frac{12\text{V}}{199\Omega} = 0.0603\text{A} = 60.3\text{mA}$$

(b) With $\varphi = 90°$ for a pure capacitor, $P_{av} = V_{rms}I_{rms}\cos 90° = 0$: no net power, for the same reason as the inductor, with the electric field now doing the storing.

(c) *On steady d.c. the frequency is zero,* so $X_C = \frac{1}{\omega C} = \infty$.

The capacitor charges up to 12V and then blocks all further flow: the steady current is **zero**.

Making Sense of the Answer: *The same capacitor passes* $60.3mA$ *of a.c. but not a whisker of d.c. The higher the frequency, the more it passes; at zero frequency it passes nothing. It is the exact mirror of the inductor, which passed d.c. freely and blocked the high frequencies.*

Think Like a Physicist: *To find what a capacitor does at any frequency, compute* $X_C = \frac{1}{\omega C}$*. Infinite at* $f = 0$ *(it blocks), small at high* f *(it passes). The reciprocal tells the whole story.*

Five examples, three letters, and one current caught apparently reading the future and then cleared of all charges. Kipanga objected, as promised, and was talked down; he now allows that a capacitor is no prophet, merely an unusually good judge of slopes, which he considers nearly as impressive.

SERIES COMBINATION CIRCUITS: RC, LR, AND RLC

In the last section we met the resistor, the inductor, and the capacitor each living alone. Real circuits string them together, and the simplest string is a **series** one, in which the same current flows through every component. We now learn to handle any series combination, and the phasor we built in the phasor section turns the job into something almost mechanical. The one new quantity we need is **impedance**, the a.c. cousin of resistance.

Impedance: the a.c. cousin of resistance

When resistors and reactances act together, *the total opposition the circuit offers to the alternating current* is called the **impedance**, written **Z** and measured in ohms. Like resistance, it is the ratio of voltage to current,

$$\mathbf{Z} = \frac{\mathbf{V_{rms}}}{\mathbf{I_{rms}}}$$

so the a.c. Ohm's law reads $V = IZ$. But impedance is more than a resistance. A **resistance** simply *dissipates*; a **reactance** simply *stores and returns*; **impedance** is the *combined effect of both*, and it *carries a phase angle as well as a size.* The crucial rule is that resistance and reactance are never added straight, because on the phasor diagram they sit at right angles to each other. They combine by Pythagoras, and that single fact,

$$Z = \sqrt{R^2 + X^2},$$

is the whole secret of every series circuit in this section.

A method that never changes

Before the particular circuits, here is the recipe they all obey. Learn it once and every series circuit, however it is dressed, yields to the same five steps.

Step 1: *Identify the components and the source: the rms (or peak) voltage, and the frequency.*

Step 2: *Find the reactances at that frequency,* $X_L = \omega L$ *and* $X_C = \frac{1}{\omega C}$.

Step 3: *Draw the phasor diagram with the current along the reference, build the impedance triangle, and read off* $Z = \sqrt{R^2 + (X_L - X_C)^2}$.

Step 4: *Apply the a.c. Ohm's law* $V_{rms} = I_{rms}Z$ *to find the unknown current or voltage.*

Step 5: *Find the phase angle from* $tan\varphi = \frac{X_L - X_C}{R}$*, and state whether the current leads or lags.*

The three series circuits we shall study are drawn below. In each, the same current threads through every component in turn, driven round the loop by an alternating source.

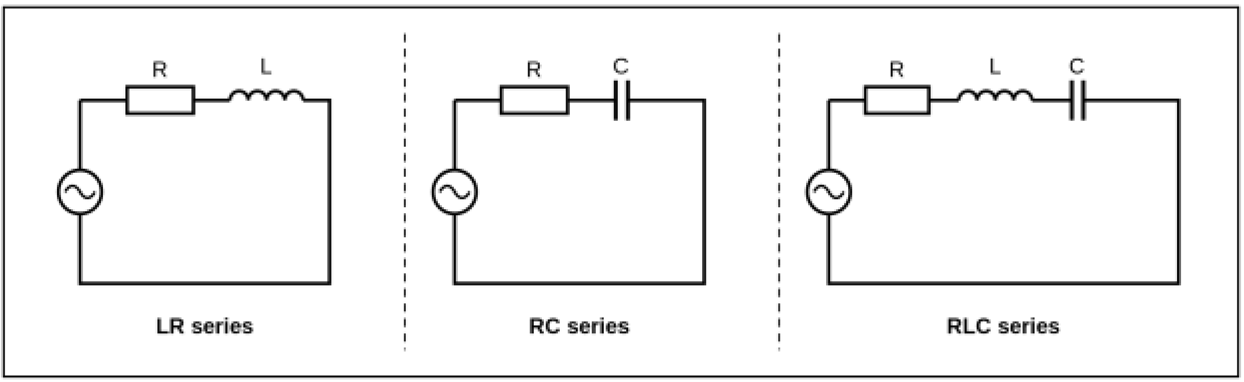


Figure: ***The three series circuits of this section:*** *a resistor with an inductor, a resistor with a capacitor, and all three together. In every case the same current flows through each component in turn, driven by an alternating source.*

The LR series circuit

Take a resistor and an inductor in series. The same current I flows through both, so we draw I along the reference. The resistor's voltage $V_R = IR$ is in phase with the current and lies along it; the inductor's voltage $V_L = IX_L$ leads the current by 90° and points straight up. The supply voltage is their phasor sum, the hypotenuse of the right-angled triangle in the figure below.

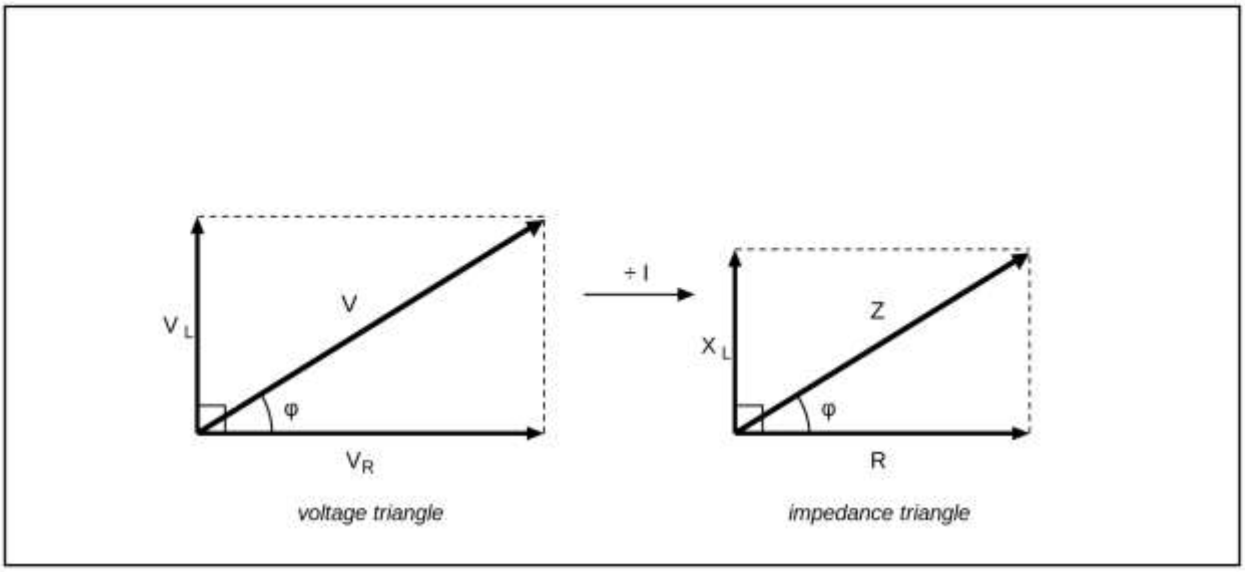


Figure: ***Adding the two voltages in an LR series circuit.*** *The resistor voltage is in phase with the current and the inductor voltage leads it by 90°, so the supply voltage V is the hypotenuse. Dividing every side by the common current I turns the voltage triangle into the impedance triangle, giving Z and the phase angle φ.*

Reading the triangle, the supply voltage is:

$$V = \sqrt{V_R^2 + V_L^2} = I\sqrt{R^2 + X_L^2}$$

Dividing through by the current gives the impedance,

$$\frac{V}{I} = Z = \sqrt{R^2 + X_L^2}$$

and the angle by which the supply voltage leads the current is given by:

$$\tan\varphi = \frac{X_L}{R} \text{ or } \varphi = \tan^{-1}\left(\frac{X_L}{R}\right)$$

The current **lags** the supply voltage; that is the mark of an inductive circuit.

This is exactly the circuit hiding inside a fluorescent tube. The tube of Chapter 12 will not run straight off the mains: left to itself the gas discharge draws more and more current until it destroys the tube. So a large inductor, the **choke** or ballast, is wired in series with it, making an LR circuit. Once the tube lights, the choke's reactance $X_L = 2\pi fL$ quietly limits the current to a safe value, dissipating almost no power of its own (a resistor large enough to do the same job would waste energy as heat). Every fluorescent tube humming in the Miono school corridor relies on this series inductor, or, in newer fittings, on an electronic circuit that imitates it.

The RC series circuit

A resistor and a capacitor in series is the mirror image. The same current flows; $V_R = IR$ lies along it, but the capacitor's voltage $V_C = IX_C$ now **lags** the current by 90°, so its phasor points straight down. The triangle is simply the LR one flipped below the current line, and the algebra follows at once:

$$Z = \sqrt{R^2 + X_C^2}, \qquad \tan\varphi = \frac{X_C}{R}$$

The only change of substance is the direction of the lean. Because the reactive voltage now points downward, the supply voltage lags the current, or, said the other way, **the current leads the supply voltage**. An inductor makes the current lag; a capacitor makes it lead.

The RLC series circuit

Now put all three in series. The same current flows through everything, so once more we draw it along the reference. The resistor's voltage lies along the current, the inductor's points up, and the capacitor's points down. Since V_L and V_C point in opposite directions, they partly cancel: the net reactive voltage is their difference $V_L - V_C$, as the figure below shows.

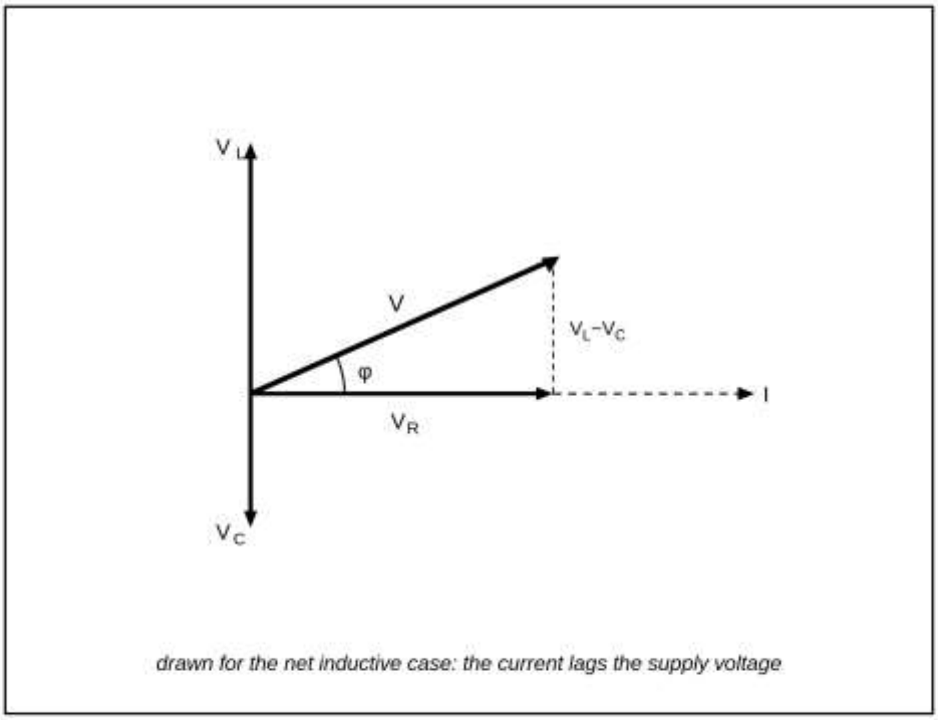


Figure: ***The phasor diagram of a series RLC circuit.*** *The inductor and capacitor voltages oppose each other, so only their difference survives, and the supply voltage V is the hypotenuse built on the resistor voltage and that net reactive voltage.*

The supply voltage is therefore the hypotenuse built on V_R and $(V_L - V_C)$, and dividing by the current gives the impedance and the phase angle:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}, \qquad \tan\varphi = \frac{X_L - X_C}{R}$$

Everything depends on which reactance wins.

- When $X_L > X_C$, the circuit is net **inductive** and the current lags, behaving like an LR circuit.

- When $X_L < X_C$, it is net **capacitive** and the current leads, behaving like an RC circuit.

And when $X_L = X_C$, the two reactances cancel exactly, the impedance shrinks to its smallest possible value $Z = R$, and the current comes into step with the voltage. That last, special case is **resonance**, and it is important enough to have its own section later in the chapter.

Power factor

The phase angle φ measures how reactive a circuit is, and one reading of the impedance triangle captures it in a single number. Since R is the side adjacent to φ and Z is the hypotenuse,

$$\cos\varphi = \frac{R}{Z}$$

This ratio is the **power factor** we first named among the pure components. It runs from 1 for a purely resistive circuit ($\varphi = 0$, all the power is taken) down to 0 for a purely reactive one ($\varphi = 90°$, none is).

$$\mathbf{0 \le power\ factor \le 1}$$

For any series circuit, $\cos\varphi = \frac{R}{Z}$, and we shall see in the power section how directly it controls the bill a workshop pays.

Five examples to put the five steps to work. Kipanga, who has come to trust triangles, is now positively eager, and has begun drawing phasor diagrams.

BINDER Example 19

In an LR series circuit the resistance is 30Ω and the inductive reactance at the working frequency is 40Ω. Find: (a) the impedance and (b) the angle by which the current lags the supply voltage.

Solution

(a) Resistance and reactance combine by Pythagoras:

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{(30\Omega)^2 + (40\Omega)^2} = 50\Omega$$

(b) The phase angle is

$$\tan\varphi = \frac{X_L}{R} = \frac{40\Omega}{30\Omega} = 1.33, \quad \varphi = \tan^{-1}1.33 = 53.1°$$

Making Sense of the Answer: *The sides* 30, 40, 50 *are the* 3-4-5 *triangle again. The impedance,* 50Ω*, is larger than either the resistance or the reactance alone, exactly as a hypotenuse must be.*

Think Like a Physicist: *Never add R and* X_L *straight; they are at right angles. Combine them with Pythagoras and read the phase from the arctangent of the ratio.*

REAL Example 20

The fluorescent tubes in the Miono school corridor each have a small, heavy component wired in series with the tube, called a choke. A new technician, replacing a dead tube, wonders whether he can save money by leaving the choke out, since "it is only a coil of wire and does nothing useful." Explain what the choke does and what would happen to the tube without it.

Solution

The choke is a large inductor, and the tube together with it forms an LR series circuit. The reason it is essential lies in the strange behaviour of the gas inside the tube: once the gas has been made to conduct, its resistance falls as more current flows, so a tube connected straight to the mains would draw an ever-larger current, overheat, and quickly destroy itself. The choke prevents this runaway. Its reactance $X_L = 2\pi fL$ rises with the current's demands and limits the current to a safe steady value, holding the tube at its proper operating point.

Why not use a plain resistor to limit the current instead? Because a resistor would dissipate a great deal of power as wasted heat, whereas the choke, being almost pure reactance, limits the current while consuming almost no power of its own. The technician's "useless coil" is in fact what keeps the tube alive and the electricity bill low. Leave it out and the new tube will flare brilliantly for a moment and then burn out.

Making Sense of the Answer: *This is the zero-power property of a pure inductor put to practical use: the choke controls the current without paying for it in heat, which a resistor never could.*

Think Like a Physicist: *A reactance is the engineer's tool of choice whenever a current must be limited without wasting energy. Resistors limit and waste; reactances limit and store.*

HOT Example 21

A $4.0\mu F$ capacitor is connected in series with an 800Ω lamp across a $9.0V$, $60Hz$ a.c. supply. Find (a) the capacitive reactance, (b) the impedance, (c) the rms current, (d) the voltage across the capacitor and across the lamp, (e) the phase angle, and (f) the average power dissipated.

Solution

(a) The capacitive reactance is:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 60Hz \times 4.0 \times 10^{-6}F} = 663\Omega$$

(b) The lamp behaves as a resistor, so the impedance is:

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{(800\Omega)^2 + (663\Omega)^2} = 1039\Omega$$

(c) The rms current follows from the a.c. Ohm's law:

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{9.0V}{1039\Omega} = 8.66mA$$

(d) The two component voltages are:

$$V_C = I_{rms}X_C = 8.66mA \times 663\Omega = 5.74V$$

$$V_R = I_{rms}R = 8.66mA \times 800\Omega = 6.93V$$

(e) The phase angle is:

$$\tan\varphi = \frac{X_C}{R} = \frac{663\Omega}{800\Omega} = 0.829, \quad \varphi = \tan^{-1}0.829 = 39.6°$$

with the current leading the supply voltage.

(f) The average power is dissipated only in the lamp:

$$P_{av} = I_{rms}^2 R = (8.66mA)^2 \times 800\Omega = 0.060W = 60mW$$

Making Sense of the Answer: *A quick check:* $\sqrt{V_R^2 + V_C^2} = \sqrt{(6.93V)^2 + (5.74V)^2} = 9.0V$, *the supply voltage, as the phasor triangle demands. The capacitor and lamp voltages do not add to* $9.0V$ *straight, because they are a quarter-cycle apart.*

Think Like a Physicist: *The component voltages in a series a.c. circuit add as phasors, not as numbers. If* V_R *and* V_C *ever seem to "add up to more than the supply", remember they are at right angles, and Pythagoras restores order.*

HOT Example 22

A resistor $R = 200\Omega$, an inductor $L = 2.0H$, and a capacitor $C = 10\mu F$ are connected in series across a $240V$, $50Hz$ supply. Find the two reactances, the impedance, the rms current, the phase angle, and the average power.

Solution

The two reactances at $50Hz$ are:

$$X_L = 2\pi fL = 2\pi \times 50Hz \times 2.0H = 628\Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50Hz \times 10 \times 10^{-6}F} = 318\Omega$$

The net reactance is $X_L - X_C = 628\Omega - 318\Omega = 310\Omega$, inductive (since $X_L > X_C$). The impedance is:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(200\Omega)^2 + (310\Omega)^2} = 369\Omega$$

The rms current and phase angle are then:

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{240V}{369\Omega} = 0.650A$$

$$\tan\varphi = \frac{X_L - X_C}{R} = \frac{310\Omega}{200\Omega} = 1.55, \quad \varphi = \tan^{-1}1.55 = 57.2°$$

and, since the circuit is net inductive, the current lags the supply voltage by $57.2°$.

The average power is dissipated only in the resistor:

$$P_{av} = I_{rms}^2 R = (0.650A)^2 \times 200\Omega = 84.5W$$

Making Sense of the Answer: *The inductor's 628Ω and the capacitor's 318Ω fight each other, and only the 310Ω that survives joins the resistance. Had the two reactances been equal, the impedance would have collapsed to the bare 200Ω of the resistor.*

Think Like a Physicist: *In an RLC circuit always subtract the reactances first, $X_L - X_C$, before reaching for Pythagoras. The sign of that difference tells you at a glance whether the current will lead or lag.*

HOT Example 23

A real coil possesses both resistance and inductance. It draws $0.50A$ from a $12V$ d.c. supply, and $0.24A$ from a $12V$, $50Hz$ a.c. supply. Find: (a) the coil's resistance, (b) its impedance on a.c., (c) its inductive reactance, and (d) its inductance.

Solution

(a) On d.c. the frequency is zero, so the reactance $X_L = \omega L = 0$ and only the resistance opposes the current. Hence:

$$R = \frac{12V}{I_{dc}} = \frac{12V}{0.50A} = 24\Omega$$

(b) On a.c. the full impedance opposes the current:

$$Z = \frac{12V}{I_{ac}} = \frac{12V}{0.24A} = 50\Omega$$

(c) Since $Z = \sqrt{R^2 + X_L^2}$, the reactance is:

$$X_L = \sqrt{Z^2 - R^2} = \sqrt{(50\Omega)^2 - (24\Omega)^2} = 43.9\Omega$$

(d) Finally the inductance, from $X_L = 2\pi fL$:

$$L = \frac{X_L}{2\pi f} = \frac{43.9\Omega}{2\pi \times 50Hz} = 0.14H$$

Making Sense of the Answer: *The trick is that d.c. sees only R while a.c. sees the whole of Z. Two simple measurements, one battery and one a.c. source, are enough to separate a coil's resistance from its inductance.*

Think Like a Physicist: *Whenever a component is tested on both d.c. and a.c., let the d.c. reading hand you R and the a.c. reading hand you Z; the reactance is then $\sqrt{Z^2 - R^2}$.*

Five circuits, one method, and not a moment's panic. Kipanga has noticed that every example ended in the same triangle, and has concluded, not unreasonably, that the whole of a.c. is Pythagoras wearing a disguise.

PARALLEL COMBINATION CIRCUITS: LR, CR, AND RLC

The series circuits of the last section made the components stand in single file, so that one and the same current had to pass through every one of them in turn. A **parallel** circuit does the opposite. It connects each component straight across the supply, between the very same two points, so that now it is the **voltage** that is shared by all of them and the current that is free to split. Imagine a daladala stand in Dar es Salaam: in a series circuit every passenger files aboard through one door, one after another; in a parallel circuit each climbs in by a door of its own, and only the conductor, counting heads, knows the total. Everything we learned about phasors still holds, but two of the rules quietly trade places.

In a series circuit the current was common and the voltages added as phasors. In a parallel circuit the **voltage is common** and it is the **currents** that add as phasors. So this time we draw the voltage as the reference phasor, work out the current in each branch on its own, and add those branch currents head to tail to find the total current drawn from the supply. That total is called the **line current**.

Here are a resistor, an inductor, and a capacitor all hung in parallel across one a.c. supply.

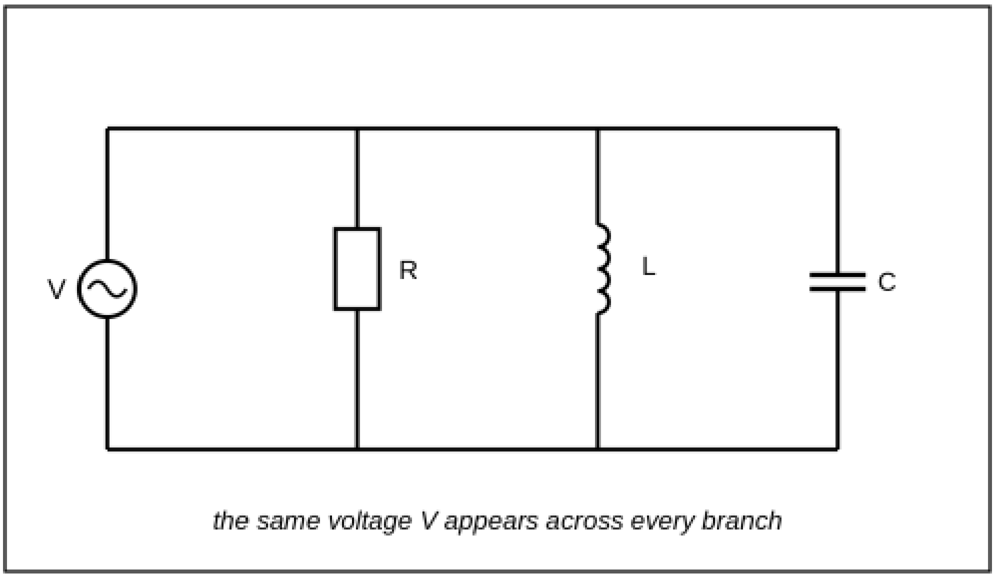


Figure: ***A resistor, an inductor, and a capacitor connected in parallel across one a.c. supply.*** *The same voltage drives all three branches, and the current the supply delivers is the sum of the three branch currents.*

One voltage, three branch currents

Because the same voltage sits across every branch, we may treat each branch entirely on its own, exactly as in the pure-circuit section, and simply read off its current. Let V be the rms supply voltage. Then:

$$I_R = \frac{V}{R}, \quad I_L = \frac{V}{X_L}, \quad I_C = \frac{V}{X_C}$$

and each branch current keeps the phase rule we proved for it earlier: the resistor's current stays **in step** with the voltage, the inductor's **lags** a quarter cycle behind, and the capacitor's **leads** a quarter cycle ahead. By Kirchhoff's current law from Chapter 12, the current the supply delivers is the sum of these three branch currents, but because they are out of step with one another we must add them as **phasors**, not as plain numbers.

Adding the branch currents

Lay the supply voltage V along the reference direction. The resistor current I_R lies right along it. The inductor current I_L hangs a quarter turn behind, drawn downwards, and the capacitor current I_C stands a quarter turn ahead, drawn upwards. The two reactive currents point in exactly opposite directions, so they fight each other, and only their difference survives.

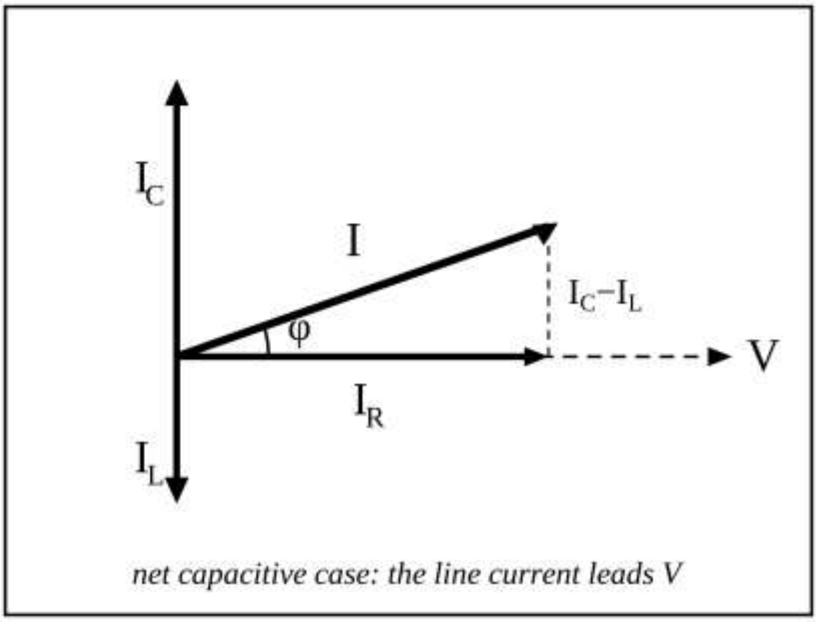


Figure: ***The branch currents drawn as phasors, with the common voltage along the reference.*** *The resistor current lies along the voltage, the capacitor current leads by a quarter cycle, and the inductor current lags by a quarter cycle. Their head-to-tail sum is the line current.*

Adding them head to tail gives a right-angled triangle once more, this time with the resistor current along the base and the net reactive current up the side. The line current is therefore:

$$I = \sqrt{I_R^2 + (I_C - I_L)^2}$$

and it makes an angle φ with the supply voltage given by:

$$\tan\varphi = \frac{I_C - I_L}{I_R},$$

leading if I_C is the larger (net capacitive) and lagging if I_L is the larger (net inductive).

Notice how the picture has turned inside out compared with the series circuit. There it was V_L that pointed up and V_C down; here it is I_C that points up and I_L down. Voltage and current have swapped their geometry, because in going from series to parallel the reactance and its reciprocal have swapped their roles.

Impedance, and a shortcut called admittance

The impedance of the whole combination is still defined the old way, as the ratio of the supply voltage to the current it drives:

$$Z = \frac{V}{I}$$

So once the line current is known the impedance follows in a single step. For parallel work there is also a neater quantity waiting in the wings. *The reciprocal of impedance* is called the **admittance**, written **Y** and measured in **siemens** (the symbol is **S**):

$$\mathbf{Y} = \frac{\mathbf{1}}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}$$

Admittance measures how **easily** current flows rather than how strongly it is opposed, and in a parallel circuit the admittances of the separate branches simply add, which is often the quickest route of all. We shall keep impedance as our main tool and reach for admittance only when it plainly saves work.

A current that almost disappears

Look once more at the phasor triangle and try a thought experiment. The inductor current points down, the capacitor current points up. Suppose we tune the frequency until the two reactances become equal, $X_L = X_C$. Then the two branch currents are equal as well, $I_L = I_C$, and being equal and opposite they cancel completely.

The net reactive current vanishes, the line current shrinks to the bare resistor current, and the impedance $Z = \frac{V}{I}$ swells towards its **largest** value.

This is the exact opposite of the series circuit, where equal reactances made the impedance collapse to a minimum and the current roar to a maximum. A series circuit at that special frequency is greedy for current; a parallel one is miserly with it. That contrast is the doorway to **resonance**, and we step through it in the next section.

Handling any parallel circuit, in five steps

Step 1: *Draw it, mark the shared voltage. The same voltage V sits across every branch; that is your reference phasor.*

Step 2: *Reactances, then branch currents. Find $X_L = \omega L$ and $X_C = \frac{1}{\omega C}$, then $I_R = \frac{V}{R}$, $I_L = \frac{V}{X_L}$, $I_C = \frac{V}{X_C}$.*

Step 3: *Place the phasors. I_R along V; I_L a quarter turn behind; I_C a quarter turn ahead.*

Step 4: *Combine. Net reactive current $I_C - I_L$, then $I = \sqrt{I_R^2 + (I_C - I_L)^2}$.*

Step 5: *Angle and impedance. $tan\varphi = \frac{I_C - I_L}{I_R}$; then $Z = \frac{V}{I}$ if it is wanted.*

Time to send some current down these branches. Kipute has been promised that parallel circuits are merely "series circuits standing on their heads," and is braced for the worst. Four circuits lie ahead, one shared voltage apiece, and at least one supply current that comes out **smaller** than the branch currents it is feeding. Mr. Akilikubwa calls that last one "the miracle of the loaves and the currents" and refuses to explain the trick until the arithmetic has been done.

BINDER Example 24

A resistor of 60Ω and a pure inductor of reactance 80Ω are connected in parallel across a $240V$ a.c. supply. Find: (a) the current in each branch, (b) the line current drawn from the supply, and (c) the phase angle between the line current and the supply voltage.

Solution

(a) The supply voltage sits across both branches. The resistor current is in phase with it, the inductor current a quarter cycle behind:

$$I_R = \frac{240V}{60\Omega} = 4A, \quad I_L = \frac{240V}{80\Omega} = 3A$$

(b) The two branch currents are $90°$ apart, so they add by Pythagoras:

$$I = \sqrt{I_R^2 + I_L^2} = \sqrt{(4A)^2 + (3A)^2} = 5A$$

(c) The phase angle follows from

$$\tan\varphi = \frac{I_L}{I_R} = \frac{3A}{4A} = 0.75, \quad \varphi = \tan^{-1}0.75 = 36.9°$$

and, the only reactive branch being the inductor, the line current **lags** the supply voltage by $36.9°$.

Making Sense of the Answer: *The familiar 3, 4, 5 triangle, but built this time from currents rather than voltages. The line current ($5A$) is larger than either branch current, because the resistor and the inductor each take their share and the supply must feed both at once.*

Think Like a Physicist: *In a parallel circuit you add currents; in a series circuit you add voltages. Same Pythagoras, different quantity. The first question to ask of any circuit is simply: which quantity is the shared one?*

BINDER Example 25

A resistor of 40Ω and a capacitor of reactance 30Ω are joined in parallel across a $120V$ a.c. supply. Find the line current and its phase relative to the supply voltage.

Solution

The branch currents are

$$I_R = \frac{120V}{40\Omega} = 3A, \quad I_C = \frac{120V}{30\Omega} = 4A$$

The capacitor current leads by $90°$, so it stands at right angles to the resistor current. The line current is:

$$I = \sqrt{I_R^2 + I_C^2} = \sqrt{(3A)^2 + (4A)^2} = 5A$$

and the phase angle is:

$$\tan\varphi = \frac{I_C}{I_R} = \frac{4A}{3A} = 1.33, \quad \varphi = \tan^{-1}1.33 = 53.1°$$

This time the reactive branch is capacitive, so the line current **leads** the supply voltage by $53.1°$.

Making Sense of the Answer: *The mirror image of the previous example. Swap the inductor for a capacitor and the only thing that changes is the direction of the lean: the line current now runs ahead of the voltage instead of behind it.*

Think Like a Physicist: *Lead or lag is settled by which reactive branch is present. Capacitor in the lead, inductor in the rear. The sizes follow the same triangle either way.*

HOT Example 26

An inductor of reactance 40Ω and a capacitor of reactance 50Ω, each of negligible resistance, are connected in parallel across a $200V$ supply. Find the current in each branch and the current drawn from the supply, and comment on the result.

Solution

Each branch carries:

$$I_L = \frac{200V}{40\Omega} = 5A, \quad I_C = \frac{200V}{50\Omega} = 4A$$

The inductor current lags the voltage by $90°$ and the capacitor current leads it by $90°$, so the two are a full $180°$ apart, in direct opposition. With no resistor branch there is no in-phase current at all, and the line current is simply their difference:

$$I = I_L - I_C = 5A - 4A = 1A$$

Because the inductor current is the larger, the surviving line current lags the supply voltage by $90°$. The impedance of the combination is:

$$Z = \frac{V}{I} = \frac{200V}{1A} = 200\Omega$$

Comment: The supply delivers a mere $1A$, and yet $5A$ surges through the inductor and $4A$ through the capacitor.

Making Sense of the Answer: *The branch currents dwarf the line current. A hidden $4A$ simply sloshes back and forth between capacitor and inductor every cycle, the capacitor handing its stored charge to the coil and the coil handing it straight back, while the supply tops up only the $1A$ of difference. The closer the two reactances, the smaller the line current and the larger the impedance, heading towards the parallel resonance of the next section.*

Think Like a Physicist: *When two reactive branches face each other with no resistor between them, subtract the currents, do not Pythagoras them; they are $180°$ apart, not $90°$. A parallel inductor and capacitor can together oppose the supply far more stubbornly than either does alone.*

HOT Example 27

A resistor of 100Ω, an inductor of $0.3H$, and a capacitor of $20\mu F$ are connected in parallel across a $240V$, $50Hz$ supply. Find (a) the three branch currents, (b) the line current, (c) the phase angle, and (d) the impedance of the combination.

Solution

(a) First the two reactances, then the three branch currents. The angular frequency is:

$$\omega = 2\pi f = 2\pi \times 50\text{Hz} = 314\text{rad/s}$$

It follows that:

$$X_L = \omega L = 314\text{rad/s} \times 0.3\text{H} = 94.2\Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{314\text{rad/s} \times 20 \times 10^{-6}\text{F}} = 159\Omega$$

So the branch currents are:

$$I_R = \frac{240\text{V}}{100\Omega} = 2.40\text{A}, \quad I_L = \frac{240\text{V}}{94.2\Omega} = 2.55\text{A}, \quad I_C = \frac{240\text{V}}{159\Omega} = 1.51\text{A}$$

(b) The resistor current is in phase; the reactive currents oppose, leaving a net reactive current $I_C - I_L = 1.51\text{A} - 2.55\text{A} = -1.04\text{A}$ (negative, hence net inductive). The line current is:

$$I = \sqrt{{I_R}^2 + (I_C - I_L)^2} = \sqrt{(2.40\text{A})^2 + (1.04\text{A})^2} = 2.62\text{A}$$

(c) The phase angle:

$$\tan\varphi = \frac{I_C - I_L}{I_R} = \frac{-1.04\text{A}}{2.40\text{A}} = -0.433, \quad \varphi = 23.4°$$

The minus sign tells us the line current **lags** the supply voltage by $23.4°$, the inductive branch having won.

(d) The impedance of the whole combination is:

$$Z = \frac{V}{I} = \frac{240\text{V}}{2.62\text{A}} = 91.6\Omega$$

Making Sense of the Answer: *The inductor draws the most current* $(2.55A)$ *because at* $50Hz$ *its reactance is the smallest; the capacitor draws the least. Their tug of war leaves the circuit slightly inductive, so the line current lags, exactly as the negative net reactive current warned.*

Think Like a Physicist: *In a parallel RLC, always test the sign of* $I_C - I_L$*. Positive means net capacitive and a leading line current; negative means net inductive and a lagging one. The resistor current sets the in-phase backbone that the reactive difference leans against.*

Four circuits dispatched, and one supply caught feeding a single amp into a pair of branches happily trading five and four amps between themselves. Kipanga, who began the section certain that the total must always be the biggest number in sight, has gone very quiet. Kipute, sensing weakness, has taken to calling the parallel inductor-and-capacitor "the circuit that prints its own current," which is not quite right, but is the sort of wrong that means he has understood it.

RESONANCE: THE CIRCUIT THAT CHOOSES A FREQUENCY

All through the series section we kept subtracting one reactance from the other, $X_L - X_C$, and all through the parallel section we watched the two reactive currents fight. A restless student eventually asks the obvious question: *what if the two were exactly equal? What if the tug of war ended in a perfect draw?* The answer turns out to be the single most useful idea in the whole of a.c. theory. It is the reason a radio can pluck one station out of the crowded air and ignore all the rest, and it goes by the name **resonance**.

You have in fact met resonance before, in a quieter form. A child on a swing, pushed gently but in time with the swing's own rhythm, rises higher and higher; pushed out of time, she barely moves. Every object that can oscillate has a natural frequency it "prefers," and when you drive it at that frequency the response grows large. We studied this for mechanical oscillators back in the chapter on simple harmonic motion. A circuit holding both an inductor and a capacitor has exactly such a natural frequency, and shows exactly the same dramatic response when driven at it.

The frequency where the reactances cancel

In a series RLC circuit the impedance is $Z = \sqrt{R^2 + (X_L - X_C)^2}$. The resistance is fixed, but the two reactances change with frequency in opposite ways: $X_L = \omega L$ grows as the frequency rises, while $X_C = \frac{1}{\omega C}$ shrinks. At low frequency the capacitor dominates; at high frequency the inductor dominates. Somewhere in between there must be one frequency at which the two are exactly equal and cancel. There the bracket vanishes, the impedance falls to its smallest possible value, just R, and the circuit behaves as though the inductor and capacitor were not present at all.

To find that frequency, set the two reactances equal:

$$\omega L = \frac{1}{\omega C}$$

Multiplying both sides by ω and dividing by L gives:

$$\omega^2 = \frac{1}{LC}, \quad \text{so} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

and since $\omega = 2\pi f$, the **resonant frequency** is:

$$\mathbf{f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

This is the circuit's own natural frequency, fixed entirely by L and C; the resistance has no say in it at all. *The frequency at which a circuit's reactances cancel and its response is greatest* is called its **resonant frequency**, and the circuit is said to be in **resonance**.

What happens at resonance in a series circuit

Imagine the series circuit driven exactly at f_0.

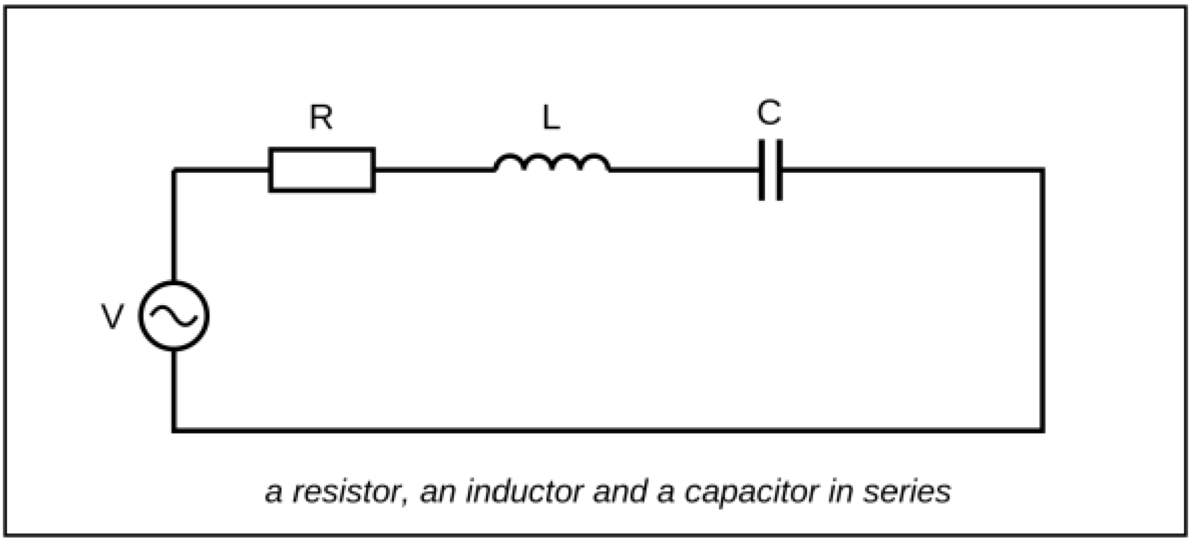


Figure: ***A resistor, an inductor and a capacitor in series across an alternating supply.*** *When the supply frequency equals the resonant frequency, the inductor's and capacitor's reactances cancel and the circuit offers only its resistance.*

Four things happen together, and each is worth stating plainly.

First, the impedance is at its minimum, $Z = R$, purely resistive.

Second, the current is at its maximum, $I = \frac{V}{R}$, limited only by the resistance.

Third, the current is exactly in step with the supply voltage; the phase angle is zero, and (as the power section will show) the circuit then draws energy as efficiently as it possibly can.

Fourth, and most surprising of all, the voltages across the inductor and the capacitor are equal in size and opposite in direction, so they cancel within the loop, and yet each one on its own can be many times larger than the supply voltage itself.

If we plot the current against the supply frequency, the result is the famous **resonance curve**: a peak that climbs to its maximum at f_0 and falls away on either side. The impedance does the mirror image, dipping to its minimum value R at f_0 and rising on both sides. A lamp placed in the circuit would glow brightest exactly at resonance and dim away as the frequency moved off it in either direction.

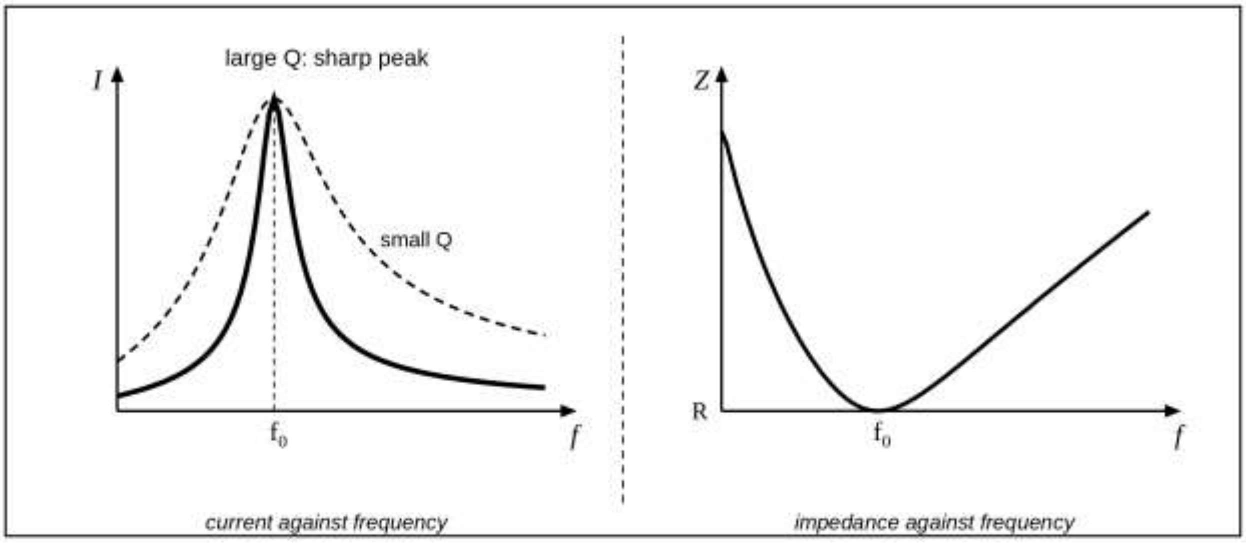


Figure: ***Left: the current rises to a sharp peak at the resonant frequency.*** *A circuit with a large Q gives a tall, narrow peak (solid); a small Q gives a low, broad one (dashed). Right: the impedance dips to its minimum value R at the same frequency.*

Sharpness, the Q-factor, and voltage magnification

How sharp is the peak? That is measured by the **quality factor**, or **Q-factor**, of the circuit:

$$\mathbf{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

A large Q means a tall, narrow peak: the circuit responds strongly, but only to a very narrow band of frequencies around f_0. A small Q gives a low, broad peak. The width of the peak is called the **bandwidth**, and it is:

$$\mathbf{\Delta f = \frac{f_0}{Q}}$$

The Q-factor carries a second meaning, just as important. At resonance the voltage across the inductor (and the equal, opposite voltage across the capacitor) is Q times the supply voltage:

$$V_L = V_C = QV$$

So a circuit with $Q = 100$ driven by a $1V$ supply develops a full $100V$ across its capacitor. This **voltage magnification** is real, occasionally dangerous, and exactly what makes a tuned circuit so sensitive.

This sharpness is the secret of tuning. The aerial of a radio picks up the signals of every station at once. Behind it sits a circuit whose capacitor can be varied; turning the tuning knob changes C and so changes f_0. When f_0 is brought to match the frequency of one station, that station drives the circuit at resonance and is magnified, while all the others, off resonance, are passed over. The higher the Q, the more sharply the set tells one station from its neighbour.

Parallel resonance: the rejector circuit

In the parallel section we noticed something and promised to come back to it. When an inductor and a capacitor are placed in parallel and the frequency is tuned so that $X_L = X_C$, the two branch currents become equal and opposite and cancel, the line current drops almost to nothing, and the impedance of the combination soars to a maximum. That is **parallel resonance**, and the inductor-capacitor pair is called a **tank circuit**, or a **rejector**.

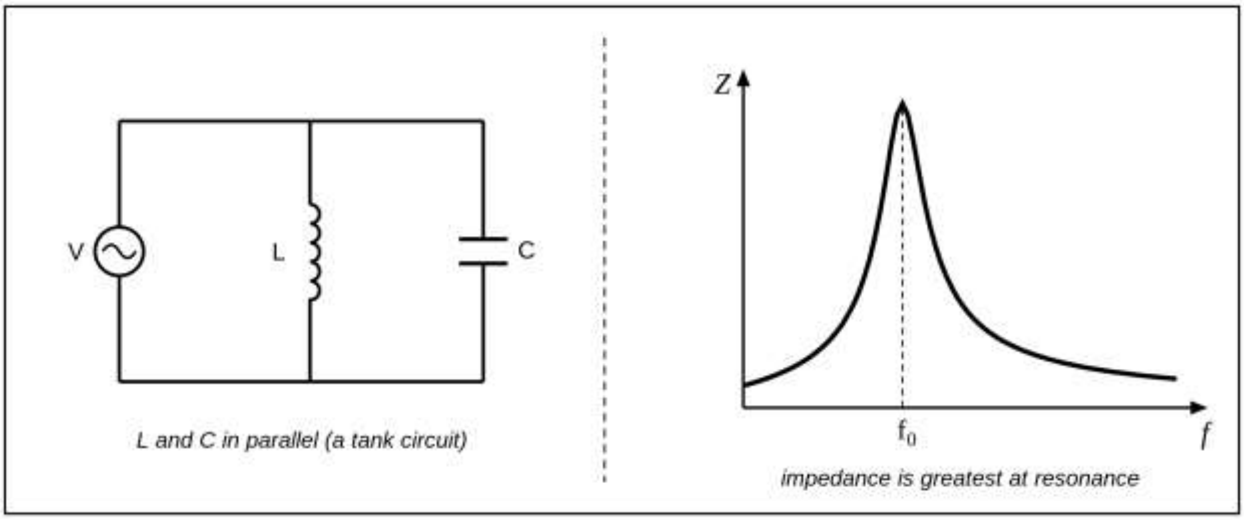


Figure: ***An inductor and a capacitor in parallel form a tank circuit.*** *At resonance the branch currents cancel, so the line current falls to a minimum while the impedance of the combination rises to a maximum, the exact opposite of the series case.*

The resonant frequency is given by the same formula, $f_0 = \frac{1}{2\pi\sqrt{LC}}$, but the behaviour is turned inside out. A series circuit at resonance has **minimum** impedance and **maximum** current; it eagerly accepts the resonant frequency and is called an **acceptor**. A parallel circuit at resonance has **maximum** impedance and **minimum** line current; it shuts that frequency out, and is called a **rejector**. Same two components, same resonant frequency, opposite job.

Resonance at a glance

Both circuits resonate at $f_0 = \frac{1}{2\pi\sqrt{LC}}$, where $X_L = X_C$.

Series (acceptor): impedance is minimum ($Z = R$), current is maximum ($I = \frac{V}{R}$), current in phase with voltage. Sharpness $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$, with $V_L = V_C = QV$ and bandwidth $\Delta f = \frac{f_0}{Q}$.

Parallel (rejector): impedance is maximum, line current is minimum. The same f_0, the opposite behaviour.

Five facts, two circuits, and one capacitor about to hold a hundred times the voltage it was offered. Time to make resonance earn its keep. Four worked examples follow.

BINDER Example 28

A coil of inductance $200\mu H$ is connected in series with a capacitor of $200pF$. Find the resonant frequency of the combination.

Solution

The resonant frequency depends only on the product LC:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{200\times10^{-6}H\times200\times10^{-12}F}} = 796kHz$$

Making Sense of the Answer: *Small inductances and capacitances give high resonant frequencies; radio work lives among microhenries and picofarads.*

Think Like a Physicist: *Only the product LC matters. To double the resonant frequency you must quarter that product, by shrinking either component.*

BINDER Example 29

A capacitor of $0.4\mu F$, a coil of inductance $0.4H$, a resistor of 10Ω and a lamp (whose resistance may be neglected) are joined in series with an alternating supply of $0.01V$ rms, whose frequency can be varied. Find

(a) the resonant frequency, (b) the current at resonance, (c) the voltage across the capacitor at resonance, and (d) the Q-factor. Describe how the lamp's brightness changes as the frequency is swept through f_0.

Solution

(a) The resonant frequency is:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.4H \times 0.4 \times 10^{-6}F}} = 398Hz$$

(b) At resonance the reactances cancel, so the impedance is just the resistance, $Z = R = 10\Omega$, and the current is:

$$I = \frac{V}{R} = \frac{0.01V}{10\Omega} = 0.001A = 1mA$$

(c) The capacitor's reactance at resonance, using $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.4H \times 0.4 \times 10^{-6}F}} = 2500rad/s$, is:

$$X_C = \frac{1}{\omega_0 C} = \frac{1}{2500rad/s \times 0.4 \times 10^{-6}F} = 1000\Omega$$

So the voltage across it is:

$$V_C = IX_C = 1 \times 10^{-3}A \times 1000\Omega = 1.0V$$

(d) The Q-factor is:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10\Omega}\sqrt{\frac{0.4H}{0.4 \times 10^{-6}F}} = 100$$

As the frequency rises from low to high, the lamp begins dim, brightens to a sharp maximum exactly at 398Hz, then fades again. It is brightest at resonance, where the current is greatest.

Making Sense of the Answer: *The supply is a mere $0.01V$, yet $1.0V$, a full hundred times more, appears across the capacitor. That is voltage magnification by the factor $Q = 100$. The inductor carries the same $1.0V$, equal and opposite, so the two cancel and the supply sees only the 10Ω resistor.*

Think Like a Physicist: *At resonance reach straight for $V_C = QV$. Here $Q = 100$ and $V = 0.01V$, so $V_C = 1.0V$ in a single line, confirming the longer calculation.*

HOT Example 30

A tuned circuit resonates at 1.0MHz with a Q-factor of 125. (a) Find its bandwidth. (b) Two stations broadcast at 1.000MHz and 1.004MHz. Can this circuit separate them?

Solution

(a) The bandwidth is the resonant frequency divided by the Q-factor:

$$\Delta f = \frac{f_0}{Q} = \frac{1.0 \times 10^6 Hz}{125} = 8kHz$$

(b) The two stations are separated by $1.004MHz - 1.000MHz = 4kHz$. Since this gap (4kHz) is smaller than the bandwidth (8kHz), both stations fall within the same resonance peak and cannot be cleanly separated. A higher Q, giving a narrower bandwidth, would be needed.

Making Sense of the Answer: *Bandwidth measures how fussy a circuit is. A bandwidth of $8kHz$ means the circuit passes a band $8kHz$ wide; stations closer together than that blur into one.*

Think Like a Physicist: *Selectivity and Q are the same idea seen from two sides. To pick apart close stations, raise Q and so narrow the bandwidth.*

HOT Example 31

A coil of inductance 2mH is connected in parallel with a capacitor of 8nF to form a tank circuit. (a) Find the resonant frequency. (b) State how the impedance and the supply current behave there, and name the circuit's role.

Solution

(a) The parallel circuit resonates at the same frequency as the series one:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{2 \times 10^{-3}H \times 8 \times 10^{-9}F}} = 39.8kHz$$

(b) At this frequency the currents in the two branches are equal and opposite, so the line current drawn from the supply falls to a minimum (ideally zero) while the impedance of the combination rises to a maximum. The circuit therefore shuts out the frequency f_0; it is acting as a rejector, or tank circuit.

Making Sense of the Answer: *The same L and C, merely rewired from series into parallel, resonate at the same frequency but do the opposite job: series accepts, parallel rejects.*

Think Like a Physicist: *The formula $f_0 = \frac{1}{2\pi\sqrt{LC}}$ is shared by both arrangements. What differs is whether the impedance bottoms out (series) or peaks (parallel) at resonance.*

And there it is: the circuit that listens for one frequency and shrugs at the rest. Kipute has gone home to announce that he can build a radio from a coil and a scrap of foil, which is very nearly true and entirely his teacher's fault. Kipanga, more careful, has noticed that the same L and C resonate at the same frequency whether wired in line or side by side, and that only the impedance cannot decide whether to vanish or to soar. He is exactly right.

THE TRANSFORMER

We opened this chapter with a quarrel: the war between one-way direct current and the back-and-forth of alternating current, and a promise that one quiet device would settle it. That device is the **transformer**, and its moment has arrived. A transformer takes an alternating voltage in at one value and hands it back out at another, larger or smaller, almost without loss and with no moving parts at all. It is the reason your phone charger can sip 5V from a 240V socket, and the reason the power born at Mtera can travel the length of the country without melting the wires on the way. It is also, as we shall see, the single reason the whole world chose alternating current.

How a transformer works: one flux, two coils

A transformer is two coils wound on the same iron core. One, the **primary**, is connected to the alternating supply; the other, the **secondary**, delivers the output. They are not joined by any wire; what links them is the iron.

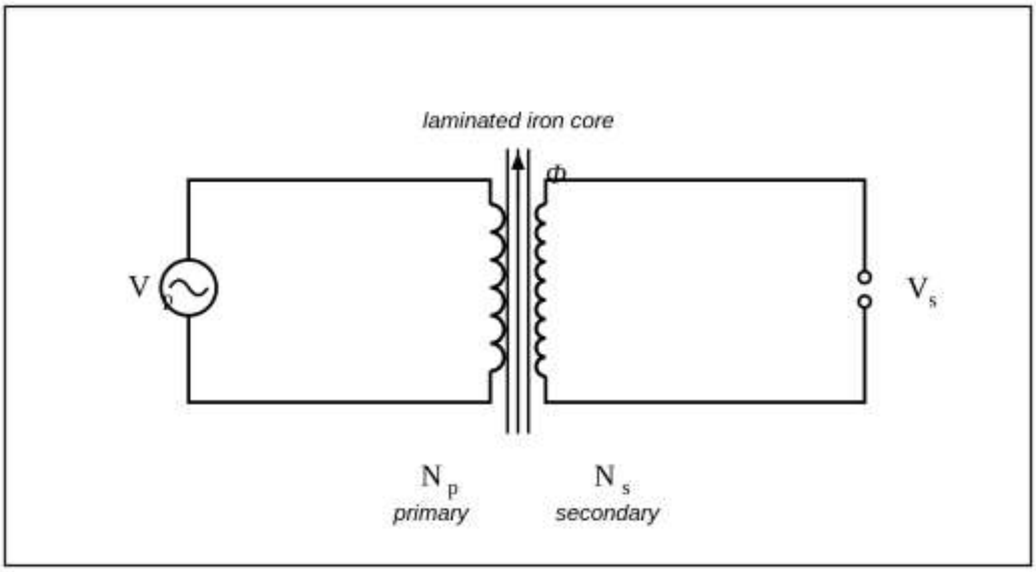


Figure: ***A transformer:*** *a primary coil and a secondary coil wound on a common laminated iron core. The alternating current in the primary sets up an alternating flux in the core, and that shared flux threads the secondary.*

Here is the chain of cause and effect. The alternating current in the primary sets up an alternating magnetic flux in the iron core. The core is a closed loop of iron, so it guides almost the whole of that flux around and through the secondary coil. Now recall the one fact we proved back in the alternator section: a coil of N turns threaded by a changing magnetic flux develops across itself a voltage equal to N times the rate at which the flux through one turn is changing. The very same changing flux Φ threads both coils, so each develops a voltage set by its own number of turns:

$$V_p = N_p\frac{d\Phi}{dt}, \qquad V_s = N_s\frac{d\Phi}{dt}$$

The rate of change of magnetic flux $\frac{d\Phi}{dt}$ is the same in both, so when we divide one equation by the other it cancels, leaving the **transformer equation**:

$$\frac{V_p}{V_s} = \frac{N_p}{N_s}$$

The voltages are in the same ratio as the turns. Give the secondary more turns than the primary ($N_s > N_p$) and it delivers a **higher** voltage; this is a **step-up** transformer. Give it fewer turns and the voltage comes out **lower**; this is a **step-down** transformer. The transformer's one trick is to trade turns for volts.

Current and power: what you gain in volts you pay in amps

A transformer cannot create energy. In an **ideal** transformer, one with no losses at all, the power fed into the primary equals the power drawn from the secondary:

$$V_pI_p = V_sI_s$$

Rearranging, and using the transformer equation, the currents come out in the **inverse** ratio of the turns:

$$\frac{I_p}{I_s} = \frac{V_s}{V_p} = \frac{N_s}{N_p}$$

This is the crucial trade. Step the voltage **up** and the current is stepped **down** in exactly the same proportion. A step-up transformer therefore delivers high voltage at low current, and that, as the last section of this topic will show, is precisely what long-distance transmission demands.

Notice one thing the transformer cannot do: work on steady direct current. A constant current makes a constant magnetic flux, and a magnetic flux that does not change induces no voltage in the secondary at all. Connect a battery to the primary and the secondary stays dead (save for a single flick at switch-on). A transformer lives entirely on the **changing** flux that only alternating current provides.

Why a real transformer falls short of perfection

A real transformer reaches only about 95% to 98% efficiency, never the full hundred. Its energy leaks away by three routes, and knowing them tells the engineer how to build a better one.

1) Copper loss

The windings are made of wire, and wire has resistance, so a current I through a winding of resistance R wastes I^2R as heat. Thicker, lower-resistance wire keeps this down.

2) Hysteresis loss

Every cycle, the alternating current drags the core's magnetisation right around a loop, the **B-H loop**, and forcing the iron round that loop costs energy. The energy lost per cycle, in each cubic metre of core, is exactly the **area enclosed by the loop**. The cure is to build the core from a soft magnetic material whose loop is as narrow as possible.

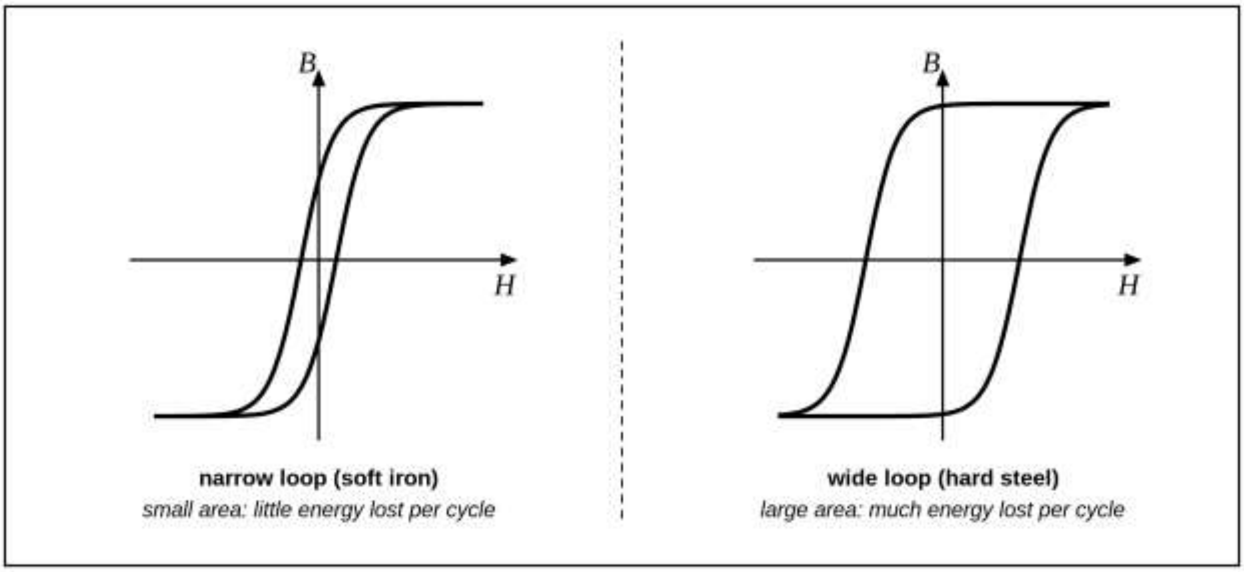


Figure: ***The B-H loop of the core material.*** *The energy turned to heat each cycle equals the area inside the loop. Soft iron (left) has a narrow loop and wastes little; hard steel (right) has a fat loop and wastes a great deal. A transformer core is therefore made of soft iron.*

This is why an examiner who asks ***"why should the core have a narrow hysteresis loop?"*** is really asking you to say: *because the loop area is the energy lost per cycle, and a narrow loop means a small area, so little heat and a more efficient transformer.*

3) Eddy-current loss

The changing flux does not only thread the coils; it also threads the iron of the core itself, and so drives little circulating currents, called **eddy currents**, round and round inside the metal. These currents heat the core for no useful purpose. The cure is to build the core not as one solid block but from many thin sheets, **laminations**, each lightly insulated from its neighbours, so the circulating currents are chopped into small, feeble loops.

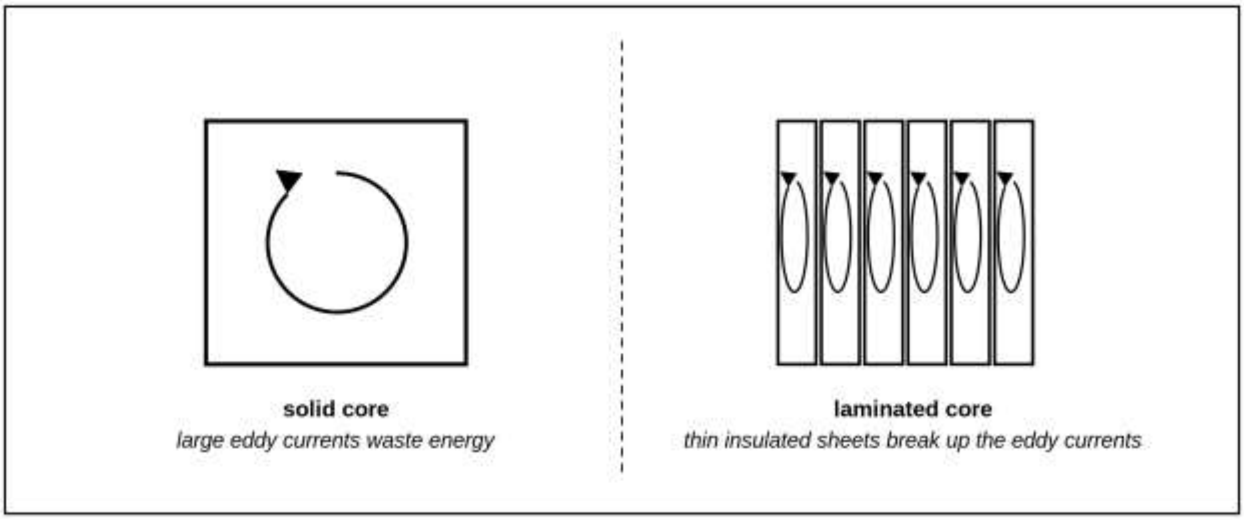


Figure: ***A solid core (left) lets one large eddy current circulate and waste energy.*** *Building the core from thin insulated laminations (right) breaks the flow into many small, weak loops, cutting the loss sharply.*

So the ideal core is soft iron of high permeability (to guide the flux well), with a narrow loop (little hysteresis) and built of thin laminations (little eddy-current loss). With these three cares taken, the small leftover gives the 95% to 98% efficiency we quoted. Efficiency itself is just:

$$\eta = \frac{P_{out}}{P_{in}} \times 100\%$$

Why the grid is alternating: from Mtera to your socket

Now the promise of the opening pays off. The alternators at the Mtera hydroelectric plant generate electricity at a few thousand volts. To carry it hundreds of kilometres, the heating loss I^2R in the lines must be kept tiny, and since the cables' resistance R is fixed, the only way is to make the current I small. By the transformer's trade, small current means high voltage. So a step-up transformer at the station lifts the voltage to the transmission level (the TANESCO grid runs at 132kV and above), the lines then carry the power at high voltage and low current with very little loss, and step-down transformers near the towns, such as at a Tanga substation, bring it back down, stage by stage, to the 240V at your wall socket.

Every link in that chain is a transformer, and a transformer works only on alternating current. That, in one sentence, is why the world's grids are a.c. and why the device cast as the hero of our opening truly earned the part.

Four worked examples to put the hero through its paces. They run from a one-line turns ratio up to the transmission sum that explains the whole grid, with the national-examination mutual-inductance item sitting in the middle. Kipute, who once tried to charge his phone straight from the wall and has the scorched memory to prove it, is unusually attentive.

BINDER Example 32

A transformer steps the 240V mains down to 12V for a doorbell. If the primary has 2000 turns, how many turns has the secondary?

Solution

The turns follow the voltages, so:

$$N_s = N_p\left(\frac{V_s}{V_p}\right) = 2000 \times \frac{12\text{V}}{240\text{V}} = 100 \text{ turns}$$

Making Sense of the Answer: *The voltage was cut to a twentieth* (12V *from* 240V)*, so the turns are cut to a twentieth as well, from* 2000 *to* 100*. Fewer turns on the secondary, lower voltage out: a step-down transformer.*

Think Like a Physicist: *The turns ratio and the voltage ratio are one and the same. Find one and you have the other, with no physics left to do.*

HOT Example 33

A step-down transformer runs a 12V, 24W lamp from the 240V mains. Assuming it is ideal, find (a) the current in the lamp (the secondary current), and (b) the current drawn from the mains (the primary current).

Solution

(a) The lamp's current follows from its power rating, $P = V_sI_s$:

$$I_s = \frac{P}{V_s} = \frac{24\text{W}}{12\text{V}} = 2\text{A}$$

(b) An ideal transformer conserves power, $V_pI_p = V_sI_s$, so:

$$I_p = \frac{V_sI_s}{V_p} = \frac{12\text{V} \times 2\text{A}}{240\text{V}} = 0.1\text{A}$$

Making Sense of the Answer: *The secondary delivers $2A$ at $12V$, the primary takes only $0.1A$ at $240V$. Both deliver the same $24W$. The transformer stepped the voltage down by twenty and stepped the current up by twenty, exactly preserving the power.*

Think Like a Physicist: *Whenever a transformer is called ideal, anchor on $V_pI_p = V_sI_s$. Voltage and current always move in opposite directions, so the watts stay put.*

REAL Example 34

A power station delivers 11MW along a line of total resistance 5Ω. Compare the power wasted as heat in the line when the power is sent at 11kV with the loss when a transformer first steps it up to 132kV.

Solution

The line current is $I = \frac{P}{V}$, and the heating loss is I^2R. At 11kV:

$$I = \frac{11 \times 10^6 W}{11 \times 10^3 V} = 1000A, \quad I^2R = (1000A)^2 \times 5\Omega = 5MW$$

At 132kV, twelve times the voltage, the current is twelve times smaller:

$$I = \frac{11 \times 10^6 W}{132 \times 10^3 V} = 83.3A, \quad I^2R = (83.3A)^2 \times 5\Omega = 35kW$$

The loss falls from 5MW (almost half the power thrown away as heat) to a mere 35kW, well under one per cent.

Making Sense of the Answer: *Raising the voltage twelve times cut the current twelve times, and the loss, which goes as I^2, fell by $12^2 = 144$ times. This single calculation is the whole case for high-voltage transmission, and so for the transformer, and so for a.c.*

Think Like a Physicist: *To move power a long way, push the voltage up and the current down. The loss I^2R rewards every extra volt handsomely, because it depends on the square of the current.*

HOT Example 35

A transformer draws 500W from the mains. Its copper loss is 12W and its iron loss (hysteresis and eddy currents together) is 8W. Find (a) the output power and (b) the efficiency.

Solution

(a) The output is the input less the losses:

$$P_{out} = P_{in} - \text{losses} = 500W - 12W - 8W = 480W$$

(b) The efficiency is:

$$\eta = \frac{P_{out}}{P_{in}} \times 100\% = \frac{480W}{500W} \times 100\% = 96\%$$

Making Sense of the Answer: *A 96% efficiency sits right in the usual range. The 12W copper loss would shrink with thicker wire; the 8W iron loss is held down by the narrow-loop core and the laminations we met above.*

Think Like a Physicist: *Efficiency is always useful output over total input. For a transformer the only thing standing between the two is the sum of copper and iron losses; account for those and the rest comes straight through.*

From a coil spinning at Mtera to the socket on your wall, every step of the journey is a transformer quietly trading volts for amps. Kipute, having grasped at last why one cannot charge a phone straight from the wall, has declared the transformer "the most polite machine in physics, because it never asks for more power than it gives." Kipanga, checking the sums, confirms it gives back all it takes save a few honest watts of warmth, and even those, he notes, are kept small by a narrow loop and a well-laminated core.

POWER IN AC CIRCUITS AND POWER FACTOR

For a plain resistor, power is the simplest thing in the world: $P = I^2R$, and there is an end of it. But the moment an inductor or a capacitor enters the circuit, the voltage and the current fall out of step, and the word "power" turns slippery. We glimpsed the strangeness back in the pure-circuits section, where a perfect inductor or capacitor was found to dissipate, on average, **no power at all**. Now we make that precise, and then do something the examiners love and the accountants love even more: we turn it into money, and explain why a workshop in Dar es Salaam can be charged more for running the very same machines.

Average power: only the in-step part counts

At every instant the power delivered is the product of the instantaneous voltage and current, $p = vi$. When the two are in step, as in a resistor, this product is always positive: energy flows steadily into the component. But when they are a quarter cycle ($\pi/2$) apart, as in a pure reactance, the product is positive for half of each cycle and negative for the other half. Energy pours in, then pours straight back out, and over a whole cycle the **average** comes to exactly zero. That is the precise reason a pure inductor or capacitor consumes no power: it borrows energy and returns it, every cycle, keeping nothing.

Take the voltage and current with a phase difference φ between them, $v = V_0 \sin\omega t$ and $i = I_0 \sin(\omega t - \varphi)$. The instantaneous power is their product:

$$p = vi = V_0 I_0 \sin\omega t \sin(\omega t - \varphi)$$

The product of two sines can be reshaped by the standard identity:

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)],$$

which turns it into:

$$p = \frac{1}{2} V_0 I_0 [\cos\varphi - \cos(2\omega t - \varphi)]$$

The first term inside the bracket is a constant; the second oscillates at twice the supply frequency, so over one complete cycle it averages to zero. The average power is therefore the constant part alone:

$$P = \frac{1}{2} V_0 I_0 \cos\varphi$$

Finally, since $V_{rms} = \frac{V_0}{\sqrt{2}}$ and $I_{rms} = \frac{I_0}{\sqrt{2}}$:

$$V_0 = V_{rms}\sqrt{2} \text{ and } I_0 = I_{rms}\sqrt{2}$$

Substituting to $P = \frac{1}{2} V_0 I_0 \cos\varphi$:

$$P = \frac{1}{2} \times V_{rms}\sqrt{2} \times I_{rms}\sqrt{2} \times \cos\varphi = \frac{1}{2} \times 2 \times V_{rms} \times I_{rms} \times \cos\varphi = V_{rms} I_{rms} \cos\varphi$$

The result is **average power** (also called the **true power)**, measured in watts:

$$\mathbf{P = V_{rms} I_{rms} \cos\varphi}$$

There is the origin of the cosφ: it is the cosine of the phase angle between voltage and current, and it drops out of the algebra the moment the two are allowed to fall out of step. For a resistor $\varphi = 0$ and $\cos\varphi = 1$, so all of the product does real work; for a pure inductor or capacitor $\varphi = 90°$ and $\cos\varphi = 0$, so none of it does, exactly confirming the zero-power result we began with. (The $\cos(2\omega t - \varphi)$ term we threw away is the energy sloshing in and out at twice the supply frequency, the very oscillation an inductor and capacitor live by.)

The power triangle: true, reactive, and apparent power

The plain product of the rms voltage and current is itself given a name. It is the **apparent power**, written S and measured in **volt-amperes** (VA), because it is what the supply must appear to provide:

$$S = V_{rms} I_{rms}$$

Of this, the part actually turned into work or heat is the **true power** $P = S\cos\varphi$, in watts (W). The remaining part, $Q = S\sin\varphi$, merely surges in and out of the inductors and capacitors and does no net work; it is the **reactive power**, measured in **volt-amperes reactive (VAR).** Because true and reactive power are at right angles, like the resistance and reactance that cause them, the three combine into a right-angled **power triangle**.

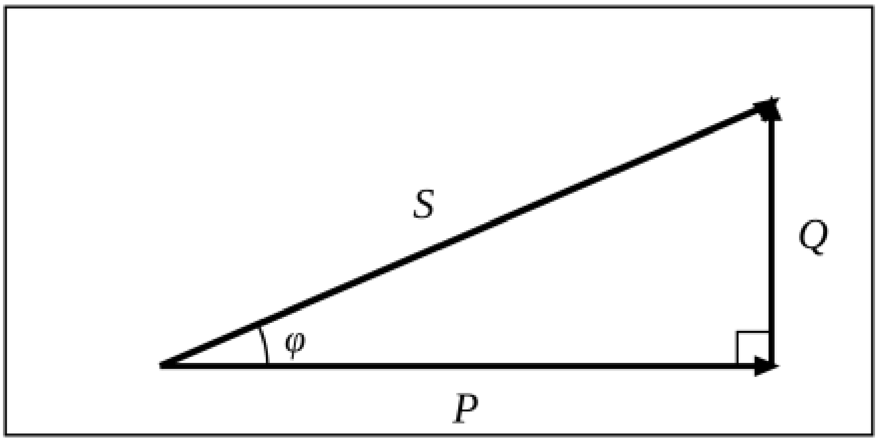


Figure: ***The power triangle.*** *The apparent power S is the hypotenuse, the true power P the base, and the reactive power Q the upright side. The angle between S and P is the same phase angle φ as in the impedance triangle.*

With S the hypotenuse:

$$S^2 = P^2 + Q^2$$

That is:

$$(V_{rms}I_{rms})^2 = (V_{rms}I_{rms}\cos\varphi)^2 + (V_{rms}I_{rms}\sin\varphi)^2$$

Power factor: how much of the apparent power works

The ratio of the true power to the apparent power is the **power factor**:

$$\textbf{power factor} = \cos\boldsymbol{\varphi} = \frac{\mathbf{P}}{\mathbf{S}} = \frac{\mathbf{R}}{\mathbf{Z}}$$

The last equality follows from the impedance triangle of the series section: there the resistance R is the side adjacent to the angle φ and the impedance Z is the hypotenuse, so the cosine of φ, adjacent over hypotenuse, is exactly $\frac{R}{Z}$.

A power factor of 1 means a purely resistive circuit, with every volt-ampere doing real work; a power factor of 0 means a purely reactive one, with none of it doing any. When the current **lags** the voltage, as in an inductive circuit, the power factor is called **lagging**; when it **leads**, as in a capacitive one, it is called **leading**.

Why a poor power factor costs money

Most industrial loads, induction motors, transformers, the chokes in fluorescent lights, are strongly inductive, and run at a power factor of only about 0.7 to 0.8 lagging. Here is the trouble. To deliver a true power P at voltage V_{rms}, the supply must push a current,

$$I_{rms} = \frac{P}{V_{rms}\cos\varphi}$$

So the smaller the power factor, the **larger** the current needed for the very same useful power. That bloated current heats the supply cables more (I^2R again), and forces the utility to install thicker cables and bigger transformers to carry it. For this reason TANESCO bills large consumers on the **apparent** power in kVA, not merely the true power in kW. A workshop with a poor power factor draws extra current it never turns into work, and pays for the privilege.

Power-factor correction: the parallel capacitor

The cure is cheap and elegant. Connect a capacitor, or a bank of them, in **parallel** with the inductive load. The capacitor draws a leading current that cancels part of the inductor's lagging reactive current, so the net reactive power drops, the power factor climbs back toward 1, and the line current shrinks, all while the load goes on doing exactly the same real work.

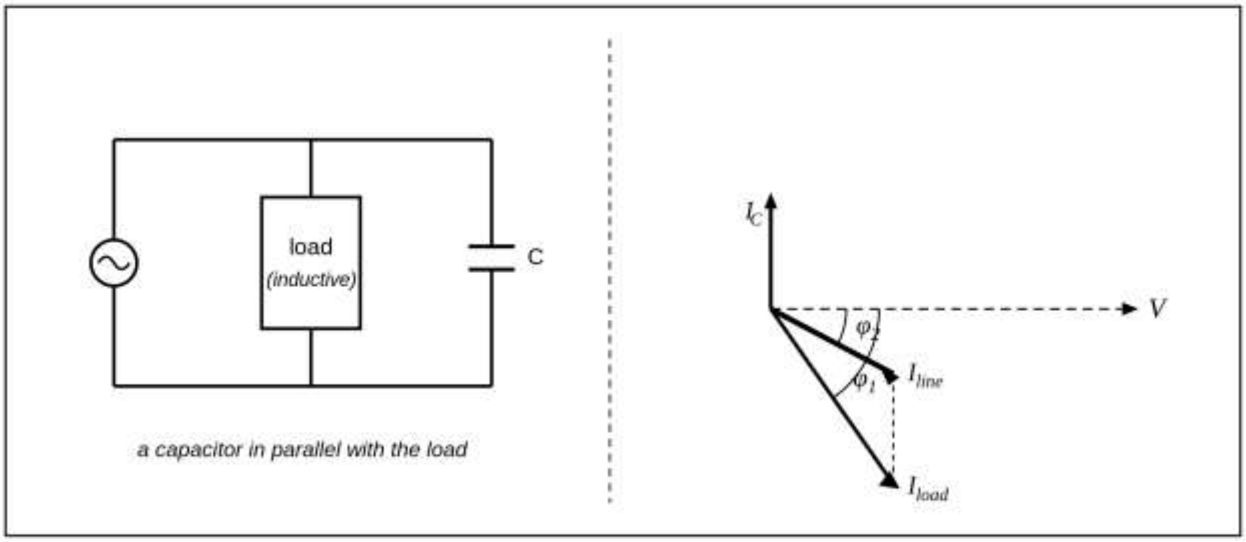


Figure: *Left: a capacitor placed in parallel with an inductive load. Right: the load current lags the voltage by a large angle; adding the capacitor's leading current pulls the line current up to a smaller angle and a smaller size, raising the power factor.*

Here is the calculation behind it. From the power triangle the reactive and true powers are linked by:

$$\tan\varphi = \frac{Q}{P}$$

So a load taking true power P at phase angle φ also draws reactive power $Q = P\tan\varphi$.

To lift the factor from $\cos\varphi_1$ to a better $\cos\varphi_2$ while holding the true power P fixed, the reactive power demanded from the supply must fall from $P\tan\varphi_1$ to $P\tan\varphi_2$, and the capacitor must make up the difference:

$$Q_C = P\tan\varphi_1 - P\tan\varphi_2 = P(\tan\varphi_1 - \tan\varphi_2)$$

A capacitor of its own draws reactive power:

$$Q_C = \frac{V_{rms}^2}{X_C} = \frac{V_{rms}^2}{\frac{1}{2\pi fC}} = V_{rms}^2 \times 2\pi fC$$

Solving for C:

$$\mathbf{C = \frac{Q_C}{2\pi f V_{rms}^2} = \frac{P(\tan\varphi_1 - \tan\varphi_2)}{2\pi f V_{rms}^2}}$$

The motor still turns, the lights still glow; only the wasteful reactive current that used to slosh all the way back to the power station is now supplied locally, by the capacitor, a few centimetres away. (The deeper reason the grid carries all this on alternating current remains the transformer of the previous section; power-factor correction is the same fight against wasted current, waged at the factory instead of the substation.)

Four worked examples, running from a one-line power factor up to a capacitor sized to save a workshop money. Kipanga, who has just learned that his uncle's welding shop is billed in kVA, is taking notes with unusual seriousness.

BINDER Example 36

A coil has resistance 12Ω and, at the working frequency, an impedance of 20Ω. Find its power factor and state whether it is leading or lagging.

Solution

The power factor is the ratio of resistance to impedance:

$$\cos\varphi = \frac{R}{Z} = \frac{12\Omega}{20\Omega} = 0.60$$

A coil is inductive, so the current lags the voltage: the power factor is 0.60 **lagging**.

Making Sense of the Answer: *Only 60% of the apparent power does real work. The other 40% is reactive, sloshing in and out of the coil's magnetic field each cycle. A bare coil is already a fairly poor power factor.*

Think Like a Physicist: *Power factor is just $\frac{R}{Z}$, the cosine of the impedance triangle's angle. Resistive part over total opposition; nothing more to memorise.*

REAL Example 37

Two workshops each draw the same useful power of 10kW from the 415V mains. The first has a power factor of 1.0; the second, full of motors, runs at 0.70 lagging. Explain, with figures, why TANESCO charges the second workshop more even though both "use the same machines."

Solution

The current each must draw is $I_{rms} = \frac{P}{V_{rms}\cos\varphi}$.

For the first workshop:

$$I_{rms} = \frac{10000W}{415V \times 1.0} = 24.1A$$

For the second workshop:

$$I_{rms} = \frac{10000W}{415V \times 0.70} = 34.4A$$

Both turn the same 10kW into useful work, but the second draws 34.4A against the first's 24.1A, about 43% more current, for nothing extra in return. That surplus current heats TANESCO's cables and ties up its transformers, so TANESCO meters the **apparent** power: the first workshop presents 10kVA, the second 14.3kVA $\left(\text{from } \frac{10kW}{0.7}\right)$. The second pays for capacity it wastes.

Making Sense of the Answer: *"The same machines" do the same real work, true, but the poor power factor makes the second workshop a heavier customer on the network. The bill follows the current, and the current follows the power factor.*

Think Like a Physicist: *Whenever someone says two loads are "the same," check the current each actually draws. Real power can be equal while apparent power, and therefore cost, differs widely.*

HOT Example 38

A 50Ω resistor, a 0.100H inductor and a 10.0μF capacitor are connected in series to a 60Hz source. The rms current in the circuit is 2.75A. Find (a) the rms voltage of the source, (b) the power factor, and (c) the average power.

Solution

(a) First the reactances at $\omega = 2\pi f = 2\pi \times 60Hz = 377rad/s$:

$$X_L = \omega L = 377rad/s \times 0.100H = 37.7\Omega, \quad X_C = \frac{1}{\omega C} = \frac{1}{377rad/s \times 10.0 \times 10^{-6}F} = 265\Omega$$

The impedance, with the reactances opposing, is:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(50\Omega)^2 + (37.7\Omega - 265\Omega)^2} = 233\Omega$$

So the source voltage is:

$$V_{rms} = I_{rms}Z = 2.75A \times 233\Omega = 641V$$

(b) The power factor is

$$\cos\varphi = \frac{R}{Z} = \frac{50\Omega}{233\Omega} = 0.215 \ \ \text{(leading)}$$

Leading, because the capacitor's reactance wins and the circuit is net capacitive.

(c) The average power is dissipated only in the resistor:

$$P = I_{rms}^2 R = (2.75A)^2 \times 50\Omega = 378W$$

Making Sense of the Answer: *The circuit is far off resonance, so its impedance* (233Ω) *is much larger than the resistance* (50Ω) *and the power factor is poor. Of the* $641V \times 2.75A = 1763VA$ *of apparent power, only* $378W$ *does real work, just as* 0.215 *of it should.*

Think Like a Physicist: *Average power is always* $I_{rms}^2 R$*, the resistor's alone, no matter how much the reactances inflate the voltage. Find Z for the current, but find R for the power.*

HOT Example 39

A Dar es Salaam workshop draws 50kW at a power factor of 0.70 lagging from the 415V, 50Hz mains. What capacitance, placed in parallel, will raise the power factor to 0.95?

Solution

The capacitor must supply the reactive power $Q_C = P(\tan\varphi_1 - \tan\varphi_2)$.

From $\cos\varphi_1 = 0.70$ we get $\tan\varphi_1 = 1.02$, and from $\cos\varphi_2 = 0.95$, $\tan\varphi_2 = 0.329$. So:

$$Q_C = 50\text{kW} \times (1.02 - 0.329) = 34.6\text{kVAR}$$

And the capacitance that provides this at 415V, 50Hz is:

$$C = \frac{Q_C}{2\pi f V_{rms}^2} = \frac{34600\text{VAR}}{2\pi \times 50\text{Hz} \times (415\text{V})^2} = 6.39 \times 10^{-4}\text{F} = 639\mu\text{F}$$

As a check, the line current falls from $I = \frac{50000\text{W}}{415\text{V}\times 0.70} = 172\text{A}$ before correction to $\frac{50000\text{W}}{415\text{V}\times 0.95} = 127\text{A}$ after, a saving of 45A for the same 50kW of work.

Making Sense of the Answer: *A* $639\mu F$ *bank trims the current by a quarter without touching the workshop's output. The capacitor supplies the motors' reactive power on the spot, so TANESCO's cables no longer have to carry it back and forth.*

Think Like a Physicist: *Correction is bookkeeping in reactive power: find* Q *before and after from* $P\tan\varphi$*, give the difference to a capacitor, and size it from* $Q_C = {V_{rms}}^2 \times 2\pi f C$*. The real power never enters the capacitor's job.*

Power, it turns out, keeps two sets of books: the watts that do the work and the volt-amperes the supply must carry to deliver them. Keep the two close together, a power factor near one, and the network is content and the bill is small. Let them drift apart and the wires run hot for nothing. Kipute has resolved to fit his uncle's welding shop with "a bucket of capacitors," and Kipanga, for once, cannot fault the idea, only recommend that he first work out the right number of microfarads.

RECTIFICATION AND THE MEASUREMENT OF AC

Alternating current is unbeatable for generating and transmitting power, as the transformer made plain. Yet almost every piece of electronics that the power finally reaches, a phone, a laptop, the radio, the CT scanner at Bugando Hospital, runs on steady **direct** current inside. Somewhere between the wall socket and the chip, the back-and-forth of a.c. must be turned into the one-way flow of d.c. The device that does it is the **rectifier**, and it is built from diodes. We close the chapter's machinery with two questions: *how do we convert a.c. into d.c., and how, since an ordinary meter is fooled by a current that keeps reversing, do we measure a.c. in the first place?*

Everything rests on one fact about the **diode**, which you met among the semiconductors: a diode is a one-way valve for current. It conducts freely when the voltage pushes it the "forward" way, and blocks almost completely when the voltage tries to push it backward. Feed alternating current into a diode and only the forward halves get through.

The half-wave rectifier

The simplest rectifier is a single diode in series with the load. During the half-cycle that drives the diode forward, current flows and the load receives the voltage; during the reverse half-cycle the diode blocks, no current flows, and the output is zero. The result is a string of positive humps with empty gaps between them.

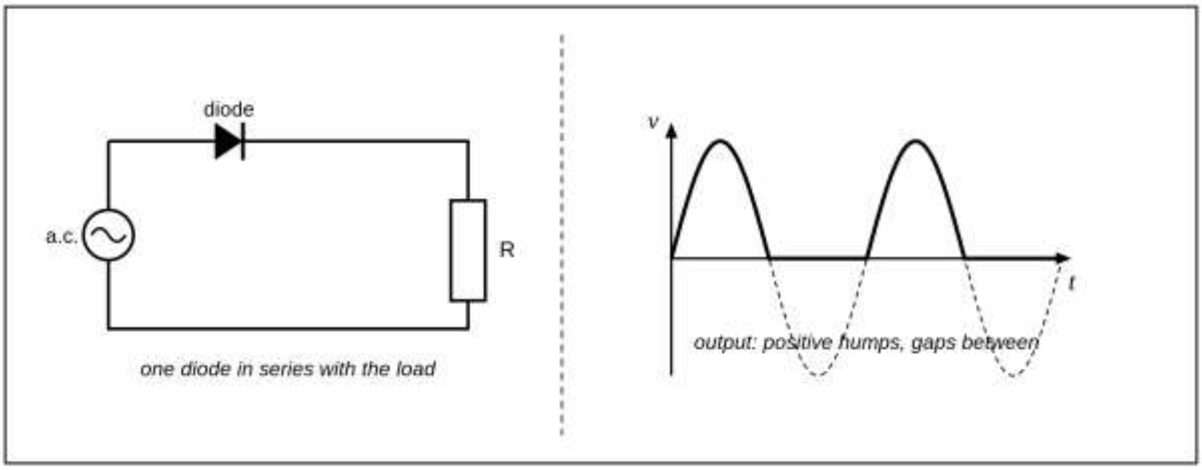


Figure: ***A half-wave rectifier:*** *a single diode in series with the load. Only the half-cycles that bias the diode forward reach the load, so the output is a row of positive humps separated by gaps where the diode blocks.*

What steady (d.c.) value does this pulsing output average to? We must average the output over a whole period. For the first half, from $\theta = 0$ to π, the output is $V_0\sin\theta$; for the second half it is zero. So:

$$V_{mean} = \frac{1}{2\pi}\int_0^{\pi} V_0\sin\theta\, d\theta = \frac{V_0}{2\pi}[-\cos\theta]_0^{\pi} = \frac{V_0}{2\pi}\times(1+1) = \frac{V_0}{\pi} = 0.318V_0$$

Therefore:

$$\mathbf{V_{mean} = \frac{V_0}{\pi} = 0.318V_0}$$

This is the average value of a half-wave rectified sine wave over a complete cycle. Half-wave rectification is simple but wasteful: it throws away every other half-cycle and leaves a very bumpy output.

The full-wave bridge rectifier

To use **both** halves of the wave, we arrange four diodes in a **bridge**. On either half-cycle, two of the four diodes conduct and steer the current through the load in the **same** direction, while the other two block. The negative halves are not discarded but flipped up to join the positive ones, so the output is an unbroken row of positive humps with no gaps.

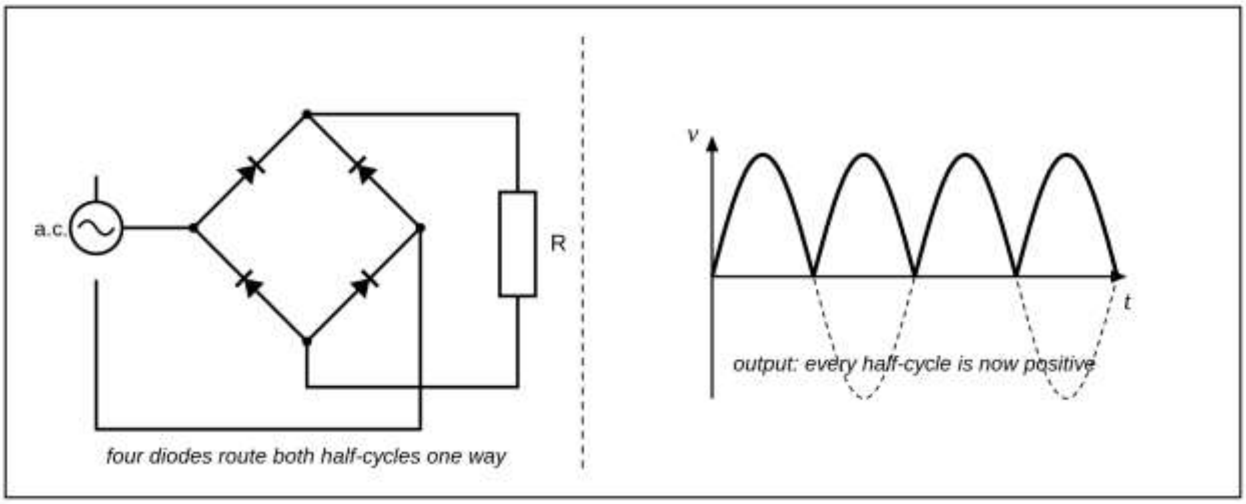


Figure: ***A full-wave bridge rectifier:*** *four diodes route both half-cycles through the load in one direction. Whichever way the supply pushes, the current crosses the load the same way, so every half-cycle appears as a positive hump.*

Now every half-cycle contributes, and the pattern repeats every π, so we average over just half a period:

$$V_{mean} = \frac{1}{\pi}\int_0^{\pi} V_0\sin\theta\, d\theta = \frac{V_0}{\pi}[-\cos\theta]_0^{\pi} = \frac{V_0}{\pi}\times 2 = \frac{2V_0}{\pi} = 0.637V_0$$

Therefore:

$$\mathbf{V_{mean} = \frac{2V_0}{\pi} = 0.637V_0},$$

exactly twice the half-wave value. The full-wave output is smoother and wastes nothing, which is why it is the arrangement found inside almost every real power supply.

Smoothing: the reservoir capacitor

Even the full-wave output still rises and falls between zero and the peak. To smooth it into something close to steady d.c., a large capacitor is connected in **parallel** with the load. It charges up to the peak of each hump, then, during the dips, discharges slowly through the load, holding the voltage up and filling in the troughs. What is left is a nearly steady voltage with only a small **ripple** riding on top.

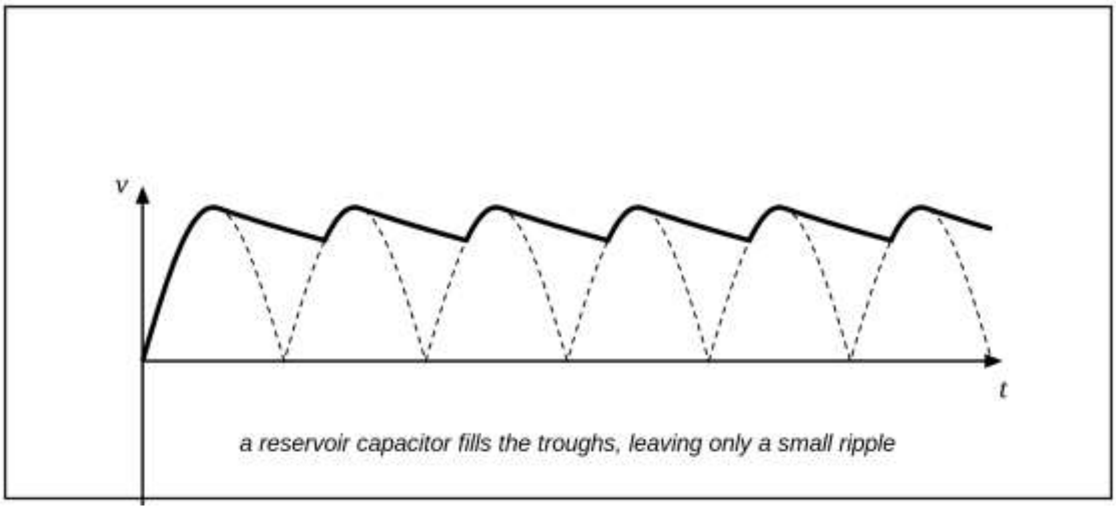


Figure: *With a reservoir capacitor across the load, the output charges to each peak and discharges only a little before the next hump arrives. The deep troughs are filled in, leaving a near-steady d.c. voltage with a small ripple.*

The larger the capacitor (and the larger the load resistance), the slower the discharge and the smaller the ripple. This is precisely how a phone charger or the d.c. supply inside the Bugando scanner turns the 240V a.c. mains, once a transformer has stepped it down, into the smooth d.c. its electronics demand.

Measuring alternating current

A **moving-coil meter**, the workhorse of the d.c. laboratory, measures the **average** current through it. On alternating current the average over a full cycle is zero, so the needle does not move: such a meter simply cannot read a.c. To measure a.c. we need an instrument that responds to the current's **size** regardless of its direction. Three designs do this.

1) **The rectifier-type meter** rectifies the current first, then lets an ordinary moving-coil meter read the mean of the rectified wave. Since that mean is not the rms value, the scale is marked up by a fixed **form factor**. For a sine wave the form factor is the ratio of the rms value to the full-wave mean, $\frac{I_{rms}}{I_{dc}} = \frac{\pi}{2\sqrt{2}} = 1.11$, so the scale reads 1.11 times the mean and thereby shows rms directly.
2) **The moving-iron meter** works by the force between pieces of iron magnetised by the current; that force depends on the **square** of the current, so the instrument responds to the average of I^2 and reads the rms value straight off, on a.c. or d.c. alike.
3) **The hot-wire meter** uses the heating of a fine wire, I^2R, which likewise depends on the square of the current; it too reads true rms, whatever the shape of the wave. Both (2 and 3 designs) have crowded, non-uniform scales, the price of squaring.

Three worked examples to put the diodes to work.

BINDER Example 40

A half-wave rectifier supplies 50V d.c. to a resistive load of 800Ω. The diode has a resistance of 25Ω. Calculate the peak a.c. voltage required.

Solution

The d.c. delivered to the load fixes the d.c. (mean) current through it:

$$I_{dc} = \frac{V_{dc}}{R} = \frac{50V}{800\Omega} = 0.0625A$$

For a half-wave rectifier the mean current is $1/\pi$ of the peak, $I_{dc} = \frac{I_0}{\pi}$, so the peak current is:

$$I_0 = \pi I_{dc} = \pi \times 0.0625A = 0.196A$$

This peak current is driven by the peak voltage through the diode and the load in series:

$$R + r = 800\Omega + 25\Omega = 825\Omega$$

It follows that:

$$V_0 = I_0(R + r) = 0.196A \times 825\Omega = 162V$$

Making Sense of the Answer: *A hefty* $162V$ *peak is needed to leave* $50V$ *of steady d.c., because half-wave rectification keeps only a thin slice of each cycle, and the mean of that slice is just* 0.318 *of the peak.*

Think Like a Physicist: *Work inward from the load. The d.c. it needs sets* I_{dc}*; the factor* π *lifts that to the peak* I_0*; and Ohm's law across the whole series path then gives the peak voltage.*

REAL Example 41

Inside a charger, a transformer steps the $240V$ mains down to a peak of $12V$, and a full-wave bridge then rectifies it. (a) Find the d.c. (mean) output voltage before smoothing. (b) State what happens when a large reservoir capacitor is added across the load.

Solution

(a) For a full-wave rectifier the mean is 0.637 of the peak:

$$V_{mean} = \frac{2V_0}{\pi} = \frac{2 \times 12V}{\pi} = 7.64V$$

(b) The capacitor charges up to the peak of each hump and discharges only slightly before the next arrives. The output therefore rises from this $7.64V$ average towards the peak $12V$, holding nearly steady at that level with only a small ripple.

Making Sense of the Answer: *Without smoothing the output averages* $7.64V$ *but swings all the way down to zero between humps; with a good capacitor it sits close to the* $12V$ *peak, which is what the charger's electronics actually want.*

Think Like a Physicist: *Rectification decides the* ***shape****; the capacitor decides the* ***smoothness****. The mean* $0.637V_0$ *is the starting point, and smoothing pushes the working voltage up toward the peak.*

HOT Example 42

An alternating current of peak value $2.0A$ flows in turn through three meters: a moving-coil meter, a rectifier-type meter, and a hot-wire meter. What does each one read?

Solution

The **moving-coil** meter reads the average current over a full cycle. For a symmetric a.c. that average is zero, so it reads:

$$\text{moving} - \text{coil reading} = 0$$

The **rectifier** meter reads the full-wave mean, $I_{dc} = \frac{2I_0}{\pi} = 0.637 \times 2.0A = 1.27A$, and its scale multiplies this by the form factor 1.11 to display the rms value:

$$\text{rectifier reading} = 1.11 \times 1.27A = 1.41A$$

The **hot-wire** meter responds to I^2 and so reads the true rms value directly:

$$I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{2.0A}{\sqrt{2}} = 1.41A$$

So the readings are 0, 1.41A and 1.41A respectively.

Making Sense of the Answer: *The rectifier meter and the hot-wire meter agree, both showing the rms value* 1.41A, *while the moving-coil meter, blind to a reversing current, reads nothing.*

Think Like a Physicist: *Ask what each meter physically responds to. Average current, and a.c. averages to zero; the square of the current, and you get rms. The rectifier meter is the in-between case, rescued by the* 1.11 *form factor.*

With that, the toolkit is complete. We can make alternating current, measure it, picture it with phasors, send it through resistors, inductors, and capacitors alone and combined, tune it to resonance, raise and lower it with transformers, reckon its power, and finally tame it into the steady d.c. that electronics drink. All that remains is to watch the whole apparatus go to work in the world, which is exactly where the chapter turns next.

APPLICATIONS OF AC THEORY

Alternating current is at its most convincing when it stops being equations and becomes objects. This closing section takes eight devices of everyday Tanzanian life and reads each one through the physics the chapter has built, from the national grid that spans the country down to the charger on the bedside table. Nothing new is introduced here; the aim is only to watch the theory already in hand earn its keep in the world.

1. The national grid

The whole chapter, in a sense, exists for this. The alternators at the Mtera and Kidatu power stations turn mechanical energy into an alternating EMF; to carry that power hundreds of kilometres without wasting it as heat in the lines, the current must be small, and so, by the transformer's trade of $\frac{V_p}{V_s}$ against turns, the voltage must be very large. A step-up transformer at the station lifts it to the transmission voltage; the line carries the power at high voltage and low current with little I^2R loss; and step-down transformers near the towns bring it back to a voltage safe for the home. The entire chain works only because the transformer works, and the transformer works only on alternating current.

The TANESCO grid transmits at 132kV, and on the newer southern interconnector at 220kV, stepping down at substations such as Tanga's to 33kV and 11kV and at last to the 240V of the wall socket. As the transmission example of the power section showed, sending the same power at 11kV instead of 132kV would multiply the line loss by 144 and melt the economics of the grid. High-voltage a.c. transmission is not a convenience but the only way bulk power can travel across a country the size of Tanzania.

2. Tuning a radio or television

Inside every receiver sits a resonant circuit, an inductor and a variable capacitor, that responds most strongly at the one frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$. The aerial picks up every station at once, but only the station whose frequency matches f_0 drives the circuit at resonance and is magnified; the rest, off resonance, are passed over. Turning the tuning knob rotates the plates of the variable capacitor, changing C and sliding f_0 from one station to the next. The sharper the resonance, the higher the circuit's Q, the more cleanly one station is separated from its neighbour.

A typical receiver pairs a coil of about 250μH with a capacitor variable from 20pF to 400pF, giving a tuning range that comfortably spans the 540kHz to 1610kHz of the AM band on which Radio Free Africa and TBC reach their listeners. The same trick, with smaller components, reaches the far higher frequencies of FM and television.

3. The choke in a fluorescent lamp

A fluorescent tube, once its gas has struck, behaves almost like a short circuit, and on the bare mains it would draw a runaway current and destroy itself in an instant. In series with it sits a **choke**, a coil of high inductance, whose job is to hold the current down. Because the choke opposes the current by its **reactance** $X_L = 2\pi fL$ rather than by resistance, it limits the current while dissipating almost no power of its own, which a plain resistor of the same limiting value never could.

At the 50Hz mains a choke of around 1H presents a reactance of some 300Ω, enough to settle the tube at its working current. This is the same ballast that lights the corridors of Miono Secondary School, and it is the inductor of the pure-circuit and series sections doing an honest day's work. Modern tubes replace the heavy iron choke with a small electronic ballast that does the same limiting at a far higher frequency.

4. Power-factor correction in industry

A workshop full of induction motors is a strongly inductive load: its current lags the voltage, its power factor is poor, and it therefore draws far more current than its real power alone would need. A bank of capacitors connected in parallel with the load supplies a leading current that cancels much of the motors' lagging reactive current, so the net reactive power falls, the power factor climbs back toward one, and the current the supply must carry drops, all without changing the useful work the motors do.

A Dar es Salaam metal workshop drawing 50kW at a power factor of 0.70 needs, as the design example of the power section found, a capacitor bank of about 640μF to reach a power factor of 0.95, trimming its line current from 172A to 127A. Because TANESCO bills industrial users on the apparent power in kVA, the bank pays for itself in lower demand charges, which is why capacitor banks hum quietly in the corner of almost every Tanzanian factory.

5. Induction heating and the induction cooker

In the transformer section the eddy currents induced in a solid core were a nuisance, fought off with laminations. Turn the same effect up deliberately and it becomes a tool. A coil carrying a high-frequency alternating current sets up a rapidly changing magnetic flux; a metal pan or workpiece placed in that flux has strong eddy currents driven round inside it, and its own resistance turns those currents straight into heat. The metal heats itself, from within, with no flame and no hot plate.

An induction cooker runs its coil at some 20kHz to 50kHz, heating only a ferromagnetic pot set on top while the glass surface beneath stays cool. On a larger scale, induction furnaces melt tonnes of steel by the same principle, and a smith's induction heater softens a bar in seconds. The loss the transformer designer works to suppress is, in the cook's hands, the whole point.

6. High-voltage supplies for hospital imaging

An X-ray tube cannot work on the mains as it stands: it needs tens of thousands of volts of steady direct current to fling electrons hard enough into its target to make X-rays. The supply that feeds it is the chapter in miniature. A step-up transformer raises the mains voltage enormously, exploiting the very turns ratio that the transformer section built; a rectifier and smoothing stage then convert that high alternating voltage into the steady high-voltage d.c. the tube demands.

The X-ray and CT equipment at Bugando Hospital runs its tubes at roughly 50kV to 120kV, voltages reached by stepping the mains up several hundredfold and then rectifying. The transformer that magnifies the voltage and the diodes that straighten it are exactly the devices of the previous two sections, here trusted with a diagnosis. The same arrangement, less dramatically, hides inside every microwave oven.

7. The metal detector

A metal detector is, at heart, a resonant LC circuit kept gently oscillating. Its search coil is the inductor; bring a piece of metal close and the metal alters the coil's inductance L, which shifts the circuit's resonant frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$ by a small but detectable amount. The electronics listen for that shift in frequency and sound the alarm. The whole instrument is the resonance section turned into a search tool.

Hand-held detectors run their coils at a few kHz up to hundreds of kHz, the lower frequencies reaching deeper and the higher ones catching smaller targets. They sweep for buried weapons at the doors of public buildings, and, in the goldfields around Geita, they help prospectors find the metal that pays. A shift in f_0 of a fraction of one per cent is all the machine needs.

We have followed electrical energy from its generation as alternating current, through its transmission and behaviour in AC circuits, to its final conversion into the direct current that powers everyday electronics. The Great Current Debate that opened the chapter is now, quietly, settled. Alternating current won the world not because it is gentler or simpler, it is neither, but because it alone can be transformed, and so alone can travel.

Yet understanding a subject is one thing; applying it confidently is another. Before we close the chapter, let us put these ideas to work in a final set of miscellaneous worked examples, drawing together the principles, techniques, and problem-solving strategies developed throughout our study of alternating current.

MISCELLANEOUS WORKED EXAMPLES ON ALTERNATING CURRENT THEORY

Example 43

(a) Explain why the EMF induced in a coil rotating at a steady speed in a uniform magnetic field varies sinusoidally with time, and state the position of the coil at which the induced EMF is (i) a maximum and (ii) zero.

(b) A rectangular coil of 50 turns, each of area $1.2 \times 10^{-2}\mathrm{m}^2$, rotates at 3000 revolutions per minute in a uniform magnetic field of flux density 0.25T. Find (i) the peak EMF generated and (ii) the rms EMF.

Solution

(a) As the coil rotates, the flux linkage through it is $\Phi = NBA\cos\theta$, where θ is the angle between the field and the normal to the coil, and since the coil turns steadily $\theta = \omega t$. The induced EMF is the rate of change of this flux linkage, $E = -\frac{d\Phi}{dt} = NBA\omega\sin\omega t$, which is sinusoidal because the rate at which the flux is cut itself varies sinusoidally as the coil turns. The EMF is therefore a maximum when the plane of the coil is parallel to the field, since the sides then cut the lines of force at the greatest rate; and it is zero when the plane of the coil is perpendicular to the field, since the sides then move along the lines of force and cut none.

(b) The frequency is $f = \frac{3000}{60} = 50\mathrm{Hz}$, and the angular speed is then:

$$\omega = 2\pi f = 2\pi \times 50\mathrm{Hz} = 314\mathrm{rad\ s^{-1}}$$

The peak EMF is:

$$E_0 = NBA\omega = 50 \times 0.25\mathrm{T} \times 1.2 \times 10^{-2}\mathrm{m}^2 \times 314\mathrm{rad\ s^{-1}} = 47.1\mathrm{V}$$

and the rms EMF is:

$$E_{rms} = \frac{E_0}{\sqrt{2}} = \frac{47.1\mathrm{V}}{\sqrt{2}} = 33.3\mathrm{V}$$

The coil generates a peak EMF of 47.1V and an rms EMF of 33.3V.

Example 44

(a) Explain what is meant by the root-mean-square (rms) value of an alternating current, and why it, rather than the peak or the mean value, is the value quoted for the mains supply and used in power calculations.

(b) An immersion heater connected to the 240V mains raises the temperature of 2.0kg of water from 20°C to 100°C in 7.0 minutes. Assuming no heat is lost to the surroundings, find (i) the power of the heater and (ii) its resistance. (Specific heat capacity of water = $4200\mathrm{J\ kg^{-1}K^{-1}}$.)

Solution

(a) The rms value of an alternating current is the value of the steady direct current that would dissipate heat in a given resistor at the same average rate as the alternating current does. It is defined in this way because the heating effect depends on the square of the current, so the average heating is governed by the mean of the square and not by the current itself; and since the alternating current spends as much time negative as positive, its ordinary average over a cycle is zero and would tell us nothing. The rms value is therefore the value quoted for the mains and used in power calculations, because it alone gives the true heating effect, so that $P = V_{rms}I_{rms}$ and the familiar d.c. formulae carry over unchanged.

(b) The heat needed to warm the water is:

$$Q = mc\Delta\theta = 2.0\mathrm{kg} \times 4200\mathrm{J\ kg^{-1}K^{-1}} \times (100 - 20)\mathrm{K} = 6.72 \times 10^5\mathrm{J}$$

Since all of it is supplied in $t = 7.0\mathrm{min} = 420\mathrm{s}$, the power is:

$$P = \frac{Q}{t} = \frac{6.72 \times 10^5\mathrm{J}}{420\mathrm{s}} = 1600\mathrm{W}$$

The resistance then follows from $P = \frac{V_{rms}^2}{R}$:

$$R = \frac{V_{rms}^2}{P} = \frac{(240\mathrm{V})^2}{1600\mathrm{W}} = 36\Omega$$

The heater has a power of 1.6kW and a resistance of 36Ω.

Example 45

(a) The instantaneous alternating current $i = I_0 \sin\omega t$ and the displacement of a body in simple harmonic motion $x = A\sin\omega t$ are described by equations of exactly the same form. State the quantities in the a.c. case that play the parts of the amplitude A and the angular frequency ω of the S.H.M., and explain the sense in which a phasor is related to the circular motion that generates S.H.M.

(b) An alternating current is given by $i = 5.0\sin(100\pi t)$A, with t in seconds. Find (i) its frequency, (ii) its rms value, and (iii) the first instant after $t = 0$ at which the instantaneous current equals its rms value.

Solution

(a) In the a.c. equation the peak current I_0 plays the part of the amplitude A, since each is the greatest value its oscillating quantity reaches, while the angular frequency ω, related to the frequency by $\omega = 2\pi f$, plays the very same part in both. A phasor is related to the circular motion in just the way S.H.M. is: S.H.M. is the projection onto a diameter of a point moving steadily round a circle, and in the same way the instantaneous a.c. value is the projection of an arrow of length I_0 rotating steadily at angular frequency ω. So the phasor is that rotating arrow, and the alternating current is its shadow.

(b) Comparing with $i = I_0 \sin\omega t$ gives $I_0 = 5.0\text{A}$ and $\omega = 100\pi \text{rad s}^{-1}$.

(i) The frequency is:

$$f = \frac{\omega}{2\pi} = \frac{100\pi \text{rad s}^{-1}}{2\pi} = 50\text{Hz}$$

(ii) The rms value is:

$$I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{5.0\text{A}}{\sqrt{2}} = 3.54\text{A}$$

(iii) The current equals its rms value when $\sin\omega t = \frac{1}{\sqrt{2}}$, that is when $\omega t = \frac{\pi}{4}$:

$$t = \frac{\pi/4}{\omega} = \frac{\pi/4}{100\pi \text{rad s}^{-1}} = \frac{1}{400}\text{s} = 2.5 \times 10^{-3}\text{s}$$

The current has frequency 50Hz, rms value 3.54A, and first reaches its rms value 2.5ms after $t = 0$.

Example 46

(a) A real coil, such as the field winding of a motor or the ballast of a fluorescent lamp, grows warm when it carries an alternating current, whereas an ideal (pure) inductor would stay cool. Explain why a real coil dissipates power while a pure inductor does not.

(b) A coil of resistance 30Ω and inductance 0.20H is connected in series with a 40Ω resistor across a 240V, 50Hz supply. Find (i) the impedance of the circuit, (ii) the current, (iii) the voltage across the coil, and (iv) the power factor.

Solution

(a) In a pure inductor the current lags the voltage by exactly a quarter-cycle, so that over each cycle the source pours energy into the magnetic field during one quarter and the field hands every joule back during the next; the average power is therefore zero and the component stays cool. A real coil, however, is wound from wire that has resistance as well as inductance, and this resistance carries the same current, so it dissipates energy at the rate I^2R as heat. The coil therefore grows warm because of its resistance and not its inductance, since only the resistive part of the coil draws real power from the supply.

(b) The inductive reactance is:

$$X_L = 2\pi fL = 2\pi \times 50\text{Hz} \times 0.20\text{H} = 62.8\Omega$$

The total resistance is $R = 30\Omega + 40\Omega = 70\Omega$, so the impedance is:

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{(70\Omega)^2 + (62.8\Omega)^2} = 94.1\Omega$$

The current is:

$$I = \frac{V}{Z} = \frac{240\text{V}}{94.1\Omega} = 2.55\text{A}$$

The voltage across the coil is the current times the coil's own impedance, $Z_{coil} = \sqrt{(30\Omega)^2 + (62.8\Omega)^2} = 69.6\Omega$:

$$V_{coil} = IZ_{coil} = 2.55A \times 69.6\Omega = 178V$$

and the power factor of the whole circuit is:

$$\cos\phi = \frac{R}{Z} = \frac{70\Omega}{94.1\Omega} = 0.744 \text{ (lagging)}$$

The impedance is 94.1Ω, the current 2.55A, the voltage across the coil 178V, and the power factor 0.744 lagging.

Example 47

(a) In a loudspeaker system a capacitor is connected in series with the small high-frequency tweeter, and an inductor in series with the large low-frequency woofer. Explain, in terms of the way reactance depends on frequency, why this arrangement sends the high notes to the tweeter and the low notes to the woofer.

(b) A 0.10H inductor and a 50μF capacitor are available. (i) Find the reactance of each at 50Hz and state which offers the greater opposition. (ii) Find the frequency at which their reactances are equal.

Solution

(a) The reactance of a capacitor is $X_C = \frac{1}{2\pi fC}$, which falls as the frequency rises, so a capacitor offers little opposition to high notes but a large opposition to low ones; placed in series with the tweeter it therefore passes the high frequencies on to it while holding the low ones back. The reactance of an inductor is $X_L = 2\pi fL$, which rises with frequency, so an inductor offers little opposition to low notes but a large opposition to high ones; placed in series with the woofer it therefore passes the low frequencies to it while choking off the high ones. In this way each note is steered to the speaker that can reproduce it best.

(b) (i) At $f = 50$Hz the reactances are:

$$X_L = 2\pi fL = 2\pi \times 50\text{Hz} \times 0.10\text{H} = 31.4\Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50\text{Hz} \times 50 \times 10^{-6}\text{F}} = 63.7\Omega$$

So the capacitor offers the greater opposition at this frequency.

(ii) The reactances are equal when $X_L = X_C$, that is $2\pi fL = \frac{1}{2\pi fC}$, giving:

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.10\text{H} \times 50 \times 10^{-6}\text{F}}} = 71.2\text{Hz}$$

At 50Hz the capacitor dominates (63.7Ω against 31.4Ω), and the two reactances become equal at 71.2Hz.

Example 48

(a) In an audio amplifier a capacitor is connected in series between one stage and the next. Explain why such a capacitor allows the alternating audio signal to pass from one stage to the next, yet blocks the steady d.c. voltage used to bias each stage.

(b) A 10μF capacitor is connected across a 50V, 50Hz supply. Find (i) the rms current it draws, (ii) the maximum energy stored in it, and (iii) the maximum charge on its plates.

Solution

(a) No charge ever crosses the gap between a capacitor's plates; a current through a capacitor means only that the plates are being charged and discharged through the connecting wires. When the alternating signal is applied the voltage is forever changing, so the plates charge and discharge continually and a current flows in the wires at every moment, and the signal passes on. When the steady bias voltage is applied, however, the capacitor charges once to that voltage and then stops, because a constant voltage calls for no further movement of charge, so the current falls to zero and the d.c. is blocked. The capacitor therefore couples the signal while isolating the bias.

(b) (i) The capacitive reactance is:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50\text{Hz} \times 10 \times 10^{-6}\text{F}} = 318\Omega$$

Thus the rms current is:

$$I_{rms} = \frac{V_{rms}}{X_C} = \frac{50V}{318\Omega} = 0.157A$$

(ii) The maximum energy is stored at the peak voltage, $V_0 = V_{rms}\sqrt{2} = 50V \times \sqrt{2}$:

$$E_{max} = \frac{1}{2}CV_0^2 = \frac{1}{2} \times 10 \times 10^{-6}F \times \left(50V\sqrt{2}\right)^2 = 0.025J$$

(iii) The maximum charge is:

$$Q_{max} = CV_0 = 10 \times 10^{-6}F \times 50V \times \sqrt{2} = 7.07 \times 10^{-4}C = 707\mu C$$

The capacitor draws 0.157A, stores at most 0.025J, and holds at most 707μC.

Example 49

(a) Show that the average value of a sinusoidal alternating current over one complete cycle is zero.
(b) A half-wave rectified current of peak value 1.5A and frequency 50Hz is passed through a copper voltameter for 20 minutes. Find (i) the average (d.c.) current and (ii) the mass of copper deposited. (Electrochemical equivalent of copper $= 3.3 \times 10^{-7}$kg C^{-1}.)

Solution

(a) Let the current be $i = I_0 \sin\omega t$. Its average over one complete period T is the area under one cycle divided by the period:

$$I_{av} = \frac{1}{T}\int_0^T I_0 \sin\omega t\, dt = \frac{I_0}{\omega T}[-\cos\omega t]_0^T = \frac{I_0}{\omega T}(1 - \cos\omega T)$$

Since $\omega T = 2\pi$:

$$I_{av} = \frac{I_0}{\omega T}(1 - \cos 2\pi) = \frac{I_0}{\omega T}(1 - 1) = 0$$

The reason is plain in the graph: the current is positive for the first half-cycle and equally negative for the second, so the two areas cancel exactly and the mean over a whole cycle is zero.

(b) A half-wave rectifier removes the negative half-cycles and leaves only the positive humps, so the average over a full cycle is that of one hump spread over the whole period:

$$I_{av} = \frac{I_0}{\pi} = \frac{1.5A}{\pi} = 0.477A$$

The charge passed in $t = 20min = 1200s$ is:

$$Q = I_{av}t = 0.477A \times 1200s = 573C$$

By Faraday's first law the mass deposited is:

$$m = zQ = 3.3 \times 10^{-7}kg\,C^{-1} \times 573C = 1.89 \times 10^{-4}kg \approx 0.19g$$

The average current is 0.477A and about 0.19g of copper is deposited.

A pure alternating current, whose average is zero, would deposit nothing, which is why electroplating must use a rectified current.

Example 50

(a) Explain why a transformer will operate on alternating current but not on a steady direct current.
(b) An ideal step-up transformer has 200 turns on its primary and 5000 turns on its secondary. The primary is connected to the 240V, 50Hz mains and the secondary supplies a load of 1.0kW. Find (i) the secondary voltage, (ii) the secondary current, and (iii) the current drawn from the mains.

Solution

(a) A transformer works by mutual induction: the changing current in the primary sets up a changing magnetic flux in the core, and this changing flux, linking the secondary, induces an EMF in it. An alternating current is forever changing, so it produces a forever-changing magnetic flux and a steady alternating EMF appears across the secondary. A direct current, once established, is constant, so the flux it produces is constant too; and since an unchanging flux induces no EMF, the secondary gives nothing, apart from a brief pulse at the instant the current is switched on or off. A transformer therefore needs the perpetual change that only alternating current can provide.

(b) (i) For an ideal transformer $\frac{V_s}{V_p} = \frac{N_s}{N_p}$, so:

$$V_s = V_p \times \frac{N_s}{N_p} = 240V \times \frac{5000}{200} = 6000V$$

(ii) The secondary current follows from the load power, $P = V_s I_s$:

$$I_s = \frac{P}{V_s} = \frac{1000W}{6000V} = 0.167A$$

(iii) An ideal transformer wastes no power, so the input power equals the output power:

$$I_p = \frac{P}{V_p} = \frac{1000W}{240V} = 4.17A$$

The secondary delivers 6000V at 0.167A, while the primary draws 4.17A from the mains.

Example 51

(a) Name the main causes of energy loss in a practical transformer, and state one way in which each is reduced.
(b) A transformer supplies 4.0kW to its load while working at an efficiency of 96%. Its copper loss at this load is 90W. Find (i) the power drawn from the supply, (ii) the total power lost in the transformer, and (iii) the iron loss.

Solution

(a) There are three main losses.

- Copper loss is the I^2R heating in the windings, and it is reduced by winding them from thick, low-resistance copper.
- Eddy-current loss is the heating by currents induced in the core itself, and it is reduced by building the core from thin, insulated laminations, so that these currents are broken into small, weak loops.
- Hysteresis loss is the energy spent in repeatedly remagnetising the core each cycle, and it is reduced by making the core of soft iron, which has a narrow hysteresis loop and so wastes little energy per cycle.

(b) (i) Efficiency is $\eta = \frac{P_{out}}{P_{in}}$, so the input power is:

$$P_{in} = \frac{P_{out}}{\eta} = \frac{4000W}{0.96} = 4167W$$

(ii) The total loss is the difference between input and output:

$$P_{loss} = P_{in} - P_{out} = 4167W - 4000W = 167W$$

(iii) The iron loss is whatever remains after the copper loss is taken out:

$$P_{iron} = P_{loss} - P_{copper} = 167W - 90W = 77W$$

The transformer draws 4167W and loses 167W in all, of which 77W is iron loss.

Example 52

(a) Explain why connecting a capacitor in parallel with an inductive load, such as a motor, improves the power factor of the supply, even though the capacitor itself consumes no power.
(b) A single-phase motor takes 12A from the 415V, 50Hz mains at a power factor of 0.60 lagging. Find (i) the true power, (ii) the apparent power, (iii) the reactive power, and (iv) the capacitance that, connected in parallel, would raise the power factor to unity.

Solution

(a) An inductive motor draws a current that lags the voltage, and this lagging current carries a large reactive component which is drawn from and returned to the supply each cycle, doing no useful work yet still having to be carried by the line. A capacitor, by contrast, draws a current that leads the voltage, so its reactive current is in antiphase with the motor's. When the capacitor is connected in parallel, its leading reactive current cancels much of the motor's lagging reactive current, so the line need carry only the small difference. The total current therefore falls and comes more nearly into phase with the voltage, so the power factor rises, even though the capacitor itself takes no real power.

(b) (i) The true power is:

$$P = VI\cos\phi = 415V \times 12A \times 0.60 = 2988W \approx 2.99kW$$

(ii) The apparent power is:

$$S = VI = 415V \times 12A = 4980VA$$

(iii) With $\cos\phi = 0.60$, $\sin\phi = \sqrt{1 - 0.60^2} = 0.80$, so the reactive power is:

$$Q = VI\sin\phi = 415V \times 12A \times 0.80 = 3984VAR$$

(iv) For unity power factor the capacitor must supply all of this reactive power, $Q_C = V^2\omega C = Q$, so:

$$C = \frac{Q}{\omega V^2} = \frac{3984VAR}{2\pi \times 50Hz \times (415V)^2} = 7.4 \times 10^{-5}F = 74\mu F$$

The motor's true, apparent and reactive powers are 2.99kW, 4980VA and 3984VAR, and a 74μF capacitor restores unity power factor.

Example 53

(a) Explain, with reference to the action of the diodes, the difference between half-wave and full-wave rectification, and state which makes the better use of the alternating input.

(b) A sinusoidal supply of peak value 12V is applied, first through a single diode (half-wave) and then through a bridge of four diodes (full-wave), to a 100Ω load. Find the mean (d.c.) output voltage and the mean output current in each case, and hence the factor by which the full-wave circuit improves the d.c. output.

Solution

(a) In half-wave rectification a single diode conducts only during those half-cycles that drive it in the forward direction and blocks the reverse half-cycles, so only one half of each cycle reaches the load and the other half is wasted. In full-wave rectification a bridge of four diodes redirects the current so that both halves of every cycle pass through the load in the same direction, the reversed half-cycles being turned the right way up instead of being thrown away. Full-wave rectification therefore makes the better use of the input, since it gives twice the mean output and a waveform that is far easier to smooth.

(b) For half-wave rectification the mean voltage is $\frac{V_0}{\pi}$:

$$V_{dc} = \frac{V_0}{\pi} = \frac{12V}{\pi} = 3.82V$$

$$I_{dc} = \frac{V_{dc}}{R} = \frac{3.82V}{100\Omega} = 38.2mA$$

For full-wave rectification the mean voltage is twice as great, $\frac{2V_0}{\pi}$:

$$V_{dc} = \frac{2V_0}{\pi} = \frac{2 \times 12V}{\pi} = 7.64V$$

$$I_{dc} = \frac{7.64V}{100\Omega} = 76.4mA$$

The full-wave circuit gives 7.64V against 3.82V, an improvement of exactly a factor of two in the d.c. output.

Example 54

(a) Explain why a moving-coil meter reads zero when an alternating current is passed through it, whereas a hot-wire meter gives a reading, and state what value of the alternating current the hot-wire meter indicates.

(b) A current $i = (3.0 + 4.0\sin\omega t)A$, a steady current with an alternating part superimposed, flows through a circuit. Find (i) the reading of a moving-coil meter and (ii) the reading of a hot-wire meter placed in the circuit.

Solution

(a) A moving-coil meter gives a deflection proportional to the instantaneous current and in a direction that depends on the current's direction. When the current alternates, the deflecting torque reverses every half-cycle, and because the pointer cannot follow the rapid reversals it settles at the average value, which for an alternating current is zero. A hot-wire meter, by contrast, responds to the heating of a wire, and since the heating is proportional to i^2 it does not depend on the current's direction, so the wire is heated by both half-cycles alike. The hot-wire meter therefore reads the root-mean-square value of the current.

(b) (i) The moving-coil meter reads the average (d.c.) current. The average of the steady part is 3.0A and the average of $4.0\sin\omega t$ over a cycle is zero, so the meter reads 3.0A.

(ii) The hot-wire meter reads the rms value, the square root of the mean of i^2. Now:

$$i^2 = (9.0 + 24\sin\omega t + 16\sin^2\omega t)A^2,$$

Thus the mean square over a cycle is:

$$9.0A^2 + 0 + 16A^2 \times \frac{1}{2} = 17A^2$$

Hence:

$$I_{rms} = \sqrt{17A^2} = 4.12A$$

The moving-coil meter reads 3.0A, while the hot-wire meter reads 4.12A.

Example 55

(a) A tuned LC circuit lies at the heart of a radio's tuning dial and of a metal detector. Explain, in terms of energy, how an LC circuit that has been set oscillating behaves like a frictionless mass on a spring, and explain why in a real circuit the oscillations gradually die away.

(b) A capacitor of 8.0μF is charged to 2.0V and then connected across an inductor of 2.0mH of negligible resistance. Find (i) the maximum energy stored in the circuit, (ii) the maximum current in the inductor, (iii) the frequency of the oscillations, and (iv) the wavelength of an electromagnetic wave of this frequency.

Solution

(a) When the charged capacitor is connected to the inductor it drives a current that builds up a magnetic field, so the energy held in the capacitor's electric field is handed over to the inductor's magnetic field; and when that field collapses it recharges the capacitor the opposite way, so the energy shuttles endlessly between the two stores. This is exactly the exchange in a mass on a spring, where energy passes to and fro between the spring's potential energy (the capacitor) and the mass's kinetic energy (the inductor), the charge playing the part of displacement and the current that of velocity. In a real circuit the wire has resistance, so a little energy is turned to heat each cycle, just as friction drains a real pendulum; the oscillations are therefore damped and slowly die away.

(b) (i) At first all the energy is in the capacitor:

$$E_{max} = \frac{1}{2}CV^2 = \frac{1}{2} \times 8.0 \times 10^{-6}F \times (2.0V)^2 = 1.6 \times 10^{-5}J = 16\mu J$$

(ii) At maximum current all of this energy is in the inductor, $\frac{1}{2}LI_0^2 = E_{max}$, so:

$$I_0 = \sqrt{\frac{2E_{max}}{L}} = \sqrt{\frac{2 \times 1.6 \times 10^{-5}J}{2.0 \times 10^{-3}H}} = 0.13A$$

(iii) The frequency of free oscillation is:

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{2.0 \times 10^{-3}H \times 8.0 \times 10^{-6}F}} \approx 1.3 \times 10^3 Hz$$

(iv) The corresponding electromagnetic wavelength is:

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8 \text{m s}^{-1}}{1.26 \times 10^3 \text{Hz}} = 2.4 \times 10^5 \text{m}$$

The circuit stores at most 16μJ and oscillates with a peak current of 0.13A at about 1.3kHz, corresponding to an electromagnetic wavelength of 2.4×10^5m, a reminder that radio tuning, at megahertz frequencies, needs far smaller L and C.

DIGGING DEEPER EXERCISE 13

EXERCISE 13A: BINDER QUESTIONS

Question 1

Explain why the mains current is said to reverse its direction one hundred times each second when its frequency is stated to be only 50Hz.

Question 2

The mains is described as a 240V supply, yet the insulation of the cables that carry it must be able to withstand a voltage of about 339V. Explain why.

Question 3

Explain, in terms of the way an inductor responds to a changing current, why the current through a pure inductor reaches its peak a quarter of a cycle later than the applied voltage.

Question 4

Explain why the opposition that a capacitor offers to an alternating current cannot be found by connecting an ordinary ohmmeter across it.

Question 5

A resistor of 30Ω is connected in series with an inductor whose reactance is 40Ω. Explain why the combined opposition to the current is not 70Ω.

Question 6

Explain why a series circuit containing a resistor, an inductor and a capacitor draws the greatest current from the supply at one particular frequency, and a smaller current at frequencies above and below it.

Question 7

Explain why a high quality factor is an advantage in the tuning circuit of a radio receiver, and state one disadvantage that would arise if the quality factor were made far too high.

Question 8

Explain why an inductor and a capacitor joined in parallel draw only a very small current from the supply at the resonant frequency, even though a large current flows back and forth within the loop itself.

Question 9

A voltmeter and an ammeter connected in an a.c. circuit read 240V and 5.0A. Explain why the power delivered to the circuit is not necessarily 1200W.

Question 10

Explain why the makers of a transformer or a generator quote its rating in kilovolt-amperes (kVA) rather than in kilowatts (kW).

Question 11

Explain why the two coils of a transformer are wound on a common iron core, and why the device would be very inefficient if the coils were merely placed side by side in air.

Question 12

Explain why the current drawn from the mains by the primary of a transformer increases when more appliances are switched on across its secondary.

Question 13

Explain why eddy currents are an unwanted source of energy loss in the core of a transformer, yet are the very effect on which the heating of a pan on an induction cooker depends.

EXERCISE 13B: REAL QUESTIONS

Question 14

A phone charger left plugged into a wall socket feels slightly warm even when no phone is connected to it. Explain why it continues to take power from the mains although it is charging nothing.

Question 15

Under the light of a fluorescent lamp running on the mains, the blades of a rotating fan sometimes appear to stand still, or to turn slowly backwards. Explain this observation.

Question 16

An electric motor, such as the one in a refrigerator or a water pump, draws a much heavier current at the instant it is switched on than when it has reached full running speed. Explain why.

Question 17

The large capacitor inside a television set or a power supply can give a serious electric shock when touched, even hours after the set has been switched off and unplugged. Explain why.

Question 18

When the switch controlling a large electromagnet or motor is opened, a bright spark jumps across the opening contacts, far brighter than the one seen when a plain lamp is switched off. Explain why.

Question 19

A simple electric wall clock driven from the mains keeps very accurate time over many months without ever being reset. Explain how the alternating supply makes this possible.

Question 20

A car battery can be recharged from a mains-operated charger, but not by connecting it to the secondary of a transformer alone. Explain why a rectifier must be included in the charger.

Question 21

A cheap multimeter measures the sinusoidal mains voltage correctly, but gives a noticeably wrong reading when used to measure the chopped output of a lamp dimmer. Explain why.

Question 22

A person who grips a wire carrying the 50Hz mains often cannot let go, whereas a brief shock from a direct-current source tends to throw the victim clear. Explain why.

Question 23

An electric arc welder is fitted with a transformer that has very many turns on its primary but only a few turns of very thick wire on its secondary. Explain why such a transformer is suited to welding.

Question 24

Lighting fitted in a bathroom or in a garden pond is often run at 12V from a transformer rather than directly at the 240V mains. Explain why this is done.

Question 25

During a power cut, mains appliances cannot be run straight from a 12V battery, but they work when the battery is connected through an inverter. Explain why the inverter is needed.

EXERCISE 13C: HOT QUESTIONS

Question 26

In a series circuit carrying an rms current of 0.50A, voltmeters read 60V across the resistor, 80V across the inductor and 50V across the capacitor. Find (a) the rms supply voltage, (b) the resistance, (c) the net reactance, (d) the phase angle between the supply voltage and the current, and (e) the average power dissipated.

Question 27

A coil draws an rms current of 2.0A from a 100V, 50Hz supply, and 1.4A from a 100V, 100Hz supply. Find the resistance and the inductance of the coil.

Question 28

An alternating current $i = 3.0\sin(100\pi t) + 4.0\sin(300\pi t)$, in amperes, flows through a 10Ω resistor. Find (a) the rms value of the current and (b) the average power dissipated in the resistor.

Question 29

A sinusoidal current of peak value 4.0A is passed through a half-wave rectifier and then through a 20Ω resistor. Find (a) the mean (d.c.) current, (b) the rms current, and (c) the average power dissipated in the resistor.

Question 30

A coil of inductance 0.20H and resistance 50Ω is connected in series with a 2.0μF capacitor across a 10V (rms), variable-frequency supply. Find (a) the resonant frequency, (b) the current at resonance, (c) the voltage across the capacitor at resonance, and (d) the quality factor of the circuit.

Question 31

A series tuned circuit is to resonate at 800kHz with a bandwidth of 8.0kHz, using a coil of inductance 0.20mH. Find (a) the quality factor, (b) the capacitance, and (c) the resistance required, and (d) state the bandwidth that would result if the resistance were doubled.

Question 32

A small power station generates 24kW at 480V. A transformer steps this up to 9.6kV for transmission along a cable of total resistance 6.0Ω. Find (a) the step-up turns ratio, (b) the current in the cable, (c) the power lost in the cable, and (d) the percentage of the generated power delivered to the far end.

Question 33

A 10kVA transformer has an iron loss of 80W and a full-load copper loss of 200W. For a load of power factor 0.80, find its efficiency (a) at full load and (b) at half load.

Question 34

A resistor of 50Ω, an inductor of reactance 25Ω and a capacitor of reactance 40Ω are connected in parallel across a 100V (rms) supply. Find (a) the current in each branch, (b) the line current and its phase angle, and (c) the reactance to which the capacitor must be changed to make the line current a minimum, and the value of that minimum current.

Question 35

A single-phase motor on the 240V, 50Hz mains takes a current of 8.0A while a wattmeter records 1.2kW. Find (a) the power factor, (b) the phase angle, (c) the reactive power, and (d) the extra current the supply must provide compared with a purely resistive load of the same true power.

Question 36

An inductor of inductance 0.50H carries an alternating current $i = 2.0\sin(100\pi t)$, in amperes. Find (a) the peak energy stored in the inductor, (b) the frequency at which the stored energy oscillates, (c) the rms current, and (d) the peak voltage across the inductor.

Question 37

A 100Ω resistor, a 0.30H inductor and a 20μF capacitor are connected in series across a 240V, 50Hz supply. Find (a) the impedance and current at 50Hz, (b) the power factor and average power, and (c) the frequency at which the power factor becomes unity, together with the current at that frequency.

Question 38

A transformer steps the 240V (rms) mains down to 9.0V (rms), whose output is full-wave rectified to charge a battery through a total resistance of 1.5Ω. Find (a) the turns ratio, (b) the peak secondary voltage, (c) the mean (d.c.) output voltage, and (d) the mean charging current.

Question 39

A coil of 500 turns, each of area 200cm², rotates at 3000 revolutions per minute in a uniform field of 0.040T and feeds a 100Ω resistor. Find (a) the peak EMF, (b) the rms EMF, (c) the rms current, and (d) the average power delivered to the resistor.

Question 40

A workshop draws 30kW at a power factor of 0.60 lagging from a 400V supply through a feeder of resistance 0.20Ω. Find (a) the line current and the power wasted in the feeder; then, after a capacitor bank raises the power factor to 0.90, find (b) the new line current and feeder loss, and (c) the percentage reduction in the feeder loss.

ANSWERS

EXERCISE 13A

1. In one complete cycle an alternating current flows first one way, falls to zero, then flows the other way and returns to zero, so it changes direction twice in every cycle. At a frequency of 50Hz the current completes 50 whole cycles in each second, and since every cycle contains two reversals the number of reversals each second is $2 \times 50 = 100$. The current therefore reverses one hundred times a second even though only 50 complete cycles, and hence a frequency of 50Hz, occur in that time.

2. The figure 240V is the rms value, the steady value that would give the same heating; it is not the greatest voltage the supply reaches. The voltage actually climbs to a peak of $V_0 = V_{rms}\sqrt{2} = 240V \times \sqrt{2} \approx 339V$ at the crest of every cycle. Since the insulation has to hold back the largest voltage that ever appears across it, it must be able to withstand this peak of about 339V and not merely the quoted rms value of 240V.

3. When the current in an inductor changes it sets up a changing magnetic flux, which induces a back-EMF that opposes the change in the current. This back-EMF is largest when the current is changing fastest, which happens as the current passes through zero, and the applied voltage must rise to its own peak to overcome it. The voltage therefore peaks while the current is still climbing through zero, so the current does not reach its own peak until a quarter of a cycle later; that is, the current lags the voltage by 90°.

4. An ordinary ohmmeter measures opposition by sending a steady direct current through the component and noting the result. A capacitor, however, blocks direct current as soon as it is fully charged, so no steady current flows and the meter simply reads an extremely high, effectively infinite, value. The true opposition of a capacitor is its reactance $X_C = \frac{1}{2\pi fC}$, which appears only with alternating current and depends on the frequency, so it cannot be obtained from the direct-current test that an ohmmeter carries out.

5. The voltage across the resistor is in phase with the current, while the voltage across the inductor reaches its peak a quarter of a cycle earlier, so the two voltages never reach their maximum values at the same instant. Because they are 90° out of phase, they must be combined as the two perpendicular sides of a right-angled triangle, giving an impedance equal to the hypotenuse, $Z = \sqrt{R^2 + X_L^2} = \sqrt{(30\Omega)^2 + (40\Omega)^2} = 50\Omega$. The combined opposition is therefore 50Ω, less than the arithmetic sum of 70Ω, because a hypotenuse is always shorter than the sum of the other two sides.

6. The impedance of the circuit is $Z = \sqrt{R^2 + (X_L - X_C)^2}$, in which the inductive reactance rises with frequency while the capacitive reactance falls. At one particular frequency the two reactances become equal and cancel, leaving the impedance at its smallest value, the resistance R alone, so the current $I = \frac{V}{Z}$ rises to a maximum. At any higher or lower frequency the reactances no longer cancel, the net reactance $(X_L - X_C)$ is no longer zero, the impedance is larger, and the current is therefore smaller.

7. A high quality factor makes the resonance curve tall and narrow, so the receiver responds strongly to the wanted station and only weakly to the stations on either side, giving the sharp tuning needed to pick one station out of many. If the quality factor were made far too high, the band of frequencies the circuit accepts would become so narrow that part of the broadcast signal would be cut off, so the reception would be weakened and distorted. A good receiver therefore needs a quality factor high enough to separate the stations but not so high as to spoil the signal it selects.

8. At the resonant frequency the inductor and the capacitor have equal reactances, so the current in the inductive branch and the current in the capacitive branch are equal in size but, being half a cycle apart in phase, opposite in direction at every instant. Where the branches rejoin these two currents very nearly cancel, so only a tiny net current is taken from the supply, even though each branch alone carries a large current that simply circulates to and fro around the loop. The parallel combination therefore behaves as a very high impedance at resonance, which is why it is used to reject that frequency.

9. The voltmeter reads the rms voltage and the ammeter reads the rms current, and their product, $V_{rms}I_{rms} = 240V \times 5.0A = 1200VA$, is only the apparent power. The power actually delivered is the true power $P = V_{rms}I_{rms}\cos\phi$, where ϕ is the phase angle between the current and the voltage, because only the part of the current in phase with the voltage does useful work. The delivered power equals 1200W only if the circuit is purely resistive, so that $\phi = 0$ and $\cos\phi = 1$; for any other load the phase difference makes it less.

10. A generator or transformer is limited by the greatest voltage its insulation can stand and the greatest current its windings can carry without overheating, and the product of these two is the apparent power, measured in volt-amperes. The useful power it actually supplies is $VI\cos\phi$, which depends on the power factor of whatever load happens to be connected, something the maker cannot know beforehand. The rating is therefore quoted in kilovolt-amperes, since that states the safe limits of voltage and current whatever the power factor of the load may be.

11. A transformer works by mutual induction, in which the changing flux set up by the primary must link the turns of the secondary if it is to induce an EMF in them. A common iron core offers an easy path that guides and concentrates almost the whole of this changing flux through the secondary coil, so the two coils are tightly coupled. If the coils were merely placed side by side in air, most of the primary's flux would spread out and miss the secondary, so very little EMF would be induced and the transformer would be extremely inefficient.

12. A transformer passes power from the primary to the secondary, and for an ideal transformer the power drawn from the mains equals the power delivered to the load, $V_pI_p = V_sI_s$. When more appliances are switched on across the secondary, the secondary current I_s rises to supply them; and since the primary voltage is fixed by the mains, the primary current I_p must rise in step to keep the input power equal to the larger output power. The transformer therefore draws more current from the mains exactly when more is demanded of its secondary.

13. Eddy currents are the currents induced in any solid piece of metal that sits in a changing magnetic flux. In a transformer the core is bathed in a strongly changing flux, so eddy currents are induced within it and waste energy as heat at the rate $P = I^2R$ in the core itself, which is why they are treated as an unwanted loss. In an induction cooker the same effect is the whole purpose: the changing flux from the coil drives eddy currents in the metal pan, and the pan's own resistance turns those currents into the heat that cooks the food.

EXERCISE 13B

14. Even with no phone connected, the primary winding is still joined across the a.c. mains, so an alternating current flows in it and sets up an alternating magnetic flux in the iron core. This continually changing flux causes hysteresis and eddy-current losses in the core, which appear as heat, while the resistance of the primary winding dissipates a little more. These iron losses continue for as long as the charger is plugged in, whether or not anything is being charged, so the charger keeps drawing a small power from the mains and grows slightly warm.

15. A fluorescent lamp gives out its light in brief bursts, brightest twice in every cycle and almost dark in between, because the power it takes, and so the light it gives, follows the square of the current and therefore peaks one hundred times a second on the 50Hz mains. If a fan blade moves on by one blade-spacing (or a whole number of them) in the time between two flashes, each flash catches a blade in the same place as the one before, so the eye sees the blades frozen; if the timing is slightly out, the blades seem to creep slowly forwards or backwards. The blades are not truly still; they are simply lit only at the instants when they have returned to almost the same position.

16. A running motor is also a generator, for as its coils turn in the magnetic field they generate a back-EMF that opposes the supply voltage, so the current drawn is fixed by the small difference between the supply and the back-EMF. At the moment of switching on the rotor is not yet turning, so there is no back-EMF, and only the low resistance and reactance of the windings limit the current, which is therefore very large. As the motor gathers speed the back-EMF grows and cancels more of the supply voltage, so the current falls to its modest running value.

17. The supply contains a large reservoir capacitor that is charged up to the peak of the rectified voltage while the set is working. A capacitor stores charge, $Q = CV$, and energy, $E = \frac{1}{2}CV^2$, and it can part with them only by driving a current through some conducting path. Once the set is switched off and unplugged there may be no such path, so the capacitor keeps its charge for a long time; if a person then bridges its terminals, the stored charge rushes out through them as a sudden current and delivers a shock.

18. An inductor opposes any change in the current through it by inducing a back-EMF proportional to the rate of change of current, $E = -L\frac{dI}{dt}$. When the switch is opened the current is forced almost instantly to zero, so $\frac{dI}{dt}$ is enormous and a very large back-EMF appears across the widening gap. This high voltage breaks down the air in the gap and drives a bright spark, which is why switching off an inductive load sparks far more fiercely than switching off a plain resistive lamp.

19. The mains completes a fixed number of cycles each second, because the supply authority holds the speed of its generators so steady that the frequency stays at 50Hz on average. A synchronous clock uses a small motor that turns in exact step with these cycles, so that it effectively counts them, and over a day the number of cycles is kept so close to $50 \times 60 \times 60 \times 24$ that the timekeeping is very accurate. Any cycles lost while the frequency dips are made up later when it is allowed to rise, so the long-term count, and hence the time shown, stays correct.

20. A battery can be charged only by a current that flows steadily into its positive terminal, that is by a direct current. If the secondary of the transformer were joined straight to the battery, the alternating current would drive charge into the battery during one half-cycle and draw the same charge out again during the next, so the average current over a cycle would be zero and the battery would gain nothing. A rectifier is therefore needed to turn the alternating current into a one-way current, so that every half-cycle delivers charge into the battery in the same direction.

21. A simple meter of this kind does not measure the rms value directly; it rectifies the waveform, measures its average value, and then multiplies by the form factor of a sine wave, 1.11, so that the scale reads the rms value of a sinusoid. This succeeds only because the mains is very nearly sinusoidal. When the waveform is chopped, as it is by a dimmer, its form factor is no longer 1.11, so the meter's fixed multiplication no longer turns the measured average into the true rms value, and the reading comes out wrong.

22. When a hand grips a live wire carrying the 50Hz mains, the alternating current passes through the muscles and makes them contract; and because the current reverses fifty times a second, the muscles are stimulated again and again and are held in a continuous clench, so the grip tightens on the wire and the person cannot release it. A direct current, by contrast, gives a single steady stimulus that tends to produce one large contraction, which often throws the victim clear of the conductor. This continual gripping is one of the reasons why alternating current at mains frequency is so dangerous.

23. A welding arc carries a very large current at a fairly low voltage, so the welder must turn the 240V mains into a supply of only a few tens of volts but of several hundred amperes. A step-down transformer, with far fewer turns on its secondary than on its primary, lowers the voltage in the ratio of the turns; and since the power is very nearly conserved, $V_pI_p = V_sI_s$, the lower secondary voltage is matched by a correspondingly larger secondary current. It is this large, low-voltage current, carried by the thick secondary wire, that melts the metal to make the weld.

24. The danger from an electric supply comes chiefly from the size of the current it can drive through the body, and for a given body resistance a lower voltage drives a smaller current. In a bathroom or beside a pond, where water lowers the resistance of the skin and makes a shock far more likely, the mains is stepped down by a transformer to a safe 12V, so that any current passing through a person would be small and harmless. The transformer also keeps the low-voltage circuit electrically separate from the mains, which adds a further measure of safety.

25. The appliances in a house are made to run on the 240V, 50Hz alternating supply, and many of them, such as motors, the transformers inside chargers, and fluorescent fittings, will not work on direct current at all. A battery, however, can provide only a low, steady direct voltage, so it cannot drive these appliances by itself. An inverter is therefore used to convert the battery's direct current into an alternating voltage of the right size and frequency, so that the appliances behave just as though they were connected to the mains.

EXERCISE 13C

26. The resistor voltage is in phase with the current, while the inductor and capacitor voltages are a quarter-cycle ahead of and behind it, so the supply voltage is their phasor sum:

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(60V)^2 + (80V - 50V)^2} = 67.1V$$

The resistance and the net reactance follow from the common current:

$$R = \frac{V_R}{I} = \frac{60V}{0.50A} = 120\Omega, \qquad X = \frac{V_L - V_C}{I} = \frac{30V}{0.50A} = 60\Omega$$

The supply voltage leads the current by:

$$\phi = \tan^{-1}\frac{V_L - V_C}{V_R} = \tan^{-1}\frac{30V}{60V} = 26.6°$$

and the average power, dissipated only in the resistor, is:

$$P = I^2R = (0.50A)^2 \times 120\Omega = 30W$$

The supply voltage is 67.1V, the resistance 120Ω, the net reactance 60Ω, the phase angle 26.6°, and the average power 30W.

27. The impedance at each frequency is the voltage divided by the current:

$$Z_1 = \frac{100V}{2.0A} = 50\Omega, \qquad Z_2 = \frac{100V}{1.4A} = 71.4\Omega$$

Since $Z^2 = R^2 + (2\pi fL)^2$ and the second frequency is double the first, the inductive reactance doubles, so subtracting the squared impedances removes R and leaves three times the first reactance squared:

$$3X_1^2 = Z_2^2 - Z_1^2 = (71.4\Omega)^2 - (50\Omega)^2 = 2598\,\Omega^2$$

where $X_1 = 2\pi(50Hz)L$, so:

$$X_1 = \sqrt{\frac{2598\,\Omega^2}{3}} = 29.4\Omega$$

The resistance and inductance then follow:

$$R = \sqrt{Z_1^2 - X_1^2} = \sqrt{(50\Omega)^2 - (29.4\Omega)^2} = 40.4\Omega$$

$$L = \frac{X_1}{2\pi \times 50Hz} = \frac{29.4\Omega}{314\,rad\,s^{-1}} = 0.094H$$

The coil has a resistance of about 40Ω and an inductance of about 0.094H.

28. The two sinusoids have different frequencies (50Hz and 150Hz), so the average of their product over a cycle is zero and their mean squares simply add. The rms current is therefore:

$$I_{rms} = \sqrt{\frac{I_1^2 + I_2^2}{2}} = \sqrt{\frac{(3.0A)^2 + (4.0A)^2}{2}} = 3.54A$$

and the average power dissipated in the resistor is:

$$P = I_{rms}^2R = (3.54A)^2 \times 10\Omega = 125W$$

The combined current has an rms value of 3.54A and dissipates 125W in the resistor.

29. A half-wave rectifier passes only the positive half-cycles. The mean (d.c.) value is that of one hump spread over a whole period:

$$I_{mean} = \frac{I_0}{\pi} = \frac{4.0A}{\pi} = 1.27A$$

Only the half-cycles that are present do any heating, and for a half-wave rectified sinusoid the rms value works out to be half the peak:

$$I_{rms} = \frac{I_0}{2} = \frac{4.0A}{2} = 2.0A$$

so the average power dissipated is:

$$P = I_{rms}^2R = (2.0A)^2 \times 20\Omega = 80W$$

The mean current is 1.27A, the rms current 2.0A, and the average power 80W.

30. The circuit resonates when the supply frequency equals the natural frequency:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.20H \times 2.0 \times 10^{-6}F}} = 252Hz$$

At resonance the reactances cancel and the impedance is just the coil's resistance, so the current is:

$$I = \frac{V}{R} = \frac{10V}{50\Omega} = 0.20A$$

The voltage across the capacitor is the current times its reactance, $X_C = \frac{1}{2\pi f_0 C} = 316\Omega$:

$$V_C = IX_C = 0.20A \times 316\Omega = 63V$$

and the quality factor, the factor by which the capacitor voltage exceeds the supply voltage, is:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{50\Omega}\sqrt{\frac{0.20H}{2.0 \times 10^{-6}F}} = 6.3$$

The circuit resonates at 252Hz, draws 0.20A, develops 63V across the capacitor, and has a quality factor of 6.3.

31. The quality factor is fixed by the ratio of the resonant frequency to the bandwidth:

$$Q = \frac{f_0}{\Delta f} = \frac{800kHz}{8.0kHz} = 100$$

The capacitance follows from $f_0 = \frac{1}{2\pi\sqrt{LC}}$:

$$C = \frac{1}{(2\pi f_0)^2 L} = \frac{1}{(2\pi \times 8.0 \times 10^5 Hz)^2 \times 2.0 \times 10^{-4} H} = 198pF$$

and the resistance from $Q = \frac{2\pi f_0 L}{R}$:

$$R = \frac{2\pi f_0 L}{Q} = \frac{2\pi \times 8.0 \times 10^5 Hz \times 2.0 \times 10^{-4} H}{100} = 10\Omega$$

Since $\Delta f = \frac{f_0}{Q}$, doubling the resistance halves the quality factor and so doubles the bandwidth to 16kHz. The circuit needs Q = 100, C = 198pF and R = 10Ω.

32. The turns ratio equals the voltage ratio:

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{480V}{9600V} = \frac{1}{20}$$

The transmission current is set by the power and the line voltage:

$$I = \frac{P}{V_s} = \frac{24000W}{9600V} = 2.5A$$

so the power wasted as heat in the cable is:

$$P_{loss} = I^2 R = (2.5A)^2 \times 6.0\Omega = 37.5W$$

and the fraction of the generated power reaching the far end is:

$$\frac{24000W - 37.5W}{24000W} \times 100\% = 99.8\%$$

The transformer steps up in the ratio 1:20, the cable carries 2.5A and wastes only 37.5W, so 99.8% of the power is delivered.

33. At full load the output power is the rated apparent power times the power factor:

$$P_{out} = 10000VA \times 0.80 = 8000W$$

Both losses act, so the efficiency is:

$$\eta = \frac{P_{out}}{P_{out} + P_{cu} + P_{Fe}} = \frac{8000W}{8000W + 200W + 80W} = 96.6\%$$

At half load the output halves, but the copper loss falls with the square of the load while the iron loss stays the same:

$$P_{cu}' = 200W \times (0.5)^2 = 50W$$

$$\eta' = \frac{4000W}{4000W + 50W + 80W} = 96.9\%$$

The efficiency is 96.6% at full load and 96.9% at half load, the half-load value being higher because the copper loss has fallen faster than the output.

34. Each branch current is the supply voltage divided by that branch's opposition:

$$I_R = \frac{100V}{50\Omega} = 2.0A, \quad I_L = \frac{100V}{25\Omega} = 4.0A, \quad I_C = \frac{100V}{40\Omega} = 2.5A$$

The resistor current is in phase, the inductor current lags by 90° and the capacitor current leads by 90°, so the line current is:

$$I = \sqrt{I_R^2 + (I_L - I_C)^2} = \sqrt{(2.0A)^2 + (4.0A - 2.5A)^2} = 2.5A$$

lagging the supply voltage by:

$$\phi = \tan^{-1}\frac{I_L - I_C}{I_R} = \tan^{-1}\frac{1.5A}{2.0A} = 36.9°$$

The line current is least when the two reactive branch currents cancel, that is when the capacitor's reactance is made equal to the inductor's, 25Ω; the line current then falls to $I_R = 2.0A$, drawn in phase with the supply. So the branch currents are 2.0A, 4.0A and 2.5A, the line current 2.5A lagging by 36.9°, and the least possible line current is 2.0A.

35. The apparent power is the product of the two meter readings:

$$S = VI = 240V \times 8.0A = 1920VA$$

so the power factor is the ratio of the true power to the apparent power:

$$\cos\phi = \frac{P}{S} = \frac{1200W}{1920VA} = 0.625, \qquad \phi = \cos^{-1}0.625 = 51.3°$$

The reactive power is:

$$Q = \sqrt{S^2 - P^2} = \sqrt{(1920VA)^2 - (1200W)^2} = 1.50 \times 10^3 VAR$$

A purely resistive load of the same true power would draw only $\frac{1200W}{240V} = 5.0A$, so the low power factor forces the supply to carry an extra 3.0A. The power factor is 0.625, the phase angle 51.3°, the reactive power 1.50kVAR, and 3.0A of extra current is drawn.

36. The peak current is $I_0 = 2.0A$, and the greatest energy is stored when the current is at its peak:

$$E_{max} = \frac{1}{2}LI_0^2 = \frac{1}{2} \times 0.50H \times (2.0A)^2 = 1.0J$$

The stored energy depends on i^2, which reaches a maximum twice in every current cycle, so the energy oscillates at twice the supply frequency, that is at $2 \times 50Hz = 100Hz$. The rms current is:

$$I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{2.0A}{\sqrt{2}} = 1.41A$$

and the peak voltage across the inductor, whose reactance is $X_L = 2\pi fL = 157\Omega$, is:

$$V_0 = I_0 X_L = 2.0A \times 157\Omega = 314V$$

The inductor stores at most 1.0J, the energy surging at 100Hz, while the rms current is 1.41A and the peak voltage 314V.

37. At 50Hz the two reactances are:

$$X_L = 2\pi fL = 2\pi \times 50Hz \times 0.30H = 94.2\Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50Hz \times 20 \times 10^{-6}F} = 159.2\Omega$$

so the impedance and current are:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(100\Omega)^2 + (94.2\Omega - 159.2\Omega)^2} = 119\Omega$$

$$I = \frac{V}{Z} = \frac{240V}{119\Omega} = 2.01A$$

The circuit is net capacitive, so the power factor leads, and the average power is:

$$\cos\phi = \frac{R}{Z} = \frac{100\Omega}{119\Omega} = 0.84, \qquad P = I^2R = (2.01A)^2 \times 100\Omega = 405W$$

The power factor becomes unity at the resonant frequency, where $X_L = X_C$:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.30H \times 20 \times 10^{-6}F}} = 65.0Hz$$

where the impedance is R alone, so the current rises to $\frac{240V}{100\Omega} = 2.40A$. At 50Hz the current is 2.01A at a leading power factor of 0.84, dissipating 405W; unity power factor occurs at 65.0Hz, where the current is 2.40A.

38. The turns ratio equals the voltage ratio:

$$\frac{N_p}{N_s} = \frac{240V}{9.0V} = 26.7$$

The peak of the secondary voltage is:

$$V_0 = V_{rms}\sqrt{2} = 9.0V \times \sqrt{2} = 12.7V$$

Full-wave rectification gives a mean (d.c.) output of $\frac{2V_0}{\pi}$:

$$V_{dc} = \frac{2V_0}{\pi} = \frac{2 \times 12.7V}{\pi} = 8.1V$$

and the mean charging current is:

$$I_{dc} = \frac{V_{dc}}{R} = \frac{8.1V}{1.5\Omega} = 5.4A$$

The turns ratio is about 27:1, the secondary peak 12.7V, the d.c. output 8.1V, and the mean charging current 5.4A.

39. The angular speed is $\omega = 2\pi \times \frac{3000}{60} = 314\,\text{rad s}^{-1}$ and the area is $A = 200cm^2 = 0.020m^2$, so the peak EMF is:

$$E_0 = NBA\omega = 500 \times 0.040T \times 0.020m^2 \times 314\,\text{rad s}^{-1} = 126V$$

The rms EMF and the rms current are:

$$E_{rms} = \frac{E_0}{\sqrt{2}} = \frac{126V}{\sqrt{2}} = 88.9V, \qquad I_{rms} = \frac{E_{rms}}{R} = \frac{88.9V}{100\Omega} = 0.889A$$

and the average power delivered to the resistor is:

$$P = I_{rms}^2 R = (0.889A)^2 \times 100\Omega = 79W$$

The generator gives a peak EMF of 126V (88.9V rms), driving 0.889A and delivering 79W to the resistor.

40. At a power factor of 0.60 the line current is set by the true power, the voltage and the power factor:

$$I_1 = \frac{P}{V\cos\phi} = \frac{30000W}{400V \times 0.60} = 125A$$

so the power wasted in the feeder is:

$$P_1 = I_1^2 R = (125A)^2 \times 0.20\Omega = 3125W$$

After the capacitor bank raises the power factor to 0.90, the line current and feeder loss become:

$$I_2 = \frac{30000W}{400V \times 0.90} = 83.3A, \qquad P_2 = I_2^2 R = (83.3A)^2 \times 0.20\Omega = 1389W$$

so the feeder loss falls by:

$$\frac{3125W - 1389W}{3125W} \times 100\% = 55.6\%$$

Correcting the power factor cuts the line current from 125A to 83.3A and the feeder loss from 3125W to 1389W, a saving of about 56%, all without changing the useful 30kW the workshop consumes.