Chapter 10

FLUID MECHANICS

INTRODUCTION

Picture this scene: It is a hot Saturday afternoon in Sinza, Dar es Salaam. A group of boys are washing a car in the yard using a garden hose connected to a tap. The water flows out gently, splashing harmlessly onto the dusty bonnet. Then one of the boys, feeling clever, presses his thumb over the end of the hose.

The water explodes outward in a thin, powerful jet. It shoots across the yard, misses the car entirely, and hits a neighbouring *mama lishe* square in the back while she is frying *sambusa*. Chaos. Shouting. A plate of *sambusa* hits the ground. The boy releases his thumb immediately, the jet dies back to a gentle flow, and he spends the rest of the afternoon apologising and buying replacement *sambusa* with money he had been saving for airtime.

But here is the physics question nobody at the scene thought to ask: why did the water speed up? The tap pressure did not change. Nobody pumped harder. The boy simply narrowed the opening. Yet the water, passing through the smaller gap, shot out with dramatically more speed. Where did the extra velocity come from?

Now picture a second scene. After the apology, the boy notices a sewing needle on the ground. He drops it into a bucket of water. It sinks, as steel should. But then his younger sister places an identical needle *flat* on the water surface, with the care of a person laying a ruler on a sleeping cat. The needle floats. Steel, denser than water, sitting on the surface as though resting on an invisible mattress. How?

And one more. That evening, the boy watches his mother pour honey into hot chai. She tilts the jar. The honey responds like a government office at lunchtime: slowly, reluctantly, as though every millimetre of progress required a committee meeting, a signed form, and three cups of chai. Meanwhile the chai swirls freely in the cup. Same gravity pulls both. One flows eagerly. The other resists every step. Why?

Three ordinary moments. Three mysteries. And behind all three lies a single, beautiful truth: **fluids have three hidden personalities.**

The first personality is **surface tension**: an invisible elastic skin on every liquid surface, pulling inward, resisting anything that tries to break through. The flat needle spreads its weight across this skin; a needle dropped point-first punctures it and sinks. Surface tension is why raindrops are spherical, why insects walk on ponds, and why water climbs up narrow tubes against gravity.

The second personality is **Bernoulli's equation**: when a fluid speeds up, its pressure drops. This single principle explains how aeroplanes generate lift, why roofs blow *off* during storms instead of being crushed *in*, and why a boy's thumb turned a harmless garden hose into a *sambusa*-destroying weapon.

The third personality is **viscosity**: internal friction between fluid layers that resists flow. It is why honey ignores gravity's polite requests, why engines in Moshi need different oil from engines in Dar es Salaam, and how Stokes' law predicts the speed at which a sphere sinks through any fluid you care to drop it into.

Welcome to fluid mechanics, the final chapter of this volume and the gateway from rigid bodies to the physics of things that flow, deform, and reshape themselves. By the end, you will understand why that boy's thumb turned a gentle stream into a jet, why his sister's needle refused to sink, and why his mother's honey took its own sweet time. You will even understand why the *sambusa* vendor was perfectly right to be angry, but wrong to blame the boy. The real culprit was Bernoulli.

PRESSURE IN FLUIDS

Before we explore the three personalities of fluids, we need to make sure we speak the same language. Pressure is the foundation on which surface tension, Bernoulli's equation, and viscosity are all built. You met these ideas at O-level; this section sharpens them for the work ahead. Think of it as tuning your instrument before the concert begins.

What is Pressure?

When you stand on soft mattress, your feet sink in. When you lie flat on the same mattress, you do not sink as much, even though your weight has not changed. The difference is not in the force, but in how that force is *spread*. Standing concentrates your entire weight on the small area of your soles. Lying down distributes the same weight over a much larger area. The quantity that captures this idea is called **pressure**.

Pressure is defined as the *force acting per unit area perpendicular to the surface:*

$$P = \frac{F}{A}$$

Where:

P = pressure (measured in pascals, Pa)

F = force perpendicular to the surface (N)

A = area over which the force acts (m^2)

Since $1\ \text{Pa} = 1\ \text{N/m}^2$, the pascal is a remarkably small unit. Atmospheric pressure, for instance, is approximately $101{,}325\ \text{Pa}$, which is why we often use kilopascals (kPa) or the atmosphere ($1\ \text{atm} = 101{,}325\ \text{Pa}$) as practical units.

One crucial property of pressure in fluids deserves emphasis: *pressure at a point in a fluid acts equally in all directions.* This is fundamentally different from a solid surface, where force acts in a specific direction. A fluid pushes upward, downward, and sideways with equal intensity at any given depth. This is why a balloon submerged in water is squeezed equally from all sides, not just from above.

Pressure at a Depth in a Fluid

Consider a fluid of density ρ at rest in an open container. We want to find the pressure at a depth h below the surface. Imagine a thin horizontal disc of area A at this depth. The fluid above the disc forms a column of height h and cross-sectional area A.

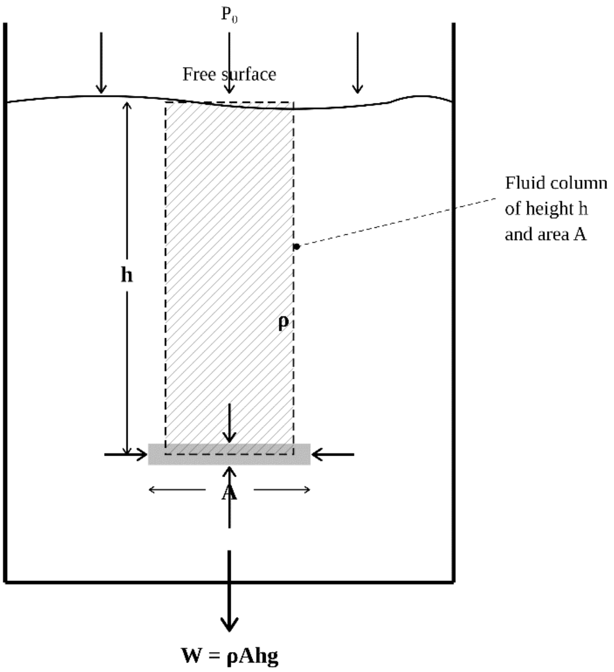


Figure: *Pressure at a depth h in a fluid of density ρ. The hatched column (height h, cross-sectional area A) has weight W = ρAhg. Arrows at the disc show that pressure acts equally in all directions. P_0 is the atmospheric pressure at the free surface.*

The volume of this column is:

$$V = Ah$$

The mass of the fluid in the column is:

$$m = \rho V = \rho A h$$

The weight of the fluid column, which acts downward on the disc, is:

$$W = mg = \rho A h g$$

This weight is spread over the area A of the disc, so the pressure due to the fluid column alone is:

$$P = \frac{W}{A} = \frac{\rho Ahg}{A}$$

$$\mathbf{P = \rho hg}$$

This is the pressure due to the fluid itself at depth h. It depends only on the depth, the density of the fluid, and the acceleration due to gravity. It does not depend on the shape of the container or the total amount of fluid. A narrow tube and a wide lake, filled with the same fluid to the same depth, produce the same pressure at the bottom.

The **total pressure** (also called **absolute pressure**) at depth h in an open container includes the atmospheric pressure P_0 acting on the surface:

$$\mathbf{P_{abs} = P_0 + \rho hg}$$

Pascal's Principle

Pascal's Principle states that *a change in pressure applied to an enclosed fluid at rest is transmitted undiminished to every point in the fluid and to the walls of the container.* In simpler terms: if you push harder at one point in a closed fluid system, every other point feels the same increase in pressure.

This principle is the foundation of all hydraulic machines. Consider a hydraulic press with a small piston of area A_1 and a large piston of area A_2, connected by an enclosed fluid. When a force F_1 is applied to the small piston, the pressure increase in the fluid is $P = \frac{F_1}{A_1}$. By Pascal's Principle, this same pressure acts on the large piston, producing an output force:

$$\mathbf{F_2} = P \times A_2 = \mathbf{F_1} \times \frac{\mathbf{A_2}}{\mathbf{A_1}}$$

Since $A_2 > A_1$, the output force F_2 is greater than the input force F_1. The hydraulic press is a force multiplier. This is exactly how the braking system of a daladala works: a small force applied by the driver's foot on the brake pedal is transmitted through brake fluid and amplified at the brake pads, producing enough force to stop a vehicle carrying passengers and their luggage.

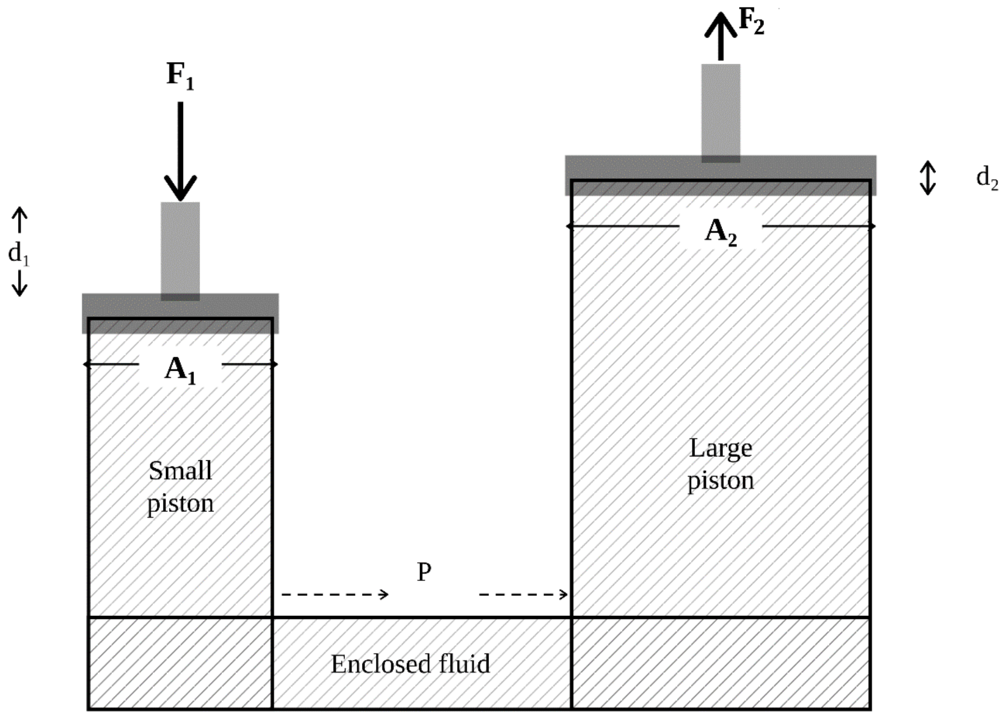


Figure: *A hydraulic press. A force F_1 applied to the small piston (area A_1) is transmitted through the enclosed fluid to the large piston (area A_2), producing a larger output force F_2. The small piston moves a distance d_1 while the large piston moves a smaller distance d_2.*

An important consequence: while the force is multiplied, **energy is conserved**. The small piston moves a larger distance d_1 while the large piston moves a smaller distance d_2, such that the work done on both sides is equal:

$$\mathbf{F_1 \times d_1 = F_2 \times d_2}$$

A hydraulic press gives you more force, but not more work. There is no free lunch in physics.

Atmospheric Pressure and Gauge Pressure

Atmospheric pressure *is the pressure exerted by the weight of the atmosphere above a given point.* At sea level, its standard value is approximately:

$$P_0 = 1.013 \times 10^5 \text{ Pa} = 101.3 \text{ kPa} = 1 \text{ atm}$$

This is the pressure that acts on every surface exposed to the atmosphere, including the surface of a liquid in an open container. It is the reason the equation of absolute pressure P_{abs} includes P_0.

Gauge pressure *is the pressure measured above atmospheric pressure.* When a tyre pressure gauge reads 200 kPa, it means the air inside the tyre is at 200 kPa above atmospheric pressure. The absolute pressure inside the tyre is therefore:

$$\mathbf{P_{abs} = P_0 + P_{gauge}} = 101.3 \text{ kPa} + 200 \text{ kPa} = 301.3 \text{ kPa}$$

In most problems in this chapter, when we write $P = h\rho g$, we are calculating gauge pressure. Always read the question carefully to determine whether atmospheric pressure should be included.

With the language of pressure now freshly sharpened, let us put it to work. The following examples are not meant to challenge you; they are meant to remind your hands how to hold the tools before we use those tools to build something extraordinary in the subtopics ahead.

BINDER Example 1

A cylindrical water storage tank in a village in Dodoma is filled to a depth of 4.0 m. Calculate the pressure exerted by the water at the base of the tank. Take the density of water as 1000 kg/m^3 and $g = 9.8 \text{ m/s}^2$.

Solution

Using:

$$P = h\rho g$$

$$P = 4.0 \text{ m} \times 1000 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 = 39{,}200 \text{ Pa} = 39.2 \text{ kPa}$$

Therefore, the pressure due to the water at the base of the tank is 39.2 kPa.

Making Sense of the Answer: *This is about 39% of atmospheric pressure. Even a modest 4-metre column of water produces significant pressure, which is why water tanks are placed on elevated platforms: the height creates the pressure that pushes water through the pipes to your tap.*

Think Like a Physicist: *The pressure at the base depends only on the depth of the water, not on the width or shape of the tank. A narrow pipe and a wide tank, both filled to* 4.0 m, *give exactly the same pressure at the bottom. This is called the hydrostatic paradox, and it surprises students every time.*

REAL Example 2

While on a school trip to Mbudya Island, **Kipanga** decides to dive into the sea. As he goes deeper, he notices a sharp, uncomfortable pain in his ears. Back on the surface, he tells Mr. Akilikubwa:

"Sir, the deeper I went, the more my ears hurt. But the water was not hitting my ears from above. The pain felt like it was coming from all sides. How can water push sideways?"

Explain why Kipanga's ears hurt more as he dived deeper, and why the pressure felt as though it came from all directions.

Solution

The pressure at a depth h in a fluid is given by $P = \rho h g$. As Kipanga dives deeper, h increases, so the pressure on his body increases.

The pain comes from all directions because pressure in a fluid at rest acts equally in all directions at a given depth. The water does not just push downward like a weight sitting on a table. It pushes sideways, upward, and at every angle with equal intensity. Kipanga's eardrums, which are vertical membranes, are pushed inward by horizontal water pressure. This is why the pain feels like it comes from everywhere, not just from above.

Making Sense of the Answer: *Pressure in a fluid increases significantly with depth. Even at just 3 metres, the additional pressure is about 30% of atmospheric pressure. Professional divers descend to 30m or more, where the water pressure alone exceeds 3 atmospheres. Without equalising the pressure in their ears (by pinching the nose and blowing gently), the eardrums can be seriously damaged.*

Think Like a Physicist: *Whenever you encounter pressure in a fluid, remember that it is not directional like force. Pressure at a point acts equally in every direction. Drawing a single downward arrow to represent fluid pressure is a common mistake. Instead, imagine arrows pointing inward from all sides.*

HOT Example 3

A hydraulic press has a small piston of radius 2.0cm and a large piston of radius 15cm. A force of 50N is applied to the small piston.

(a) Find the force exerted by the large piston.

(b) If the small piston is pushed down by 20cm, find the distance the large piston moves upward.

(c) Calculate the work done by the input force and the work done by the output force. Comment on your result.

Solution

(a) Finding the output force:

Area of small piston:

$$A_1 = \pi r_1^2 = \pi \times (0.020\text{ m})^2 = 1.257 \times 10^{-3}\text{ m}^2$$

Area of large piston:

$$A_2 = \pi r_2^2 = \pi \times (0.15\text{ m})^2 = 7.069 \times 10^{-2}\text{ m}^2$$

By Pascal's Principle, pressure is transmitted equally:

$$\frac{F_2}{A_2} = \frac{F_1}{A_1}$$

$$F_2 = F_1 \times \frac{A_2}{A_1}$$

$$F_2 = 50\text{ N} \times \frac{7.069 \times 10^{-2}\text{ m}^2}{1.257 \times 10^{-3}\text{ m}^2} = 2812.5\text{N}$$

The force exerted by the large piston is 2812.5N.

(b) Since the fluid is incompressible, the volume of fluid pushed down by the small piston equals the volume displaced upward at the large piston:

$$A_1 d_1 = A_2 d_2$$

$$d_2 = d_1 \times \frac{A_1}{A_2}$$

$$d_2 = 0.20\text{m} \times \frac{1.257 \times 10^{-3}\text{ m}^2}{7.069 \times 10^{-2}\text{ m}^2} = 3.56 \times 10^{-3}\text{ m} = 3.56\text{mm}$$

The distance moved by large piston is 3.56×10^{-3} m or 3.56mm.

(c) Comparing work done:

Work done by the input force:

$$W_1 = F_1 \times d_1 = 50\text{ N} \times 0.20\text{ m} = 10.0\text{J}$$

Work done by the output force:

$$W_2 = F_2 \times d_2 = 2812.5\text{ N} \times 3.56 \times 10^{-3}\text{ m} = 10.0\text{J}$$

Comment: The work done is the same on both sides. The hydraulic press multiplies force, but it does not multiply energy. Energy is conserved.

Making Sense of the Answer: *A small* 50N *force (about the weight of 5kg) produced an output force of over* 2800N *(about the weight of 280kg). That is a force multiplication factor of 56.25, which is simply the ratio of the piston areas. But the large piston moved only 3.56mm while the small piston moved 200mm. Enormous force, tiny movement. This is the trade-off at the heart of every hydraulic machine, from car jacks to the brakes on a daladala.*

Think Like a Physicist: *Whenever a machine multiplies force, always ask: what is the cost? In levers, the cost is distance. In hydraulic systems, the cost is also distance. In gears, the cost is angular speed. Nature never gives you more force for free. Conservation of energy is the referee that keeps all machines honest.*

That completes our brief review of pressure. The tools are sharpened: $\mathrm{P} = \frac{\mathrm{F}}{\mathrm{A}}$ tells us what pressure is, $\mathrm{P} = \mathrm{h\rho g}$ tells us how pressure builds with depth, and Pascal's Principle tells us how pressure is transmitted through enclosed fluids. These three ideas will reappear throughout the chapter, sometimes quietly in the background, sometimes at the centre of a derivation.

In the next section, we meet the first hidden personality of fluids: **surface tension**. We begin at the molecular level, where the physics of why a needle can float on water and why raindrops are spherical has been hiding all along.

SURFACE TENSION: MOLECULAR THEORY AND SURFACE ENERGY

We now meet the first hidden personality of fluids. In the introduction, we watched a steel needle float on water and asked: *how can something denser than water refuse to sink?* The answer lives at the surface of the liquid, where molecules behave differently from those buried deep inside. This subtopic builds the explanation from the molecular level upward, and by the end of it, the floating needle will make perfect sense.

The Concept of Surface Tension

You have seen this personality before, even if you did not know its name. Water droplets on a freshly waxed car form nearly perfect spheres instead of spreading flat. A steel sewing needle, placed carefully on a still water surface, floats even though steel is nearly eight times denser than water. Insects called pond skaters walk across ponds without breaking the surface. In every case, the liquid surface behaves as though it is covered by a thin, invisible elastic skin that resists being stretched or broken.

This behaviour is called **surface tension**. It is not caused by an actual membrane. There is no physical skin on the surface of water. The effect arises entirely from the forces between molecules, as we will see later. But the elastic-skin analogy is useful because it captures the key observation: the surface of a liquid resists any attempt to increase its area, just as a stretched rubber sheet resists being pulled further.

To define surface tension precisely, imagine a straight line drawn on the surface of a liquid, dividing it into two halves. The molecules on one side of the line pull on the molecules on the other side with a force that acts along the surface and perpendicular to the line. Surface tension is defined as this *force per unit length:*

$$\boldsymbol{\gamma = \frac{F}{l}}$$

where γ is the coefficient of surface tension, F is the force acting along the surface perpendicular to the line, and l is the length of the line.

The SI unit of surface tension is N/m (newtons per metre), which is equivalent to $\mathrm{Nm^{-1}}$. Its dimensions are:

$$\gamma = \frac{\mathrm{MLT^{-2}}}{\mathrm{L}} = \mathrm{MT^{-2}}$$

An important distinction: **surface tension** *is the physical phenomenon (the tendency of a liquid surface to contract),* while the **coefficient of surface tension ($\boldsymbol{\gamma}$)** *is the numerical value that measures how strong this tendency is.*

Molecular Theory of Surface Tension

Why does the surface behave like a stretched skin? The answer lies in the forces between molecules.

Every molecule in a liquid attracts its neighbours. This *attractive force between molecules of the same substance* is called a **cohesive force**. The range of this force is very short, typically a few molecular diameters (about 10^{-9}m). Beyond this range, the force is effectively zero.

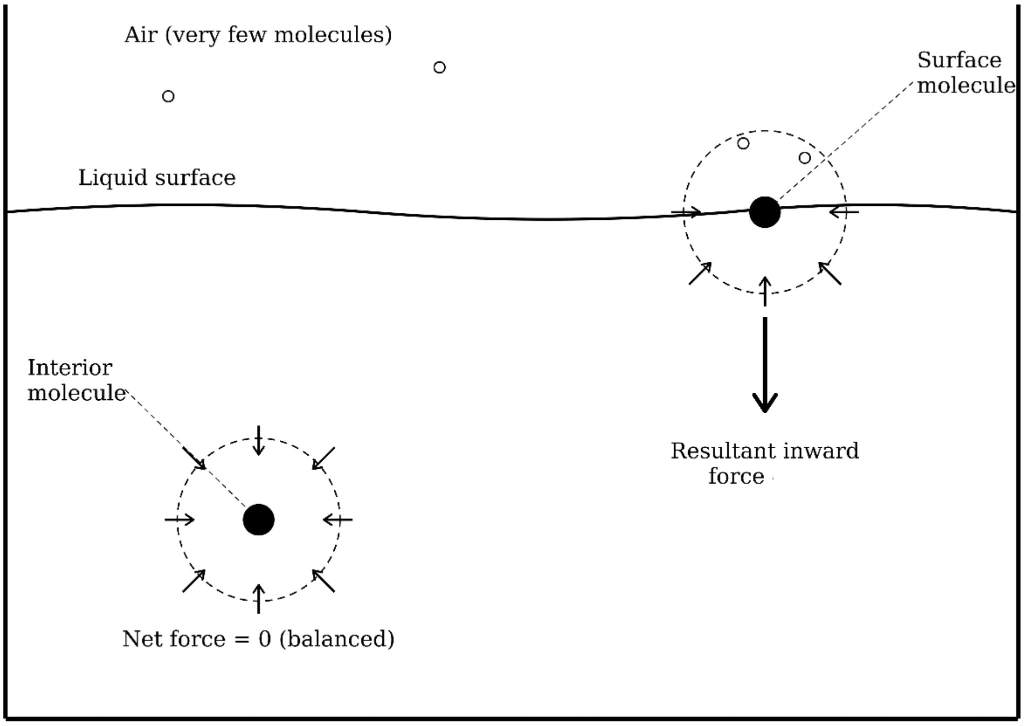


Figure: ***Molecular forces in a liquid.*** *Interior molecule (deep inside) is pulled equally in all directions by its neighbours; the net force is zero. Surface molecule has very few neighbours above; the unbalanced cohesive forces produce a net inward force pulling it into the bulk of the liquid.*

Now consider what happens to molecules at different positions in the liquid.

Interior molecule (deep inside the liquid): This molecule is surrounded by other molecules on all sides. The cohesive forces pull it equally in every direction. The net force on the molecule is zero. It is in equilibrium and can move freely within the liquid.

Surface molecule: This molecule has neighbours below it and to its sides, but very few above it (only the sparse molecules of air or vapour). The cohesive forces from below and from the sides are not balanced by forces from above. The result is a **net inward force** pulling the molecule downward, into the bulk of the liquid.

This net inward force on every surface molecule has two important consequences:

First consequence: Surface molecules are pulled inward, which means the liquid surface tends to **contract to the smallest possible area**. For a given volume of liquid, the shape with the smallest surface area is a sphere. This is why free-falling liquid drops are spherical (in the absence of air resistance distorting them). It is not that something pushes the drop into a sphere; rather, the surface contracts until no further contraction is possible, and that minimum-area shape is a sphere.

Second consequence: Because surface molecules are pulled inward, they must have been **pulled away from their neighbours** compared to molecules in the bulk. Work was done against cohesive forces to bring them to the surface. This means surface molecules have **higher potential energy** than interior molecules. The liquid surface is therefore a region of stored energy, which is why it resists being stretched: stretching the surface means bringing more molecules from the interior to the surface, which requires work.

This is the molecular origin of surface tension. The “elastic skin” (surface tension) is *really the collective pull of many (billions) surface molecules trying to* ***minimise the surface area*** *and* ***reduce their excess potential energy***.

A common misconception is that surface tension is merely a description, not a real force. This is wrong. Surface tension is measurable, it does work, and it can support the weight of objects (like a needle or an insect). It is as real as the tension in a stretched rubber band. The difference is that a rubber band has material connecting its parts, while the “membrane” on a liquid surface is made of nothing but molecular forces.

Surface Energy

Since surface molecules have higher potential energy than interior molecules, any increase in surface area requires energy. This energy is called **surface energy**.

Consider stretching a film of liquid on a horizontal rectangular frame with a sliding wire of length l, as shown in the following figure.

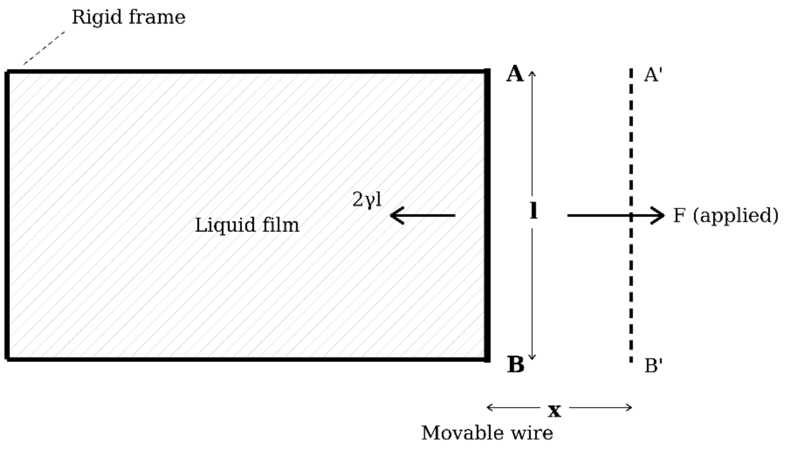


Figure: ***Stretching a liquid film on a rectangular frame.*** *The movable wire AB of length l is held in place against the surface tension force 2γl by an applied force F. When pulled to position A'B' through a distance x, the work done is W=2γlx.*

The liquid film has two free surfaces (upper and lower), both in contact with the wire. Each surface exerts a force of γl on the wire, pulling it inward (trying to contract the film). The total surface tension force on the wire is therefore:

$$F = 2\gamma l$$

To hold the wire in place, an equal and opposite external force must be applied. If the wire is now pulled outward through a distance x (slowly, so kinetic energy is negligible), the work done against the surface tension force is:

$$W = F \times x = 2\gamma l \times x$$

The increase in surface area is $\Delta A = 2lx$ (because both the upper and lower surfaces increase by lx each). The work done per unit increase in surface area is:

$$\delta = \frac{W}{\Delta A} = \frac{2\gamma lx}{2lx}$$

$$\boldsymbol{\delta = \gamma}$$

This is an elegant result: the **surface energy** (*work done per unit area in extending the liquid surface*) **is numerically equal to the coefficient of surface tension**. This means surface tension can be interpreted in two equivalent ways: *as a force per unit length (Nm^{-1}), or as energy per unit area (Jm^{-2})*. Both descriptions are correct and both have the same numerical value.

For **a single liquid surface** (such as *the surface of a liquid drop* or an *air bubble in liquid*), the work done to increase the surface area by ΔA is:

$$\mathbf{W = \gamma \Delta A}$$

For *a soap film* or any thin film with **two free surfaces**, the work done is:

$$\mathbf{W = 2\gamma \Delta A}$$

The factor of 2 appears because a soap film always has an inner surface and an outer surface, and both must be stretched.

Factors Affecting Surface Tension

Surface tension is not a fixed property. It depends on the conditions:

1. Nature of the liquid (strength of intermolecular forces)

This is the most fundamental factor. Surface tension exists because cohesive forces pull surface molecules inward. The stronger these forces, the stronger the inward pull, and the higher the surface tension. Mercury, whose atoms are held together by strong metallic bonds, has a very high surface tension ($\gamma \approx 0.47\text{Nm}^{-1}$). Water, with its strong hydrogen bonds, has $\gamma \approx 0.073\text{Nm}^{-1}$. Ethanol, with weaker hydrogen bonds, has $\gamma \approx 0.022\text{Nm}^{-1}$. Oils and organic solvents, held together only by weak van der Waals forces, have even lower values (typically 0.020 to 0.030Nm^{-1}).

2. Temperature

The surface tension of most liquids **decreases** with increasing temperature. As temperature rises, molecules move faster and the average distance between them increases. The cohesive forces weaken, and the surface contracts less strongly. This is why hot water wets surfaces more easily than cold water, and why washing with warm water is more effective than washing with cold water.

3. Impurities and dissolved substances

Adding substances to a liquid can either increase or decrease its surface tension. Soap and detergents **decrease** the surface tension of water dramatically (from 0.073Nm^{-1} to about 0.025Nm^{-1}), which is why soapy water spreads more easily and is better at wetting surfaces and removing grease. Detergent molecules have one end that attracts water and one end that repels it; they wedge themselves into the surface layer and weaken the cohesive forces between water molecules, reducing the surface tension. On the other hand, dissolving salt in water **increases** its surface tension slightly, because the dissolved ions strengthen the attractive forces between water molecules near the surface. This is why saltwater forms slightly larger droplets than freshwater.

4. Nature of the medium above the surface

Surface tension depends not only on the liquid itself but also on what is above it. The surface tension of water in contact with air is 0.073Nm^{-1}, but the surface tension of water in contact with oil is different, because oil molecules above the surface exert some attractive force on the water molecules, partially compensating the inward pull. In general, the greater the difference in intermolecular forces between the liquid and the medium above it, the higher the surface tension at the interface.

5. Electrification

When an electric charge is placed on the surface of a liquid, the surface tension **decreases**. The charged surface molecules repel each other, which opposes the inward pull of cohesive forces and makes the surface easier to stretch. This effect is used in electrospraying, where a high voltage causes a liquid to break into a fine mist of tiny droplets.

With these factors, the theory is now in place. You now understand what surface tension is, why it exists at the molecular level, and how it connects to surface energy. Before the next section arrives, let us anchor these ideas with worked examples that put the formulas to work.

BINDER Example 4

A rectangular wire frame has a sliding wire of length 15cm that can move freely along the frame. The frame is dipped in a soap solution and a thin film forms across it. Calculate the force required to hold the sliding wire in equilibrium. Take the surface tension of the soap solution as 0.025Nm^{-1}.

Solution

The soap film has two free surfaces (upper and lower), both in contact with the sliding wire. The total surface tension force pulling the wire inward is:

$$\text{F} = 2\gamma\text{l}$$

$$\text{F} = 2 \times 0.025\text{Nm}^{-1} \times 0.15\text{m} = 7.5 \times 10^{-3}\text{N}$$

Therefore, the force required to hold the sliding wire in equilibrium is $7.5 \times 10^{-3}\text{N}$ (or 7.5mN).

Making Sense of the Answer: *This is a tiny force, less than the weight of a one-gram mass. Yet it is enough to visibly pull the wire inward when released. Surface tension forces are small by everyday standards, but they dominate at small scales, which is why they matter for drops, bubbles, needles, and insects, but not for swimming pools or oceans.*

Think Like a Physicist: *The factor of 2 appears because the film has two surfaces. Forgetting this factor is one of the most common errors in surface tension problems. Whenever you see the word "**film**," immediately think: **two surfaces**. Whenever you see "**drop**" or "**bubble in liquid**," think: **one surface**. Whenever you see "**soap bubble in air**," think: **two surfaces** (inner and outer). Get this right, and half the battle is won!*

BINDER Example 5

A soap film is formed on a rectangular frame measuring 8.0cm by 5.0cm. The surface tension of the soap solution is 0.030Nm^{-1}. Calculate the surface energy of the film.

Solution

The area of one surface of the film is:

$$\text{A} = 8.0 \times 10^{-2}\text{m} \times 5.0 \times 10^{-2}\text{m} = 4.0 \times 10^{-3}\text{m}^2$$

Since a soap film has two free surfaces, the total surface area is:

$$\Delta\text{A} = 2\text{A} = 2 \times 4.0 \times 10^{-3}\text{m}^2 = 8.0 \times 10^{-3}\text{m}^2$$

The surface energy (work stored in the film) is:

$$\text{W} = \gamma\Delta\text{A}$$

$$\text{W} = 0.030\text{Nm}^{-1} \times 8.0 \times 10^{-3}\text{m}^2 = 2.4 \times 10^{-4}\text{J}$$

Therefore, the surface energy of the soap film is $2.4 \times 10^{-4}\text{J}$ (or 0.24mJ).

Making Sense of the Answer: *This is an extremely small amount of energy, far less than the energy needed to lift a grain of rice. But remember: this energy is spread over a very thin film. Per unit area, the energy density is significant enough to hold the film together and resist stretching. Scale matters in physics.*

Think Like a Physicist: *Notice that we used $W = \gamma\Delta A$ (not $W = 2\gamma\Delta A$) because we already accounted for the two surfaces when we calculated $\Delta A = 2A$. You can use either approach, but never count the factor of 2 twice. Be consistent.*

REAL Example 6

While walking past a garage in Dar es Salaam, **Kipute** notices that a small amount of engine oil has leaked onto a puddle of rainwater. Instead of staying as a drop, the oil spreads out into a thin, shimmering film that covers a large area of the puddle.

"That is strange," she says to **Mr. Akilikubwa**, who happens to be nearby. *"On the ground, oil stays as a thick puddle. But on water, it spreads out thin. Why does oil behave differently on water than on a solid surface?"*

Explain why oil spreads on water but not on a dry concrete surface.

Solution

Oil spreads on water because doing so **reduces the total surface energy** of the system.

When oil sits on water, there are three surfaces to consider: the oil-air surface, the oil-water interface, and the water-air surface. The surface tension of water is much higher than the surface tension of oil. When oil spreads, it replaces the high-energy water-air surface with two lower-energy surfaces (oil-air and oil-water). The total surface energy is thus decreases, so spreading is energetically favourable. The system moves spontaneously toward the lower-energy state.

On a dry concrete surface, the situation is different. Concrete has a relatively low surface energy, and there is no high-energy water surface to be replaced. Spreading the oil would increase the total surface area without a corresponding decrease in surface energy. So the oil stays as a thick puddle, minimising its own surface area.

Making Sense of the Answer: *The oil is not "attracted" to water. It spreads because spreading lowers the total energy. Nature always moves toward lower energy when it can. This is the same principle that makes a ball roll downhill or a compressed spring expand.*

Think Like a Physicist: *Whenever you see a liquid spreading or contracting on a surface, ask: **does spreading reduce or increase the total surface energy?** If it reduces the total energy, the liquid spreads. If it increases the total energy, the liquid contracts into a drop. This single question explains an enormous range of wetting and non-wetting phenomena.*

HOT Example 7

A rectangular wire frame of length 10cm and width 6.0cm is dipped in a soap solution of surface tension 0.030Nm^{-1} and then removed. A thin soap film forms across the frame. The sliding wire along one of the shorter sides is now pulled outward by a distance of 4.0cm.

(a) Calculate the force exerted by the film on the sliding wire.

(b) Calculate the work done in pulling the wire through the distance of 4.0cm.

(c) Calculate the increase in surface area and verify that the work done equals $\gamma \times$ (increase in total surface area).

Solution

(a) The sliding wire has length $\text{l} = 6.0 \times 10^{-2}\text{m}$. The soap film has two surfaces, so the total surface tension force on the wire is:

$$\text{F} = 2\gamma\text{l}$$

$$\text{F} = 2 \times 0.030\text{Nm}^{-1} \times 6.0 \times 10^{-2}\text{m} = 3.6 \times 10^{-3}\text{N}$$

Therefore, the force exerted by the soap film on the sliding wire is $3.6 \times 10^{-3}\text{N}$ (or 3.6mN).

(b) The wire is pulled through a distance $\text{x} = 4.0 \times 10^{-2}\text{m}$. The work done against the surface tension force is:

$$\text{W} = \text{F} \times \text{x}$$

$$\text{W} = 3.6 \times 10^{-3}\text{N} \times 4.0 \times 10^{-2}\text{m} = 1.44 \times 10^{-4}\text{J}$$

Therefore, the work done in pulling the wire is $1.44 \times 10^{-4}\text{J}$.

(c) The increase in area of one surface is:

$$\Delta\text{A}_{\text{one}} = \text{l} \times \text{x} = 6.0 \times 10^{-2}\text{m} \times 4.0 \times 10^{-2}\text{m} = 2.4 \times 10^{-3}\text{m}^2$$

The total increase in surface area (two surfaces) is:

$$\Delta\text{A}_{\text{total}} = 2 \times 2.4 \times 10^{-3}\text{m}^2 = 4.8 \times 10^{-3}\text{m}^2 = \text{Increase in total surface area}$$

The work done calculated from surface energy is:

$$\text{W} = \gamma \times \Delta\text{A}_{\text{total}}$$

$$\text{W} = 0.030\text{Nm}^{-1} \times 4.8 \times 10^{-3}\text{m}^2$$

$$\text{W} = 1.44 \times 10^{-4}\text{J}$$

This matches the work calculated in part (b) exactly. The two approaches are consistent, confirming that surface tension (force per unit length) and surface energy (energy per unit area) are two faces of the same quantity.

Making Sense of the Answer: *The work done* (1.44×10^{-4}*J) is extremely small. You could do this much work by lifting a grain of rice about 1.5cm. Yet this tiny amount of energy is what holds soap films together and allows them to form the beautiful, fragile structures that children blow into bubbles. At this scale, surface tension is the dominant force.*

Think Like a Physicist: *Part (c) is the heart of this problem. It shows that the force approach* ($W = Fx$) *and the energy approach* ($W = \gamma\Delta A$) *give identical results. This is not a coincidence. It is a consequence of the fact that* γ *is simultaneously a force per unit length and an energy per unit area. Whenever you can verify a result by two independent methods, do it. Agreement between methods is one of the strongest checks in physics.*

That brings us to the end of this subtopic. We have built surface tension from the ground up: from the observation of floating needles and spherical drops, through the molecular theory that explains why surface molecules are special, to the mathematics of surface energy that will power everything ahead.

But surface tension does more than make needles float and drops spherical. It creates excess pressure inside bubbles and curved surfaces, and it pulls liquids up narrow tubes against gravity. These consequences are the subjects of the next two subtopics. We will derive the excess pressure inside bubbles and drops. And later, we will explore capillarity and the angle of contact. Both are built directly on the ideas of this section.

EXCESS PRESSURE IN CURVED SURFACES AND BUBBLES

In the previous subtopic, we learned that surface tension causes a liquid surface to behave like a stretched elastic membrane that tries to contract to the smallest possible area. But *what happens when a surface is forced to be curved, as in a bubble or a drop?* The stretched membrane pulls inward, and for the bubble to hold its shape, the pressure on the concave side (pressure inside) must be greater than the pressure on the convex side (pressure outside). This difference is called **excess pressure**, and it is the mathematical heart of surface tension.

Why Curved Surfaces Have Excess Pressure

Think about inflating a balloon. When the balloon is deflated, the rubber is relaxed and there is no extra pressure inside. The moment you start blowing, the rubber stretches and curves outward. The stretched rubber pulls inward (toward the centre), trying to contract. For the balloon to remain inflated, the air pressure inside must be greater than the atmospheric pressure outside. The difference between the inside and outside pressures is what keeps the balloon expanded against the inward pull of the rubber.

A curved liquid surface behaves in exactly the same way. The surface tension acts along the surface, pulling it inward, trying to flatten it. For the surface to remain curved, there must be a pressure difference across it: the pressure on the **concave side** (the inside of the curve) is always greater than the pressure on the **convex side** (the outside of the curve).

The smaller the radius of curvature, the greater this excess pressure. This is why it is harder to start blowing a balloon than to continue inflating it once it is already large: the small initial radius requires a large excess pressure, while the larger radius of the inflated balloon requires less.

We will now derive the excess pressure for three geometries: an air bubble inside a liquid, a soap bubble in air, and a liquid drop.

Excess Pressure Inside an Air Bubble in a Liquid (One Surface)

Consider a small air bubble of radius r formed inside a liquid of surface tension γ. The bubble has only **one** liquid surface (the boundary between the air inside and the liquid outside).

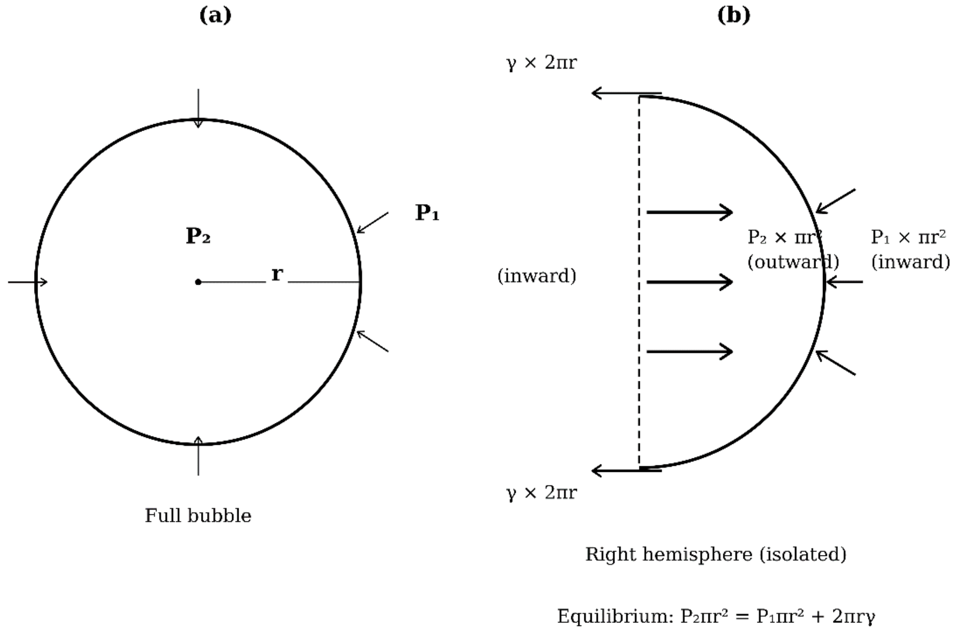


Figure: ***(a) An air bubble of radius r inside a liquid****. The internal pressure P_2 is greater than the external pressure P_1.* ***(b) The right hemisphere isolated for force analysis.*** *The outward force $P_2\pi r^2$ on the flat face is balanced by the inward force $P_1\pi r^2$ on the curved surface and the surface tension force $2\pi r\gamma$ along the equatorial rim.*

To derive the excess pressure, consider the equilibrium of one hemisphere of the bubble. Imagine slicing the bubble in half along its equator. The flat circular face exposed by the cut has area πr^2.

The forces acting on this hemisphere are:

1. The pressure force from outside (pushing the hemisphere inward):

$$F_{\text{outside}} = P_1 \times \pi r^2$$

2. The pressure force from inside (pushing the hemisphere outward):

$$F_{\text{inside}} = P_2 \times \pi r^2$$

3. The surface tension force along the circumference of the cut (pulling the hemisphere inward, trying to contract the bubble). The circumference is $2\pi r$, and the surface tension acts along this entire length:

$$F_\gamma = \gamma \times 2\pi r$$

For the hemisphere to be in equilibrium, the outward force must equal the total inward forces:

$$P_2 \times \pi r^2 = P_1 \times \pi r^2 + \gamma \times 2\pi r$$

Dividing both sides by πr^2:

$$P_2 = P_1 + \frac{2\gamma}{r} \text{ or } P_2 - P_1 = \frac{2\gamma}{r}$$

Therefore, the excess pressure inside an air bubble in a liquid is:

$$\mathbf{\Delta P = \frac{2\gamma}{r}}$$

where $\Delta P = P_2 - P_1$ is the difference between the pressure inside and outside the bubble.

This result tells us two important things. First, the pressure inside the bubble is always greater than the pressure outside ($P_2 > P_1$). Second, the excess pressure is inversely proportional to the radius: the smaller the bubble, the greater the excess pressure. A tiny bubble has extremely high internal pressure relative to its surroundings.

Excess Pressure Inside a Soap Bubble in Air (Two Surfaces)

A soap bubble floating in air is different from an air bubble in liquid. A soap bubble is a thin film of soapy water with air on both sides. This means it has **two** free surfaces: an inner surface and an outer surface. Each surface contributes a surface tension force.

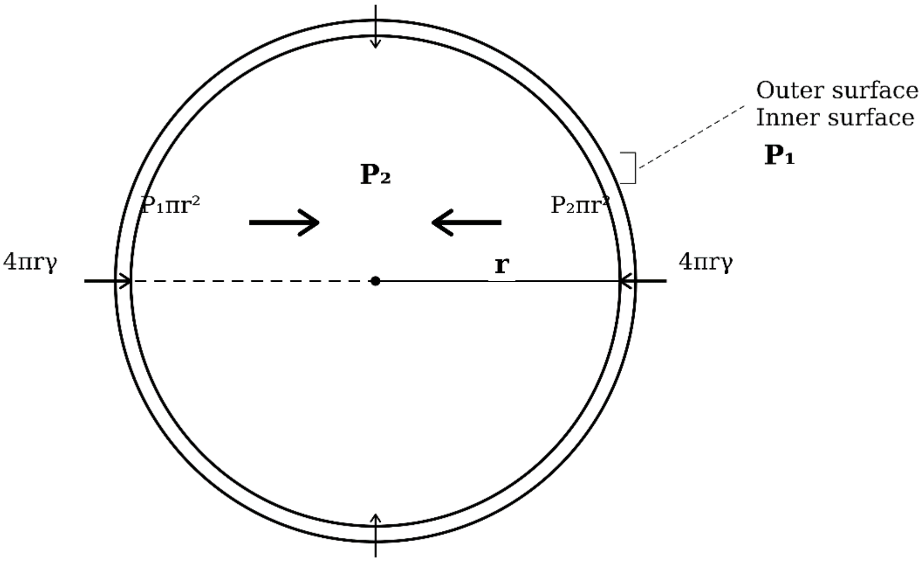


Figure: *Equilibrium of one hemisphere of a soap bubble in air. The bubble has two surfaces (inner and outer), so the total surface tension force along the equator is $4\pi r\gamma$, double that of a single-surface bubble.*

The derivation follows the same approach as in the previous derivation, except the surface tension force is doubled:

$$F_\gamma = 2 \times \gamma \times 2\pi r = 4\pi r\gamma$$

For equilibrium of one hemisphere:

$$P_2 \times \pi r^2 = P_1 \times \pi r^2 + 4\pi r\gamma$$

Dividing both sides by πr^2:

$$P_2 = P_1 + \frac{4\gamma}{r}$$

$$P_2 - P_1 = \frac{4\gamma}{r}$$

Therefore, the excess pressure inside a soap bubble in air is:

$$\boldsymbol{\Delta P = \frac{4\gamma}{r}}$$

The factor of 4 (instead of 2) is entirely due to the two surfaces. This is one of the most frequently tested distinctions in examinations: an air bubble in liquid has excess pressure $2\gamma/r$ (one surface), while a soap bubble in air has excess pressure $4\gamma/r$ (two surfaces). Confusing the two is a common and costly mistake.

Excess Pressure Inside a Liquid Drop (One Surface)

A liquid drop (such as a raindrop falling through air) has only **one** surface: the outer boundary between the liquid inside and the air outside. There is no air cavity inside a drop; it is solid liquid throughout.

The derivation is identical to that of an air bubble in a liquid, and the result is the same:

$$\boldsymbol{\Delta P = \frac{2\gamma}{r}}$$

where P_2 is the pressure inside the drop and P_1 is the atmospheric pressure outside.

It is important to be clear about the distinction between the three cases:

- An **air bubble in liquid** has one liquid surface. Excess pressure: $\Delta P = \frac{2\gamma}{r}$.
- A **soap bubble in air** has two surfaces (inner and outer). Excess pressure: $\Delta P = \frac{4\gamma}{r}$.
- A **liquid drop** has one surface. Excess pressure: $\Delta P = \frac{2\gamma}{r}$.

Coalescing and Breaking of Drops and Bubbles

When drops or bubbles combine or break apart, the total surface area changes. Since surface energy equals $\gamma \times$ (surface area), any change in surface area involves a transfer of energy. Understanding this energy change is essential for problems involving coalescing drops, breaking drops, and the common interface between two bubbles.

Coalescing of liquid drops in vacuum

Consider two liquid drops of radii r_1 and r_2 that coalesce to form a single drop of radius R under isothermal conditions in vacuum. Since the total volume of liquid is conserved:

$$\frac{4}{3}\pi r_1^3 + \frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi R^3$$

Simplifying:

$$r_1^3 + r_2^3 = R^3$$

If the drops are identical ($r_1 = r_2 = r$), then $2r^3 = R^3$, giving $R = r\sqrt[3]{2}$.

For the surface energy analysis, the initial total surface area is $4\pi r_1^2 + 4\pi r_2^2$ and the final surface area is $4\pi R^2$. Since a sphere has the minimum surface area for a given volume, the combined drop has less total surface area than the two separate drops. The process releases surface energy. By conservation of energy, this released energy appears as heat, slightly raising the temperature of the drop.

For the special case where we need R in terms of r_1 and r_2 using surface energy conservation (isothermal, in vacuum), the surface energies must balance:

$$4\pi r_1^2\gamma + 4\pi r_2^2\gamma = 4\pi R^2\gamma$$

$$\boldsymbol{R = \sqrt{r_1^2 + r_2^2}}$$

For n identical drops of radius r coalescing into one drop of radius R:

$$\mathrm{n} \times 4\pi \mathrm{r}^2\gamma = 4\pi \mathrm{R}^2\gamma$$

giving $R^2 = nr^2$, so $\boldsymbol{R = r\sqrt{n}}$.

Be careful! The volume conservation formula ($\mathrm{R}^3 = \mathrm{r}_1^3 + \mathrm{r}_2^3$) and the surface energy formula ($\mathrm{R}^2 = \mathrm{r}_1^2 + \mathrm{r}_2^2$) give different results for R. Which one applies depends on the problem. If the coalescing happens in vacuum under isothermal conditions with no other energy input, the surface energy formula applies. If the problem simply states that two drops merge into one and asks for the new radius, use volume conservation. Always read the question carefully.

Breaking of a large liquid drop into smaller drops

When a large drop of radius R breaks into n identical smaller drops, each of radius r, the total volume is conserved:

$$\frac{4}{3}\pi \mathrm{R}^3 = \mathrm{n} \times \frac{4}{3}\pi \mathrm{r}^3$$

This gives $\mathrm{R}^3 = \mathrm{nr}^3$, so $\mathbf{n = R^3/r^3.}$

The total surface area increases (many small drops have more combined surface area than one large drop), so work must be done. The work done in breaking the drop is:

$$\mathrm{W} = \gamma \times \Delta\mathrm{A} = \gamma \times (\mathrm{n} \times 4\pi\mathrm{r}^2 - 4\pi\mathrm{R}^2)$$

$$\mathrm{W} = 4\pi\gamma(\mathrm{nr}^2 - \mathrm{R}^2)$$

Since $\mathrm{n} = \mathrm{R}^3/\mathrm{r}^3$, we can substitute:

$$\mathrm{W} = 4\pi\gamma\left(\frac{\mathrm{R}^3}{\mathrm{r}^3} \times \mathrm{r}^2 - \mathrm{R}^2\right)$$

$$\mathbf{W = 4\pi\gamma R^2\left(\frac{R}{r} - 1\right)}$$

Since $\mathrm{R} > \mathrm{r}$, the factor $(\mathrm{R}/\mathrm{r} - 1)$ is positive, confirming that work must be done to break a large drop into smaller ones. This energy goes into creating the additional surface area.

Interesting! If all this energy is absorbed by the liquid itself (no heat loss to the surroundings), it causes a slight temperature rise. If the mass of the liquid is m, the density is ρ, and the specific heat capacity is c, then the temperature rise ΔT can be found from:

$$\mathrm{W} = \mathrm{mc}\Delta\mathrm{T}$$

$$4\pi\gamma\mathrm{R}^2\left(\frac{\mathrm{R}}{\mathrm{r}} - 1\right) = \frac{4}{3}\pi\mathrm{R}^3\rho\mathrm{c}\Delta\mathrm{T}$$

$$\boldsymbol{\Delta \mathrm{T} = \frac{3\gamma}{\rho c}\left(\frac{1}{r} - \frac{1}{R}\right)}$$

Coalescing of two soap bubbles (radius of common interface)

When two soap bubbles of different radii r_1 and r_2 (with $\mathrm{r}_1 < \mathrm{r}_2$) come into contact, a common interface forms between them. The smaller bubble has a higher internal pressure (since excess pressure is inversely proportional to radius), so the common interface bulges into the larger bubble.

Let P_A be the atmospheric pressure outside both bubbles, and let P_1 and P_2 be the pressures inside the small and large bubbles respectively.

For the small bubble: $\mathrm{P}_1 - \mathrm{P}_\mathrm{A} = \frac{4\gamma}{\mathrm{r}_1}$

For the large bubble: $\mathrm{P}_2 - \mathrm{P}_\mathrm{A} = \frac{4\gamma}{\mathrm{r}_2}$

The pressure difference across the common interface is:

$$\mathrm{P}_1 - \mathrm{P}_2 = \frac{4\gamma}{\mathrm{r}_1} - \frac{4\gamma}{\mathrm{r}_2} = 4\gamma\left(\frac{1}{\mathrm{r}_1} - \frac{1}{\mathrm{r}_2}\right)$$

If R is the radius of curvature of the common interface, and the interface is itself a soap film with two surfaces:

$$P_1 - P_2 = \frac{4\gamma}{R}$$

Equating:

$$\frac{4\gamma}{R} = 4\gamma\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\frac{1}{R} = \frac{1}{r_1} - \frac{1}{r_2} = \frac{r_2 - r_1}{r_1 r_2}$$

$$\mathbf{R} = \frac{\mathbf{r_1 r_2}}{\mathbf{r_2 - r_1}}$$

Since $r_2 > r_1$, R is positive. The interface curves into the larger bubble, as expected.

The theory of this section is now complete. We have derived the excess pressure for all three geometries and the key results for coalescing and breaking of drops and bubbles. Let us now test these ideas with worked examples.

BINDER Example 8

An air bubble of radius 2.5cm is formed inside a liquid whose surface tension is $25 \times 10^{-3}\mathrm{Nm^{-1}}$. Calculate the excess pressure inside the bubble.

Solution

An air bubble in liquid has one surface, so:

$$\Delta P = \frac{2\gamma}{r} = \frac{2 \times 25 \times 10^{-3}\mathrm{Nm^{-1}}}{2.5 \times 10^{-2}\mathrm{m}} = 2.0\mathrm{Nm^{-2}}$$

Therefore, the excess pressure inside the air bubble is $2.0\mathrm{Nm^{-2}}$ (or 2.0Pa).

Making Sense of the Answer: *This is a very small pressure, about 0.002% of atmospheric pressure. The bubble is relatively large ($2.5cm$ radius), which is why the excess pressure is small. A bubble one thousand times smaller (radius $0.025mm$) would have an excess pressure one thousand times larger: $2000Pa$. This is the inverse relationship at work.*

Think Like a Physicist: *Before calculating, always identify the geometry.* ***Is it an air bubble in liquid (one surface, $\Delta P = 2\gamma/r$), a soap bubble in air (two surfaces, $\Delta P = 4\gamma/r$), or a liquid drop (one surface, $\Delta P = 2\gamma/r$)?*** *This single decision determines whether you use 2 or 4 in the numerator. Get it wrong, and the entire answer is wrong.*

BINDER Example 9

A soap bubble has a diameter of 4.0mm. Calculate the pressure inside it if the atmospheric pressure is $1.0 \times 10^5\mathrm{Pa}$ and the surface tension of the soap solution is $2.8 \times 10^{-2}\mathrm{Nm^{-1}}$.

Solution

A soap bubble in air has two surfaces, so the excess pressure is:

$$\Delta P = \frac{4\gamma}{r} = \frac{4 \times 2.8 \times 10^{-2}\mathrm{Nm^{-1}}}{2.0 \times 10^{-3}\mathrm{m}} = 56\mathrm{Pa}$$

The total pressure inside the bubble is:

$$P_{\text{inside}} = P_A + \Delta P$$

$$P_{\text{inside}} = 1.0 \times 10^5\mathrm{Pa} + 56\mathrm{Pa} = 1.00056 \times 10^5\mathrm{Pa}$$

Therefore, the total pressure inside the soap bubble is $1.00056 \times 10^5\mathrm{Pa}$.

Making Sense of the Answer: *The excess pressure* ($56Pa$) *is tiny compared to atmospheric pressure* ($100{,}000Pa$)*. Yet this small difference is what keeps the bubble inflated. If the excess pressure were zero, the bubble would have no reason to maintain its spherical shape and would collapse.*

Think Like a Physicist: *The question asks for "the pressure inside," not "the excess pressure." These are different quantities. The excess pressure is* $\Delta P = 56Pa$*. The total pressure inside is* $P_A + \Delta P = 1.00056 \times 10^5 Pa$*. Always check whether the question asks for the excess or the total.*

BINDER Example 10

Two soap bubbles of radii 3.0cm and 4.0cm are made from the same soap solution. They coalesce to form a single bubble. Assuming the process is isothermal, find the radius of the new bubble.

Solution

For soap bubbles coalescing isothermally, the total surface energy is conserved. Since each bubble has two surfaces:

$$2 \times 4\pi r_1^2\gamma + 2 \times 4\pi r_2^2\gamma = 2 \times 4\pi R^2\gamma$$

The factor $2 \times 4\pi\gamma$ cancels from all terms:

$$r_1^2 + r_2^2 = R^2$$

$$R = \sqrt{r_1^2 + r_2^2} = \sqrt{(3.0\text{cm})^2 + (4.0\text{cm})^2} = 5.0\text{cm}$$

Therefore, the radius of the new bubble formed after coalescence is 5.0cm.

Making Sense of the Answer: *The new bubble* ($R = 5.0cm$) *is larger than either original bubble, as expected.*

Think Like a Physicist: *The assumption "isothermal" is critical. It means no heat is gained or lost, so all the surface energy of the original bubbles is transferred to the new bubble. Without this assumption, some energy could be lost as heat, and the calculation would be different.*

REAL Example 11

At a birthday party, **Kipanga** tries to blow up a balloon. He notices something strange: the balloon is very hard to blow at the beginning, but once it starts expanding, it becomes much easier.

He complains to Mr. Akilikubwa: *"Sir, I almost passed out trying to start this balloon. But now it inflates easily. Why is the first breath so hard?"*

Explain why it is harder to begin inflating a balloon than to continue inflating it once it has already expanded.

Solution

The excess pressure required to inflate a curved surface is given by $\Delta P = 4\gamma/r$ (since the rubber membrane, like a soap bubble, has two surfaces). When the balloon is small (just starting to inflate), the radius r is very small. Since the excess pressure is inversely proportional to r, a small radius means a large excess pressure is needed.

As Kipanga continues blowing and the balloon expands, the radius r increases. The required excess pressure $\Delta P = 4\gamma/r$ decreases. This is why the balloon feels easier to inflate once it is already partially expanded: less pressure difference is needed to maintain and increase the larger radius.

In addition, the rubber of the balloon is initially stiff and resists stretching. Once it begins to stretch, the elastic restoring force per unit extension decreases (rubber does not obey Hooke's law at large extensions), making further inflation easier.

Therefore, the first breath is hardest because both the small radius (requiring high excess pressure) and the initial stiffness of the rubber work against Kipanga.

Making Sense of the Answer: *This is why balloon vendors at Kariakoo market always pre-stretch their balloons before selling them. Pre-stretching increases the initial radius and softens the rubber, making the balloon much easier for a child to inflate. Physics at the marketplace.*

Think Like a Physicist: *The formula $\Delta P = 4\gamma/r$ predicts that as r approaches zero, the excess pressure approaches infinity. In practice, this does not happen because the balloon material has a finite thickness and the formula breaks down at very small radii. But the trend is real: smaller radius means harder to inflate.*

HOT Example 12

A large spherical drop of mercury of radius 2.0mm breaks into 1000 identical smaller drops. The surface tension of mercury is 0.472Nm^{-1} and its specific heat capacity is $140\text{Jkg}^{-1}\text{K}^{-1}$. The density of mercury is 13600kgm^{-3}.

(a) Find the radius of each small drop.

(b) Calculate the work done in breaking the large drop.

(c) If all the work done appears as heat in the mercury, find the rise in temperature.

Solution

(a) By conservation of volume:

$$\frac{4}{3}\pi R^3 = n \times \frac{4}{3}\pi r^3$$

$$R^3 = nr^3$$

$$r = \frac{R}{\sqrt[3]{n}} = \frac{2.0 \times 10^{-3}\text{m}}{\sqrt[3]{1000}} = 2.0 \times 10^{-4}\text{m}$$

Therefore, the radius of each small drop is 2.0×10^{-4}m (or 0.20mm).

(b) The change in surface area is:

$$\Delta A = n \times 4\pi r^2 - 4\pi R^2$$

$$\Delta A = 1000 \times 4\pi \times (2.0 \times 10^{-4}\text{m})^2 - 4\pi \times (2.0 \times 10^{-3}\text{m})^2 = 4.524 \times 10^{-4}\text{m}^2$$

The work done is:

$$W = \gamma \times \Delta A = 0.472\text{Nm}^{-1} \times 4.524 \times 10^{-4}\text{m}^2 = 2.14 \times 10^{-4}\text{J}$$

Therefore, the work done in breaking the large drop is 2.14×10^{-4}J.

(c) The mass of mercury is:

$$m = \rho \times \frac{4}{3}\pi R^3 = 13{,}600\text{kgm}^{-3} \times \frac{4}{3}\pi \times (2.0 \times 10^{-3}\text{m})^3 = 4.557 \times 10^{-4}\text{kg}$$

Using $W = mc\Delta T$:

$$\Delta T = \frac{W}{mc} = \frac{2.14 \times 10^{-4}\text{J}}{4.557 \times 10^{-4}\text{kg} \times 140\text{Jkg}^{-1}\text{K}^{-1}} = 3.35 \times 10^{-3}\text{K}$$

Therefore, the rise in temperature of the mercury is 3.35×10^{-3}K (or approximately 0.003K).

Making Sense of the Answer: *The temperature rise is extraordinarily small, about three thousandths of a degree. This makes physical sense: the surface energy involved in breaking a 2mm drop is tiny, and mercury has significant thermal mass even in small quantities. In practice, this temperature rise would be undetectable by ordinary thermometers. But the calculation demonstrates that surface energy is real and can, in principle, be converted to thermal energy.*

Think Like a Physicist: *This problem combines three topics: surface tension (surface area change), energy conservation (work equals surface energy gained), and thermal physics (energy equals $mc\Delta T$). Multi-topic problems like this are frequently tested in examinations. The key strategy is to solve each part as a separate sub-problem and feed results forward: part (a) gives r, which is needed in part (b); part (b) gives W, which is needed in part (c). Never try to do everything in one equation.*

HOT Example 13

Two soap bubbles of radii 10mm and 30mm are made from the same soap solution of surface tension 0.030Nm^{-1}. They coalesce so that they share a common interface.

(a) Find the radius of curvature of the common interface.

(b) Determine whether the interface curves into the larger or smaller bubble.

Solution

(a) Finding the radius of the common interface:

Interpreting data:

$r_1 = 10\text{mm} = 10 \times 10^{-3}\text{m}$ (smaller bubble), $r_2 = 30\text{mm} = 30 \times 10^{-3}\text{m}$ (larger bubble)

The radius of curvature of the common interface is:

$$R = \frac{r_1 r_2}{r_2 - r_1} = \frac{10 \times 10^{-3}\text{m} \times 30 \times 10^{-3}\text{m}}{30 \times 10^{-3}\text{m} - 10 \times 10^{-3}\text{m}} = 15 \times 10^{-3}\text{m} = 15\text{mm}$$

Therefore, the radius of curvature of the common interface is 15mm.

(b) The smaller bubble ($r_1 = 10\text{mm}$) has a higher internal pressure than the larger bubble ($r_2 = 30\text{mm}$), because excess pressure is inversely proportional to radius:

$$P_1 - P_A = \frac{4\gamma}{r_1} = \frac{4 \times 0.030}{10 \times 10^{-3}} = 12\text{Pa}$$

$$P_2 - P_A = \frac{4\gamma}{r_2} = \frac{4 \times 0.030}{30 \times 10^{-3}} = 4.0\text{Pa}$$

Since $P_1 > P_2$, the higher pressure inside the smaller bubble pushes the common interface outward into the larger bubble. Therefore, the common interface **curves into the larger bubble**, bulging toward its centre.

Making Sense of the Answer: *The common interface radius (15mm) is larger than the smaller bubble but smaller than the larger one. This makes physical sense: the interface is gently curved, not tightly bent, because the pressure difference between the two bubbles ($12 - 4 = 8Pa$) is relatively small compared to the excess pressure in either individual bubble.*

Think Like a Physicist: *The formula $R = r_1 r_2/(r_2 - r_1)$ only works when $r_2 \neq r_1$. If both bubbles have the same radius, they have the same internal pressure, so there is no pressure difference across the interface. The common surface is flat (infinite radius of curvature). Mathematically, as $r_1 \to r_2$, the denominator approaches zero and $R \to \infty$, which correctly describes a flat surface.*

That completes this beautiful subtopic. We have derived the excess pressure for all three standard geometries, explored the energy changes when drops and bubbles coalesce or break apart, and derived the radius of the common interface between two coalescing soap bubbles.

In the next subtopic, we apply surface tension to one of its most visible consequences: the rise (or depression) of liquids in narrow tubes. This phenomenon is called **capillarity**, and it connects surface tension to something you can see with your own eyes every time water climbs up a paper towel or mercury dips below the level in a glass tube.

CAPILLARITY AND ANGLE OF CONTACT

Surface tension does not only make needles float and drops spherical. It also pulls liquids up narrow tubes against gravity, and pushes other liquids down. If you have ever watched water climb up a paper towel, or noticed that mercury in a glass thermometer sits slightly below the level you would expect, you have already seen this phenomenon at work. It is called **capillarity**, and it connects the invisible molecular forces we discussed earlier to something you can measure with a ruler and see with your own eyes. By the end of this subtopic, you will understand exactly why water rises, why mercury falls, and how a simple glass tube can be used to measure the surface tension of any liquid.

Angle of Contact

When a liquid meets a solid surface, the liquid surface near the contact point is not flat. It curves, either upward or downward, depending on how strongly the liquid molecules are attracted to the solid compared to how strongly they are attracted to each other. The angle that describes this curvature is called the **angle of contact** (or **contact angle**), and it is denoted by θ.

Precisely, the angle of contact *is the angle between the solid surface and the tangent to the liquid surface at the point of contact,* ***measured through the liquid****.*

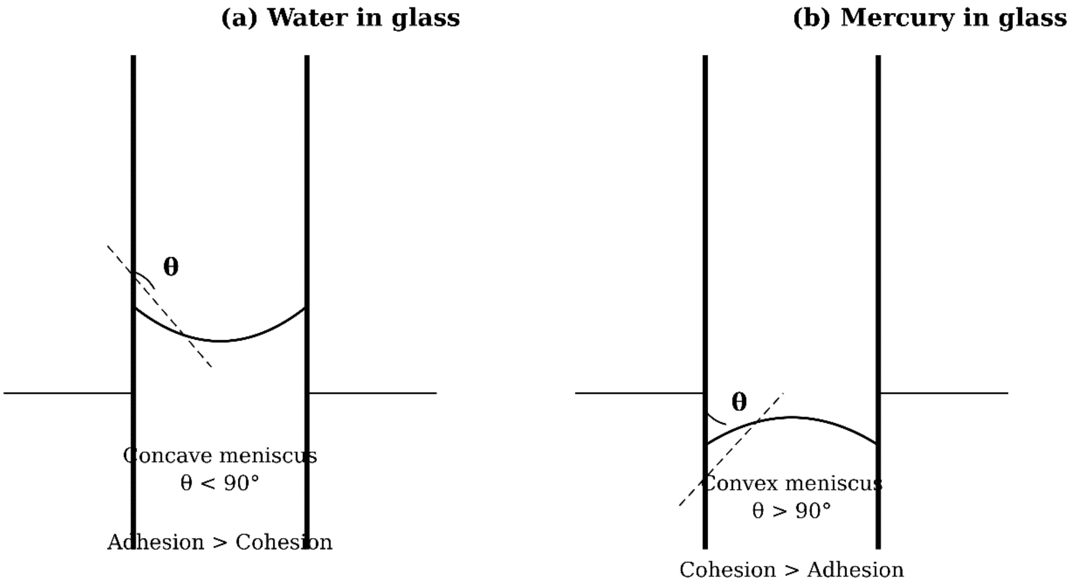


Figure: ***(a) Water in a glass tube:*** *the meniscus is concave because adhesive forces between water and glass are stronger than cohesive forces between water molecules. The angle of contact* $\theta < 90^\circ$. ***(b) Mercury in a glass tube:*** *the meniscus is convex because cohesive forces between mercury atoms are stronger than adhesive forces between mercury and glass. The angle of contact* $\theta > 90^\circ$.

The shape of the meniscus depends on the competition between two types of intermolecular forces:

1) **Adhesive forces** *are the attractive forces between molecules of different substances* (for example, between water molecules and glass molecules).
2) **Cohesive forces** *are the attractive forces between molecules of the same substance* (for example, between water molecules and other water molecules).

When **adhesion is stronger than cohesion** (as with water and glass), the liquid molecules are attracted to the solid surface more strongly than to each other. The liquid climbs up the wall, wetting it, and the meniscus is **concave** (curves upward). The angle of contact is $\theta < 90^\circ$. For water on clean glass, $\theta \approx 0^\circ$.

When **cohesion is stronger than adhesion** (as with mercury and glass), the liquid molecules are attracted to each other more strongly than to the solid surface. The liquid pulls away from the wall, and the meniscus is **convex** (curves downward). The angle of contact is $\theta > 90^\circ$. For mercury on glass, $\theta \approx 135^\circ$.

Factors Affecting the Angle of Contact

The angle of contact is not a fixed number. Like surface tension itself, it depends on several conditions. Understanding these factors will help you interpret experimental results and predict how different liquids behave on different surfaces.

1. Nature of the liquid and the solid surface

This is the most fundamental factor. The angle of contact depends on the relative strengths of adhesive and cohesive forces, which are determined by the chemical nature of both the liquid and the solid. Water on clean glass gives $\theta \approx 0^\circ$ (strong adhesion), but water on a waxed surface gives $\theta > 90^\circ$ (weak adhesion). Mercury on glass gives $\theta \approx 135^\circ$, but mercury on clean copper gives a much smaller angle because mercury adheres more strongly to copper than to glass.

2. Nature of the medium above the liquid

The angle of contact is not a property of the liquid alone. It also depends on what sits above the liquid surface. Water in contact with air has a different angle of contact from water in contact with oil vapour, because the molecules of the medium above the surface exert their own forces on the surface molecules of the liquid, partially altering the balance between adhesion and cohesion.

3. Impurities in the liquid

Adding impurities to the liquid changes the intermolecular forces at the surface, which alters the angle of contact. In general, adding a detergent or surfactant decreases the angle of contact of water on most surfaces, which is precisely why detergent helps water spread and wet surfaces more effectively.

4. Temperature

The angle of contact generally increases with increasing temperature. As temperature rises, the increased molecular motion weakens the adhesive forces at the liquid-solid interface, making the liquid less willing to cling to the solid surface.

Capillary Rise

Here is one of those moments in physics where something very simple leads to something very beautiful. Take a narrow glass tube, dip it vertically into a beaker of water, and watch. The water inside the tube quietly rises above the level of the water outside, and stays there, motionless, as though gravity has forgotten about it. It has not. The water is being held up by surface tension, and the balance between these two forces produces a formula of remarkable elegance.

This phenomenon is called **capillary rise**, and the narrow tube is called a **capillary tube**.

The rise happens because the adhesive forces between the liquid and the glass pull the liquid upward along the inner wall of the tube. Surface tension provides the upward force. The liquid continues to rise until the upward surface tension force is exactly balanced by the weight of the liquid column that has been lifted. At that point, the liquid stops rising and the system reaches equilibrium.

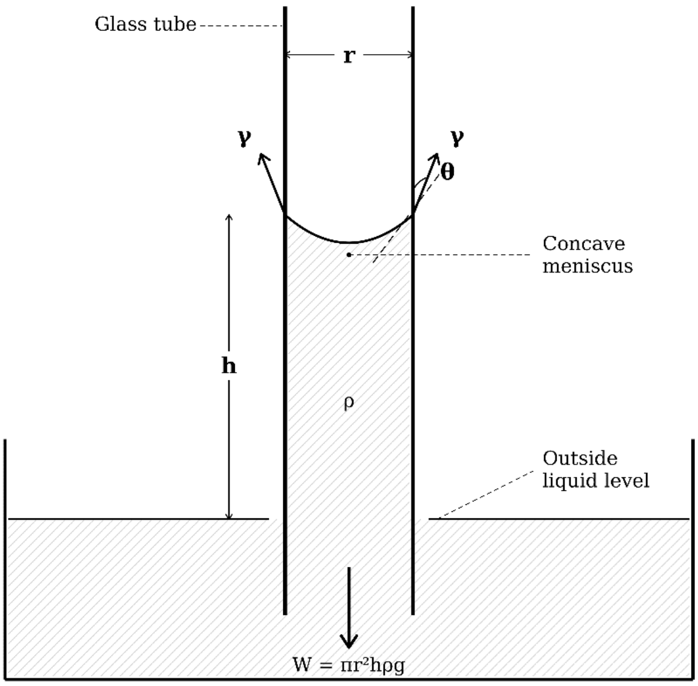


Figure: *Capillary rise of a liquid in a narrow tube of internal radius r. The surface tension* γ *acts along the contact perimeter* $2\pi r$ *at an angle* θ *to the vertical. The vertical component of the total surface tension force,* $\gamma \times 2\pi r \times \cos\theta$*, supports the weight* $W = \pi r^2 h\rho g$ *of the liquid column of height h.*

Derivation of the capillary rise formula:

The surface tension γ acts along the entire circumference of the contact line between the liquid and the inner wall of the tube. The length of this contact line is $2\pi r$, where r is the internal radius of the tube.

The surface tension force acts at an angle θ to the vertical (along the tangent to the meniscus at the contact point). Only the vertical component of this force supports the weight of the liquid column. The total upward force is:

$$F_{up} = \gamma \times 2\pi r \times \cos\theta$$

The weight of the liquid column of height h, radius r, and density ρ is:

$$W = \pi r^2 h\rho g$$

At equilibrium, the upward surface tension force equals the weight of the liquid column:

$$\gamma \times 2\pi r \times \cos\theta = \pi r^2 h\rho g$$

Solving for h:

$$\mathbf{h = \frac{2\gamma\cos\theta}{\rho g r}}$$

This is the **capillary rise formula**, and it deserves a moment of appreciation. Look at what it tells us:

1. The height h is **inversely proportional to the radius** r of the tube. The narrower the tube, the higher the liquid rises. This is why capillary effects are dramatic in very thin tubes but invisible in wide containers.
2. The height depends on $\cos\theta$. For liquids that wet the surface ($\theta < 90°$, so $\cos\theta > 0$), h is positive: the liquid rises. For liquids that do not wet the surface ($\theta > 90°$, so $\cos\theta < 0$), h is negative: the liquid is depressed below the outside level. One formula handles both rise and depression. The sign of $\cos\theta$ does all the work.
3. The height does not depend on the shape of the tube, only on its internal radius. A tube with a circular cross-section and a tube with a square cross-section of the same effective radius give the same capillary rise.

Correction for meniscus volume: The derivation above assumes the liquid column is a perfect cylinder, but the meniscus at the top is curved, adding a small extra volume of liquid. For very narrow tubes where this extra volume matters, the corrected formula for the height of rise is:

$$\mathbf{h = \frac{2\gamma\cos\theta}{\rho g r} - \frac{r}{3}}$$

The correction term $\frac{r}{3}$ is only significant when the tube radius is comparable to the height of rise. However, for most A-level problems, the simple formula is sufficient.

Practical Applications of Capillary Rise

Capillary action is not just a laboratory curiosity. It operates quietly and powerfully in everyday life. Here are the most important applications you should know:

1. Water transport in soil

Water rises from the water table to the roots of plants through capillary action in the tiny spaces between soil particles. Without this, crops would only grow where roots reach the water table directly.

2. Oil in a lamp wick

In a kerosene lamp, the liquid fuel rises from the reservoir up through the narrow fibres of the wick to the flame at the top. Capillary rise does the lifting; gravity would keep the kerosene at the bottom.

3. Ink in blotting paper and paper towels

When you touch blotting paper to a drop of ink, the ink spreads rapidly through the narrow spaces between the fibres. This is capillary action pulling the liquid into every available gap.

4. Ink in a fountain pen

The narrow channel inside a fountain pen nib draws ink from the reservoir to the writing tip by capillary action.

5. Construction

Damp-proof courses in buildings prevent water from rising through brick walls by capillary action. Without them, moisture would climb up from the foundation and damage the walls.

Capillary Depression

When a capillary tube is dipped into a liquid that does not wet the glass ($\theta > 90°$), the liquid inside the tube is **depressed** below the outside level. Mercury in a glass tube is the classic example.

The physics is exactly the same as capillary rise, but the sign of $\cos\theta$ reverses everything. Since $\theta > 90°$, $\cos\theta$ is negative, and the formula gives a negative value of h:

$$h = \frac{2\gamma\cos\theta}{\rho g r}$$

The magnitude of h gives the depth of depression below the outside level.

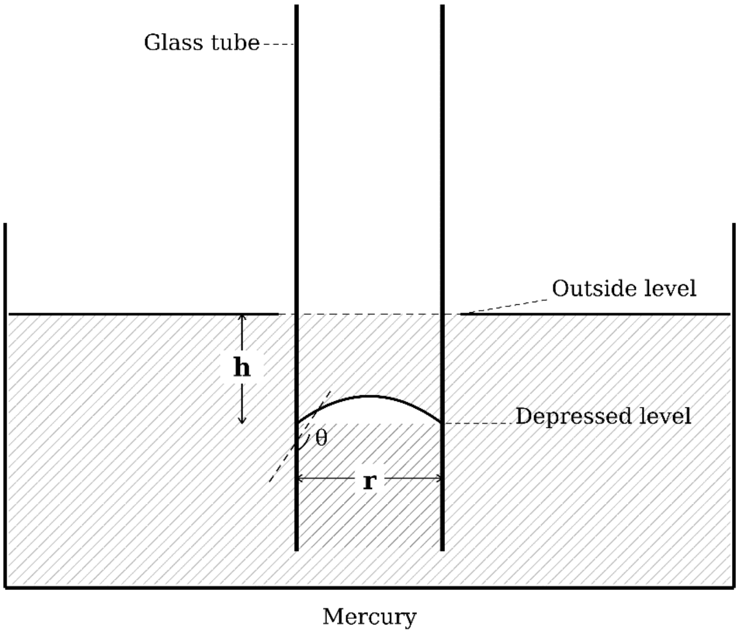


Figure: *Capillary depression of mercury in a glass tube. The mercury level inside the tube is depressed by a height h below the outside level. The meniscus is convex with an angle of contact θ > 90°. The same formula h = 2γcosθ/(ρgr) applies, with cosθ negative, giving a negative h (depression).*

The convex meniscus means the pressure just below the curved surface inside the tube is greater than the atmospheric pressure. This higher pressure pushes the mercury down until equilibrium is reached. There is no need to memorise a separate formula for depression. The capillary rise formula handles everything; the sign of $\cos\theta$ automatically tells you whether the liquid rises or falls.

Measuring Surface Tension by the Capillary Method

The capillary rise formula can be rearranged to find the surface tension of a liquid. If you can measure the height of rise, you can calculate γ:

$$\boldsymbol{\gamma = \frac{\rho g r h}{2\cos\theta}}$$

This is the basis of the **capillary tube method** for measuring surface tension, and it is one of the simplest and most elegant experiments in fluid mechanics. Here is how it works:

A clean capillary tube of known internal radius r is clamped vertically and dipped into the liquid whose surface tension is to be measured. The liquid rises inside the tube. Once the liquid has settled, the height h of the liquid above the outside level is measured using a travelling microscope. The density ρ of the liquid is measured separately (or looked up). If the angle of contact is known (for water on clean glass, $\theta \approx 0°$, so $\cos\theta \approx 1$), then γ is calculated directly.

Sources of error and how to minimise them

Every experiment has enemies. In the capillary method, the main ones are:

1. Tube not perfectly vertical

If the tube is tilted, the measured height h is less than the true vertical rise, giving a surface tension value that is too low. To avoid this, clamp the tube carefully with a retort stand and use a spirit level to check.

2. Tube not perfectly clean

Grease, dust, or chemical residue on the inner wall changes the angle of contact, sometimes dramatically. A greasy glass tube will not give $\theta = 0°$ for water. To avoid this, clean the tube thoroughly with chromic acid or distilled water before the experiment, and handle it only by the outer surface.

3. Non-uniform internal radius

If the tube is wider at one point and narrower at another, the rise depends on where the meniscus sits. To avoid this, use a high-quality capillary tube with a uniform bore, and measure the internal radius at several points.

4. Temperature change during the experiment

Surface tension decreases with temperature. If the room temperature changes during the experiment, γ changes too. To avoid this, take all measurements quickly, or work in a temperature-controlled environment.

5. Difficulty reading the meniscus

The exact position of the meniscus can be hard to judge with the naked eye. To avoid this, use a travelling microscope to read the height, and always read from the bottom of the meniscus (for concave menisci) or the top (for convex menisci).

Now let us put these ideas to work with some carefully chosen examples.

BINDER Example 14

A capillary tube of internal radius 0.25mm is dipped vertically into water. Calculate the height to which water rises in the tube. Take the surface tension of water as $7.2 \times 10^{-2}\text{Nm}^{-1}$, the density of water as 1000kgm^{-3}, the angle of contact as $0°$, and $g = 9.8\text{ms}^{-2}$.

Solution

Using the capillary rise formula:

$$h = \frac{2\gamma\cos\theta}{\rho g r}$$

$$h = \frac{2 \times 7.2 \times 10^{-2}\text{Nm}^{-1} \times \cos 0°}{1000\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 2.5 \times 10^{-4}\text{m}} = 5.88 \times 10^{-2}\text{m}$$

Therefore, water rises to a height of 5.88×10^{-2}m (or approximately 5.9cm) in the capillary tube.

Making Sense of the Answer: *Nearly 6cm of rise in a tube only 0.5mm wide. This is a substantial height, easily visible to the naked eye. In an even narrower tube (say,* 0.05mm *radius), the water would rise five times higher: about 29cm. This is why capillary effects are dramatic in very thin tubes but invisible in ordinary drinking glasses.*

Think Like a Physicist: *The angle of contact for water on clean glass is essentially* $0°$*, which means* $\cos\theta = 1$ *and the formula simplifies to* $h = 2\gamma/(\rho g r)$*. Many problems give* $\theta = 0°$ *for water on glass. If the angle of contact is not given in a problem, assume* $0°$ *unless stated otherwise.*

BINDER Example 15

Water rises in a capillary tube to a height of 2.0cm. What will be the height of rise in a capillary tube of one-third the radius?

Solution

From the capillary rise formula, $h = \frac{2\gamma\cos\theta}{\rho g r}$.

For the same liquid, same temperature, and same solid surface, γ, θ, ρ, and g are all constant. Therefore:

$$h \propto \frac{1}{r}$$

This means:

$$h_1 r_1 = h_2 r_2$$

If $r_2 = \frac{r_1}{3}$. Then:

$$h_2 = h_1 \times \frac{r_1}{r_2} = 2.0\text{cm} \times \frac{r_1}{\frac{r_1}{3}} = 2.0\text{cm} \times 3 = 6.0\text{cm}$$

Therefore, water rises to a height of 6.0cm in the narrower tube.

Making Sense of the Answer: *One-third the radius gives three times the height. This is the inverse proportionality at work: $h \propto 1/r$. If you halve the radius, the height doubles. If you reduce the radius to one-tenth, the height increases tenfold. This scaling law is why capillary action is powerful in the microscopic channels of plant xylem but unnoticeable in a wide bucket.*

Think Like a Physicist: *The relation $h_1 r_1 = h_2 r_2$ is one of the most useful shortcuts in capillary problems. It avoids the need to know γ, ρ, θ, or g. Whenever a problem compares two capillary tubes with the same liquid, reach for this relation first.*

BINDER Example 16

A glass capillary tube of internal radius 0.75mm is dipped vertically into mercury. Given that the surface tension of mercury is 0.54Nm^{-1}, the density of mercury is 13600kgm^{-3}, the angle of contact of mercury with glass is 135°, and $g = 9.8\text{ms}^{-2}$, calculate the depression of mercury in the tube.

Solution

Using the capillary formula:

$$h = \frac{2\gamma\cos\theta}{\rho g r}$$

$$h = \frac{2 \times 0.54\text{Nm}^{-1} \times \cos 135°}{13{,}600\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 7.5 \times 10^{-4}\text{m}} h = -7.64 \times 10^{-3}\text{m}$$

The negative sign confirms that mercury is depressed below the outside level. Therefore, the depression of mercury in the glass tube is 7.64×10^{-3}m (or approximately 7.6mm).

Making Sense of the Answer: *A depression of about* 7.6mm *in a tube of radius* 0.75mm. *This is small but significant in precision instruments. Mercury thermometers and barometers use glass tubes, and the capillary depression introduces a systematic error in the readings. Manufacturers correct for this by calibrating the instruments against known standards.*

Think Like a Physicist: *The formula is identical for rise and depression. The sign of $\cos\theta$ does all the work: positive for $\theta < 90°$ (rise), negative for $\theta > 90°$ (depression). There is no need to memorise a separate formula for mercury. One formula, two behaviours. That is good physics.*

REAL Example 17

After a rainy afternoon, Kipanga spills water on the classroom table. He grabs a paper towel to wipe it up and watches the water rush upward through the towel the moment it touches the puddle. Impressed, he tries the same thing with a sheet of waxed paper from his lunch wrapper. The water sits on the waxed surface and refuses to climb at all.

Explain why water rises through a paper towel but not through waxed paper.

Solution

A paper towel is made of loosely woven plant fibres with tiny air gaps between them. These gaps act as capillary tubes. Since paper fibres are hydrophilic (water-attracting), the adhesive forces between water and the fibres are stronger than the cohesive forces within the water. The angle of contact is very small ($\theta \approx 0°$), and capillary rise draws the water upward through the narrow gaps against gravity.

Waxed paper, on the other hand, is coated with a hydrophobic (water-repelling) layer. The cohesive forces within the water are stronger than the adhesive forces between water and the waxy surface. As result, the angle of contact is large ($\theta > 90°$), which means $\cos\theta$ is negative. The capillary formula gives a negative height which means that the water would be depressed, not raised. Since the water is not attracted to the waxed surface, it does not enter the tiny gaps and instead remains on top as a puddle.

Making Sense of the Answer: *The same liquid, the same gravity, the same physics. The only difference is the surface. Paper fibres attract water (adhesion wins), so capillary rise pulls the water in. Wax repels water (cohesion wins), so the water stays out. This is why waterproof jackets are waxed or coated: they turn every fibre gap from a capillary channel into a barrier.*

Think Like a Physicist: *The angle of contact is the gatekeeper. If* $\theta < 90°$, *the liquid enters and rises. If* $\theta > 90°$, *the liquid stays out and may even be depressed. Whenever you see a liquid being absorbed or repelled by a material, ask: is the contact angle acute or obtuse? That single question answers everything.*

HOT Example 18

The same capillary tube is used to measure the rise of water and the depression of mercury. Water rises to a height of 9.0cm, while mercury is depressed by 3.4cm in the same tube.

Given that the angle of contact for water with glass is 0° and for mercury with glass is 135°, and the density of mercury is 13600kgm^{-3} and the density of water is 1000kgm^{-3}, find the ratio of the surface tension of mercury to that of water.

Solution

For water: $h_w = 9.0\text{cm}$, $\theta_w = 0°$, $\rho_w = 1000\text{kgm}^{-3}$

For mercury: $h_m = 3.4\text{cm}$ (depression), $\theta_m = 135°$, $\rho_m = 13600\text{kgm}^{-3}$

From the capillary formula, for each liquid:

$$\gamma = \frac{\rho g r h}{2\cos\theta}$$

Since the same tube is used, r and g are the same for both. Taking the ratio:

$$\frac{\gamma_m}{\gamma_w} = \frac{\rho_m \times h_m \times \cos\theta_w}{\rho_w \times h_w \times \cos\theta_m}$$

$$\frac{\gamma_m}{\gamma_w} = \frac{46,240}{-6364}$$

Taking the magnitude (since we are comparing strengths of surface tension):

$$\frac{\gamma_m}{\gamma_w} = 7.27$$

Therefore, the ratio of the surface tension of mercury to that of water is approximately $7.3:1$. (Mercury has about 7 times the surface tension of water.)

Making Sense of the Answer: *This ratio is consistent with known values:* $\gamma_{\text{mercury}} \approx 0.47\text{Nm}^{-1}$ *and* $\gamma_{\text{water}} \approx 0.073\text{Nm}^{-1}$, *giving a ratio of about 6.4. Our experimental result of 7.3 is in the right ballpark. The discrepancy arises from experimental uncertainties: imperfect angles of contact, non-uniform tube radius, and temperature variations.*

Think Like a Physicist: *This is a ratio problem. Whenever the same apparatus is used for two different liquids, set up the ratio to cancel common factors* (r, g)*. This eliminates the need to know the tube radius, which may be difficult to measure precisely. Ratio methods are a powerful experimental strategy: they turn absolute measurements (which are hard) into relative measurements (which are easier and more reliable).*

That completes this subtopic. Up to this point, the first hidden personality of fluids is now fully developed: from the molecular theory of why surface tension exists, through the excess pressure it creates inside bubbles and drops, to the capillary rise and depression it produces in narrow tubes.

In the next subtopic, we leave static fluids behind and step into the world of moving fluids. We are going to introduce the types of fluid flow, the Reynolds number, and the equation of continuity. So the next subtopic is the gateway to the crown jewel of this chapter: **Bernoulli's equation**.

TYPES OF FLUID FLOW AND EQUATION OF CONTINUITY

Up to this point, every fluid we have studied has been sitting still. Surface tension, excess pressure, and capillary rise are all properties of fluids at rest. But fluids in the real world move. Water flows through pipes, blood flows through arteries, and air flows over aircraft wings. The moment a fluid begins to move, new physics enters the picture, and we need new language to describe it.

This section introduces that language. We will classify fluids, distinguish between orderly and chaotic flow, and derive a powerful equation that tells us exactly what happens to the speed of a fluid when it is forced through a narrow opening. By the end, you will have the tools to explain why a garden hose shoots faster when you cover the end with your thumb, and you will be ready for the crown jewel of this chapter: Bernoulli's equation.

Compressible and Incompressible Fluids

When you squeeze a balloon, the air inside is forced into a smaller space. The same number of molecules now occupy a smaller volume, so the density of the air increases. Air, like all gases, is a **compressible fluid**: its *density changes when pressure is applied.*

Now try squeezing a sealed syringe full of water. Push as hard as you like. The plunger barely moves. The water refuses to be compressed. Liquids are very nearly **incompressible**: their *density remains essentially constant no matter how much pressure you apply.*

This distinction matters enormously for flowing fluids. If a fluid is compressible, its density varies from place to place inside the pipe, and the mathematics becomes considerably more complex. If the fluid is incompressible, its density is the same everywhere, and the equations simplify beautifully.

In this chapter, unless stated otherwise, we deal exclusively with **incompressible fluids**. This is a reasonable assumption for all liquids and for gases moving at low speeds (below the speed of sound).

Viscous and Non-Viscous Fluids

Imagine two tall glasses on a table. One is filled with water, the other with honey. You tilt both glasses at the same angle. The water rushes to the rim instantly, eager to flow. The honey crawls forward reluctantly, as though every millimetre of progress is a personal struggle.

Both liquids are being pulled by the same gravity. Both are free to flow. Yet honey resists flowing in a way that water does not. Why?

The answer is that when honey flows, its layers do not all move at the same speed. The layer touching the glass wall does not move at all (it sticks to the surface). The layer just above it moves a little. The next layer moves a little faster. And so on, until the layer at the centre moves fastest. Each layer drags on the layer next to it, slowing it down, like a stack of heavy blankets being pulled sideways: the bottom blanket is pinned to the bed, and each blanket above it must overcome the friction of the one below before it can move.

This dragging between adjacent layers is called **internal friction**, and the property of a fluid that measures how strongly its layers resist sliding over one another is called **viscosity**. A fluid with high viscosity (honey, engine oil, glycerine) resists flowing strongly because its layers drag heavily on each other. A fluid with low viscosity (water, air, petrol) flows easily because the dragging between layers is weak.

A **viscous fluid** *is one in which this internal friction is significant.* A **non-viscous fluid** (also called an **inviscid fluid**) *is one in which the internal friction is so small that it can be ignored.* No real fluid is perfectly non-viscous, but water and air come close enough that, for many problems, the assumption of zero viscosity gives excellent results.

Why does this matter for flowing fluids? Because viscosity drains energy. In a viscous fluid, some of the fluid's kinetic energy is continuously converted into heat by the friction between layers. Consequently, the fluid gradually loses speed unless an external force (such as a pump or gravity) keeps pushing it. In a non-viscous fluid, no energy is lost to internal friction. Hence, the total mechanical energy of the fluid is conserved as it flows, which is the key assumption behind Bernoulli's equation.

An **ideal fluid** *is one that is both incompressible and non-viscous, and whose flow is irrotational* (meaning the fluid elements do not spin about their own axes as they move along). Ideal fluids do not exist in nature, but the concept is extraordinarily useful. It allows us to derive clean, exact equations that describe real fluid behaviour to a very good approximation under many practical conditions. Think of the ideal fluid as the physicist's perfectly smooth surface or frictionless pulley: a simplification that captures the essential physics while removing the complications that would otherwise make the mathematics unmanageable.

Steady (Laminar) Flow and Turbulent Flow

Open a tap very slightly and let water trickle out. The stream is smooth, glassy, and predictable. Every drop follows the same path as the drop before it. Now open the tap fully. The stream becomes rough, noisy, and chaotic, splashing in all directions.

These are the two faces of fluid flow, and understanding the difference between them is essential.

Steady (laminar) flow

When a fluid flows slowly and smoothly, every particle that passes through a given point follows exactly the same path as every particle that passed through that point before it. The flow is orderly, layered, and predictable. This type of flow is called **steady flow** or **laminar flow** (from the Latin *lamina*, meaning a thin layer).

To picture laminar flow, imagine thin sheets of fluid sliding smoothly over one another, like pages in a book being pushed sideways. The path followed by any individual fluid particle is called a **streamline**. In laminar flow, streamlines are smooth, continuous curves, and they never cross one another. If two streamlines crossed at a point, a fluid particle arriving at that point would face two possible velocities at the same instant, which is physically impossible. Thus, the non-crossing of streamlines is not just a convention; it is a requirement of physical consistency.

Furthermore, in steady flow, the velocity of the fluid at any given point does not change with time. Different points may have different velocities (for example, the fluid moves faster in a narrow section of pipe than in a wide section), but at each fixed point, the velocity remains constant as long as the flow is steady.

Turbulent flow

When the flow speed increases beyond a certain limit, the orderly laminar pattern collapses. The fluid particles begin to move chaotically, with rapidly fluctuating velocities and unpredictable paths. Eddies and vortices form, fluid layers mix violently, and the streamline picture breaks down entirely. This is **turbulent flow**.

You have seen turbulence many times: the swirling water behind a rock in a fast river, the chaotic smoke rising from a campfire after it breaks into curls and twists, the bumpy ride on an aircraft passing through rough air. In every case, the flow has become too fast or too disturbed to remain orderly.

Turbulent flow dissipates far more energy than laminar flow at the same average speed. Consequently, turbulence is undesirable in many engineering applications: pipeline designers, aircraft engineers, and even blood vessel surgeons all work hard to keep flow laminar wherever possible.

The Reynolds Number

A natural question arises: *how fast is too fast? At what point does laminar flow give way to turbulence?* The answer depends not on the speed alone, but on a combination of the fluid's speed, density, viscosity, and the size of the pipe. The quantity that captures all of these factors in a single number is called the **Reynolds number**, denoted by $\mathrm{R_e}$:

$$\mathbf{R_e = \frac{\rho v r}{\eta}}$$

where ρ is the density of the fluid ($\mathrm{kgm^{-3}}$), v is the speed of flow ($\mathrm{ms^{-1}}$), r is the radius of the tube (m), and η is the coefficient of viscosity of the fluid ($\mathrm{Nsm^{-2}}$).

The Reynolds number is a **dimensionless quantity** (it has no units), and it predicts the flow regime:

$\mathrm{R_e} < 2000$: the flow is **laminar** (smooth and orderly).

$R_e > 3000$: the flow is **turbulent** (chaotic and irregular).

$2000 < R_e < 3000$: the flow is **unstable** (it may switch unpredictably between laminar and turbulent).

What does this number actually tell you? Physically, the Reynolds number is the ratio of **inertial forces** to **viscous forces** in the fluid.

$$R_e = \frac{\text{Inertial forces}}{\text{Viscous forces}}$$

Inertial forces are the forces associated with the fluid's momentum; they tend to keep the fluid moving and, if disturbed, to amplify the disturbance into turbulence. Viscous forces are the internal friction forces; they tend to damp out disturbances and restore order. When viscous forces dominate (small R_e), disturbances are suppressed and the flow stays laminar. When inertial forces dominate (large R_e), disturbances grow and the flow becomes turbulent.

This explains several everyday observations. A thick, viscous fluid like honey almost always flows in a laminar manner, because its high viscosity gives it a small Reynolds number even at moderate speeds. Water in a narrow pipe at low speed is also laminar, but the same water in a wide pipe at high speed becomes turbulent, because the increased speed and radius push the Reynolds number above 3000. Blood flow in healthy arteries is normally laminar ($R_e \approx 1000$), but if the artery narrows due to plaque buildup, the speed increases at the constriction, the Reynolds number rises, and the flow may become turbulent, producing the characteristic "murmur" that a doctor hears through a stethoscope.

Critical velocity

The speed at which the flow regime changes from laminar to turbulent is called the **critical velocity**, denoted by v_c. *It is the maximum speed at which laminar flow can be maintained in a given system.*

We can find its value directly from the Reynolds number. At the transition point, R_e equals the critical value (approximately 2000 for flow through a pipe). Setting $R_e = 2000$ and solving for v:

$$R_e = \frac{\rho v_c r}{\eta}$$

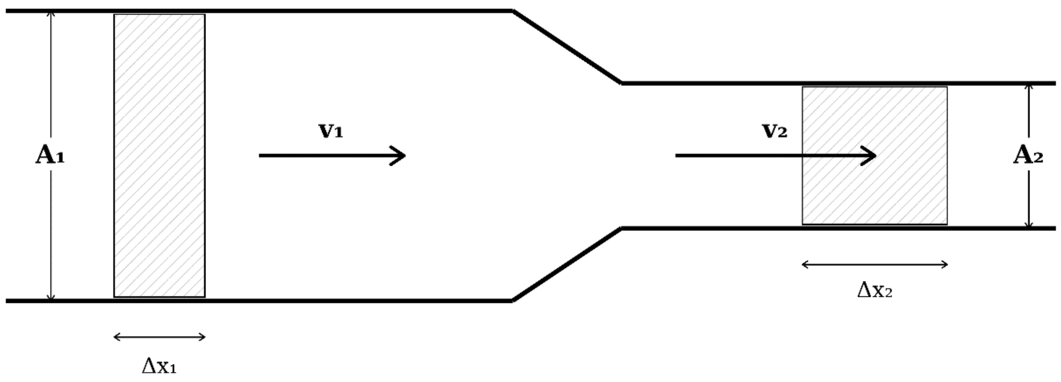
$$\mathbf{v_c} = \frac{R_e \times \eta}{\rho r} = \frac{\mathbf{2000 \times \eta}}{\boldsymbol{\rho} \mathbf{r}}$$

This formula reveals three important facts. First, a more viscous fluid (larger η) has a higher critical velocity, which is why honey can flow quite fast and still remain laminar. Second, a denser fluid (larger ρ) has a lower critical velocity, because its inertial forces overpower its viscous forces more easily. Third, a wider pipe (larger r) has a lower critical velocity, which is why turbulence appears more readily in large rivers than in narrow tubes.

The Equation of Continuity

We now derive one of the most important and most useful results in fluid mechanics. It is called the **equation of continuity**, and it is nothing more than the **law of conservation of mass** applied to a flowing fluid. The question it answers is deceptively simple: *if a fluid flows through a pipe that changes width, what happens to the speed?*

You already know the answer intuitively. When you partially cover the end of a garden hose with your thumb, the water shoots out faster. When a river narrows between two rocks, the current speeds up. When traffic on a highway is forced from three lanes into one, each car must move faster to maintain the overall flow. The equation of continuity makes this intuition precise and mathematical.



Fluid flows from left to right

Figure: *Steady flow through a pipe of varying cross-section. At the wider section (area A_1), the fluid moves with velocity v_1. At the narrower section (area A_2), the fluid moves with velocity v_2. The shaded elements represent equal volumes of fluid passing each section in time Δt. By conservation of mass, $A_1 v_1 = A_2 v_2$.*

Derivation

Consider an incompressible fluid flowing steadily through a pipe that narrows from cross-sectional area A_1 to cross-sectional area A_2. Since the fluid is incompressible, its density ρ is the same everywhere in the pipe.

In a small time interval Δt, the fluid at the wide section moves a distance $\Delta x_1 = v_1 \Delta t$, sweeping out a volume:

$$\Delta V_1 = A_1 \times \Delta x_1 = A_1 v_1 \Delta t$$

The mass of fluid entering the wide section in time Δt is therefore:

$$\Delta m_1 = \rho A_1 v_1 \Delta t$$

Similarly, at the narrow section, the fluid moves a distance $\Delta x_2 = v_2 \Delta t$, and the mass leaving in time Δt is:

$$\Delta m_2 = \rho A_2 v_2 \Delta t$$

Now here is the key physical argument. The fluid is not leaking, not being created, and not being destroyed. Every drop that enters the wide end must eventually leave through the narrow end. Therefore, the mass entering per unit time must equal the mass leaving per unit time:

$$\Delta m_1 = \Delta m_2$$

$$\rho A_1 v_1 \Delta t = \rho A_2 v_2 \Delta t$$

Since the fluid is incompressible, ρ is the same on both sides. The time interval Δt is also the same. Both cancel, leaving:

$$\mathbf{A_1 v_1 = A_2 v_2}$$

This is the **equation of continuity** for an incompressible fluid. It states that *the product of cross-sectional area and fluid velocity is the same at every point along the pipe.*

The quantity $Q = Av$ is called the **volume flow rate or volume flux**: *the volume of fluid passing a given cross-section per unit time*. Its SI unit is $m^3 s^{-1}$. The equation of continuity therefore says:

$$\mathbf{Q = A_1 v_1 = A_2 v_2 = constant}$$

Take a moment to appreciate what this equation is telling you. Where the pipe narrows (A decreases), the speed must increase (v goes up) to keep the product Av constant. Where the pipe widens (A increases), the speed must decrease. No extra force is needed. No pump is required. The speed change is an automatic consequence of mass conservation: the same volume of fluid must pass through every cross-section in the same time, so where the area is smaller, the fluid must move faster to compensate.

This is precisely why the boy in the introduction turned a gentle hose into a powerful jet by pressing his thumb over the end. His thumb reduced the exit area. Consequently, the continuity equation demanded that the exit velocity increase. The tap pressure did not change. Conservation of mass did all the work.

Note for compressible fluids: If the fluid is compressible (such as a gas at high speed), the density differs at each cross-section, and the general form of the continuity equation becomes:

$$\boldsymbol{\rho_1 A_1 v_1 = \rho_2 A_2 v_2}$$

For all problems in this chapter, we use the incompressible form $A_1v_1 = A_2v_2$.

With the language of fluid flow now in place, let us put it to work.

BINDER Example 19

Water flows steadily through a horizontal pipe that narrows from a cross-sectional area of 40cm^2 to 10cm^2. If the velocity of water in the wider section is 2.0ms^{-1}, find the velocity in the narrower section.

Solution

Given: $A_1 = 40\text{cm}^2 = 40 \times 10^{-4}\text{m}^2$, $A_2 = 10\text{cm}^2 = 10 \times 10^{-4}\text{m}^2$, $v_1 = 2.0\text{ms}^{-1}$

Applying the equation of continuity for an incompressible fluid:

$$A_1v_1 = A_2v_2$$

$$v_2 = \frac{A_1v_1}{A_2} = \frac{40 \times 10^{-4}\text{m}^2 \times 2.0\text{ms}^{-1}}{10 \times 10^{-4}\text{m}^2} = 8.0\text{ms}^{-1}$$

Therefore, the velocity of water in the narrower section is 8.0ms^{-1}.

Making Sense of the Answer: *The area decreased by a factor of 4, and consequently the velocity increased by the same factor. This is the continuity equation in its purest form: area down, speed up, in exact inverse proportion. The volume flow rate (*$Q = 8.0 \times 10^{-3} m^3 s^{-1}$*) is the same in both sections, as conservation of mass requires.*

Think Like a Physicist: *The equation of continuity is a ratio equation. If the area halves, the speed doubles. If the area drops to one-tenth, the speed increases tenfold. Always check your answer against this proportionality before moving on.*

REAL Example 20

Kipanga is helping his mother water the garden using a hose connected to the outdoor tap. The water flows out gently from the open end of the hose. Then Kipanga places his thumb over the end, covering most of the opening and leaving only a small gap. The water immediately shoots out as a fast, thin jet that reaches much further across the garden.

Explain why partially covering the end of the hose causes the water to exit at a higher speed.

Solution

When the hose end is fully open, the water exits through the full cross-sectional area of the hose opening. When Kipanga places his thumb over the end, the effective exit area is reduced to the small gap between his thumb and the hose rim.

The tap delivers a fixed volume flow rate Q to the hose. By the equation of continuity, $Q = Av$, where A is the exit area and v is the exit velocity. Since the tap setting has not changed, Q remains constant. Consequently, reducing A causes v to increase in the same proportion.

As a result, the water exits as a fast, narrow jet that travels a greater horizontal distance before gravity brings it to the ground.

Making Sense of the Answer: *No extra force is applied. The tap pressure has not changed. The speed increase is entirely a consequence of conservation of mass: the same volume of water must pass through a smaller opening in the same time, and therefore it must move faster.*

Think Like a Physicist: *This example connects directly to the introduction of the chapter, where a boy's thumb turned a gentle hose into a sambusa-destroying weapon. The continuity equation explains what happens to the speed. Bernoulli's equation, which we meet next, will explain what happens to the pressure.*

HOT Example 21

The cylindrical tube of a spray pump has a cross-sectional area of 8.0cm^2. At one end, the tube is fitted with a nozzle containing 40 fine holes, each of diameter 0.10mm. If the liquid flows inside the main tube at 1.5m per minute, find the speed of ejection of the liquid through the holes.

Solution

Given: Area of main tube $A_1 = 8.0\text{cm}^2 = 8.0 \times 10^{-4}\text{m}^2$

Velocity in main tube: $v_1 = 1.5\text{m/min} = \frac{1.5}{60}\text{ms}^{-1} = 0.025\text{ms}^{-1}$

Each hole has diameter $d = 0.10\text{mm} = 1.0 \times 10^{-4}\text{m}$, hence radius $r = 0.50 \times 10^{-4}\text{m}$.

Area of one hole:

$$a = \pi r^2 = \pi \times (0.50 \times 10^{-4}\text{m})^2 = \pi \times 2.5 \times 10^{-9}\text{m}^2 = 7.854 \times 10^{-9}\text{m}^2$$

Total exit area through all 40 holes:

$$A_2 = 40 \times a = 40 \times 7.854 \times 10^{-9}\text{m}^2 = 3.142 \times 10^{-7}\text{m}^2$$

Applying the equation of continuity:

$$A_1 v_1 = A_2 v_2$$

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{8.0 \times 10^{-4}\text{m}^2 \times 0.025\text{ms}^{-1}}{3.142 \times 10^{-7}\text{m}^2} = 63.7\text{ms}^{-1}$$

Therefore, the speed of ejection of the liquid through the holes is approximately 64ms^{-1}.

Making Sense of the Answer: *The liquid enters the tube at a leisurely* 0.025ms^{-1} *and exits the holes at approximately* 64ms^{-1} *(about* 230kmh^{-1}*). This represents a speed amplification factor of over 2500, which is consistent with the area ratio: the total exit area is roughly 2500 times smaller than the tube area. This enormous amplification is what makes spray nozzles effective: a slow, steady push produces a fine, high-speed mist.*

Think Like a Physicist: *The most common error in problems involving multiple holes is forgetting to multiply the area of one hole by the total number of holes. The exit area is the combined area of all 40 holes, not the area of a single hole. Always read the geometry carefully before applying the continuity equation.*

That completes the foundations of fluid flow. We have classified fluids, distinguished laminar from turbulent flow, and derived the equation of continuity from conservation of mass. We now know that when an incompressible fluid speeds up through a constriction, it must obey $A_1 v_1 = A_2 v_2$. But a deeper question remains unanswered: *what happens to the* ***pressure*** *when the speed changes?* The answer is one of the most beautiful results in all of physics, and it is waiting in the next subtopic.

BERNOULLI'S PRINCIPLE: DERIVATION AND MEANING

This is the section the entire chapter has been building toward. Every concept we have developed so far: pressure, surface tension, and the equation of continuity has been preparing the ground for what comes next. Bernoulli's equation is to fluid mechanics what Newton's second law is to particle mechanics: it is the central organising principle, the equation that ties everything together, and one of the most beautiful results in all of physics.

The idea behind it is deceptively simple. When a fluid speeds up, its pressure drops. When it slows down, its pressure rises. Speed and pressure trade with each other, like two ends of a seesaw. This trade is not random; it follows an exact mathematical law, and that law is Bernoulli's equation.

Physical Intuition Before the Mathematics

Before we touch a single equation, let us build the intuition that will make the mathematics feel natural rather than mysterious.

Think about energy. You already know from mechanics that energy is conserved: it can change form (kinetic to potential, potential to kinetic), but the total cannot increase or decrease unless an external force does work. A ball rolling down a hill converts potential energy into kinetic energy. A ball rolling up a hill does the reverse. At every point, the total mechanical energy is the same.

Now apply this thinking to a fluid flowing through a pipe. A small parcel of fluid moving through the pipe carries three forms of energy:

1) **Kinetic energy**, because it is moving. The faster it moves, the more kinetic energy it has.
2) **Gravitational potential energy**, because it may be at some height above a reference level. The higher it is, the more potential energy it has.
3) **Pressure energy**, and this is the new idea. In a fluid, the surrounding fluid exerts pressure on the parcel from all sides. This pressure can do work on the parcel (pushing it forward) or the parcel can do work against the pressure (pushing surrounding fluid out of the way). The energy associated with this pressure is called pressure energy, and it equals the pressure multiplied by the volume of the parcel.

Here is the key insight. If the fluid is ideal (incompressible, non-viscous, and flowing steadily), no energy is lost to friction and no energy is added by a pump. Consequently, the total energy per unit volume which is the sum of pressure energy, kinetic energy, and gravitational potential energy must remain constant along any streamline. If the kinetic energy increases (because the fluid speeds up), something else must decrease to compensate. That something is the pressure energy. Hence, where the fluid moves faster, the pressure is lower.

Think of it as an energy budget. The fluid has a fixed total budget of energy per unit volume. If it spends more on speed (kinetic energy), it has less to spend on pressure. If it spends more on height (potential energy), it has less for both speed and pressure. The budget is fixed; only the allocation changes.

This is Bernoulli's principle:

For a steady, incompressible, non-viscous flow, the sum of the pressure, the kinetic energy per unit volume, and the gravitational potential energy per unit volume is constant along a streamline.

Now let us derive it rigorously.

Full Derivation of Bernoulli's Equation

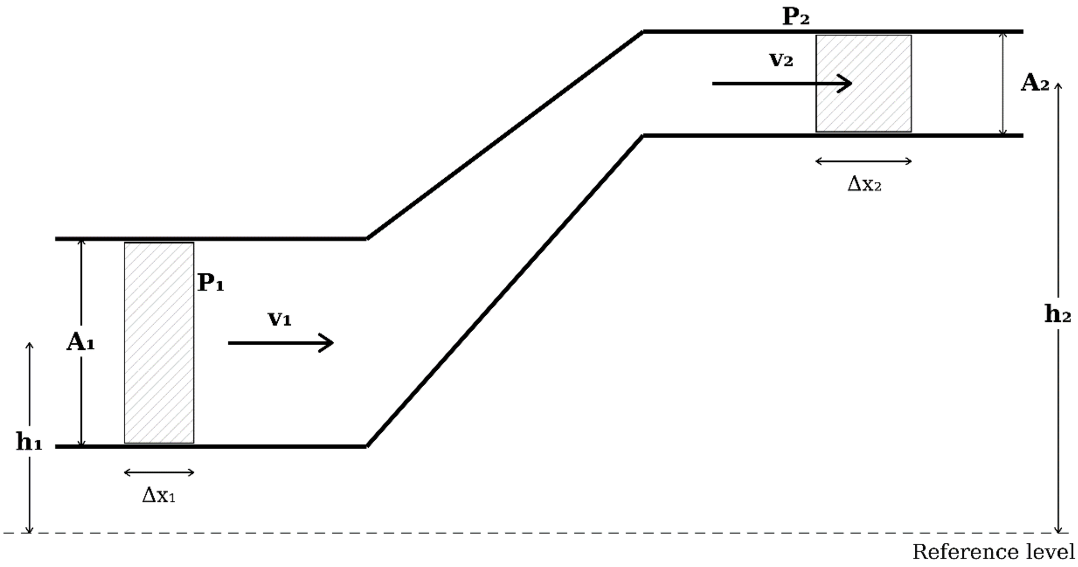


Figure: ***Fluid flowing through a non-uniform pipe that narrows and rises.*** *At the lower section (area A_1, velocity v_1, pressure P_1, height h_1), a fluid element of length Δx_1 is shown. At the upper section (area A_2, velocity v_2, pressure P_2, height h_2), the corresponding element has length Δx_2. Heights are measured from the horizontal reference level.*

Consider an ideal fluid (incompressible, non-viscous, irrotational) flowing steadily through a pipe of varying cross-section that is not horizontal. At the lower end, the pipe has cross-sectional area A_1, and the fluid flows with velocity v_1 at pressure P_1, at a height h_1 above a chosen reference level. At the upper end, the corresponding quantities are A_2, v_2, P_2, and h_2.

In a small time interval Δt, a small volume of fluid ΔV enters the pipe at the lower end and an equal volume ΔV leaves at the upper end. By the equation of continuity, these volumes are equal because the fluid is incompressible:

$$\Delta V = A_1 v_1 \Delta t = A_2 v_2 \Delta t$$

The mass of this fluid element is:

$$\Delta m = \rho \Delta V$$

Work done on the fluid by the surrounding pressure:

At the lower end, the fluid behind the element pushes it forward with a force $F_1 = P_1 A_1$. This force moves the element through a distance $\Delta x_1 = v_1 \Delta t$. The work done by this force is:

$$W_1 = P_1 A_1 \times \Delta x_1 = P_1 A_1 \times v_1 \Delta t = P_1 \Delta V$$

At the upper end, the fluid ahead of the element resists the incoming fluid with a force $F_2 = P_2 A_2$, directed backward. The element moves forward through $\Delta x_2 = v_2 \Delta t$, doing work against this resistance:

$$W_2 = P_2 A_2 \times \Delta x_2 = P_2 A_2 \times v_2 \Delta t = P_2 \Delta V$$

The net work done on the fluid element by the pressure forces is:

$$W_{net} = W_1 - W_2 = P_1 \Delta V - P_2 \Delta V = \mathbf{(P_1 - P_2)\Delta V}$$

Change in kinetic energy of the fluid element:

$$\Delta KE = \frac{1}{2}\Delta m\, v_2^2 - \frac{1}{2}\Delta m\, v_1^2 = \frac{1}{2}\rho \Delta V\, v_2^2 - \frac{1}{2}\rho \Delta V\, v_1^2 = \boldsymbol{\frac{1}{2}\rho \Delta V\left(v_2^2 - v_1^2\right)}$$

Change in gravitational potential energy of the fluid element:

$$\Delta PE = \Delta m\, g\, h_2 - \Delta m\, g\, h_1 = \rho \Delta V\, g\, h_2 - \rho \Delta V\, g\, h_1 = \boldsymbol{\rho \Delta V\, g(h_2 - h_1)}$$

Applying the work-energy theorem:

The net work done on the fluid element equals the total change in its mechanical energy (kinetic plus potential):

$$W_{net} = \Delta KE + \Delta PE$$

$$(P_1 - P_2)\Delta V = \frac{1}{2}\rho \Delta V(v_2^2 - v_1^2) + \rho \Delta V\, g(h_2 - h_1)$$

Dividing every term by ΔV (which is non-zero):

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1)$$

Rearranging so that all quantities at point 1 are on the left and all quantities at point 2 are on the right:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Since points 1 and 2 are any two points along the same streamline, this equation holds for all points. We can therefore write it in its general form:

$$\mathbf{P + \frac{1}{2}\rho v^2 + \rho g h = constant\ (along\ a\ streamline)}$$

This is **Bernoulli's equation**. Each term has the dimensions of pressure (Nm^{-2} or Pa), and each has a clear physical meaning:

P is the **static pressure,** which is the actual fluid pressure at that point.

$\frac{1}{2}\rho v^2$ is the **dynamic pressure,** which is the kinetic energy per unit volume. It represents the pressure equivalent of the fluid's motion.

$\rho g h$ is the **hydrostatic pressure,** which is the gravitational potential energy per unit volume. It represents the pressure equivalent of the fluid's height.

Bernoulli's equation says that the sum of these three pressures is constant along a streamline. If one increases, at least one of the others must decrease to maintain the balance.

Special case: horizontal flow. If the pipe is horizontal, $h = 0$ ($h_1 = h_2$), the gravitational terms cancel, and Bernoulli's equation reduces to:

$$P + \frac{1}{2}\rho v^2 = \text{constant}$$

This is the form you will use most often. It says, directly and unambiguously*: in a horizontal flow, where the speed is high, the pressure is low; where the speed is low, the pressure is high.* This single statement explains an astonishing range of phenomena, from how aeroplanes fly to why shower curtains attack you.

Conditions and Limitations

Bernoulli's equation is powerful, but it is not universal. It was derived under specific assumptions, and it is valid only when those assumptions are reasonably satisfied. Understanding the limitations is just as important as knowing the equation itself, because applying Bernoulli's equation where it does not apply leads to wrong answers and wrong physics.

1. The fluid must be incompressible

The density ρ must remain constant throughout the flow. This is an excellent assumption for liquids and for gases at speeds well below the speed of sound (roughly below 100ms^{-1} for air). For high-speed gas flows (aircraft near or above the speed of sound), compressibility effects become important, and Bernoulli's equation in this simple form no longer applies.

2. The fluid must be non-viscous

The derivation assumed no energy loss due to internal friction. In reality, all fluids have some viscosity, and viscous losses cause the pressure to drop along the pipe even at constant speed. For fluids with low viscosity (water, air) flowing through short pipes, the viscous losses are small and Bernoulli's equation gives good results. For highly viscous fluids (honey, engine oil) or very long pipes, viscous losses dominate and Bernoulli's equation significantly overestimates the pressure.

3. The flow must be steady

The velocity at each point must not change with time. Bernoulli's equation does not apply to rapidly changing flows, such as the flow during the opening or closing of a valve, or the pulsating flow of blood from a beating heart (although it can be applied approximately to the average flow between heartbeats).

4. The flow must be irrotational

The fluid elements must not rotate about their own axes as they move along. This condition is usually satisfied in flows far from solid boundaries, but is violated near walls and in vortices.

6. The equation applies along a streamline

Bernoulli's equation relates quantities at two points on the same streamline. It does not, in general, relate quantities at points on different streamlines, unless additional conditions (such as irrotationality throughout the entire flow) are met.

A common misconception: Students sometimes state that "Bernoulli's equation proves that faster fluid always has lower pressure." This is an overstatement. Bernoulli's equation says that along a streamline in a steady, incompressible, non-viscous flow, higher speed corresponds to lower pressure. It does not apply to compressible flows, unsteady flows, or comparisons between different streamlines. A supersonic aircraft does not have vacuum around its wings, despite the very high air speed, because at those speeds the air is compressible and the simple form of Bernoulli's equation breaks down.

With the equation derived and its boundaries understood, let us now see it in action.

BINDER Example 22

Water enters a house through a horizontal pipe of inner diameter 2.0cm at an absolute pressure of $4.0 \times 10^5\text{Pa}$ and a flow velocity of 4.0ms^{-1}. The pipe leads to a bathroom on the second floor, 5.0m higher, where the inner diameter narrows to 1.0cm. Find:

(a) The flow velocity in the bathroom pipe.

(b) The water pressure in the bathroom.

Take the density of water as 1000kgm^{-3} and $g = 9.8\text{ms}^{-2}$.

Solution

(a) By the equation of continuity:

$$A_1 v_1 = A_2 v_2$$

For circular pipes, $A = \pi\left(\frac{d}{2}\right)^2$, so the area ratio equals the square of the diameter ratio:

$$\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2} = \frac{(2.0\text{cm})^2}{(1.0\text{cm})^2} = 4$$

$$v_2 = v_1 \times \frac{A_1}{A_2} = 4.0\text{ms}^{-1} \times 4 = 16\text{ms}^{-1}$$

Therefore, the flow velocity in the bathroom pipe is 16ms^{-1}.

(b) Applying Bernoulli's equation between the entry point (point 1, ground level) and the bathroom (point 2, height 5.0m):

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Taking the ground level as the reference ($h_1 = 0$, $h_2 = 5.0\text{m}$) and rearranging for P_2:

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) + \rho g(h_1 - h_2)$$

Substituting values:

$$P_2 = 4 \times 10^5\text{Pa} + \frac{1}{2} \times 1000\text{kgm}^{-3} \times ((4\text{ms}^{-1})^2 - (16\text{ms}^{-1})^2) + 1000\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times (0 - 5\text{m})$$

$$P_2 = 2.31 \times 10^5\text{Pa}$$

Therefore, the water pressure in the bathroom is $2.31 \times 10^5\text{Pa}$.

Making Sense of the Answer: *The pressure dropped from* $4.0 \times 10^5 Pa$ *to* $2.31 \times 10^5 Pa$*, a reduction of about 42%. Two effects caused this drop: the increase in speed (from* $4.0ms^{-1}$ *to* $16ms^{-1}$*, which consumed* $1.20 \times 10^5 Pa$ *of pressure energy) and the increase in height (5.0m, which consumed* $0.49 \times 10^5 Pa$*). The speed increase was the dominant factor. This is why water pressure on upper floors of buildings is noticeably lower than on the ground floor, especially when multiple taps are open simultaneously.*

Think Like a Physicist: *This problem uses both the continuity equation (to find* v_2*) and Bernoulli's equation (to find* P_2*). Many Bernoulli problems require both equations working together: continuity provides the velocity, and Bernoulli provides the pressure. Always start with continuity to find the unknown velocity before applying Bernoulli.*

BINDER Example 23

Water flows steadily through a horizontal pipe that narrows from a cross-sectional area of 25cm^2 to 10cm^2. The pressure in the wider section is $6.0 \times 10^4\text{Pa}$, and the velocity there is 2.0ms^{-1}. Find the pressure at the constriction. Take $\rho = 1000\text{kgm}^{-3}$.

Solution

First, find the velocity at the constriction using the continuity equation:

$$v_2 = v_1 \times \frac{A_1}{A_2} = 2.0\text{ms}^{-1} \times \frac{25\text{cm}^2}{10\text{cm}^2} = 5.0\text{ms}^{-1}$$

Since the pipe is horizontal ($h_1 = h_2$), Bernoulli's equation reduces to:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Rearranging for P_2:

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2)$$

$$P_2 = 6.0 \times 10^4\text{Pa} + \frac{1}{2} \times 1000\text{kgm}^{-3} \times ((2.0\text{ms}^{-1})^2 - (5.0\text{ms}^{-1})^2) = 4.95 \times 10^4\text{Pa}$$

Therefore, the pressure at the constriction is $4.95 \times 10^4\text{Pa}$.

Making Sense of the Answer: *The speed increased from* $2.0ms^{-1}$ *to* $5.0ms^{-1}$*, and the pressure dropped from* $6.0 \times 10^4 Pa$ *to* $4.95 \times 10^4 Pa$*. Speed up, pressure down; exactly as Bernoulli predicts. The pressure drop (*$1.05 \times 10^4 Pa$*) is about 17.5% of the original pressure, which is significant but not dramatic. In a more extreme constriction, the pressure drop would be much larger.*

Think Like a Physicist: *For horizontal pipes, Bernoulli's equation simplifies to* $P + \frac{1}{2}\rho v^2 = constant$*. This is the form to reach for whenever the pipe is horizontal or the height difference is negligible. The gravitational term only matters when there is a significant change in elevation.*

REAL Example 24

Kipute notices something strange every time she takes a shower. When she turns on the water, the shower curtain moves inward toward her, as if being sucked in. It does not blow outward, as she would expect from the spray hitting it. The harder the water flows, the more the curtain moves inward.

Explain why a shower curtain moves inward when the shower is running.

Solution

When the shower is running, a stream of water and entrained air moves rapidly downward between Kipute and the curtain. By Bernoulli's equation for horizontal flow ($\mathrm{P} + \frac{1}{2}\rho \mathrm{v}^2 = \text{constant}$), the rapidly moving air on the inner side of the curtain has a lower pressure than the still air on the outer side.

This creates a pressure difference across the curtain: higher pressure on the outside (where the air is stationary), lower pressure on the inside (where the air is moving with the shower stream). Consequently, the net force on the curtain is directed inward, pushing it toward the shower. The faster the water flows, the greater the air speed on the inner side, the greater the pressure difference, and the more strongly the curtain is pushed inward.

Therefore, the curtain is not being "sucked" inward. It is being **pushed** inward by the higher atmospheric pressure on the outer side, because the moving air on the inner side has reduced pressure.

Making Sense of the Answer: *This is Bernoulli's equation at work in your bathroom. The moving air has lower pressure; the still air has higher pressure. The pressure difference pushes the curtain toward the region of lower pressure. Many people describe this as "suction," but there is no suction force in physics. There is only a pressure difference, and the net force always pushes from high pressure toward low pressure.*

Think Like a Physicist: *Whenever you see an object being "pulled toward" a fast-moving fluid, ask:* ***is the pressure on the fast side lower than on the slow side?*** *If yes, the object is being pushed by the pressure difference, not pulled by some mysterious suction. This reasoning explains why two ships sailing close together in parallel tend to be drawn toward each other, why a flag flutters, and why roofs blow off during storms (the fast wind above creates lower pressure than the still air inside the house, so the roof is pushed upward from below).*

HOT Example 25

Water flows through a pipe system consisting of three sections. In the first section, the cross-sectional area is $30\mathrm{cm}^2$, the velocity is $3.0\mathrm{ms}^{-1}$, and the pressure is $2.0 \times 10^5\mathrm{Pa}$. The pipe then narrows to $15\mathrm{cm}^2$ in the second section, which is at the same height. Finally, the pipe rises $4.0\mathrm{m}$ and widens to $20\mathrm{cm}^2$ in the third section.

Find the velocity and pressure in the second and third sections. Take $\rho = 1000\mathrm{kgm}^{-3}$ and $\mathrm{g} = 9.8\mathrm{ms}^{-2}$.

Solution

Finding the velocity in the second section:

$$\mathrm{v}_2 = \mathrm{v}_1 \times \frac{\mathrm{A}_1}{\mathrm{A}_2} = 3.0\mathrm{ms}^{-1} \times \frac{30\mathrm{cm}^2}{15\mathrm{cm}^2} = 6.0\mathrm{ms}^{-1}$$

Finding the pressure in the second section:

The first and second sections are at the same height, so Bernoulli's equation reduces to:

$$\mathrm{P}_2 = \mathrm{P}_1 + \frac{1}{2}\rho(\mathrm{v}_1^2 - \mathrm{v}_2^2)$$

$$\mathrm{P}_2 = 2.0 \times 10^5\mathrm{Pa} + \frac{1}{2} \times 1000\mathrm{kgm}^{-3} \times ((3.0\mathrm{ms}^{-1})^2 - (6.0\mathrm{ms}^{-1})^2) = 1.865 \times 10^5\mathrm{Pa}$$

Therefore, in the second section, the velocity is $\mathbf{6.0ms^{-1}}$ and the pressure is $\mathbf{1.865 \times 10^5Pa}$.

Finding the velocity in the third section:

$$\mathrm{v}_3 = \mathrm{v}_1 \times \frac{\mathrm{A}_1}{\mathrm{A}_3} = 3.0\mathrm{ms}^{-1} \times \frac{30\mathrm{cm}^2}{20\mathrm{cm}^2} = 4.5\mathrm{ms}^{-1}$$

Finding the pressure in the third section:

Applying Bernoulli's equation between section 1 (height $\mathrm{h}_1 = 0$) and section 3 (height $\mathrm{h}_3 = 4.0\mathrm{m}$):

$$P_3 = P_1 + \frac{1}{2}\rho(v_1^2 - v_3^2) + \rho g(h_1 - h_3)$$

$P_3 = 2 \times 10^5\text{Pa} + \frac{1}{2} \times 1000\text{kgm}^{-3} \times ((3\text{ms}^{-1})^2 - (4.5\text{ms}^{-1})^2) + 1000\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times (0 - 4\text{m})$

$P_3 = 1.552 \times 10^5\text{Pa}$

Therefore, in the third section, the velocity is 4.5ms^{-1} and the pressure is $1.552 \times 10^5\text{Pa}$.

Making Sense of the Answer: *Follow the pressure as it drops through the system:* $2.0 \times 10^5\text{Pa} \rightarrow 1.865 \times 10^5\text{Pa} \rightarrow 1.552 \times 10^5\text{Pa}$. *Each drop has a clear physical cause. The first drop (from section 1 to 2) is caused entirely by the speed increase (the sections are at the same height). The second drop (from section 1 to 3) is caused by both the speed increase and the height increase. The height contribution (*39200Pa*) is actually larger than the speed contribution (*5625Pa*) in this case, which shows that elevation changes can dominate over speed changes when the pipe rises significantly.*

Think Like a Physicist: *In multi-section problems, always apply Bernoulli's equation between section 1 and each subsequent section directly, rather than going step by step from 1 to 2 and then from 2 to 3. Going directly from 1 to 3 avoids accumulating rounding errors from intermediate calculations. The constant in Bernoulli's equation is the same at all points on the streamline, so you can compare any two points directly.*

That completes the derivation and first applications of Bernoulli's equation. But the true power of Bernoulli's equation lies in its applications to real devices and real phenomena. In the next section, we apply it to five celebrated devices. Each application takes the abstract equation and turns it into something you can see, build, and use.

APPLICATIONS OF BERNOULLI'S PRINCIPLE

Bernoulli's equation would be beautiful even if it had no practical applications. But it does. It has so many that an entire engineering discipline (fluid dynamics) is built on its foundations. In this section, we take the abstract equation from the previous section and put it to work in five celebrated devices and phenomena. Each one demonstrates a different face of the same principle: where speed goes up, pressure goes down.

Torricelli's Theorem (Fluid Flowing from a Tank)

Imagine a large open tank filled with water to a height h above a small hole (orifice) in the side wall near the bottom. When the plug is removed, water rushes out of the hole. How fast does it emerge?

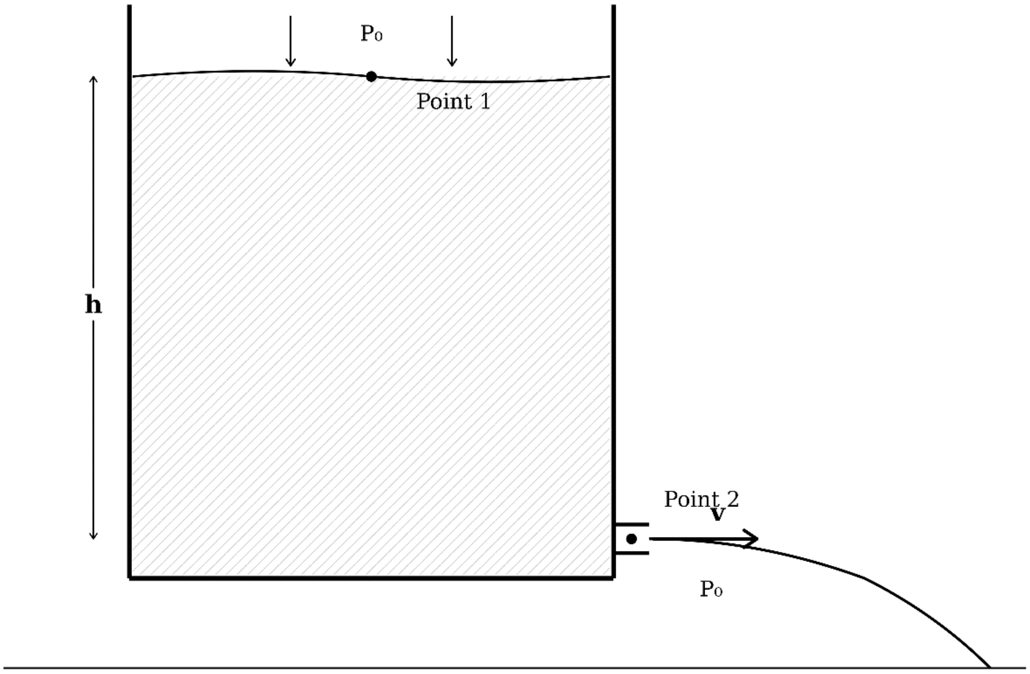


Figure: ***A tank of water with an orifice near the base.*** *Point 1 is at the water surface (velocity ≈ 0, pressure P₀). Point 2 is at the orifice (velocity v, pressure P₀). Both points are open to the atmosphere. The depth of water above the orifice is h.*

Apply Bernoulli's equation between the water surface (point 1) and the orifice (point 2). Both the surface and the orifice are open to the atmosphere, so $P_1 = P_2 = P_0$ (atmospheric pressure). Take the orifice as the reference level, so $h_2 = 0$ and $h_1 = h$.

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho g h = P_0 + \frac{1}{2}\rho v_2^2 + 0$$

Now here is a crucial simplification. The tank is large and the orifice is small. Consequently, the water level drops very slowly, so $v_1 \approx 0$. The atmospheric pressure terms cancel on both sides. What remains is:

$$\rho g h = \frac{1}{2}\rho v_2^2$$

Cancelling ρ and solving for v_2:

$$v_2 = \sqrt{2gh}$$

Hence, the speed of efflux is:

$$\mathbf{v = \sqrt{2gh}}$$

This is **Torricelli's theorem**. It states that *the speed of efflux of a liquid from an orifice at depth h below the surface is the same as the speed that a freely falling body would acquire after falling through the same height h*. This is a remarkable result: the water shoots out of the hole as fast as if it had fallen from the surface in free fall.

Time to empty the tank

If the tank has a cross-sectional area A and the orifice has area a, we can find how long it takes to empty the tank completely. By the equation of continuity, the rate at which water leaves through the orifice equals the rate at which the water level drops: $Av_1 = av_2$ with $A_1 = A, A_2 = a$.

$$A\frac{dh}{dt} = -a\sqrt{2gh}$$

The negative sign indicates that h decreases with time. Separating variables and integrating from initial height h to zero:

$$-\frac{A}{a\sqrt{2g}}\int_h^0 h^{-1/2}\,dh = \int_0^t dt$$

$$\mathbf{t} = \frac{2A}{a}\sqrt{\frac{h}{2g}} = \frac{\mathbf{A}}{\mathbf{a}}\sqrt{\frac{\mathbf{2h}}{\mathbf{g}}}$$

Notice that the water emerging from the orifice is a horizontal projectile. It leaves the orifice with horizontal velocity $v = \sqrt{2gh}$ and then follows a parabolic trajectory under gravity, exactly like the projectiles we studied in Chapter 6. The horizontal range of the water jet can be calculated using projectile equations once the height of the orifice above the ground is known.

The Venturi Meter

The venturi meter is a device used to measure the speed of a fluid flowing through a pipe. It works by creating a constriction in the pipe, measuring the resulting pressure difference, and using Bernoulli's equation to calculate the flow speed.

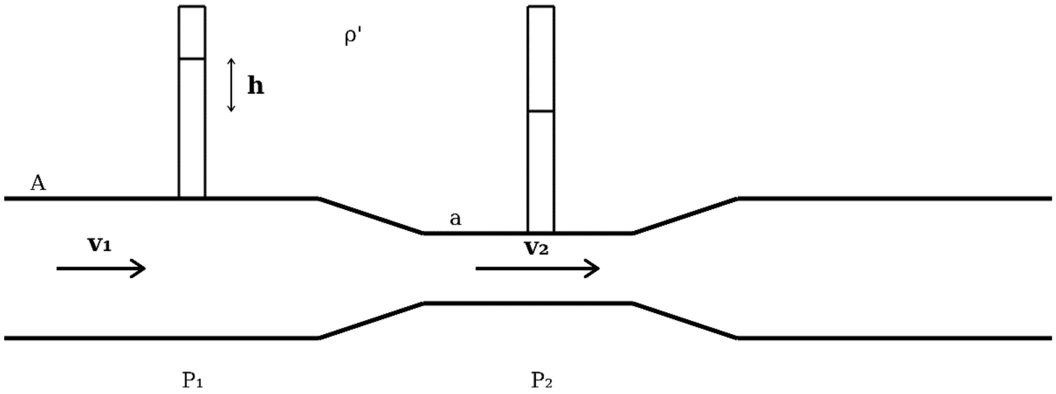


Figure: ***A venturi meter.*** *Fluid flows from the wide section (area A, velocity v_1, pressure P_1) through the narrow throat (area a, velocity v_2, pressure P_2). The manometer liquid of density ρ' shows a level difference h, from which the flow speed is calculated.*

A liquid of density ρ flows through a horizontal pipe from a wide section of area A (where the velocity is v_1 and the pressure is P_1) to a narrow throat of area a (where the velocity is v_2 and the pressure is P_2). A U-shaped manometer containing a liquid of density ρ' is connected between the two sections.

By the equation of continuity:

$$v_2 = v_1 \times \frac{A}{a}$$

By Bernoulli's equation for horizontal flow:

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$$

The pressure difference $P_1 - P_2$ is measured by the manometer. If the manometer liquid levels differ by a height h:

$$P_1 - P_2 = \rho' g h$$

Substituting $v_2 = v_1 A/a$ into the Bernoulli result and equating with the manometer reading:

$$\rho' g h = \frac{1}{2}\rho\left(v_1^2 \times \frac{A^2}{a^2} - v_1^2\right) = \frac{1}{2}\rho v_1^2\left(\frac{A^2}{a^2} - 1\right) = \frac{1}{2}\rho v_1^2\left(\frac{A^2 - a^2}{a^2}\right)$$

Solving for v_1:

$$v_1 = a\sqrt{\frac{2\rho' g h}{\rho(A^2 - a^2)}}$$

The volume flow rate is then:

$$\mathbf{Q} = Av_1 = \mathbf{Aa}\sqrt{\frac{\mathbf{2\rho' g h}}{\mathbf{\rho(A^2 - a^2)}}}$$

The venturi meter is widely used in practice. Water suppliers, such as DAWASCO uses venturi meters to measure water flow through Dar es Salaam's distribution pipes. Fuel injection systems in engines use the venturi principle to mix air and fuel in the correct proportions. Medical devices use miniature venturi meters to measure airflow in respiratory equipment.

Aerofoil Lift (Dynamic Lift)

This is perhaps the most celebrated application of Bernoulli's principle: the explanation of how an aeroplane generates the lift force that holds it in the air.

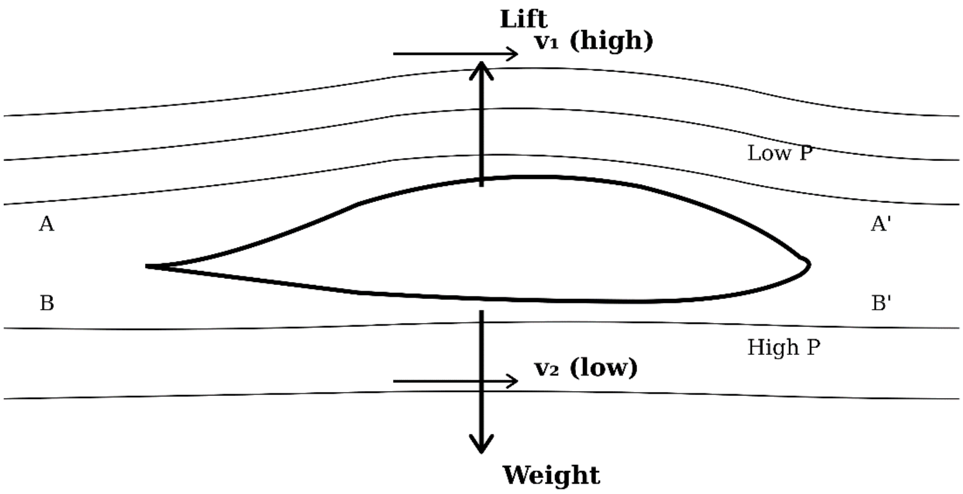


Figure: *Cross-section of an aerofoil. Streamlines above the wing are closer together (faster air, lower pressure). Streamlines below are more widely spaced (slower air, higher pressure). The pressure difference creates an upward lift force.*

An aircraft wing is shaped so that its upper surface is longer and more curved than its lower surface. When the wing moves through the air (or equivalently, when air flows past the wing), the air stream splits at the front edge. The air travelling over the longer upper surface must cover a greater distance in the same time as the air travelling under the shorter lower surface.

By the equation of continuity, if the air above must move faster, the streamlines above the wing are more closely spaced (equivalent to a narrower "pipe"). Consequently, $v_1 > v_2$, where v_1 is the speed above the wing and v_2 is the speed below.

By Bernoulli's equation ($P + \frac{1}{2}\rho v^2 = \text{constant}$), the faster air above the wing has lower pressure (P_1) than the slower air below (P_2). This pressure difference creates a net upward force on the wing. This force is called **dynamic lift**.

If the wing has area A, the lift force is:

$$\mathbf{F} = (P_2 - P_1) \times A = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\rho}\mathbf{A}\left(\mathbf{v}_1^2 - \mathbf{v}_2^2\right)$$

For the aircraft to fly level, the lift force must equal the weight of the aircraft:

$$F = Mg$$

$$\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\rho}\mathbf{A}\left(\mathbf{v}_1^2 - \mathbf{v}_2^2\right) = \mathbf{Mg}$$

This is why heavily loaded aircraft need longer runways: greater weight requires greater lift, which requires higher speed, which requires more distance to achieve during takeoff. It is also *why aircraft fly faster at higher altitudes:* the air density ρ is lower at high altitude, so the speed must be greater to generate the same lift.

The Magnus effect (spinning ball)

When a ball spins as it moves through the air, it drags the air around it in the direction of spin. On one side, the air dragged by the spin adds to the oncoming airflow, increasing the air speed. On the opposite side, the spin opposes the airflow, decreasing the air speed. By Bernoulli's principle, the side with faster air has lower pressure than the side with slower air. Consequently, a net force pushes the ball sideways, toward the low-pressure side.

This is the **Magnus effect**, and it explains *why a spinning football curves in flight, why a cricket bowler can make the ball swing*, and *why a table tennis player can make the ball dip or swerve with topspin or sidespin.*

The Atomiser (Sprayer)

An atomiser converts a liquid into a fine spray using Bernoulli's principle. It is used in perfume bottles, insecticide sprayers, paint spray guns, and carburettors.

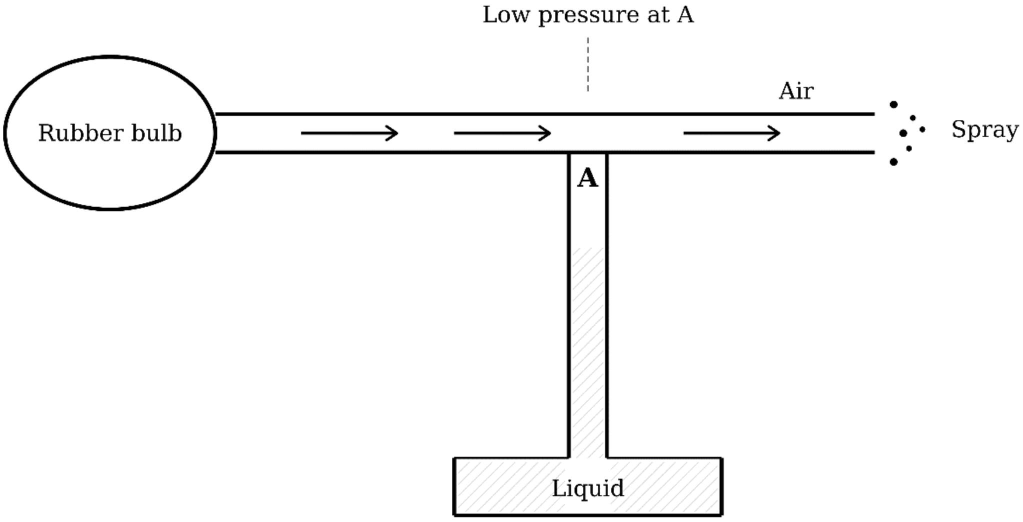


Figure: ***An atomiser.*** *Fast-moving air from the rubber bulb passes over point A, creating low pressure. Atmospheric pressure on the liquid surface pushes the liquid up the vertical tube, where it emerges as a fine spray.*

When the rubber bulb is squeezed, air rushes through the narrow horizontal tube at high speed. At point A, where the vertical tube meets the horizontal tube, the high-speed air creates a region of low pressure (by Bernoulli's principle). The atmospheric pressure acting on the liquid surface in the container is now greater than the pressure at point A. As a result, this pressure difference pushes the liquid up the vertical tube. When it reaches point A, the fast-moving air stream breaks it into fine droplets, producing a spray.

The Pitot Tube

The pitot tube measures the velocity of a fluid (or the speed of an object moving through a fluid) by comparing the stagnation pressure with the static pressure.

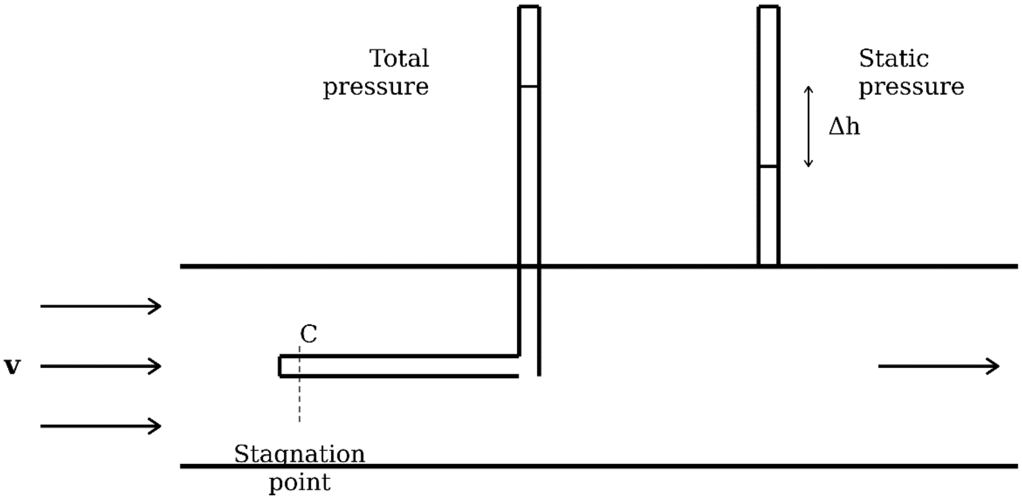


Figure: ***A pitot tube in a fluid stream.*** *The stagnation tube faces the flow and measures total pressure. The static tube is perpendicular to the flow and measures static pressure. The height difference Δh gives the fluid velocity.*

At the stagnation point (where the fluid is brought to rest), the velocity is zero. Applying Bernoulli's equation between a point in the undisturbed flow (velocity v, static pressure P) and the stagnation point (velocity 0, total pressure P_T):

$$P + \frac{1}{2}\rho v^2 = P_T$$

$$P_T - P = \frac{1}{2}\rho v^2$$

If the pressure difference is measured by a manometer showing a height difference Δh:

$$\frac{1}{2}\rho v^2 = \rho' g\Delta h$$

$$\mathbf{v} = \sqrt{\frac{2\rho' g\Delta h}{\rho}}$$

where ρ' is the density of the manometer liquid and ρ is the density of the flowing fluid. If the flowing fluid itself is used as the manometer liquid ($\rho' = \rho$), this simplifies to:

$$\mathbf{v} = \sqrt{2g\Delta h}$$

Pitot tubes are used in aircraft to measure airspeed, in weather stations to measure wind speed, and in industrial pipelines to monitor flow rates.

The theory has been served; now it is time for the worked examples to earn their keep. The following problems bring Bernoulli's applications from the textbook into the real world, and they will test whether you have truly understood the principle or merely memorised the formula. Get your pen ready!

BINDER Example 26

A cylindrical water tank of radius 1.0m rests on a platform 5.0m above the ground. The tank is filled with water to a height of 5.0m. A small circular plug of area $1.0 \times 10^{-4}\text{m}^2$ is removed from an orifice at the bottom of the tank.

(a) Find the speed at which water flows from the orifice.

(b) Find the horizontal distance from the base of the platform to the point where the water jet strikes the ground. Take $g = 9.8\text{ms}^{-2}$.

Solution

(a) By Torricelli's theorem, the speed of efflux from an orifice at depth h below the water surface is:

$$v = \sqrt{2gh}$$

$$v = \sqrt{2 \times 9.8\text{ms}^{-2} \times 5.0\text{m}} = 9.9\text{ms}^{-1}$$

Therefore, the speed of efflux is 9.9ms^{-1}.

(b) Horizontal range of the jet:

The water leaves the orifice horizontally at $v = 9.9\text{ms}^{-1}$, at a height of 5.0m above the ground (the height of the platform). This is a projectile problem. The time to fall to the ground is:

$$H = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 5.0\text{m}}{9.8\text{ms}^{-2}}} = 1.01\text{s}$$

The horizontal distance is:

$$x = v \times t = 9.9\text{ms}^{-1} \times 1.01\text{s} = 10.0\text{m}$$

Therefore, the water jet strikes the ground at a horizontal distance of approximately 10.0m from the base of the platform.

Making Sense of the Answer: *The water shoots out at nearly* $10ms^{-1}$ *and lands* $10m$ *away. This is a substantial distance, roughly the length of a classroom. The problem beautifully connects Bernoulli's equation (Torricelli gives the exit speed) with Chapter 6 (projectile motion gives the range). Physics builds on itself.*

Think Like a Physicist: *Torricelli's theorem is a projectile problem in disguise. The exit speed* $v = \sqrt{2gh}$ *is identical to the speed an object would reach after falling freely through height h. Once the water leaves the orifice, it is a projectile. Always check whether a Torricelli problem has a second part involving projectile motion.*

BINDER Example 27

A venturi meter is fitted in a horizontal pipe carrying water ($\rho = 1000\text{kgm}^{-3}$). The cross-sectional areas of the wide section and the throat are 25cm^2 and 10cm^2 respectively. A mercury manometer ($\rho = 13600\text{kgm}^{-3}$) connected between the two sections shows a level difference of 3.0cm. Find the flow speed in the wide section and the volume flow rate. Take $g = 9.8\text{ms}^{-2}$.

Solution

Given: $A = 25\text{cm}^2 = 25 \times 10^{-4}\text{m}^2$, $a = 10\text{cm}^2 = 10 \times 10^{-4}\text{m}^2$, $\rho = 1000\text{kgm}^{-3}$, $\rho' = 13{,}600\text{kgm}^{-3}$, $h = 3.0\text{cm} = 3.0 \times 10^{-2}\text{m}$

Using the venturi meter formula:

$$v_1 = a\sqrt{\frac{2\rho' gh}{\rho(A^2 - a^2)}} = \sqrt{\frac{2 \times 13{,}600\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 3.0 \times 10^{-2}\text{m}}{1000\text{kgm}^{-3}((25 \times 10^{-4}\text{m}^2)^2 - (10 \times 10^{-4}\text{m}^2)^2)}} = 1.23\text{ms}^{-1}$$

The volume flow rate is:

$$Q = Av_1 = 25 \times 10^{-4}\text{m}^2 \times 1.23\text{ms}^{-1} = 3.08 \times 10^{-3}\text{m}^3\text{s}^{-1}$$

Therefore, the flow speed in the wide section is 1.23ms^{-1} and the volume flow rate is $3.08 \times 10^{-3}\text{m}^3\text{s}^{-1}$ (or approximately 3.1 litres per second).

Making Sense of the Answer: *A tiny mercury level difference of just* 3.0cm *reveals a flow speed of* 1.23ms^{-1} *and a flow rate of about 3 litres per second. The venturi meter works because mercury is 13.6 times denser than water, so a small height difference corresponds to a large pressure difference. This sensitivity is* ***why mercury manometers are preferred for measuring small pressure differences in water flow.***

Think Like a Physicist: *The venturi meter formula looks intimidating, but it is built from just two equations you already know: the continuity equation (*$Av_1 = av_2$*) and Bernoulli's equation (*$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$*). In an examination, you can derive the result from scratch rather than memorising the final formula. Derivation is always safer than memorisation.*

BINDER Example 28

A pitot tube is placed in a stream of water ($\rho = 1000\text{kgm}^{-3}$) flowing through a horizontal pipe. The pitot tube is connected to a mercury manometer ($\rho = 13600\text{kgm}^{-3}$), which shows a height difference of 8mm. Calculate the speed of the water flow. Take $g = 9.8\text{ms}^{-2}$.

Solution

Given: $\rho = 1000\text{kgm}^{-3}$, $\rho' = 13{,}600\text{kgm}^{-3}$, $\Delta h = 8\text{mm} = 8 \times 10^{-3}\text{m}$, $g = 9.8\text{ms}^{-2}$

At the stagnation point, the fluid is brought to rest. Applying Bernoulli's equation between the undisturbed flow (velocity v, static pressure P) and the stagnation point (velocity 0, total pressure P_T):

$$P + \frac{1}{2}\rho v^2 = P_T$$

The pressure difference $P_T - P = \frac{1}{2}\rho v^2$ is measured by the mercury manometer:

$$\frac{1}{2}\rho v^2 = \rho' g\Delta h$$

Solving for v:

$$v = \sqrt{\frac{2\rho' g\Delta h}{\rho}}$$

$$v = \sqrt{\frac{2 \times 13{,}600\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 8 \times 10^{-3}\text{m}}{1000\text{kgm}^{-3}}} = 1.46\text{ms}^{-1}$$

Therefore, the speed of the water flow is approximately 1.5ms^{-1}.

Making Sense of the Answer: *A tiny mercury difference of just* 8mm *reveals a flow speed of* 1.5ms^{-1}*. This high sensitivity is why pitot tubes use a denser manometer liquid: mercury amplifies small pressure* *differences into readable height differences. If water were used as the manometer liquid instead, the same flow speed would produce a height difference of about* 11cm *which is readable, but less precise.*

Think Like a Physicist: *The pitot tube formula* $v = \sqrt{2\rho' g \Delta h/\rho}$ *looks similar to Torricelli's* $v = \sqrt{2gh}$, *and for good reason: both come from Bernoulli's equation with one point at rest. The difference is that Torricelli has the fluid itself providing the height, while the pitot tube uses a separate manometer liquid.*

REAL Example 29

During a physics demonstration, Mr. Akilikubwa holds two sheets of paper parallel to each other, about 3cm apart, and blows a stream of air between them. Kipute expects the papers to fly apart, but instead they move **toward** each other. The whole class gasps.

Explain why the two paper sheets move toward each other when air is blown between them.

Solution

When Mr. Akilikubwa blows air between the sheets, the air between them moves at high speed. By Bernoulli's equation for horizontal flow ($\mathrm{P} + \frac{1}{2}\rho \mathrm{v}^2 = \text{constant}$), the rapidly moving air between the sheets has a lower pressure than the still air on the outer sides of the sheets.

This creates a pressure difference across each sheet: higher atmospheric pressure on the outer side (where the air is stationary) and lower pressure on the inner side (where the air is moving). Consequently, each sheet experiences a net inward force, pushing it toward the other sheet.

Therefore, the papers are not being "pulled" together. They are being **pushed** together by the higher atmospheric pressure on their outer surfaces, because the fast-moving air between them has reduced the pressure on their inner surfaces.

Making Sense of the Answer: *This is one of the simplest and most convincing demonstrations of Bernoulli's principle. It requires only two sheets of paper and a pair of lungs. The same physics explains* ***why a shower curtain moves inward when the shower is running, why roofs blow off during storms*** *(fast wind above the roof creates lower pressure than the still air inside the house), and why* ***two ships sailing close together in parallel tend to drift toward each other.***

Think Like a Physicist: ***Whenever an object appears to be "attracted" toward a region of fast-moving fluid, the real mechanism is a pressure difference.*** *The fast fluid has low pressure; the still fluid has high pressure. The net force pushes the object from the high-pressure side toward the low-pressure side. There is no "suction" force in physics, only pressure differences.*

HOT Example 30

A cylindrical tank of radius 0.50m is filled with water to a height of 4.0m. A plug is removed from a small hole of area $2.0 \times 10^{-4}\mathrm{m}^2$ at the base of the tank.

(a) Find the initial speed of efflux.

(b) Find the initial volume flow rate.

(c) Show that the time to empty the tank completely is given by $\mathrm{t} = \frac{\mathrm{A}}{\mathrm{a}}\sqrt{\frac{2\mathrm{h}}{\mathrm{g}}}$ and calculate this time. Take $\mathrm{g} = 9.8\mathrm{ms}^{-2}$.

Solution

(a) By Torricelli's theorem:

$$\mathrm{v} = \sqrt{2\mathrm{gh}} = \sqrt{2 \times 9.8\mathrm{ms}^{-2} \times 4.0\mathrm{m}} = 8.85\mathrm{ms}^{-1}$$

Therefore, the initial speed of efflux is $8.85\mathrm{ms}^{-1}$.

(b) Initial volume flow rate:

$$\mathrm{Q} = \mathrm{av} = 2.0 \times 10^{-4}\mathrm{m}^2 \times 8.85\mathrm{ms}^{-1} = 1.77 \times 10^{-3}\mathrm{m}^3\mathrm{s}^{-1}$$

Therefore, the initial volume flow rate is $1.77 \times 10^{-3}\mathrm{m}^3\mathrm{s}^{-1}$ (about 1.8 litres per second).

(c) Let h be the height of water at any time t. The speed of efflux at this instant is $v = \sqrt{2gh}$. By the equation of continuity, the rate at which the volume decreases in the tank equals the rate at which water exits through the orifice:

$$-A\frac{dh}{dt} = a\sqrt{2gh}$$

The negative sign arises because h is decreasing with time. Separating variables:

$$\frac{dh}{\sqrt{h}} = -\frac{a}{A}\sqrt{2g}\,dt$$

Integrating from initial height $h_0 = h$ (at $t = 0$) to $h = 0$ (at t):

$$\int_h^0 h^{-1/2}\,dh = -\frac{a}{A}\sqrt{2g}\int_0^t dt$$

$$\left[2h^{1/2}\right]_h^0 = -\frac{a}{A}\sqrt{2g} \times t$$

$$0 - 2\sqrt{h} = -\frac{a}{A}\sqrt{2g} \times t$$

$$t = \frac{2A\sqrt{h}}{a\sqrt{2g}} = \frac{A}{a}\sqrt{\frac{2h}{g}}$$

Hence:

$$t = \frac{A}{a}\sqrt{\frac{2h}{g}}$$

Substituting values:

$$t = \frac{\pi \times (0.50m)^2}{2.0 \times 10^{-4}m^2} \times \sqrt{\frac{2 \times 4.0m}{9.8ms^{-2}}} = 3548s$$

Therefore, the time to empty the tank completely is 3548s (or about 59 minutes).

Making Sense of the Answer: *Nearly an hour to empty a tank that is only 4m tall through a hole the size of a small coin. This makes physical sense: as the tank empties, the water level drops, the pressure at the orifice decreases, the exit speed decreases (by Torricelli), and the flow rate slows down. The tank empties quickly at first and progressively more slowly toward the end. This is why the time is proportional to* $\sqrt{h}$*, not to h: the square root reflects the decelerating nature of the emptying process.*

Think Like a Physicist: *This problem combines Torricelli's theorem (from Bernoulli), the equation of continuity, and calculus (separation of variables and integration). It is exactly the kind of multi-step problem that examinations favour. The strategy is to write the differential equation first, then separate variables, then integrate, then substitute. Each step is straightforward; the challenge is knowing the sequence.*

HOT Example 31

An aircraft has a total wing area of $120m^2$ and a total mass of 20000kg. During level flight, the speed of air over the upper surface of the wings is $70ms^{-1}$ and under the lower surface is $60ms^{-1}$. Take the density of air as $1.2kgm^{-3}$ and $g = 9.8ms^{-2}$.

(a) Find the dynamic lift on the wings.

(b) Determine whether the aircraft can maintain level flight.

Solution

(a) Dynamic lift is given by:

$$F = \frac{1}{2}\rho A(v_1^2 - v_2^2)$$

where $v_1 = 70ms^{-1}$ (upper surface, faster) and $v_2 = 60ms^{-1}$ (lower surface, slower).

$$F = \frac{1}{2} \times 1.2\text{kgm}^{-3} \times 120\text{m}^2 \times ((70\text{ms}^{-1})^2 - (60\text{ms}^{-1})^2) = 93600\text{N}$$

Therefore, the dynamic lift on the wings is 93600N (or 93.6kN).

(b) The weight of the aircraft is:

$$W = Mg = 20{,}000\text{kg} \times 9.8\text{ms}^{-2} = 196000\text{N}$$

Comparing lift and weight:

$$F = 93600\text{N} < W = 196000\text{N}$$

The dynamic lift (93.6kN) is less than the weight (196kN). Therefore, the aircraft **cannot** maintain level flight at these air speeds.

(The pilot would need to increase the speed, increase the angle of attack, or extend the wing flaps to generate additional lift.)

Making Sense of the Answer: *The lift is only about 48% of the weight needed. A speed difference of* 10ms^{-1} *between the upper and lower surfaces is not enough for this heavy aircraft. If the upper surface speed were increased to about* 95ms^{-1} *(with the lower surface at* 60ms^{-1}*), the lift would match the weight. This shows why commercial aircraft need high takeoff speeds and long runways.*

Think Like a Physicist: *The lift force depends on the difference of the squares of the speeds,* $(v_1^2 - v_2^2) = (v_1 + v_2)(v_1 - v_2)$*, not simply on the difference of the speeds,* $(v_1 - v_2)$*. This means that if both speeds increase (the aircraft moves faster overall), the lift also increases because of the extra factor* $(v_1 + v_2)$*, even if the speed difference* $(v_1 - v_2)$ *stays the same. This is the physics behind the rule that* ***faster aircraft generate more lift.***

The worked examples have had their say, and Bernoulli's principle has proven itself across tanks, pipes, wings, and sprayers. In the next subtopic, we leave the ideal fluid behind and confront reality: fluids that resist flowing, fluids with internal friction, fluids with **viscosity**. The third hidden personality of fluids is about to step forward.

VISCOSITY AND NEWTON'S LAW OF VISCOSITY

In the previous subtopics, we treated fluids as though they had no internal friction. We assumed they flowed without resistance, without energy loss, without any layer dragging on any other layer. That was the ideal fluid, and it gave us Bernoulli's beautiful equation. But now it is time to face reality. Real fluids resist flowing. Honey pours slowly. Engine oil clings to your fingers. Even water, which seems to flow so freely, slows down and stops if you do not keep pushing it.

This resistance to flow is called **viscosity**, and it is the third hidden personality of fluids. We introduced the idea briefly when we discussed types of fluid flow. Now we give it a precise definition, a physical law, and a mathematical derivation.

The Concept of Viscosity

Pour honey from a spoon. It moves slowly, reluctantly, as though something inside it is fighting every millimetre of motion. Now pour water from the same spoon. It falls instantly, almost eagerly.

Both liquids are being pulled by the same gravitational force. The difference is that honey has much stronger **internal friction** between its layers. When honey flows, the layer touching the spoon does not move at all (it sticks to the surface). The layer just above it moves a tiny bit. The next layer moves a little faster, and so on. Each layer drags backward on the layer above it, resisting its motion. The result is that the whole body of honey moves slowly, as though every layer is fighting a battle with its neighbours.

This internal friction between adjacent layers of a moving fluid is called **viscous drag**, and the property of the fluid that measures how strongly its layers resist sliding over one another is called **viscosity**.

Water has viscosity too, but it is about 10,000 times weaker than that of honey. As a result, the layers of water slide over each other almost effortlessly, and the water flows quickly.

The Velocity Profile

When a fluid flows through a pipe, it does not move as a single block. Instead, it moves in layers, each at a different speed. Understanding this layered structure is essential before we can write any equation.

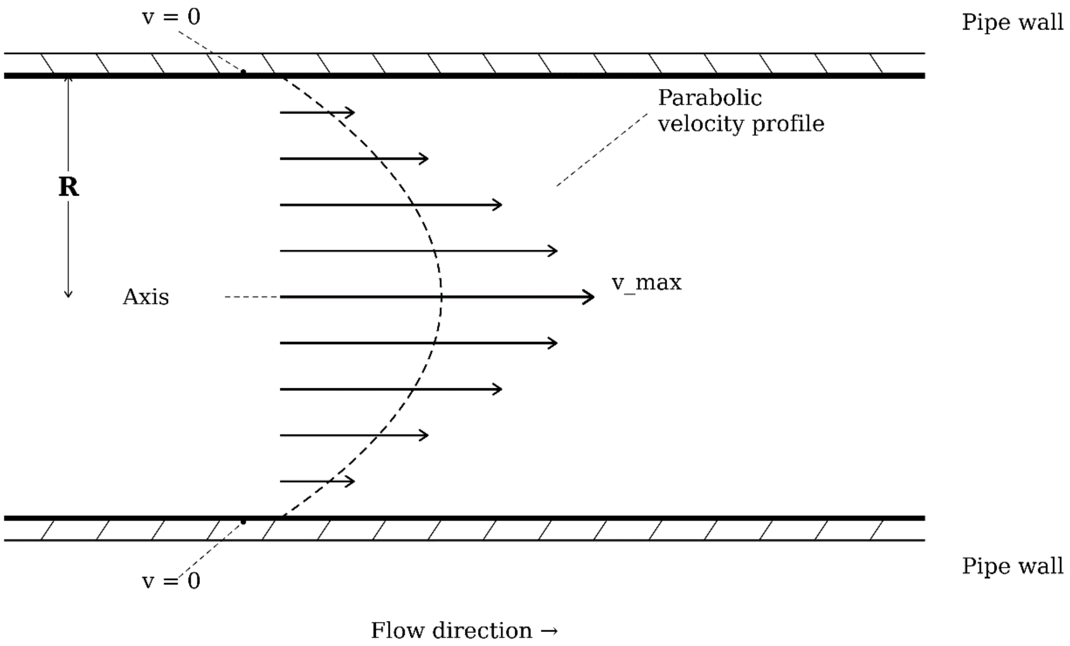


Figure: *Velocity profile of a viscous fluid flowing through a pipe of radius R. The velocity is zero at the pipe wall (no-slip condition) and maximum at the centre. The arrows represent the velocity at each layer. The dashed curve connecting the arrow tips traces a parabola.*

Here is what happens, layer by layer:

At the pipe wall: The fluid layer in direct contact with the wall has **zero velocity**. It is stuck to the wall and does not move at all. This is called the **no-slip condition**, and it is one of the most important facts in fluid mechanics. It holds for all real fluids on all solid surfaces.

At the centre: The fluid layer at the very centre of the pipe has the **maximum velocity**. No wall is nearby to slow it down.

Between the wall and the centre: Every layer has an intermediate velocity, increasing smoothly from zero at the wall to maximum at the centre.

This smooth variation of velocity from zero at the wall to maximum at the centre is called the **velocity profile**. For laminar flow in a circular pipe, the velocity profile has the shape of a **parabola**. The velocity at a distance r from the centre of a pipe of radius R is:

$$\mathbf{v(r) = v_{max}\left(1 - \frac{r^2}{R^2}\right)}$$

At the centre ($r = 0$), $v = v_{max}$. At the wall ($r = R$), $v = 0$. The parabolic shape is a direct consequence of the viscous forces between the layers.

Newton’s Law of Viscosity

Now we make the concept quantitative. We want to answer the question: *how large is the viscous force between two adjacent layers of fluid, and what does it depend on?*

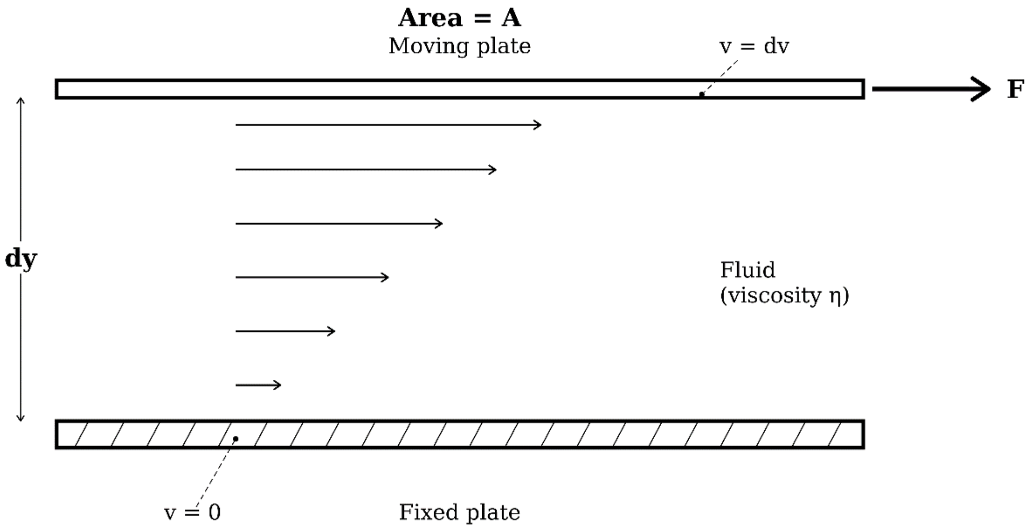


Figure: *The setup for Newton's law of viscosity. A fluid of viscosity η fills the gap of thickness dy between a fixed plate (bottom, v = 0) and a moving plate (top, v = dv, area A). A force F is required to maintain the motion. The fluid layers between the plates have velocities increasing linearly from zero at the fixed plate to dv at the moving plate.*

Consider two adjacent layers of fluid, each of area A, separated by a small perpendicular distance dy. The upper layer moves at velocity $\mathrm{v} + \mathrm{dv}$ and the lower layer moves at velocity v. The difference in their velocities is dv.

Defining the velocity gradient

The quantity that describes how rapidly the velocity changes as you move perpendicular to the flow is:

$$\frac{\mathrm{dv}}{\mathrm{dy}}$$

This is called the **velocity gradient** (or **rate of shear strain**). It tells you how fast the layers are sliding over one another. A large velocity gradient means the layers are sliding quickly; a small velocity gradient means they are sliding slowly. Its unit is $(\mathrm{ms}^{-1})/\mathrm{m} = \mathrm{s}^{-1}$.

Identifying what the viscous force depends on

Newton observed experimentally that the viscous force F between two adjacent layers depends on two factors:

1) The **area of contact** A between the layers. If you double the area, you double the friction, just as a wider brake pad produces more friction than a narrow one. Therefore:

$$\mathrm{F} \propto \mathrm{A}$$

2) The **velocity gradient** $\frac{\mathrm{dv}}{\mathrm{dy}}$. If the layers are sliding faster over each other (larger $\frac{\mathrm{dv}}{\mathrm{dy}}$), the friction is greater. If they are sliding slowly (smaller $\frac{\mathrm{dv}}{\mathrm{dy}}$), the friction is weaker. Therefore:

$$\mathrm{F} \propto \frac{\mathrm{dv}}{\mathrm{dy}}$$

Combining the proportionalities

Since F is proportional to both A and $\frac{\mathrm{dv}}{\mathrm{dy}}$:

$$\mathrm{F} \propto \mathrm{A} \times \frac{\mathrm{dv}}{\mathrm{dy}}$$

Introducing the constant of proportionality

To convert the proportionality into an equation, we introduce a constant η (the Greek letter eta) that depends on the nature of the fluid:

$$F = \eta A\frac{dv}{dy}$$

This equation is **Newton's law of viscosity**. It can be stated in words as follows:

The viscous force F between two adjacent layers of a fluid is directly proportional to the area of contact A between the layers and to the velocity gradient dv/dy perpendicular to the direction of flow.

The constant of proportionality **η** is called the **coefficient of viscosity** of the fluid.

Let us now examine each quantity in the equation carefully.

- F is the viscous force (in N) between two adjacent layers. It acts tangentially, parallel to the layers, opposing the relative motion between them.
- A is the area of contact between the layers (in m^2).
- dv/dy is the velocity gradient (in s^{-1}).
- η is the coefficient of viscosity of the fluid. It is a measure of how strongly the fluid resists the sliding of its layers.

Deriving the Units of the Coefficient of Viscosity

The unit of η can be derived from Newton's law by making η the subject of the formula.

Starting from:

$$F = \eta A\frac{dv}{dy}$$

Rearranging to isolate η:

$$\eta = \frac{F}{A \times \frac{dv}{dy}}$$

Now substituting the SI units of each quantity:

$$[\eta] = \frac{[F]}{[A] \times [dv/dy]}$$

$$[\eta] = \frac{N}{m^2 \times s^{-1}}$$

$$[\eta] = \frac{N}{m^2} \times s$$

$$[\eta] = N \cdot s \cdot m^{-2} = Nsm^{-2}$$

This can also be written as $Pa \cdot s$ (pascal-second), since $1Pa = 1Nm^{-2}$, or as $kgm^{-1}s^{-1}$, since $1N = 1kgms^{-2}$:

$$[\eta] = kgms^{-2} \times s \times m^{-2} = kgm^{-1}s^{-1}$$

In the older CGS system, the unit of viscosity is the **poise** (P), named after Jean-Léonard-Marie Poiseuille:

$$1\,\text{poise} = 1\,\text{dyne} \cdot s \cdot cm^{-2} = 0.1\,Nsm^{-2}$$

The commonly used subunit is the **centipoise** (cP): $1\ cP = 10^{-3}\ Nsm^{-2}$.

Representative values at room temperature: Water has $\eta \approx 1.0 \times 10^{-3} Nsm^{-2}$ (about 1 centipoise), blood has $\eta \approx 3 \times 10^{-3} Nsm^{-2}$, glycerine has $\eta \approx 1.5 Nsm^{-2}$, and honey has $\eta \approx 2$ to $10 Nsm^{-2}$ depending on temperature. Air has $\eta \approx 1.8 \times 10^{-5} Nsm^{-2}$, which is roughly 50 times less viscous than water.

Newtonian and Non-Newtonian Fluids

Fluids that obey Newton's law of viscosity (where η is constant at a given temperature and does not depend on the velocity gradient) are called **Newtonian fluids**. Water, air, glycerine, and most simple liquids are Newtonian.

Fluids whose viscosity changes with the velocity gradient are called **Non-Newtonian fluids**. Blood, paint, toothpaste, and ketchup are non-Newtonian. Ketchup, for example, becomes less viscous when you shake the bottle (this is called **shear thinning**), which is why tapping the bottle makes it pour more easily. Cornstarch mixed with water becomes more viscous when you stir it hard (this is called **shear thickening**), which is why you can run across a pool of cornstarch mixture but sink if you stand still.

For all problems in this chapter, we deal only with Newtonian fluids.

Factors Affecting Viscosity

The viscosity of a fluid is not a fixed universal constant. It depends on the conditions, and understanding these factors is essential for solving problems and interpreting experiments.

1. Temperature (the dominant factor)

For liquids, viscosity **decreases** with increasing temperature. As the temperature rises, molecules move faster and the intermolecular forces that cause the internal friction are more easily broken. Consequently, the layers slide over each other more easily, and the fluid flows more freely. This is why honey flows much faster when warmed, and why engine oil is thinner on a hot day than on a cold morning.

For gases, the opposite is true: viscosity **increases** with increasing temperature. In a gas, viscosity arises not from intermolecular forces (which are negligible) but from the transfer of momentum between layers by molecules bouncing back and forth. At higher temperatures, molecules move faster and transfer more momentum between layers, increasing the friction. Hence, hot air is slightly more viscous than cold air.

2. Nature of the fluid

Different fluids have vastly different viscosities at the same temperature. Liquids with strong hydrogen bonds (like glycerine) or long-chain molecules (like engine oil) have high viscosity because their molecules cling to each other and resist sliding. Liquids with weak intermolecular forces (like petrol or acetone) have low viscosity.

3. Pressure (usually negligible for liquids)

For most liquids, viscosity is nearly independent of pressure under normal conditions. For gases, viscosity is essentially independent of pressure at moderate pressures but may change at very high pressures.

The theory has been laid out with care. Now it is time to see whether it holds up under the pressure of a well-crafted problem. The following examples will test your ability to apply Newton’s law of viscosity, and the last one will demand that you derive, integrate, and calculate all in one go.

BINDER Example 32

A flat glass plate of area 0.05m^2 is pulled with a constant velocity of 0.5ms^{-1} over a layer of glycerine 2mm thick on a flat surface. The coefficient of viscosity of glycerine is 1.5Nsm^{-2}. Calculate the force required to maintain the motion.

Solution

Given: $\text{A} = 0.05\text{m}^2$, $\text{v} = 0.5\text{ms}^{-1}$, $\text{dy} = 2\text{mm} = 2 \times 10^{-3}\text{m}$, $\eta = 1.5\text{Nsm}^{-2}$

The bottom layer of glycerine is in contact with the fixed flat surface, so its velocity is zero (no-slip condition). The top layer is in contact with the moving glass plate, so its velocity equals the plate velocity: 0.5ms^{-1}. The velocity gradient is therefore:

$$\frac{\text{dv}}{\text{dy}} = \frac{\text{v}_{\text{top}} - \text{v}_{\text{bottom}}}{\text{dy}} = \frac{0.5\text{ms}^{-1} - 0}{2 \times 10^{-3}\text{m}} = 250\text{s}^{-1}$$

Applying Newton’s law of viscosity:

$$\text{F} = \eta\text{A}\frac{\text{dv}}{\text{dy}}$$

$$\text{F} = 1.5\text{Nsm}^{-2} \times 0.05\text{m}^2 \times 250\text{s}^{-1} = 18.75\text{N}$$

Therefore, the force required to pull the glass plate at constant velocity is 18.75N.

Making Sense of the Answer: *Nearly 19 newtons to slide a plate over a 2mm layer of glycerine. That is roughly the weight of a 2kg bag of sugar. Glycerine is a highly viscous liquid* ($\eta = 1.5Nsm^{-2}$)*, which is why the force is so substantial. If the glycerine were replaced by water* ($\eta = 10^{-3}Nsm^{-2}$)*, the required force would be 1500 times smaller: about* $0.013N$*, the weight of a single coin.*

Think Like a Physicist: *The velocity gradient dv/dy is the engine of this calculation. If the glycerine layer were thicker (say 4mm instead of 2mm), the velocity gradient would halve, and so would the force. The thicker the fluid layer, the more gently the velocity changes from zero to v, and the weaker the viscous drag. This is* ***why thick layers of lubricant reduce friction more effectively than thin layers.***

REAL Example 33

On a cold morning in Moshi, Kipanga tries to pour cooking oil from a bottle. The oil barely moves. His mother heats the bottle gently in warm water for a minute, and the oil pours freely.

Explain why warming the oil makes it flow more easily.

Solution

The viscosity of a liquid decreases with increasing temperature. At low temperature, the molecules of the cooking oil have relatively low kinetic energy. The intermolecular attractive forces between the oil molecules are strong enough to resist the sliding of adjacent layers over one another. Consequently, the viscous drag between layers is large, the velocity gradient is small for a given applied force, and the oil flows slowly.

When the bottle is warmed, the temperature of the oil rises. The oil molecules gain kinetic energy and move more vigorously. As a result, the intermolecular forces are more easily broken, the resistance between adjacent layers decreases, and the coefficient of viscosity η drops. The same gravitational force now produces a larger velocity gradient, and the oil flows faster.

Therefore, warming the oil reduces its viscosity, which reduces the internal friction between layers, which allows the oil to flow more freely under the same gravitational force.

Making Sense of the Answer: *This is* ***why mechanics warm engine oil before draining it from a car:*** *warm oil flows out quickly; cold oil clings stubbornly to the engine walls. It is also* ***why the chocolate industry controls temperature precisely during manufacturing:*** *at the wrong temperature, chocolate flows too slowly to fill moulds, or too quickly to hold its shape.*

Think Like a Physicist: *For liquids, higher temperature means lower viscosity. For gases, higher temperature means higher viscosity. The mechanisms are completely different. In liquids, viscosity comes from intermolecular forces (which weaken at higher temperature). In gases, viscosity comes from molecular momentum transfer (which increases at higher temperature). Always state which mechanism applies when explaining a temperature effect on viscosity.*

HOT Example 34

A flat circular disc of radius 15cm rotates on a thin film of oil of thickness 1.5mm placed between the disc and a flat surface. The coefficient of viscosity of the oil is 0.80Nsm^{-2}. If the torque required to maintain a steady rotation of 60 revolutions per minute is τ, show that:

$$\tau = \frac{\pi\eta\omega R^4}{2d}$$

and calculate τ.

Solution

The disc rotates on the oil film. The oil layer touching the fixed surface has zero velocity (no-slip condition). The oil layer touching the rotating disc moves with the disc. Between these two layers, the velocity varies linearly from zero at the bottom to $v = \omega r$ at the top (at radial distance r from the centre). We derive the torque by considering an infinitesimal annular ring and integrating.

Defining the annular ring

Consider a thin annular ring on the disc at radius r from the centre, with width dr. The area of this ring is:

$$dA = 2\pi r\,dr$$

Finding the velocity at radius r

At radius r, the linear velocity of the disc surface is:

$$v = \omega r$$

where ω is the angular velocity of the disc.

Finding the velocity gradient across the oil film

The oil beneath this ring has the disc moving at $v = \omega r$ on top and the fixed surface (velocity = 0) on the bottom, separated by oil film thickness d. The velocity gradient is:

$$\frac{dv}{dy} = \frac{\omega r - 0}{d} = \frac{\omega r}{d}$$

Applying Newton's law of viscosity to find the force on the ring

The viscous force on this annular ring is:

$$dF = \eta \times dA \times \frac{dv}{dy}$$

Substituting the expressions for dA and $\frac{dv}{dy}$ from above:

$$dF = \eta \times 2\pi r\, dr \times \frac{\omega r}{d}$$

Rearranging:

$$dF = \frac{2\pi\eta\omega}{d} \times r^2\, dr$$

Finding the torque from this ring

This viscous force acts tangentially at radius r. The torque (moment of force) contributed by this ring about the axis of rotation is:

$$d\tau = r \times dF$$

$$d\tau = r \times \frac{2\pi\eta\omega}{d} \times r^2\, dr$$

$$d\tau = \frac{2\pi\eta\omega}{d} \times r^3\, dr$$

Integrating over the entire disc

The total torque is obtained by integrating from $r = 0$ (centre) to $r = R$ (edge):

$$\tau = \int_0^R \frac{2\pi\eta\omega}{d} r^3\, dr$$

The constants $\frac{2\pi\eta\omega}{d}$ come outside the integral:

$$\tau = \frac{2\pi\eta\omega}{d} \int_0^R r^3\ dr = \frac{2\pi\eta\omega}{d} \left[\frac{r^4}{4}\right]_0^R = \frac{2\pi\eta\omega}{d} \left(\frac{R^4}{4} - \frac{0^4}{4}\right) = \frac{\pi\eta\omega R^4}{2d}$$

Hence:

$$\boldsymbol{\tau = \frac{\pi\eta\omega R^4}{2d}}$$

Numerical calculation:

$$\omega = 2\pi \times \frac{60}{60} = 2\pi\, \mathrm{rad\, s^{-1}} = 6.283\, \mathrm{rad\, s^{-1}}$$

Substituting values:

$$\tau = \frac{\pi \times 0.80\mathrm{Nsm^{-2}} \times 6.283\, \mathrm{rad\, s^{-1}} \times (0.15\mathrm{m})^4}{2 \times 1.5 \times 10^{-3}\mathrm{m}} = 2.665\mathrm{Nm}$$

Therefore, the torque required to maintain the steady rotation is approximately 2.7Nm.

Making Sense of the Answer: *A torque of* 2.7*Nm to spin a disc on a thin oil film. For comparison, the torque needed to open a typical door handle is about 1 to* 2*Nm. The oil film is only* 1.5*mm thick, which* *creates a steep velocity gradient and a substantial viscous resistance. A thicker oil film would reduce the torque; a more viscous oil would increase it.*

Think Like a Physicist: *This derivation follows a beautiful pattern that appears throughout physics: divide the object into infinitesimal elements (here, annular rings), write the expression for each element, and integrate over the whole object. You met this exact technique with moment of inertia in rotational mechanics. The integration step is where calculus earns its keep in physics.*

That completes the formal treatment of viscosity. But viscosity does more than drag on plates and discs. When it acts inside a cylindrical pipe, it controls the rate at which a viscous fluid flows through the pipe. The equation that describes this is Poiseuille's formula, and it is waiting in the next subtopic. After that, we meet Stokes' law, which describes the viscous drag on a sphere falling through a fluid, and which leads to one of the most elegant ideas in fluid mechanics: **terminal velocity**.

POISEUILLE'S FORMULA

In the previous subtopic, we learned that viscosity is the internal friction that resists the flow of a fluid. We derived Newton's law of viscosity, which tells us the force between two adjacent layers. But a natural question follows: *if you know the viscosity of a fluid and the dimensions of the pipe, can you predict how much fluid will flow through the pipe per second?*

The answer is yes, and the equation that does this is called **Poiseuille's formula** (sometimes called the Hagen-Poiseuille equation). It is one of the most practically useful results in fluid mechanics, and it connects the flow rate to the pipe radius, the pressure difference driving the flow, the viscosity of the fluid, and the length of the pipe. Doctors use it to understand blood flow. Engineers use it to design pipelines. Chemists use it to measure viscosity. By the end of this section, you will be able to do all three.

Setting up the problem

Consider a viscous, incompressible fluid flowing steadily through a horizontal cylindrical pipe of internal radius R and length L. The pressure at the entry end is P_1 and at the exit end is P_2, with $P_1 > P_2$. The pressure difference $\Delta P = P_1 - P_2$ is what drives the flow.

We want to find the total volume of fluid that passes through the pipe per unit time (the volume flow rate Q). To do this, we will consider a thin cylindrical shell of fluid inside the pipe, find the velocity of that shell, and then integrate over all shells from the centre to the wall.

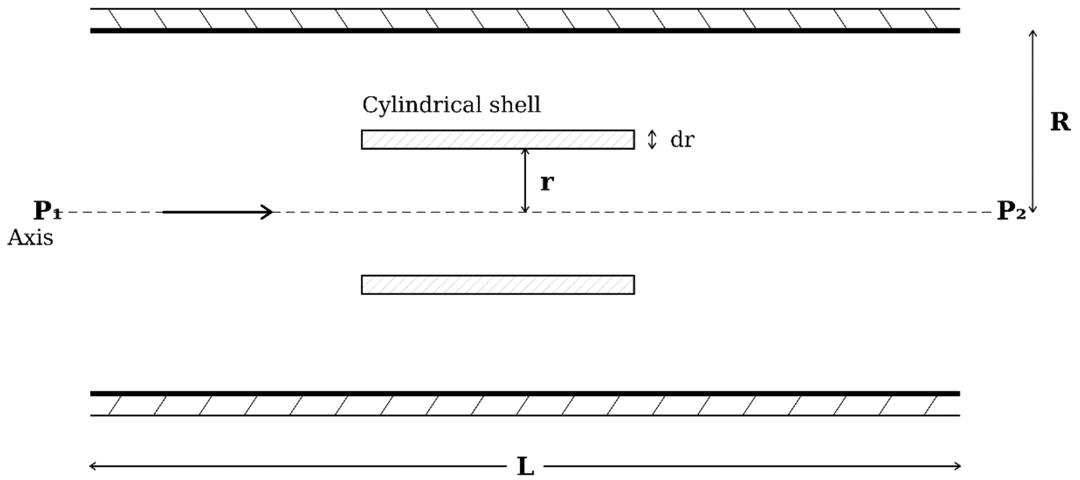


Figure: *Cross-section (side view) of a horizontal cylindrical pipe of radius R and length L. The fluid flows from left (pressure P_1) to right (pressure P_2). A thin cylindrical shell of fluid at distance r from the axis, with thickness dr, is highlighted. The velocity profile is parabolic: maximum at the axis, zero at the wall.*

Deriving the velocity profile

Identifying the cylindrical shell

Inside the pipe, consider a thin cylindrical shell of fluid at distance r from the central axis, with thickness dr and length L. This shell is a hollow cylinder of inner radius r and outer radius $r + dr$.

Writing the driving force on the shell

The pressure difference $\Delta P = P_1 - P_2$ acts on the cross-sectional area πr^2 of the fluid cylinder of radius r. The net forward force due to the pressure difference on this inner cylinder is:

$$F_{pressure} = (P_1 - P_2) \times \pi r^2 = \Delta P \times \pi r^2$$

Writing the viscous (retarding) force on the shell

The viscous force acts on the curved surface of this cylinder. By Newton's law of viscosity, the viscous force on a surface of area A is $F = \eta A \left(\frac{dv}{dy}\right)$. For a cylindrical surface at radius r, the area is $A = 2\pi rL$, and the velocity gradient in the radial direction is $\frac{dv}{dr}$ (note: $\frac{dv}{dr}$ is negative because v decreases as r increases from the centre to the wall). The viscous retarding force is:

$$F_{viscous} = -\eta \times 2\pi rL \times \frac{dv}{dr}$$

The negative sign ensures that $F_{viscous}$ is positive (opposing the flow), since $\frac{dv}{dr}$ is negative.

Applying the condition for steady flow

For steady (non-accelerating) flow, the driving force equals the retarding force:

$$F_{pressure} = F_{viscous}$$

$$\Delta P \times \pi r^2 = -\eta \times 2\pi rL \times \frac{dv}{dr}$$

Simplifying and separating variables

Dividing both sides by πr:

$$\Delta P \times r = -\eta \times 2L \times \frac{dv}{dr}$$

Rearranging to isolate dv:

$$dv = -\frac{\Delta P}{2\eta L} \times r\, dr$$

Integrating to find the velocity at radius r

Integrating both sides. The left side integrates from some velocity v (at radius r) to 0 (at the wall, $r = R$, where the no-slip condition gives $v = 0$). The right side integrates from r to R:

$$\int_v^0 dv = -\frac{\Delta P}{2\eta L}\int_r^R r\, dr$$

$$0 - v = -\frac{\Delta P}{2\eta L}\left[\frac{r^2}{2}\right]_r^R = -\frac{\Delta P}{2\eta L} \times \frac{R^2 - r^2}{2} = -\frac{\Delta P(R^2 - r^2)}{4\eta L}$$

$$-v = -\frac{\Delta P(R^2 - r^2)}{4\eta L}$$

$$\mathbf{v(r) = \frac{\Delta P}{4\eta L}(R^2 - r^2)}$$

This is the **velocity profile** for laminar flow in a cylindrical pipe. It confirms that the profile is parabolic as v depends on r^2. At the centre $(r = 0)$, the velocity is maximum:

$$v_{max} = \frac{\Delta P\, R^2}{4\eta L}$$

At the wall $(r = R)$, the velocity is zero, as required by the no-slip condition.

Deriving the volume flow rate (Poiseuille's formula)

Now we calculate the total volume of fluid flowing through the pipe per unit time by integrating the velocity over the entire cross-section.

Finding the volume flow rate through the thin shell

The thin cylindrical shell at radius r with thickness dr has cross-sectional area:

$$dA = 2\pi r\, dr$$

The volume of fluid passing through this shell per unit time is:

$$dQ = v(r) \times dA = \frac{\Delta P}{4\eta L}(R^2 - r^2) \times 2\pi r\, dr$$

$$dQ = \frac{\pi \Delta P}{2\eta L}(R^2 r - r^3)\, dr$$

Integrating over the entire cross-section

The total volume flow rate is obtained by integrating from $r = 0$ to $r = R$:

$$Q = \int_0^R \frac{\pi \Delta P}{2\eta L}(R^2 r - r^3)\, dr$$

Taking the constants outside the integral and integrating the rest:

$$Q = \frac{\pi \Delta P}{2\eta L}\int_0^R (R^2 r - r^3)\ dr = \frac{\pi \Delta P}{2\eta L}\left(R^2\left[\frac{r^2}{2}\right]_0^R - \left[\frac{r^4}{4}\right]_0^R\right) = \frac{\pi \Delta P}{2\eta L}\left(\frac{R^4}{2} - \frac{R^4}{4}\right) = \frac{\pi \Delta P}{2\eta L} \times \frac{R^4}{4}$$

Hence:

$$\mathbf{Q = \frac{\pi \Delta P R^4}{8\eta L}}$$

This is **Poiseuille's formula**. It can be stated in words as follows:

The volume flow rate Q of a viscous, incompressible fluid in steady laminar flow through a horizontal cylindrical pipe of radius R and length L is directly proportional to the pressure difference ΔP and to the fourth power of the radius, and inversely proportional to the viscosity η and the length of the pipe.

Physical meaning and consequences

Poiseuille's formula deserves careful study. Every factor in the equation tells a physical story.

1. **The R^4 dependence is extraordinary.** The flow rate depends on the **fourth power** of the radius. This means that a small change in radius produces a dramatic change in flow rate. If you double the radius of the pipe, the flow rate increases by a factor of $2^4 = 16$. If you halve the radius, the flow rate drops to $1/16$ of its original value. This is why even a slight narrowing of a blood vessel (as in atherosclerosis) causes a devastating reduction in blood flow.
2. **The dependence on ΔP is linear.** Double the pressure difference, and the flow rate doubles. This is the simplest factor and the most intuitive: push harder, get more flow.
3. **The dependence on η is inverse.** A more viscous fluid flows more slowly through the same pipe under the same pressure. This is why honey drips slowly from a tap while water gushes.
4. **The dependence on L is inverse.** A longer pipe means more viscous resistance, so less flow. Double the length, halve the flow rate.

Average velocity: The average velocity of the fluid across the entire cross-section can be found by dividing the volume flow rate by the cross-sectional area:

$$\mathbf{v_{avg}} = \frac{Q}{\pi R^2} = \mathbf{\frac{\Delta P R^2}{8\eta L}}$$

Comparing this with the maximum velocity at the centre:

$$\mathbf{v_{max}} = \frac{\Delta P R^2}{4\eta L} = \mathbf{2 \times v_{avg}}$$

Therefore, in Poiseuille flow, the maximum velocity is exactly **twice** the average velocity.

Exponential Decay of Fluid Level

Here is a problem that combines Poiseuille's formula with calculus in a way that produces one of the most elegant results in the entire chapter. It is worth savouring.

A cylindrical tank of cross-sectional area A_T is filled with liquid to an initial height h_0. The liquid drains out slowly through a long, narrow horizontal tube (a capillary) of radius R and length L attached at the base of the tank. The liquid has viscosity η. We want to find how the liquid level h changes with time.

This is different from the Torricelli emptying problem we solved earlier. In that problem, the orifice was short and wide, so viscosity was negligible and the exit speed depended only on height (Torricelli's theorem). Here, the tube is long and narrow, so viscous resistance dominates and the flow rate is governed by Poiseuille's formula.

Derivation

Writing the flow rate through the capillary

At any instant, the liquid in the tank stands at height h above the tube. The pressure difference driving the flow through the capillary is the hydrostatic pressure at the base of the tank:

$$\Delta P = \rho g h$$

By Poiseuille's formula, the volume flow rate through the capillary is:

$$Q = \frac{\pi \Delta P R^4}{8\eta L} = \frac{\pi \rho g h R^4}{8\eta L}$$

Relating the flow rate to the rate of change of height

The volume of liquid leaving the tank per unit time equals the rate at which the liquid level drops multiplied by the tank area:

$$Q = -A_T \frac{dh}{dt}$$

The negative sign appears because h is decreasing as liquid drains out.

Equating the two expressions for Q

$$-A_T \frac{dh}{dt} = \frac{\pi \rho g R^4}{8\eta L} \times h$$

Separating variables

$$\frac{dh}{h} = -\frac{\pi \rho g R^4}{8\eta L A_T} dt$$

The group of constants on the right is a single positive number. Let us call it c:

$$\mathbf{c = \frac{\pi \rho g R^4}{8\eta L A_T}}$$

So the equation becomes:

$$\frac{dh}{h} = -c\,dt$$

Integrating both sides

The left side is integrated from h_0 (at $t = 0$) to h (at time t). The right side is integrated from 0 to t:

$$\int_{h_0}^{h} \frac{dh'}{h'} = -c\int_0^t d\,t'$$

$$[\ln h']_{h_0}^{h} = -c[t']_0^t$$

$$\ln h - \ln h_0 = -ct$$

$$\ln\left(\frac{h}{h_0}\right) = -ct$$

$$\log_e\left(\frac{h}{h_0}\right) = -ct$$

$$\mathbf{h = h_0\, e^{-ct}}$$

where $c = \frac{\pi\rho g R^4}{8\eta L A_T}$.

This is an **exponential decay**. The liquid level does not drop linearly, nor does it follow the square-root law we found for Torricelli emptying. Instead, it decays exponentially: the level drops quickly at first (when h is large and the driving pressure is high) and progressively more slowly as h approaches zero (when the driving pressure weakens).

In principle, the tank never fully empties: the exponential function approaches zero but never reaches it. In practice, the level becomes negligibly small after a few time constants ($t \approx \frac{3}{c}$ to $\frac{5}{c}$).

The quantity $\frac{1}{c}$ is the **time constant** of the system:

$$\tau = \frac{1}{c} = \frac{8\eta L A_T}{\pi\rho g R^4}$$

After one time constant ($t = \tau$), the level has dropped to $\frac{h_0}{e} \approx 0.37h_0$, or about 37% of its initial value.

Why this matters: This result connects fluid mechanics to the same exponential decay that appears in radioactive decay, RC circuit discharge, and Newton's law of cooling. The mathematics is identical because the underlying structure is the same: the rate of decrease of a quantity is proportional to its current value. Whenever you see $\frac{dh}{dt} \propto -h$, the solution is always $h = h_0 e^{-ct}$.

Conditions for Poiseuille's Formula to Apply

Poiseuille's formula was derived under specific assumptions, and it is valid only when these are satisfied:

1. The flow must be laminar

Poiseuille's formula does not apply to turbulent flow. The Reynolds number must be below 2000 for Poiseuille's formula to be applicable. This means for any speed below critical velocity v_c, the Poiseuille's formula applies. For speeds above v_c, the flow is turbulent and Poiseuille's formula is no longer valid.

2. The fluid must be incompressible

The density must be constant throughout the pipe.

3. The fluid must be Newtonian

The viscosity η must be constant and independent of the velocity gradient.

4. The flow must be steady

The velocity at each point must not change with time.

5. The pipe must be rigid and cylindrical

Flexible or irregularly shaped pipes are not described by this formula.

6. The pipe must be long enough

Near the entrance of a pipe, the velocity profile is still developing and is not yet parabolic. Poiseuille's formula applies only in the fully developed region, far enough from the entrance. This entry length is typically about 50 to 100 pipe diameters.

It is time to put this powerful formula to work. The following examples range from straightforward substitution to clinical medicine, and each one reveals a different face of Poiseuille's equation.

BINDER Example 35

Water ($\eta = 1.0 \times 10^{-3}\mathrm{Nsm^{-2}}$) flows through a horizontal pipe of radius $5\mathrm{mm}$ and length $2\mathrm{m}$ under a pressure difference of $400\mathrm{Pa}$. Calculate:

(a) The volume flow rate.

(b) The velocity of the water at the centre of the pipe.

(c) The average velocity of the water.

Solution

Given: $\eta = 1.0 \times 10^{-3}\mathrm{Nsm^{-2}}$, $\mathrm{R} = 5\mathrm{mm} = 5 \times 10^{-3}\mathrm{m}$, $\mathrm{L} = 2\mathrm{m}$, $\Delta\mathrm{P} = 400\mathrm{Pa}$

(a) Applying Poiseuille's formula:

$$\mathrm{Q} = \frac{\pi\Delta\mathrm{P}\mathrm{R}^4}{8\eta\mathrm{L}}$$

$$\mathrm{Q} = \frac{\pi \times 400\mathrm{Pa} \times (5 \times 10^{-3}\mathrm{m})^4}{8 \times 1.0 \times 10^{-3}\mathrm{Nsm^{-2}} \times 2\mathrm{m}} = 4.909 \times 10^{-5}\mathrm{m^3s^{-1}}$$

Therefore, the volume flow rate is $4.909 \times 10^{-5}\mathrm{m^3s^{-1}}$ (about 49 millilitres per second).

(b) The velocity at the centre of the pipe is the maximum velocity which is given by:

$$\mathrm{v_{max}} = \frac{\Delta\mathrm{P}\mathrm{R}^2}{4\eta\mathrm{L}} = \frac{400\mathrm{Pa} \times (5 \times 10^{-3}\mathrm{m})^2}{4 \times 1.0 \times 10^{-3}\mathrm{Nsm^{-2}} \times 2\mathrm{m}} = 1.25\mathrm{ms^{-1}}$$

Therefore, the velocity at the centre of the pipe is $1.25\mathrm{ms^{-1}}$.

(c) The average velocity is given by:

$$\mathrm{v_{avg}} = \frac{\mathrm{v_{max}}}{2} = \frac{1.25\mathrm{ms^{-1}}}{2} = 0.625\mathrm{ms^{-1}}$$

Alternatively, from the flow rate:

$$\mathrm{v_{avg}} = \frac{\mathrm{Q}}{\pi\mathrm{R}^2} = \frac{4.91 \times 10^{-5}\mathrm{m^3s^{-1}}}{\pi \times (5 \times 10^{-3}\mathrm{m})^2} = 0.625\mathrm{ms^{-1}}$$

Both methods give the same answer, confirming the result.

Therefore, the average velocity is $0.625\mathrm{ms^{-1}}$.

Making Sense of the Answer: *The maximum velocity* ($1.25ms^{-1}$) *is exactly twice the average* ($0.625ms^{-1}$)*, as Poiseuille's theory predicts. The flow rate of about 49 millilitres per second through a pipe only 1cm in diameter is substantial. If the radius were halved to* $2.5mm$ *(keeping everything else the same), the flow rate would drop by a factor of* $2^4 = 16$*, from 49 to about 3 millilitres per second. The fourth-power dependence is ruthless.*

Think Like a Physicist: *It is always helpful to verify your answer by confirming that* $v_{max} = 2v_{avg}$*. If this relationship does not hold, you have made an error somewhere.*

BINDER Example 36

Two pipes of equal length are connected in series. Pipe A has radius R and pipe B has radius 2R. If the same fluid flows through both pipes under steady conditions, find the ratio of the pressure drop across pipe A to the pressure drop across pipe B.

Solution

In series connection, the volume flow rate Q is the same through both pipes (by conservation of mass: what flows out of A flows into B).

From Poiseuille's formula, rearranged to give the pressure drop:

$$\Delta\mathrm{P} = \frac{8\eta\mathrm{LQ}}{\pi\mathrm{R}^4}$$

For pipe A (radius R):

$$\Delta\mathrm{P_A} = \frac{8\eta\mathrm{LQ}}{\pi\mathrm{R}^4}$$

For pipe B (radius 2R):

$$\Delta P_B = \frac{8\eta LQ}{\pi(2R)^4} = \frac{8\eta LQ}{\pi \times 16R^4} = \frac{8\eta LQ}{16\pi R^4}$$

Taking the ratio:

$$\frac{\mathbf{\Delta P_A}}{\mathbf{\Delta P_B}} = \frac{\frac{8\eta LQ}{\pi R^4}}{\frac{8\eta LQ}{16\pi R^4}} = \frac{16\pi R^4}{\pi R^4} = \mathbf{16}$$

Therefore, the pressure drop across pipe A is **16 times** the pressure drop across pipe B.

Making Sense of the Answer: *Pipe A is narrower, so it offers far more resistance to flow. The factor of 16 comes directly from the fourth-power dependence:* $(2R/R)^4 = 16$*. In a series system, the narrow pipe is the bottleneck. Most of the total pressure drop occurs across the narrowest section, just as most of the voltage drop in a series electrical circuit occurs across the largest resistor.*

Think Like a Physicist: *Poiseuille's formula has a beautiful analogy with Ohm's law in electricity. The pressure difference* ΔP *is analogous to voltage, the flow rate* Q *is analogous to current, and the quantity* $8\eta L/(\pi R^4)$ *plays the role of resistance. Pipes in series add their resistances; pipes in parallel share the flow. This analogy is widely used in medical physics to model the circulatory system.*

REAL Example 37

Kipute reads in a biology textbook that atherosclerosis (the buildup of fatty plaque inside arteries) can reduce the internal radius of an artery by 25%. She calculates the effect on blood flow using Poiseuille's formula and is shocked by the result.

Assuming all other factors remain constant, what percentage decrease in blood flow did Kipute find, and why was it so surprising?

Solution

Let the original radius be R and the reduced radius be $R' = 0.75R$ (a 25% reduction).

By Poiseuille's formula, $Q \propto R^4$ when all other factors are constant. The ratio of the new flow rate to the original is:

$$\frac{Q'}{Q} = \frac{R'^4}{R^4} = \frac{(0.75R)^4}{R^4} = (0.75)^4 = 0.316$$

Therefore:

$$Q' = 0.316Q$$

$$\text{Percentage decrease } = \frac{Q - Q'}{Q} \times 100\% = \left(\frac{Q - 0.316Q}{Q}\right) \times 100\% = 68.4\%$$

The percentage decrease in blood flow is 68.4%.

Reason: Kipute expected a moderate decrease since the radius only reduced by 25%. However, the reduction in blood flow much higher than 25% suggesting that the blood flow depends on the **fourth power of the radius**, not just the radius itself.

Making Sense of the Answer: *A 25% narrowing of the artery causes a 68% drop in blood flow. This is the devastating power of the fourth-power law. A modest narrowing produces a catastrophic reduction in flow. This is precisely why atherosclerosis is so dangerous: by the time the patient notices symptoms (reduced blood supply to the heart or brain), the artery may have already lost more than half its flow capacity. Poiseuille's formula explains why cardiologists take even small blockages very seriously.*

Think Like a Physicist: *The* R^4 *dependence means that percentage changes in radius are amplified dramatically in the flow rate. A 10% reduction in radius gives* $(0.90)^4 = 0.656$*, a 34% drop in flow. A 50% reduction gives* $(0.50)^4 = 0.0625$*, a 94% drop. This is why the body has powerful mechanisms to regulate artery diameter: even tiny changes have enormous consequences.*

HOT Example 38

A viscous liquid flows through a horizontal tube of length 50cm and internal radius 1mm under a pressure difference of $3 \times 10^4\text{Pa}$. The volume of liquid collected in 10 minutes is 500cm^3.

(a) Calculate the coefficient of viscosity of the liquid.

(b) Find the velocity of the liquid at a distance of 0.5mm from the axis.

(c) Find the velocity at the wall.

Solution

(a) Given: $\mathrm{L = 50cm = 0.50m,\ R = 1mm = 1.0 \times 10^{-3}m,\ \Delta P = 3 \times 10^4Pa}$

Volume collected: $\mathrm{V = 500cm^3 = 500 \times 10^{-6}m^3 = 5.0 \times 10^{-4}m^3}$

Time: $\mathrm{t = 10\ min = 600s}$

Volume flow rate:

$$\mathrm{Q = \frac{V}{t} = \frac{5.0 \times 10^{-4}m^3}{600s} = 8.33 \times 10^{-7}m^3s^{-1}}$$

Rearranging Poiseuille's formula to make η the subject:

$$\mathrm{Q = \frac{\pi \Delta P R^4}{8\eta L}}$$

$$\mathrm{\eta = \frac{\pi \Delta P R^4}{8QL}}$$

Substituting:

$$\mathrm{\eta = \frac{\pi \times 3 \times 10^4Pa \times (1.0 \times 10^{-3}m)^4}{8 \times 8.33 \times 10^{-7}m^3s^{-1} \times 0.50m} = 2.83 \times 10^{-2}Nsm^{-2}}$$

Therefore, the coefficient of viscosity of the liquid is $\mathrm{2.83 \times 10^{-2}Nsm^{-2}}$.

(b) Using the velocity profile formula:

$$\mathrm{v(r) = \frac{\Delta P}{4\eta L}(R^2 - r^2)}$$

With $\mathrm{r = 0.5mm = 5.0 \times 10^{-4}m}$:

$$\mathrm{v = \frac{3 \times 10^4Pa}{4 \times 2.83 \times 10^{-2}Nsm^{-2} \times 0.50m} \times ((1.0 \times 10^{-3}m)^2 - (5.0 \times 10^{-4}m)^2) = 0.398ms^{-1}}$$

Therefore, the velocity at 0.5mm from the axis is approximately $\mathrm{0.40ms^{-1}}$.

(c) At the wall, $\mathrm{r = R}$:

$$\mathrm{v(R) = \frac{\Delta P}{4\eta L}(R^2 - R^2) = \frac{\Delta P}{4\eta L} \times 0 = 0}$$

Therefore, the velocity at the wall is **zero**.

Making Sense of the Answer: *The viscosity of about* $0.0283Nsm^{-2}$ *places this liquid between water* ($0.001Nsm^{-2}$) *and glycerine* ($1.5Nsm^{-2}$)*; perhaps a light oil or a dilute sugar solution. The velocity at mid-radius* ($0.40ms^{-1}$) *is 75% of the maximum velocity. This makes sense from the parabolic profile: at half the radius,* $r^2/R^2 = 0.25$*, so* $v = v_{max}(1 - 0.25) = 0.75 \times v_{max}$*. And the velocity at the wall is exactly zero, as the no-slip condition demands.*

Think Like a Physicist: *Poiseuille's formula is frequently used in reverse: measure the flow rate, and calculate the viscosity. This is the principle behind the capillary viscometer, one of the standard laboratory instruments for measuring viscosity. The key experimental quantities are the volume collected, the time, the pressure difference, and the tube dimensions. The tube radius must be measured very carefully, because it enters the formula as* R^4*: a 5% error in measuring R produces a 20% error in* η*.*

That completes Poiseuille's formula. In the next subtopic, we turn to a different problem: *what happens when a solid object moves through a viscous fluid?* The answer is **Stokes' law**, and it leads to the concept of **terminal velocity**, where gravity and viscous drag reach a perfect, elegant balance.

STOKES' LAW AND TERMINAL VELOCITY

Drop a steel ball into a jar of honey. It sinks, but slowly, far more slowly than it would fall through air. Drop the same ball into water. It sinks faster, but still not as fast as it would in free fall. Now drop a tiny grain of sand into the same honey. It sinks so slowly that you can barely see it move.

In every case, the fluid pushes back against the falling object, resisting its motion. This resistance is viscous drag, and the equation that describes it for a sphere moving through a viscous fluid is called **Stokes' law**. It leads to one of the most elegant ideas in fluid mechanics: **terminal velocity**, the constant speed at which an object falls when the drag force exactly balances the net downward force.

Stokes' Law

Derivation

When a sphere moves through a viscous fluid, the fluid must part to let it through and then close up again behind it. This disturbs many layers of fluid near the sphere, and all of these layers exert viscous drag on the sphere. The total drag force F depends on three quantities:

1) The **coefficient of viscosity η** of the fluid; because a more viscous fluid exerts more drag.
2) The **radius r** of the sphere; because a larger sphere disturbs more fluid.
3) The **velocity v** of the sphere; because a faster sphere shears the fluid layers more rapidly.

We can write:

$$F = k\,\eta^{a}\,r^{b}\,v^{c}$$

where k is a dimensionless constant, and a, b, c are powers to be determined. We find these powers using **dimensional analysis**.

Writing the dimensions of each quantity:

$$[F] = MLT^{-2}, [\eta] = ML^{-1}T^{-1}, [r] = L, [v] = LT^{-1}$$

Substituting into the dimensional equation:

$$[F] = [\eta]^{a}\,[r]^{b}\,[v]^{c}$$

$$MLT^{-2} = (ML^{-1}T^{-1})^{a} \times L^{b} \times (LT^{-1})^{c}$$

$$MLT^{-2} = M^{a}\,L^{-a}\,T^{-a} \times L^{b} \times L^{c}\,T^{-c}$$

$$MLT^{-2} = M^{a} \times L^{-a+b+c} \times T^{-a-c}$$

Equating the powers of M, L, and T on both sides:

For M: $\mathbf{1 = a}$

For T: $-2 = -a - c; -2 = -1 - c;$ $\mathbf{c = 1}$

For L: $1 = -a + b + c; 1 = -1 + b + 1;$ $\mathbf{b = 1}$

Substituting the powers back into the formula:

$$F = k\,\eta^{1}\,r^{1}\,v^{1} = k\eta rv$$

Dimensional analysis tells us that the drag force is proportional to ηrv, but it cannot determine the dimensionless constant k. The full mathematical solution (due to Sir George Gabriel Stokes, 1851, obtained by solving the fluid flow equations around a sphere) gives:

$$k = 6\pi$$

Therefore:

$$\mathbf{F = 6\pi\eta rv}$$

This is **Stokes' law**. It can be stated in words as follows:

The viscous drag force on a sphere moving slowly through a viscous fluid is directly proportional to the coefficient of viscosity of the fluid, the radius of the sphere, and the velocity of the sphere.

The force F acts in the direction opposite to the motion of the sphere: if the sphere is falling downward, the drag force acts upward.

Conditions for Stokes' law to apply

1. The sphere must be smooth and rigid

Rough or deformable objects experience different drag.

2. The flow must be laminar

Stokes' law applies only when the Reynolds number for the sphere is small (typically $R_e < 1$). At higher speeds, the flow around the sphere becomes turbulent and the drag increases much faster than Stokes' law predicts.

3. The fluid must extend far in all directions

The walls of the container must be far enough from the sphere that they do not affect the flow pattern. If the sphere is falling through a narrow tube, wall effects modify the drag.

4. The sphere must not be accelerating rapidly

Stokes' law describes the drag in steady or quasi-steady motion.

Terminal Velocity

To understand terminal velocity, study the following figure:

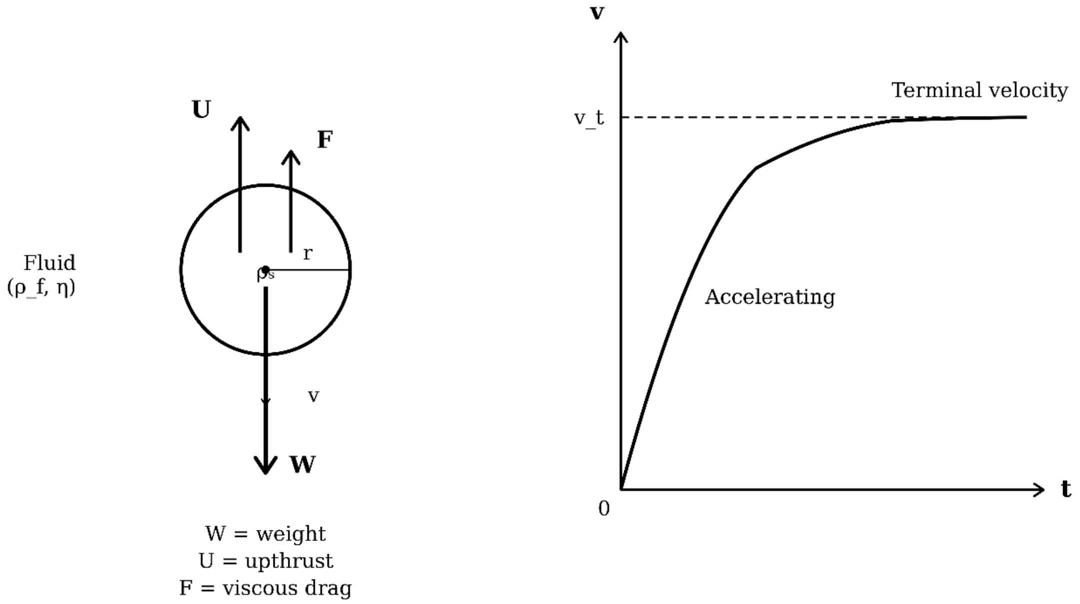


Figure: ***(Left)*** *A sphere of radius r and density* ρ_s *falling through a viscous fluid of density* ρ_f *and viscosity* η*. Three forces act on the sphere: weight W (downward), upthrust U (upward), and viscous drag F (upward). At terminal velocity, W = U + F.* ***(Right)*** *The velocity-time graph for the falling sphere. The velocity increases from zero, with decreasing acceleration, and approaches the terminal velocity* v_t *asymptotically.*

When a sphere is released from rest in a viscous fluid, three forces act on it as it falls:

1) **Weight** W, acting downward:

$$W = \frac{4}{3}\pi r^3 \rho_s g$$

where ρ_s is the density of the sphere.

2) **Upthrust (buoyancy)** U, acting upward. By Archimedes' principle, the upthrust equals the weight of fluid displaced:

$$U = \frac{4}{3}\pi r^3 \rho_f g$$

where ρ_f is the density of the fluid.

3) **Viscous drag** F, acting upward (opposing the downward motion). By Stokes' law:

$$F = 6\pi\eta r v$$

The net downward force at any instant is:

$$F_{net} = W - U - F = \frac{4}{3}\pi r^3 \rho_s g - \frac{4}{3}\pi r^3 \rho_f g - 6\pi\eta r v$$

$$\boldsymbol{F_{net} = \frac{4}{3}\pi r^3 (\rho_s - \rho_f) g - 6\pi\eta r v}$$

Now watch what happens as the sphere falls:

At the moment of release ($v = 0$): The drag force is zero (because $v = 0$). The net downward force is $W - U = \frac{4}{3}\pi r^3(\rho_s - \rho_f)g$, which is the effective weight. The sphere accelerates downward.

As the sphere speeds up: The velocity v increases, so the drag force $F = 6\pi\eta rv$ increases. The net downward force decreases. The acceleration decreases.

At terminal velocity: The sphere reaches a speed v_t at which the drag force has grown large enough to exactly balance the effective weight. The net force is zero. The acceleration is zero. The sphere continues to fall at a constant velocity. This constant velocity is the **terminal velocity**.

Derivation of the terminal velocity formula:

At terminal velocity, $F_{net} = 0$:

$$W - U - F = 0$$

$$\frac{4}{3}\pi r^3\rho_s g - \frac{4}{3}\pi r^3\rho_f g - 6\pi\eta r v_t = 0$$

Rearranging to isolate the drag term:

$$6\pi\eta r v_t = \frac{4}{3}\pi r^3(\rho_s - \rho_f)g$$

Dividing both sides by $6\pi\eta r$:

$$v_t = \frac{4\pi r^3(\rho_s - \rho_f)g}{3 \times 6\pi\eta r} = \frac{4r^3(\rho_s - \rho_f)g}{18\eta r} = \frac{4r^2(\rho_s - \rho_f)g}{18\eta} = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

Hence:

$$\mathbf{v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}}$$

This is the **terminal velocity formula** for a sphere falling through a viscous fluid.

Let us examine what it tells us.

1. **The terminal velocity is proportional to r^2.** Larger spheres fall faster. A sphere of twice the radius has four times the terminal velocity. This is why large raindrops fall faster than small ones, and why fine sand settles much more slowly than coarse gravel in water.
2. **The terminal velocity is proportional to $(\rho_s - \rho_f)$.** The greater the density difference between the sphere and the fluid, the faster the sphere falls. A steel ball sinks quickly through water because the density difference is large. A plastic ball barely sinks because the density difference is small. If $\rho_s = \rho_f$, the terminal velocity is zero: the sphere is neutrally buoyant and does not sink at all.
3. **The terminal velocity is inversely proportional to η.** The more viscous the fluid, the slower the sphere falls. A steel ball sinks slowly through honey but quickly through water.

Measuring Viscosity Using Stokes' Law

The terminal velocity formula can be rearranged to measure the viscosity of a liquid:

$$\eta = \frac{2r^2(\rho_s - \rho_f)g}{9v_t}$$

This is the basis of the **falling sphere viscometer**. A sphere of known radius r and density ρ_s is dropped into a tall column of the liquid whose viscosity is to be measured. The sphere is allowed to accelerate until it reaches terminal velocity. The terminal velocity v_t is then measured by timing the sphere as it falls through a known distance between two marks on the column (placed far enough from the top and bottom to ensure the sphere is at terminal velocity and not affected by the ends). The density ρ_f of the liquid is measured separately. The viscosity η is then calculated from the formula.

Sources of error include: the sphere may not be perfectly spherical (gives incorrect r), the sphere may not have reached terminal velocity when timing begins (gives v_t too high), wall effects from the container (use a container at least 10 times wider than the sphere), temperature variations during the experiment (viscosity changes with temperature), and air bubbles clinging to the sphere (changes effective density and radius).

Now the equations have been derived, the physics has been explained, and all three forces have been identified. What remains is to see whether these ideas can survive contact with real numbers. The following examples will test your understanding of Stokes' law and terminal velocity, and the last one carries a twist that separates those who have memorised the formula from those who have understood the physics.

BINDER Example 39

A steel ball of radius 2mm and density 7800kgm^{-3} is dropped into glycerine of density 1260kgm^{-3} and viscosity 1.5Nsm^{-2}. Calculate the terminal velocity of the ball. Take $g = 9.8\text{ms}^{-2}$.

Solution

Given: $r = 2\text{mm} = 2 \times 10^{-3}\text{m}$, $\rho_s = 7800\text{kgm}^{-3}$, $\rho_f = 1260\text{kgm}^{-3}$, $\eta = 1.5\text{Nsm}^{-2}$, $g = 9.8\text{ms}^{-2}$

Applying the terminal velocity formula:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

$$v_t = \frac{2 \times (2 \times 10^{-3}\text{m})^2 \times (7800 - 1260)\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 1.5\text{Nsm}^{-2}} = 0.038\text{ms}^{-1}$$

Therefore, the terminal velocity of the steel ball in glycerine is approximately 0.038ms^{-1} (or 3.8cms^{-1}).

Making Sense of the Answer: *About $3.8cm$ per second, walking speed for an ant, not for a steel ball. Glycerine is extremely viscous ($\eta = 1.5Nsm^{-2}$, about 1500 times more viscous than water), so even a dense steel ball sinks very slowly through it. In water ($\eta = 10^{-3}Nsm^{-2}$), the same ball would have a terminal velocity about 1500 times larger (roughly $57ms^{-1}$), which is far too fast for Stokes' law to apply (the flow would be turbulent). This illustrates an important point: Stokes' law works best for small spheres in highly viscous fluids.*

Think Like a Physicist: *Always check whether the Stokes' law conditions are satisfied. The Reynolds number for this ball is $R_e = \rho_f v_t(2r)/\eta = 1260 \times 0.038 \times 0.004/1.5 \approx 0.13$, which is well below 1. The flow is laminar, and Stokes' law is valid. If R_e came out much larger than 1, the answer would be unreliable.*

BINDER Example 40

In a falling sphere viscometer experiment, a glass sphere of radius 1.5mm and density 2500kgm^{-3} is dropped into a tall column of oil of density 900kgm^{-3}. The sphere reaches terminal velocity and falls through 20cm in 4.2s. Calculate the coefficient of viscosity of the oil. Take $g = 9.8\text{ms}^{-2}$.

Solution

Given: $r = 1.5\text{mm} = 1.5 \times 10^{-3}\text{m}$, $\rho_s = 2500\text{kgm}^{-3}$, $\rho_f = 900\text{kgm}^{-3}$, $d = 20\text{cm} = 0.20\text{m}$, $t = 4.2\text{s}$

First, find the terminal velocity:

$$v_t = \frac{d}{t} = \frac{0.20\text{m}}{4.2\text{s}} = 0.0476\text{ms}^{-1}$$

Rearranging the terminal velocity formula to find η:

$$\eta = \frac{2r^2(\rho_s - \rho_f)g}{9v_t}$$

$$\eta = \frac{2 \times (1.5 \times 10^{-3}\text{m})^2 \times (2500 - 900)\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 0.0476\text{ms}^{-1}} = 0.165\text{Nsm}^{-2}$$

Therefore, the coefficient of viscosity of the oil is approximately 0.17Nsm^{-2}.

Making Sense of the Answer: *A viscosity of $0.17Nsm^{-2}$ is typical of a moderately viscous oil; thicker than water ($10^{-3}Nsm^{-2}$) but much thinner than glycerine ($1.5Nsm^{-2}$). The terminal velocity of about $4.8cms^{-1}$ through $20cm$ in $4.2s$ is slow enough for the flow to be laminar, confirming that Stokes' law is applicable.*

Think Like a Physicist: *In the viscometer experiment, the timing must begin only after the sphere has reached terminal velocity. If you start timing too early (while the sphere is still accelerating), the measured velocity will be lower than the true terminal velocity, and the calculated viscosity will be too high. This is* *why the two timing marks are placed well below the surface of the liquid, leaving a run-up distance for the sphere to reach terminal velocity.*

REAL Example 41

During a rainy evening in Dar es Salaam, Kipanga and Kipute stand on the balcony watching the rain. Kipanga notices that some drops are tiny and seem to float down gently, while others are large and hit the ground hard. He asks Kipute: *"If gravity is the same for all drops, why do the big ones fall so much faster?"*

Kipute smiles. She has just finished studying Stokes' law. What was her explanation?

Solution

Each raindrop reaches a terminal velocity at which the viscous drag from the air exactly balances its effective weight. By the terminal velocity formula:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

The density difference $(\rho_s - \rho_f)$, the gravitational acceleration g, and the viscosity η of air are the same for all raindrops. The only factor that varies from drop to drop is the radius r.

Since the terminal velocity is proportional to r^2, a drop with twice the radius has four times the terminal velocity.

The r^2 dependence of terminal velocity is explained by the fact that the weight of a sphere grows as r^3, while the Stokes drag grows only as r $\left(\frac{r^3}{r} = r^2\right)$, so larger spheres need a higher velocity before the drag can balance the weight.

Making Sense of the Answer: *A fine mist drop ($r = 0.1mm$) has a terminal velocity of about $0.3ms^{-1}$ in air, gentle enough to feel like it is floating. A heavy raindrop ($r = 2mm$) has a terminal velocity of about 6 to $9ms^{-1}$ which is fast enough to sting your skin. The r^2 law explains the entire spectrum of raindrop behaviour, from hovering mist to pelting monsoon rain.*

Think Like a Physicist: *Strictly speaking, Stokes' law applies only for very small Reynolds numbers ($R_e < 1$). Large raindrops fall fast enough that the flow around them becomes turbulent, and the actual drag is higher than Stokes' law predicts. In practice, this means large raindrops have slightly lower terminal velocities than the Stokes formula gives. Nevertheless, the qualitative conclusion that, bigger drops fall faster because v_t increases with r remains correct.*

HOT Example 42

Two spheres, P and Q, are made of the same material and are dropped simultaneously into a large tank of viscous liquid. Sphere P has radius r and sphere Q has radius 3r. Both spheres eventually reach terminal velocity.

(a) Find the ratio of the terminal velocity of Q to that of P.

(b) If the viscous drag on sphere P at terminal velocity is F_P, find the viscous drag on sphere Q at terminal velocity in terms of F_P.

(c) Sphere P reaches terminal velocity after falling 5cm. Estimate whether sphere Q reaches terminal velocity in a shorter or longer distance, and explain why.

Solution

(a) Both spheres are made of the same material, so ρ_s is the same. Both fall through the same fluid, so ρ_f and η are the same. By the terminal velocity formula:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

Since all factors except r are common:

$$v_t \propto r^2$$

$$\frac{\mathbf{v_Q}}{\mathbf{v_P}} = \frac{(3r)^2}{r^2} = \frac{9r^2}{r^2} = \mathbf{9}$$

Therefore, the terminal velocity of Q is **9 times** that of P.

(b) At terminal velocity, the drag force equals the effective weight (weight minus upthrust):

$$F_{drag} = \frac{4}{3}\pi r^3(\rho_s - \rho_f)g$$

This is proportional to r^3:

$$\frac{\mathbf{F_Q}}{\mathbf{F_P}} = \frac{(3r)^3}{r^3} = \frac{27r^3}{r^3} = \mathbf{27}$$

Therefore, the viscous drag on Q at terminal velocity is $27F_P$.

(c) Sphere Q is heavier (27 times the effective weight of P) and has a higher terminal velocity (9 times that of P). It accelerates under a net force that starts at 27 times the effective weight of P and decreases as the drag builds up. Although the drag force grows as the sphere speeds up, Q must accelerate to a much higher terminal velocity ($9v_P$) before equilibrium is reached.

The distance needed to reach terminal velocity depends on the acceleration phase. A heavier, faster sphere needs more distance to reach its higher terminal velocity, (for the same reason that a heavy truck needs a longer distance to reach cruising speed than a bicycle).

Therefore, sphere Q reaches terminal velocity after falling a **longer distance** than sphere P.

Making Sense of the Answer: *Part (a) gives the* r^2 *law for velocity. Part (b) gives the* r^3 *law for drag at terminal velocity. Part (c) requires physical reasoning beyond the formula: the question is not about the magnitude of the terminal velocity but about how long it takes to get there. This distinction separates formula-users from physics-thinkers.*

Think Like a Physicist: *Part (b) has a subtle trap. Students sometimes use Stokes' law directly ($F = 6\pi\eta rv$) and substitute the terminal velocities, getting $F_Q/F_P = (3r \times 9v_P)/(r \times v_P) = 27$. This gives the correct answer, but the cleaner approach is to recognise that at terminal velocity, drag equals effective weight, and effective weight is proportional to* r^3*. Both methods must give the same answer, if they do not, you have made an error.*

HOT Example 43

A thin square metal plate of side 10cm and mass 50g is released from rest between two large, fixed, vertical surfaces. The gap between each surface and the plate is 0.5mm, and each gap is filled with oil of viscosity 0.20Nsm^{-2}. The plate slides vertically downward under gravity.

Kipanga watches it descend and expects it to accelerate like any falling object. But after a very short time, the plate stops accelerating and falls at a constant speed. ***What is this speed, and why does the plate reach a constant velocity?***

Solution

The plate falls under gravity and is resisted by viscous drag from the oil on both sides. After a brief initial acceleration, the plate reaches a constant velocity (terminal velocity) when the net force on it is zero.

Three forces act on the plate:

Weight (downward): $W = mg = 0.050\text{kg} \times 9.8\text{ms}^{-2} = 0.49\text{N}$

Viscous drag from the left oil layer (upward): By Newton's law of viscosity, the drag force from one oil layer is $F_1 = \eta A\frac{v}{d}$, where A is the area of the plate, v is the plate velocity, and d is the gap thickness. The velocity gradient is $\frac{v}{d}$ because the plate moves at v and the fixed wall is at rest.

Viscous drag from the right oil layer (upward): $F_2 = \eta A\frac{v}{d}$

Since there are two oil layers (one on each side of the plate), the total viscous drag is:

$$F_{total} = 2\eta A\frac{v}{d}$$

At terminal velocity, the total drag equals the weight:

$$W = 2\eta A\frac{v_t}{d}$$

Rearranging for v_t:

$$v_t = \frac{Wd}{2\eta A}$$

Substituting values:

$A = (0.10\text{m})^2 = 0.01\text{m}^2$, $d = 0.5\text{mm} = 5 \times 10^{-4}\text{m}$, $\eta = 0.20\text{Nsm}^{-2}$

$$v_t = \frac{0.49\text{N} \times 5 \times 10^{-4}\text{m}}{2 \times 0.20\text{Nsm}^{-2} \times 0.01\text{m}^2}$$

$$v_t = \frac{2.45 \times 10^{-4}}{4.0 \times 10^{-3}} = 0.061\text{ms}^{-1}$$

Therefore, the terminal velocity of the plate is approximately 0.061ms^{-1} (about 6.1cms^{-1}).

The plate reaches a constant velocity because the viscous drag force is proportional to velocity ($F \propto v$). As the plate accelerates from rest, v increases, so the drag increases. The acceleration decreases. Eventually, the drag grows large enough to exactly balance the weight. At that point, the net force is zero, the acceleration is zero, and the plate falls at constant terminal velocity.

Making Sense of the Answer: *About* 6cm *per second through a* 0.5mm *oil gap. This is extremely slow for a falling object, which shows how effectively a thin layer of viscous oil resists motion. If the gaps were doubled to* 1mm, *the velocity gradient would halve, the drag would halve, and the terminal velocity would double to about* 12cms^{-1}. *This is the same physics behind viscous dampers used in door closers and shock absorbers.*

Think Like a Physicist: *This problem is the flat-plate version of Stokes' terminal velocity for a sphere. The logic is identical: identify all forces, set the net force to zero, solve for velocity. The factor of 2 in the drag comes from having oil on both sides of the plate. Missing this factor of 2 is the most common error in this type of problem.*

That completes the treatment of Stokes' law and terminal velocity. With this subtopic, the three hidden personalities of fluids: surface tension, the pressure-speed trade-off of Bernoulli, and viscosity have all been revealed. Let us now assemble the key applications.

APPLICATIONS OF FLUID MECHANICS

If you have made it this far, you have earned something remarkable: you now understand the three hidden personalities of fluids. You know that surface tension makes drops spherical, needles float, and liquids climb up narrow tubes. You know that Bernoulli's equation trades speed for pressure, lifting aircraft, emptying tanks, and attacking shower curtains. You know that viscosity makes honey crawl, blood flow obey Poiseuille, and falling spheres reach terminal velocity.

This section gathers all the major applications in one place. It is not a new theory section. It is a concise, organised reference that pulls together everything you have already learned, application by application. These applications appear frequently in examinations, and having them assembled in one place will save you time during revision.

Applications of Surface Tension

1. Insects walking on water (pond skaters)

A pond skater's legs are coated with a waxy, hydrophobic substance. When the insect stands on water, its legs press down on the surface but do not break through. The surface tension of water acts along the contact line around each leg, providing an upward force that supports the insect's weight. The insect is light enough that this upward force is sufficient. A human, with far greater weight relative to foot area, would break the surface instantly.

2. Floating needle

A steel needle placed gently on the surface of still water floats, even though steel is about eight times denser than water. The needle is supported not by buoyancy (which would be far too small) but by surface tension acting along the contact line where the water surface meets the needle. The water surface bends downward slightly under the needle, and the vertical components of the surface tension forces along this contact line balance the needle's weight. Adding a drop of detergent reduces the surface tension, and the needle sinks immediately.

3. Spherical liquid drops

In the absence of external forces (or when forces are balanced, as in free fall or in space), liquid drops are perfectly spherical. This is because surface tension pulls every part of the surface inward, minimising the surface area. Among all shapes with a given volume, a sphere has the smallest surface area. Consequently, a sphere is the shape of lowest surface energy, and surface tension drives every drop toward it.

4. Detergents and soap

Detergent molecules have a hydrophilic (water-attracting) head and a hydrophobic (water-repelling) tail. When dissolved in water, they wedge themselves into the surface layer with their tails pointing outward. This disrupts the cohesive forces between water molecules and dramatically reduces the surface tension (from about 0.073Nm^{-1} to about 0.025Nm^{-1}). As a result, the soapy water spreads more easily, wets surfaces more completely, and penetrates into grease and dirt, loosening them for removal.

5. Waterproofing of fabrics

Waterproof fabrics are treated with hydrophobic coatings that increase the angle of contact between water and the fabric fibres to well above $90°$. Consequently, instead of being absorbed by capillary action (which requires $\theta < 90°$), water beads up on the surface and rolls off. The same principle is at work on waxed car surfaces and on the leaves of lotus plants.

6. Capillary action in plants

Water rises from the roots to the leaves of plants partly through capillary rise in the narrow xylem vessels. The adhesive forces between water and the hydrophilic walls of the xylem pull the water upward. Capillary rise alone, however, can account for only about 1 metre of rise. In tall trees, the dominant mechanism is transpiration pull: evaporation of water from the leaf surfaces creates a negative pressure at the top of the water column, pulling it upward like a continuous chain.

7. Water absorption by soil

Water from rainfall or irrigation is held in the tiny spaces between soil particles by capillary forces. Without this, water would simply drain downward under gravity, and plant roots would have no moisture to absorb between rainfalls. Farmers sometimes break up the top layer of soil (tilling) to increase the size of the air gaps, which reduces capillary rise from the wet subsoil and prevents evaporation from the surface.

8. Oil spreading on water

When a drop of oil falls on a water surface, it spreads into a thin film. Water has high surface tension (0.073Nm^{-1}) and oil has low surface tension ($\approx 0.03\text{Nm}^{-1}$). Replacing the high-energy water-air interface with a lower-energy oil-air interface reduces the total surface energy. Consequently, spreading is energetically favourable, and it continues until the oil forms a very thin film.

9. Lung surfactants

The alveoli (tiny air sacs) in the lungs are lined with a thin film of liquid. During exhalation, the alveoli shrink, and without intervention, the surface tension of this film would collapse them completely. The body produces a substance called **pulmonary surfactant** that reduces the surface tension of the alveolar fluid, preventing collapse and allowing the alveoli to re-inflate easily on the next breath. Premature babies sometimes lack sufficient surfactant, leading to respiratory distress syndrome.

10. Blotting paper and paper towels

The narrow spaces between the plant fibres in paper act as capillary tubes. When the paper touches a liquid, capillary rise draws the liquid into these spaces rapidly. The fibres are hydrophilic, so the angle of contact is small and the capillary rise is strong. Waxed paper, by contrast, has hydrophobic fibres, so the contact angle exceeds $90°$ and the liquid is not absorbed.

Applications of Fluid Flow

1. Torricelli's theorem and tank drainage

The speed of efflux from an orifice in a tank is $v = \sqrt{2gh}$, where h is the depth of liquid above the orifice. This result is used in the design of water tanks, dams, and irrigation systems to predict flow rates and drainage times. Combined with projectile motion, it predicts the horizontal range of the water jet emerging from the orifice.

2. Venturi meter

A constriction in a pipe creates a speed increase (by the equation of continuity) and a corresponding pressure decrease (by Bernoulli's equation). By measuring the pressure difference between the wide section and the throat using a manometer, the flow speed and volume flow rate can be calculated. Venturi meters are used by water utilities (such as DAWASCO in Dar es Salaam) to monitor water distribution, in fuel injection systems to mix air and fuel, and in respiratory equipment to measure airflow.

3. Aerofoil lift (dynamic lift)

An aircraft wing is shaped so that air flows faster over the curved upper surface than over the flatter lower surface. By Bernoulli's equation, the pressure above the wing is lower than below. This pressure difference creates an upward lift force $F = \frac{1}{2}\rho A(v_1^2 - v_2^2)$. The same principle applies to helicopter rotor blades, wind turbine blades, and racing car spoilers (which are inverted aerofoils designed to push the car downward for better grip).

4. Magnus effect (spinning ball)

A spinning ball moving through air drags the air around it in the direction of spin. On one side, the spin adds to the oncoming airflow (higher speed, lower pressure). On the opposite side, the spin opposes the airflow (lower speed, higher pressure). The pressure difference deflects the ball sideways. This is how footballers curve free kicks, cricket bowlers swing the ball, and table tennis players create topspin and sidespin.

5. Atomiser and sprayer

Fast-moving air blown through a horizontal tube creates a low-pressure region at the junction with a vertical tube dipping into a liquid reservoir. The atmospheric pressure on the liquid surface pushes the liquid up the vertical tube, where it meets the fast airstream and is broken into a fine spray. This principle is used in perfume bottles, insecticide sprayers, paint spray guns, and carburettors in older engines.

6. Pitot tube

A tube facing directly into a fluid stream brings the fluid to rest at its tip (the stagnation point). By Bernoulli's equation, the pressure at this point equals the static pressure plus $\frac{1}{2}\rho v^2$. Comparing the stagnation pressure with the static pressure (measured by a side opening) gives the fluid velocity. Pitot tubes are mounted on aircraft to measure airspeed, in weather stations to measure wind speed, and in industrial pipelines to monitor flow rates.

7. Chimney effect (stack effect)

Hot gases inside a chimney are less dense than the cooler air outside. The pressure difference drives the hot gases upward through the chimney. By continuity, fresh air is drawn in at the base to replace them. A taller chimney creates a larger pressure difference and a stronger draft. This same effect ventilates buildings with tall atriums and drives the airflow in cooling towers.

8. Blood flow in the circulatory system

Blood flow through arteries and veins obeys the equation of continuity and Poiseuille's formula. The total cross-sectional area of capillaries is much larger than that of the aorta, so blood flows slowly in capillaries (allowing time for gas exchange) even though it flows quickly in the aorta. Arterial narrowing (atherosclerosis) reduces flow dramatically because of the R^4 dependence in Poiseuille's formula.

9. Hydraulic systems (Pascal's principle)

Pressure applied to a confined fluid is transmitted equally in all directions. This is the basis of hydraulic brakes, hydraulic lifts, and hydraulic presses. A small force applied to a small piston creates the same pressure as a large force on a large piston. Consequently, the force is amplified by the ratio of the piston areas. Every daladala bus in Tanzania uses hydraulic brakes that rely on this principle.

10. Sedimentation and particle separation

In a mixture of particles of different sizes suspended in a fluid, larger particles reach higher terminal velocities (because $v_t \propto r^2$ by Stokes' law) and settle to the bottom faster. This is used in mining to separate mineral particles by size, in blood tests (erythrocyte sedimentation rate, ESR, measures how fast red blood cells settle in a tube), and in water treatment plants to remove suspended solids.

That completes the applications assembly. This section serves as your quick-reference map: when an examination question asks you about a specific application, you know exactly where to find the theory and the worked example that goes with it.

Two sections remain. The miscellaneous worked examples will test your ability to combine concepts from different parts of the chapter. The Digging Deeper exercise will push you further. The finish line is in sight.

MISCELLANEOUS WORKED EXAMPLES ON FLUID MECHANICS

Example 44

(a) Distinguish between cohesive forces and adhesive forces. Using these concepts, explain why water forms a concave meniscus in a glass tube while mercury forms a convex meniscus.

(b) A soap bubble of radius 5cm and surface tension 0.03Nm^{-1} is blown at the end of a tube at sea level where atmospheric pressure is $1.01 \times 10^5\text{Pa}$. Calculate the total pressure inside the bubble.

Solution

(a) Cohesive forces are attractive forces between molecules of the same substance while adhesive forces are attractive forces between molecules of different substances.

In a glass tube containing water, the adhesive forces between water molecules and glass molecules are stronger than the cohesive forces between water molecules. Consequently, water molecules are pulled toward the glass wall, causing the liquid to climb upward along the wall. This produces a concave meniscus with an angle of contact less than 90°.

In a glass tube containing mercury, the cohesive forces between mercury atoms are stronger than the adhesive forces between mercury and glass. Hence, mercury molecules are pulled away from the glass wall, and the liquid curves downward near the wall. This produces a convex meniscus with an angle of contact greater than 90°.

(b) A soap bubble has two surfaces. The excess pressure is:

$$\Delta P = \frac{4\gamma}{r} = \frac{4 \times 0.03\text{Nm}^{-1}}{0.05\text{m}} = 2.4\text{Pa}$$

The total pressure inside the bubble is:

$$P = P_{atm} + \Delta P = 1.01 \times 10^5\text{Pa} + 2.4\text{Pa} = 1.01024 \times 10^5\text{Pa}$$

Example 45

(a) A student applies Bernoulli's equation to calculate the pressure in a pipe carrying hot, thick engine oil at high speed through a very long industrial pipeline. Explain three reasons why Bernoulli's equation may give an inaccurate result in this situation.

(b) Water flows through a horizontal pipe that narrows from an internal diameter of 8cm to 4cm. The pressure in the wider section is $3.0 \times 10^5\text{Pa}$ and the flow speed there is 1.5ms^{-1}. Find the pressure in the narrower section. Take $\rho = 1000\text{kgm}^{-3}$.

Solution

(a) Bernoulli's equation assumes three conditions that are violated in this situation:

First, the fluid must be non-viscous. Engine oil is highly viscous, so significant energy is lost to internal friction between layers as it flows. Consequently, the pressure drops along the pipe even without any change in speed or height, which Bernoulli's equation does not account for.

Second, the fluid must be incompressible. Hot engine oil may experience density changes due to temperature variations along the long pipeline, violating the constant-density assumption.

Third, the flow must be steady. In a long industrial pipeline, fluctuations in pump output or valve positions may cause unsteady flow, where velocities at a given point change with time.

Therefore, Bernoulli's equation is unreliable for this situation because the fluid is viscous, possibly compressible at varying temperatures, and the flow may not be steady.

(b) By the continuity equation:

$$v_2 = v_1 \times \frac{d_1^2}{d_2^2} = 1.5\text{ms}^{-1} \times \frac{(8\text{cm})^2}{(4\text{cm})^2} = 6.0\text{ms}^{-1}$$

For horizontal flow, Bernoulli's equation gives:

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) = 3.0 \times 10^5\text{Pa} + \frac{1}{2} \times 1000\text{kgm}^{-3} \times ((1.5\text{ms}^{-1})^2 - (6.0\text{ms}^{-1})^2)$$

$$P_2 = 2.83 \times 10^5\text{Pa}$$

Example 46

(a) Explain why a large spherical drop of liquid has less surface energy per unit volume than a small drop of the same liquid.

(b) A liquid drop of radius R and surface tension γ is sprayed into n identical smaller drops. Derive an expression for the work done in the spraying process. If a drop of mercury of radius 1mm ($\gamma = 0.5\text{Nm}^{-1}$) is broken into 8 equal drops, calculate the work done.

Solution

(a) The surface area of a sphere is $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$. The surface-to-volume ratio is:

$$\frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{r}$$

This ratio is inversely proportional to r. A larger drop has a smaller surface-to-volume ratio, and therefore less surface energy per unit volume. A smaller drop has a larger ratio and more surface energy per unit volume. This is why energy must be supplied to break a large drop into smaller ones.

(b) By conservation of volume: $\frac{4}{3}\pi R^3 = n \times \frac{4}{3}\pi r^3$, giving $r = R/n^{1/3}$.

The work done equals the increase in surface energy:

$$W = \gamma(n \times 4\pi r^2 - 4\pi R^2) = 4\pi\gamma\left(\frac{nR^2}{n^{2/3}} - R^2\right) = 4\pi\gamma R^2\left(n^{1/3} - 1\right)$$

Hence:

$$\mathbf{W = 4\pi\gamma R^2\left(n^{1/3} - 1\right)}$$

For the mercury drop: $R = 1 \times 10^{-3}\text{m}$, $\gamma = 0.5\text{Nm}^{-1}$, $n = 8$

$$W = 4\pi \times 0.5\text{Nm}^{-1} \times (10^{-3}\text{m})^2 \times \left(8^{1/3} - 1\right) = 4\pi \times 0.5\text{Nm}^{-1} \times 10^{-6}\text{m}^2 \times (2 - 1)$$

$$W = 6.28 \times 10^{-6}\text{J}$$

Example 47

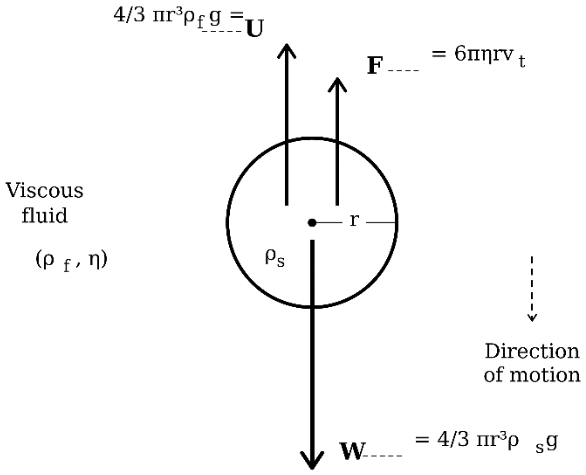
(a) A steel ball is dropped into a tall jar of glycerine. Initially it accelerates, but after a short distance it moves at constant velocity. Explain why the ball stops accelerating, and draw a well labelled diagram showing all the forces acting on the ball when it has reached constant velocity.

(b) A tank of water rests on a platform 3m above the ground. The water surface is 4m above the base of the tank. A small hole is opened at the base. Find the speed of the emerging water and the horizontal distance from the platform where the jet hits the ground. Take $g = 9.8\text{ms}^{-2}$.

Solution

(a) When the ball is first released, its velocity is zero and the viscous drag force ($F = 6\pi\eta rv$) is also zero. The net downward force is the weight minus the upthrust, so the ball accelerates. As the ball speeds up, the viscous drag increases (because drag is proportional to velocity). Consequently, the net downward force decreases and the acceleration decreases. Eventually, the drag becomes large enough to exactly balance the effective weight. At that point, the net force is zero, the acceleration is zero, and the ball falls at a constant velocity called the **terminal velocity**.

At terminal velocity, three forces act: weight $W = \frac{4}{3}\pi r^3\rho_s g$ downward, upthrust $U = \frac{4}{3}\pi r^3\rho_f g$ upward, and viscous drag $F = 6\pi\eta rv_t$ upward. The condition is $W = U + F$. So the diagram is:



At terminal velocity:
W = U + F

(b) By Torricelli's theorem:

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 4} = \sqrt{78.4} = \mathbf{8.85ms^{-1}}$$

The jet emerges horizontally from a height $H = 3m$ above the ground. The time to fall:

$$t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 3m}{9.8ms^{-2}}} = \sqrt{0.612s^2} = 0.782s$$

Horizontal range:

$$x = vt = 8.85ms^{-1} \times 0.782s = 6.92m$$

Therefore, the jet hits the ground approximately 6.9m from the base of the platform.

Example 48

(a) When a flat plate is dragged across a thin layer of oil on a surface, the force required increases if the plate area increases, the plate speed increases, or the oil layer becomes thinner. Using Newton's law of viscosity, explain why each of these three changes increases the required force.

(b) Water ($\eta = 1.0 \times 10^{-3}Nsm^{-2}$, $\rho = 1000kgm^{-3}$) flows through a horizontal pipe of length 2m and internal radius 3mm at a volume flow rate of $2.0 \times 10^{-5}m^3s^{-1}$. Find the pressure difference between the ends of the pipe and the maximum velocity of the water.

Solution

(a) By Newton's law of viscosity, the viscous force is $F = \eta A\left(\frac{dv}{dy}\right)$, where A is the contact area, $\frac{dv}{dy}$ is the velocity gradient, and η is the viscosity.

Increasing plate area: The force is directly proportional to A. A larger plate has more surface in contact with the oil, so more fluid layers are sheared simultaneously. Consequently, the total viscous resistance increases.

Increasing plate speed: The velocity gradient $\frac{dv}{dy} = \frac{v}{d}$ increases when the plate speed v increases (with oil thickness d unchanged). A steeper velocity gradient means the layers are sliding over each other faster, producing greater friction.

Thinner oil layer: Reducing the thickness d increases the velocity gradient $\frac{dv}{dy} = \frac{v}{d}$ because the same velocity change occurs over a shorter distance. A steeper gradient means stronger friction between layers.

(b) From Poiseuille's formula, $Q = \frac{\pi \Delta P R^4}{8\eta L}$, rearranging:

$$\Delta P = \frac{8\eta LQ}{\pi R^4} = \frac{8 \times 1.0 \times 10^{-3}\mathrm{Nsm^{-2}} \times 2\mathrm{m} \times 2.0 \times 10^{-5}\mathrm{m^3s^{-1}}}{\pi \times (3 \times 10^{-3}\mathrm{m})^4}$$

$$\mathbf{\Delta P = 1257Pa}$$

The maximum velocity (at the centre):

$$\mathbf{v_{max}} = \frac{\Delta P R^2}{4\eta L} = \frac{1257\mathrm{Pa} \times (3 \times 10^{-3}\mathrm{m})^2}{4 \times 10^{-3}\mathrm{Nsm^{-2}} \times 2\mathrm{m}} = \mathbf{1.41ms^{-1}}$$

Example 49

(a) A student claims that a soap bubble with twice the radius has twice the excess pressure. Identify the error in this claim and provide the correct relationship.

(b) Two soap bubbles of radii 2cm and 4cm coalesce under isothermal conditions to form a new bubble. Find the radius of the new bubble and the change in surface energy. Take $\gamma = 0.025\mathrm{Nm^{-1}}$.

Solution

(a) The student's claim is incorrect. The excess pressure inside a soap bubble is $\Delta P = 4\gamma/r$, which is **inversely** proportional to the radius. A bubble with twice the radius has **half** the excess pressure, not twice. The student has confused direct and inverse proportionality.

(b) Under isothermal conditions, surface energy is conserved:

$$2 \times 4\pi r_1^2\gamma + 2 \times 4\pi r_2^2\gamma = 2 \times 4\pi R^2\gamma$$

$$R^2 = r_1^2 + r_2^2 = (0.02\mathrm{m})^2 + (0.04\mathrm{m})^2 = 4 \times 10^{-4}\mathrm{m^2} + 16 \times 10^{-4}\mathrm{m^2} = 20 \times 10^{-4}\mathrm{m^2}$$

$$\mathbf{R} = \sqrt{20 \times 10^{-4}\mathrm{m^2}} = \mathbf{4.47 \times 10^{-2}m = 4.47cm}$$

Change in surface area (two surfaces):

$$\Delta A = 8\pi(R^2 - r_1^2 - r_2^2) = 8\pi(20 - 4 - 16) \times 10^{-4}\mathrm{m^2} = 0$$

Since $\Delta A = 0$, the change in surface energy is **zero** under isothermal conditions, (which is consistent with the energy conservation assumption used in the derivation).

Example 50

(a) Explain what is meant by "streamline flow" and why streamlines can never cross each other.

(b) A venturi meter is designed to measure the flow rate of water in a pipe of internal diameter 10cm. The available manometer uses mercury ($\rho = 13600\mathrm{kgm^{-3}}$). If the throat diameter is chosen as 4cm and the mercury level difference reads 5cm, calculate the volume flow rate. Take $\rho = 1000\mathrm{kgm^{-3}}$ and $g = 9.8\mathrm{ms^{-2}}$.

Solution

(a) Streamline flow (laminar flow) is a type of fluid flow in which every particle passing through a given point follows exactly the same path as every previous particle that passed through that point. The velocity at any fixed point remains constant with time.

Streamlines cannot cross because at a crossing point, a fluid particle would have two velocities simultaneously (one along each streamline). This is physically impossible. Therefore, the non-crossing of streamlines is a requirement of steady flow.

(b) Given: $A = \pi(0.05)^2 = 7.854 \times 10^{-3}\text{m}^2$, $a = \pi(0.02)^2 = 1.257 \times 10^{-3}\text{m}^2$, $h = 0.05\text{m}$

$$A^2 - a^2 = (7.854 \times 10^{-3}\text{m}^2)^2 - (1.257 \times 10^{-3}\text{m}^2)^2 = 6.011 \times 10^{-5}\text{m}^4$$

$$v_1 = a\sqrt{\frac{2\rho' gh}{\rho(A^2 - a^2)}} = 1.257 \times 10^{-3}\text{m}^2 \times \sqrt{\frac{2 \times 13600\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 0.05\text{m}}{1000\text{kgm}^{-3} \times 6.011 \times 10^{-5}\text{m}^4}}$$

$$v_1 = 0.592\text{ms}^{-1}$$

$$Q = Av_1 = 7.854 \times 10^{-3}\text{m}^2 \times 0.592\text{ms}^{-1} = 4.65 \times 10^{-3}\text{m}^3\text{s}^{-1}$$

Therefore, the volume flow rate is $4.65 \times 10^{-3}\text{m}^3\text{s}^{-1}$.

Example 51

(a) A student uses the falling sphere method to measure the viscosity of glycerine. She uses a large steel ball (radius 5mm) and gets a very different answer from her classmate who uses a small ball (radius 0.5mm). Explain which student is more likely to get an accurate result, and why the other student's result is unreliable.

(b) Two spheres, X and Y, are made of materials with densities 8000kgm^{-3} and 3000kgm^{-3} respectively. Sphere X has radius r and sphere Y has radius 2r. Both are dropped into the same viscous liquid of density 1000kgm^{-3}. Find the ratio of their terminal velocities.

Solution

(a) The student using the small ball (0.5mm) is more likely to get an accurate result. Stokes' law ($F = 6\pi\eta rv$), on which the falling sphere method is based, is valid only when the flow around the sphere is laminar, which requires the Reynolds number to be less than 1.

A large ball falls faster (terminal velocity is proportional to r^2) and has a larger characteristic length. Both factors increase the Reynolds number. Consequently, the flow around the large ball may become turbulent, in which case Stokes' law underestimates the actual drag, the measured terminal velocity is lower than Stokes' law predicts, and the calculated viscosity is incorrect.

The small ball, with its lower speed and smaller radius, maintains a low Reynolds number and keeps the flow laminar. Therefore, its result is more reliable.

(b) Terminal velocity: $v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$

For sphere X: $v_X = \frac{2r^2(8000-1000)g}{9\eta} = \frac{2r^2 \times 7000g}{9\eta}$

For sphere Y: $v_Y = \frac{2(2r)^2(3000-1000)g}{9\eta} = \frac{2 \times 4r^2 \times 2000g}{9\eta} = \frac{8r^2 \times 2000g}{9\eta}$

$$\frac{v_X}{v_Y} = \frac{2 \times 7000}{8 \times 2000} = \frac{7}{8}$$

Therefore, $v_X : v_Y = 7:8$.

Example 52

(a) Explain, using Bernoulli's principle, why it is dangerous to stand close to the edge of a railway platform when a high-speed train passes through.

(b) Water is supplied to a building through a horizontal pipe of radius R and length L at a certain flow rate Q. The water company needs to double the flow rate but can only increase the pressure by 50%. By what factor must the pipe radius be increased? State the engineering implication of your answer.

Solution

(a) When a high-speed train passes, the air between the person and the train is dragged along and moves at high speed. By Bernoulli's equation, fast-moving air has lower pressure than the still air behind the person.

This pressure difference creates a net force pushing the person toward the train. The faster the train, the greater the speed difference, the larger the pressure drop, and the stronger the force.

(b) From Poiseuille's formula: $Q = \frac{\pi \Delta P R^4}{8\eta L}$

Since η and L are unchanged, $Q \propto \Delta P \times R^4$.

Let the new radius be R' and the new pressure be $1.5\Delta P$:

$$\frac{Q'}{Q} = \frac{1.5\Delta P \times R'^4}{\Delta P \times R^4}$$

$$2 = 1.5 \times \left(\frac{R'}{R}\right)^4$$

$$\left(\frac{R'}{R}\right)^4 = \frac{2}{1.5} = \frac{4}{3}$$

$$\frac{R'}{R} = \left(\frac{4}{3}\right)^{1/4} = 1.075$$

The radius must increase by only about **7.5%**. This illustrates the extraordinary sensitivity of the R^4 law: a tiny increase in pipe radius produces a huge increase in flow capacity. So the engineering implication is that replacing old pipes with slightly wider ones is far more cost-effective than installing more powerful pumps to increase pressure.

Example 53

(a) A student fills a capillary tube with water and holds it horizontally. The water does not flow out. Explain why, using the concept of surface tension.

(b) A capillary tube of internal radius 0.3mm is dipped vertically into a liquid of surface tension 0.05Nm^{-1}, density 800kgm^{-3}, and angle of contact 30°. Calculate the height of the liquid in the tube. If the tube is only 15cm long and open at both ends, describe what happens. Take $g = 9.8\text{ms}^{-2}$.

Solution

(a) At each open end of the horizontal capillary tube, the water surface forms a meniscus. Surface tension acts along the circumference of this meniscus, creating a pressure difference (excess pressure) across the curved surface. This excess pressure is sufficient to support the weight of the small column of water inside the narrow tube against gravity. The narrower the tube, the stronger this effect. Consequently, the water remains held in place.

(b) Using the capillary rise formula:

$$h = \frac{2\gamma\cos\theta}{\rho g r} = \frac{2 \times 0.05\text{Nm}^{-1} \times \cos 30°}{800\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 3 \times 10^{-4}\text{m}} = 0.0368\text{m}$$

$$\mathbf{h = 3.68cm}$$

Since the tube is 15cm long (much longer than 3.68cm), the liquid rises to 3.68cm and stops. The meniscus forms normally at this height.

Understand! If the tube were shorter than 3.68cm, the liquid would rise to the top of the tube but would **not overflow**. Instead, the meniscus at the top would adjust its radius of curvature (becoming flatter) until the excess pressure across the meniscus is just sufficient to support the liquid column equal to the tube length. The angle of contact would increase from 30° to a value that satisfies $\cos\theta' = \frac{\rho g r L_{\text{tube}}}{2\gamma}$.

Example 54

(a) Explain, using Bernoulli's principle, how an atomiser works.

(b) An aeroplane of mass 5000kg has wings of total area 40m^2. During takeoff, the speed of air over the upper surface is 80ms^{-1}. Find the minimum speed of air under the lower surface for the plane to become airborne. Take $\rho_{\text{air}} = 1.2\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Solution

(a) When the rubber bulb is squeezed, air rushes through the horizontal tube at high speed. At the junction with the vertical tube, the fast-moving air creates a region of low pressure (by Bernoulli's principle: higher speed means lower pressure). The atmospheric pressure on the liquid surface in the container is now greater than the pressure at the junction. This pressure difference pushes the liquid up the vertical tube. When it reaches the junction, the fast airstream breaks it into fine droplets, producing a spray.

(b) For the plane to become airborne, the lift force must equal the weight:

$$F = Mg$$

$$\frac{1}{2}\rho A(v_1^2 - v_2^2) = Mg$$

$$v_1^2 - v_2^2 = \frac{2Mg}{\rho A} = \frac{2 \times 5000\text{kg} \times 9.8\text{ms}^{-2}}{1.2\text{kgm}^{-3} \times 40\text{m}^2} = 2042\text{m}^2\text{s}^{-2}$$

$$v_2 = \sqrt{v_1^2 - 2042\text{m}^2\text{s}^{-2}} = \sqrt{(80\text{ms}^{-1})^2 - 2042\text{m}^2\text{s}^{-2}} = 66\text{ms}^{-1}$$

Therefore, the minimum speed of air under the wing is approximately 66ms^{-1}.

Example 55

(a) A laboratory technician measures the viscosity of an oil using the falling sphere method and obtains $\eta = 0.8\text{Nsm}^{-2}$. However, she notices that the glass sphere she used was not perfectly spherical and that the oil temperature rose by 3°C during the experiment. Explain how each of these factors would affect her result, and whether the true viscosity is likely to be higher or lower than her measured value.

(b) Glycerine ($\eta = 1.5\text{Nsm}^{-2}$, $\rho = 1260\text{kgm}^{-3}$) flows through a horizontal pipe of radius 5mm and length 0.5m. If the pressure difference between the ends is 2×10^4Pa. Calculate the volume flow rate, the average velocity, and the Reynolds number. Comment on the type of flow.

Solution

(a) **Non-spherical shape:** Stokes' law assumes a perfect sphere. A non-spherical object experiences greater drag than a sphere of the same volume because its irregular shape disrupts the flow more. Greater drag means a lower terminal velocity. Since $\eta = \frac{2r^2(\rho_s - \rho_f)g}{9v_t}$, a lower v_t gives a higher calculated viscosity. Therefore, the non-spherical shape causes the measured viscosity to be **too high**.

Temperature rise: The viscosity of a liquid decreases with increasing temperature. If the oil warmed by 3°C during the experiment, the viscosity at the end was lower than at the start. The sphere fell through oil of varying viscosity, and the measured terminal velocity reflects an average viscosity that is lower than the viscosity at the initial temperature. Therefore, the measured value of 0.8Nsm^{-2} is likely **lower** than the true viscosity at the starting temperature.

The two errors act in opposite directions: the shape error makes the result too high, while the temperature error makes it too low. The net effect depends on which error dominates.

(b) By Poiseuille's formula:

$$Q = \frac{\pi \Delta P R^4}{8\eta L} = \frac{\pi \times 2 \times 10^4\text{Pa} \times (5 \times 10^{-3}\text{m})^4}{8 \times 1.5\text{Nsm}^{-2} \times 0.5\text{m}}$$

$$\mathbf{Q = 6.54 \times 10^{-6}m^3s^{-1}}$$

The average velocity:

$$\mathbf{v_{avg}} = \frac{Q}{\pi R^2} = \frac{6.54 \times 10^{-6}\text{m}^3\text{s}^{-1}}{\pi \times (5 \times 10^{-3}\text{m})^2} = \mathbf{0.0833ms^{-1}}$$

Reynolds number:

$$\mathbf{R_e} = \frac{\rho_f v_{avg} \times 2R}{\eta} = \frac{1260\text{kgm}^{-3} \times 0.0833\text{ms}^{-1} \times 0.01\text{m}}{1.5\text{Nsm}^{-2}} = \mathbf{0.70}$$

Since $R_e = 0.70$, which is far below 2000, the flow is **laminar**.

Example 56

(a) Explain why the viscosity of liquids decreases with increasing temperature while the viscosity of gases increases with increasing temperature.

(b) A water tank has two identical small holes, one at depth h below the surface and another at depth 4h. Find the ratio of: (i) the speeds of efflux, and (ii) the volume flow rates from the two holes.

Solution

(a) In liquids, viscosity arises from intermolecular attractive forces that resist the sliding of adjacent layers. As temperature increases, molecules gain kinetic energy and overcome these forces more easily. Consequently, the resistance between layers decreases and viscosity drops.

In gases, viscosity arises from the transfer of momentum between layers by molecules moving randomly back and forth. As temperature increases, molecules move faster and transfer more momentum between layers. As a result, the friction between layers increases and viscosity rises.

The two mechanisms are fundamentally different: intermolecular forces in liquids versus momentum transfer in gases.

(b) By Torricelli's theorem, $v = \sqrt{2gh'}$ where h' is the depth below the surface.

(i) $\frac{v_1}{v_2} = \frac{\sqrt{2gh}}{\sqrt{2g(4h)}} = \frac{\sqrt{h}}{\sqrt{4h}} = \frac{1}{2}$

(ii) Both holes are identical (same area a), so $Q = av$:

$$\frac{Q_1}{Q_2} = \frac{v_1}{v_2} = \frac{1}{2}$$

The deeper hole has twice the efflux speed and twice the volume flow rate.

Example 57

(a) A rectangular wire frame with a sliding wire is dipped in a soap solution, forming a thin film. When the sliding wire is released, it moves inward (toward the frame) on its own. Explain why the wire moves, and show mathematically that the force per unit length pulling the wire equals the work done per unit area in stretching the film.

(b) A soap film is formed on a rectangular wire frame with a sliding wire of length 5cm. The surface tension of the soap solution is 0.03Nm^{-1}. The sliding wire is pulled outward through 2cm. Calculate the work done and the force required to hold the wire in the new position.

Solution

(a) The soap film has two surfaces, and surface tension acts along each surface, pulling every part of the film toward minimum area. The sliding wire is free to move, so the surface tension forces on it are unopposed. These forces pull the wire inward, reducing the film area and hence the surface energy. The wire moves because the system spontaneously reduces its surface energy.

To show the equivalence: the surface tension force on the wire of length l (two surfaces) is $F = 2\gamma l$. If the wire moves a distance dx, the work done is $dW = F\,dx = 2\gamma l\,dx$. The increase in surface area (two surfaces) is $dA = 2l\,dx$. Therefore, the work done per unit area is:

$$\frac{dW}{dA} = \frac{2\gamma l\,dx}{2l\,dx} = \gamma$$

This equals the surface tension (force per unit length), confirming that surface tension and surface energy are numerically equal.

(b) The soap film has two surfaces. The increase in total area:

$$\Delta A = 2 \times l \times x = 2 \times 0.05\text{m} \times 0.02\text{m} = 2.0 \times 10^{-3}\text{m}^2$$

Work done:

$$W = \gamma \times \Delta A = 0.03\text{Nm}^{-1} \times 2.0 \times 10^{-3}\text{m}^2 = 6.0 \times 10^{-5}\text{J}$$

Force required (two surfaces):

$$F = 2\gamma l = 2 \times 0.03\text{Nm}^{-1} \times 0.05\text{m} = 3.0 \times 10^{-3}\text{N}$$

Example 58

Kipanga has been thinking. He announces to **Mr. Akilikubwa** that he has calculated the perfect speed at which a person could swim through honey without getting tired, because the viscous drag at low speed is small.

Mr. Akilikubwa raises an eyebrow. *"Tell me your calculation."*

Kipanga says: *"Honey has viscosity* $\eta \approx 5Nsm^{-2}$ *and density* $\rho_f = 1400kgm^{-3}$*. A person of mass* $70kg$*, volume* $0.07m^3$*, and effective radius* $0.15m$ *can be modelled as a sphere. By Stokes' law, the drag at* $v = 0.1ms^{-1}$ *would be about* $F = 6\pi\eta rv = 6\pi \times 5 \times 0.15 \times 0.1 = 1.41N$*. This is a tiny force. So swimming slowly through honey should be easy."*

Mr. Akilikubwa smiles. *"You have made at least three errors. Find them."*

Identify Kipanga's three errors and, for one of them, calculate the correct value to show how it changes the conclusion.

Solution

Error 1: A human body is not a sphere. Stokes' law applies to smooth spheres. A human body has arms, legs, and an irregular shape that creates far more drag than a sphere of the same volume. The drag on a real human body is many times larger than Stokes' formula predicts.

Error 2: The buoyancy changes everything. The density of the human body ($\rho_s = m/V = 70/0.07 = 1000\text{kgm}^{-3}$) is less than the density of honey (1400kgm^{-3}). Consequently, the person would float in honey, not sink. There is no need to swim; the upthrust exceeds the weight. The net upward force is:

$$F_{net} = V(\rho_f - \rho_s)g = 0.07\text{m}^3 \times (1400 - 1000)\text{kgm}^{-3} \times 9.8\text{ms}^{-2} = 274\text{N}$$

This is a substantial upward force, much larger than the person's weight. So the person would bob on the surface of the honey.

Hence, the buoyancy error changes the entire conclusion.

Error 3: Stokes' law requires laminar flow with $R_e < 1$. The Reynolds number for a person moving at 0.1ms^{-1} through honey is:

$$R_e = \frac{\rho_f v(2r)}{\eta} = \frac{1400\text{kgm}^{-3} \times 0.1\text{ms}^{-1} \times 0.30\text{m}}{5\text{Nsm}^{-2}} = 8.4$$

This is well above 1, so Stokes' law does not apply. The actual drag would be much larger than Kipanga calculated.

Kipanga's plan fails on three counts: the wrong shape, the wrong density comparison, and the wrong flow regime. Physics does not reward wishful thinking.

If you have survived all 15 miscellaneous examples with your confidence intact, congratulations! You have been squeezed through constrictions like water in a venturi meter, broken into smaller drops like mercury under pressure, and dragged through viscous resistance like a steel ball in glycerine. But you emerged faster, stronger, and with lower pressure on your ego.

Now comes the Digging Deeper exercise. Fifty-five questions. No worked solutions to peek at. No Mr. Akilikubwa to rescue you. Just you, the formulas, and whatever understanding you have built over the course of this chapter. Kipanga would panic. Kipute would calmly prepare her pen and calculator, with a confident smile. Be like Kipute.

DIGGING DEEPER EXERCISE 10

EXERCISE 10A: BINDER QUESTIONS

Question 1

Using the molecular theory of surface tension, explain why a molecule at the surface of a liquid has higher potential energy than a molecule deep inside the liquid. Hence explain why liquid surfaces tend to contract to the smallest possible area.

Question 2

A student places a steel needle flat on water and it floats. She then adds a drop of liquid soap to the water and the needle sinks immediately. Explain both observations using the concept of surface tension.

Question 3

Two glass tubes of different radii are dipped vertically into the same beaker of water. The water rises to different heights in the two tubes. Explain why the water rises higher in the narrower tube, and state the relationship between the height of rise and the tube radius.

Question 4

Explain the difference between a concave meniscus and a convex meniscus. In each case, state whether adhesive or cohesive forces dominate, and give one example of a liquid-solid combination that produces each type.

Question 5

A fluid flows steadily through a pipe that narrows at one section. Explain, using the equation of continuity, why the fluid speeds up at the narrow section even though no additional force is applied.

Question 6

Using Bernoulli's equation, explain why the pressure of a fluid decreases when its speed increases in a horizontal pipe. Your explanation should refer to the energy budget of the fluid.

Question 7

Explain why a ball spinning about a vertical axis curves sideways as it moves through the air. Name this effect and identify which side of the ball has lower pressure.

Question 8

Using the concept of viscosity, explain why the velocity of a fluid in a pipe is zero at the wall and maximum at the centre. Include a sketch of the velocity profile for laminar flow in a cylindrical pipe.

Question 9

Explain why the viscosity of a liquid decreases with increasing temperature while the viscosity of a gas increases with increasing temperature. In each case, identify the molecular mechanism responsible.

Question 10

A steel ball is dropped into a tall jar of glycerine. Describe, with reference to the forces acting on the ball, how its velocity changes from the moment of release until it reaches the bottom of the jar. Sketch the velocity-time graph.

EXERCISE 10B: REAL QUESTIONS

Question 11

A pond skater (an insect) can walk on the surface of still water, but a coin placed gently on the same surface sinks. Explain why the insect is supported but the coin is not, even though both are denser than water.

Question 12

During the rainy season in Dar es Salaam, water droplets on a freshly waxed car form small, nearly spherical beads, but on an unwaxed car they spread into flat patches. Explain the difference using the concepts of angle of contact and surface tension.

Question 13

A farmer in Dodoma notices that after tilling the top layer of soil, the soil beneath stays moist for longer than untilled soil. Using the concept of capillarity, explain why breaking up the top layer helps retain moisture.

Question 14

During a storm in Mtwara, the roof of a house is blown upward even though the wind blows horizontally across the top of the roof. Explain how a horizontal wind can produce a vertical (upward) force on the roof.

Question 15

A mechanic in Dar es Salaam pours engine oil through a narrow funnel on a hot afternoon. His colleague in Moshi, working in cold morning air, pours the same brand of oil through an identical funnel but finds it flows much more slowly. Using the concepts of viscosity and Poiseuille's formula, explain why the same oil flows at different rates in the two locations.

Question 16

Explain why commercial aircraft fly faster at high altitude (where air is thinner) than at low altitude, even though the engines produce roughly the same thrust.

Question 17

A doctor listens to a patient's chest with a stethoscope and hears a faint "whooshing" sound (murmur) in one of the arteries. Using the concepts of Reynolds number and fluid flow, explain what this sound indicates about the condition of that artery.

Question 18

Kipanga notices that during a heavy downpour, large raindrops hit the ground hard while fine drizzle seems to float down gently. He claims that gravity must be pulling the large drops harder. Kipute disagrees and says the difference is about terminal velocity, not gravitational force. Evaluate both claims and explain who is correct.

Question 19

A person standing on a railway platform feels a strong pull toward a high-speed train as it passes. Explain this effect using Bernoulli's principle and suggest why safety lines are painted well back from the platform edge.

Question 20

In a hospital, an intravenous (IV) drip delivers saline solution to a patient through a thin tube. Explain why raising the IV bag higher above the patient increases the flow rate, and identify which formula(s) governs the flow through the tube.

Question 21

Explain why a soap bubble always takes a spherical shape, regardless of how it was blown. Your answer should refer to surface energy.

Question 22

A student sips a thick milkshake through a narrow straw and finds it very difficult. She switches to a wider straw and the milkshake flows much more easily. Using Poiseuille's formula, explain why the wider straw makes such a large difference.

Question 23

Oil tanker ships sometimes spill oil onto the ocean surface. Explain, using the concept of surface energy, why the oil spreads to form a thin film rather than staying as a thick blob.

Question 24

Fine dust particles from a construction site remain suspended in the air for hours, while coarse sand grains settle to the ground within seconds. Using Stokes' law and terminal velocity, explain this difference.

EXERCISE 10C: HOT QUESTIONS

Question 25

An air bubble of radius 1mm is formed at the bottom of a lake at a depth of 10m. The surface tension of water is 0.073Nm^{-1} and atmospheric pressure is $1.01 \times 10^5\text{Pa}$. Calculate:

(a) the excess pressure inside the bubble, and

(b) the total pressure inside the bubble.

Question 26

Two soap bubbles of radii 4cm and 6cm ($\gamma = 0.04\text{Nm}^{-1}$) come into contact and share a common surface. Calculate:

(a) the radius of curvature of the common surface, and

(b) the pressure difference across the common surface.

Question 27

A spherical drop of mercury of radius 3mm ($\gamma = 0.5\text{Nm}^{-1}$, $\rho = 13{,}600\text{kgm}^{-3}$, $c = 140\text{Jkg}^{-1}\text{K}^{-1}$) breaks into 27 identical smaller drops. Calculate:

(a) the radius of each small drop,

(b) the work done in the process, and

(c) the temperature rise of the mercury if all the work appears as heat.

Question 28

A capillary tube of internal radius 0.4mm is dipped vertically into water ($\gamma = 0.072\text{Nm}^{-1}$, $\rho = 1000\text{kgm}^{-3}$).

(a) Calculate the height of rise.

(b) If the capillary tube is only 3cm long, explain what happens to the water level and the meniscus shape. Take $g = 9.8\text{ms}^{-2}$.

Question 29

Mercury ($\gamma = 0.5\text{Nm}^{-1}$, $\rho = 13{,}600\text{kgm}^{-3}$, $\theta = 138°$) is poured into a glass U-tube with one arm of internal radius 2mm and the other of internal radius 6mm. Calculate the difference in mercury levels in the two arms. Take $g = 9.8\text{ms}^{-2}$.

Question 30

Water flows through a horizontal pipe that narrows from a diameter of 12cm to 6cm. The pressure in the wider section is $2.5 \times 10^5\text{Pa}$ and the flow speed there is 2ms^{-1}. Calculate:

(a) the flow speed in the narrow section,

(b) the pressure in the narrow section, and

(c) the volume flow rate. Take $\rho = 1000\text{kgm}^{-3}$.

Question 31

A large open water tank has its base 8m above the ground. The water level is 5m above the base. A small hole is opened at the base of the tank. Calculate:

(a) the speed of efflux,

(b) the time for the water jet to reach the ground, and

(c) the horizontal distance from the base of the tank where the jet strikes the ground. Take $\text{g} = 9.8\text{ms}^{-2}$.

Question 32

A venturi meter has a pipe diameter of 20cm and a throat diameter of 8cm. Water ($\rho = 1000\text{kgm}^{-3}$) flows through the pipe. A mercury manometer ($\rho = 13600\text{kgm}^{-3}$) connected between the pipe and the throat shows a level difference of 4cm. Calculate the volume flow rate in litres per second. Take $\text{g} = 9.8\text{ms}^{-2}$.

Question 33

A student claims: *"By Bernoulli's equation, a supersonic aircraft flying at* $600ms^{-1}$ *must have near-zero pressure around its wings, since the dynamic pressure* $\frac{1}{2}\rho v^2$ *at that speed is enormous."* Identify two errors in this reasoning and explain why the student's conclusion is wrong.

Question 34

Water ($\eta = 1.0 \times 10^{-3}\text{Nsm}^{-2}$) flows through a horizontal pipe of internal radius 2cm and length 5m under a pressure difference of 500Pa. Calculate:

(a) the volume flow rate, and

(b) the new flow rate if the radius is doubled while all other factors remain constant.

Question 35

Two horizontal pipes of equal length L and radii R and 2R are connected in parallel. Find the ratio of the volume flow rate through the wider pipe to that through the narrower pipe.

Question 36

A steel ball of radius 1.5mm ($\rho = 7800\text{kgm}^{-3}$) falls through oil ($\rho = 900\text{kgm}^{-3}$) and reaches terminal velocity. The ball is timed falling through 25cm in 5.2s. Calculate:

(a) the terminal velocity,

(b) the coefficient of viscosity of the oil, and

(c) the Reynolds number.

Comment on whether Stokes' law is valid. Take $\text{g} = 9.8\text{ms}^{-2}$.

Question 37

Two students each measure the excess pressure inside a bubble. Student A measures an air bubble of radius 2mm inside water ($\gamma = 0.073\text{Nm}^{-1}$). Student B measures a soap bubble of the same radius in air ($\gamma = 0.025\text{Nm}^{-1}$). Student A gets a higher value and claims her bubble has more surface tension. Evaluate this claim by calculating both excess pressures and identifying the real reason for the difference.

Question 38

A pitot tube is mounted on an aircraft flying through air of density 0.9kgm^{-3}. The pitot tube is connected to a water manometer ($\rho = 1000\text{kgm}^{-3}$) which shows a level difference of 18cm. Calculate the airspeed of the aircraft.

Question 39

A water company supplies water through a pipe of radius R at a flow rate Q. Due to population growth, the company needs to triple the flow rate. If the pressure and pipe length cannot be changed:

(a) By what factor must the radius increase?

(b) If replacing the pipe costs Tsh 50,000 per centimetre of diameter increase, and the original diameter is 10cm, estimate the cost of the upgrade.

Question 40

A soap film is formed on a rectangular frame with a sliding wire of length 8cm. The wire is in equilibrium under the action of the surface tension force and a weight hanging from it. If $\gamma = 0.03\text{Nm}^{-1}$, calculate:

(a) the force exerted by the film on the wire, and

(b) the mass of the weight required to keep the wire in equilibrium. Take $g = 9.8\text{ms}^{-2}$.

Question 41

Water flows upward through a vertical pipe that widens from radius 2cm at the bottom to 4cm at the top. The bottom and top sections are 3m apart. The flow speed at the bottom is 4ms^{-1} and the pressure there is $3 \times 10^5\text{Pa}$. Find the pressure at the top. Take $\rho = 1000\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 42

A cylindrical tank of radius 0.6m is filled with water to a height of 2m. Water drains through a small hole of area 1cm^2 at the bottom. Derive an expression for the time to empty the tank and calculate this time. Take $g = 9.8\text{ms}^{-2}$.

Question 43

An aircraft of mass 15,000kg has a total wing area of 50m^2. The air speed over the upper surface of the wings is 100ms^{-1}. Calculate the minimum speed of air under the lower surface required for the aircraft to maintain level flight. Take $\rho_{\text{air}} = 1.2\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 44

Glycerine ($\eta = 1.5\text{Nsm}^{-2}$) is poured onto a flat horizontal surface. A circular disc of radius 20cm and mass 2kg is placed on the glycerine layer, which has a thickness of 1mm. The disc is pushed horizontally at a constant velocity of 0.3ms^{-1}. Calculate the force required to maintain this velocity.

Question 45

Three identical soap bubbles, each of radius r, coalesce under isothermal conditions to form a single bubble. Show that the radius of the new bubble is $R = r\sqrt{3}$, and find the ratio of the excess pressure in the original bubbles to that in the new bubble.

Question 46

Water rises to a height of 4.5cm in a capillary tube of radius r_1. When the same tube is dipped into a different liquid of density 800kgm^{-3}, surface tension 0.052Nm^{-1}, and angle of contact 30°, the liquid rises to 6.2cm. Find:

(a) the radius r_1 of the tube, and

(b) the surface tension of water if its angle of contact with glass is 0°. Take $\rho_{\text{water}} = 1000\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 47

A sphere of radius r and density 1.5ρ is released from rest in a liquid of density ρ and viscosity η. (a) Show that the terminal velocity is $v_t = \frac{r^2\rho g}{9\eta}$.

(b) Find the ratio of the terminal velocity of this sphere to that of a sphere of radius $2r$ and density 3ρ falling through the same liquid.

Question 48

A horizontal pipe carries water at $3ms^{-1}$. At one point, the pipe develops a small leak (a hole of area $0.5cm^2$ in the top surface). Calculate the speed at which water emerges from the leak and the volume of water lost per minute. Take the pressure inside the pipe as 2×10^5Pa and atmospheric pressure as 1.01×10^5Pa. Take $\rho = 1000kgm^{-3}$.

Question 49

A viscous liquid flows through a horizontal pipe of radius R and length L. The flow rate is Q. The same liquid is then passed through a system of four identical pipes, each of radius $R/2$ and length L, connected in parallel. Compare the total flow rate through the parallel system with Q, assuming the same pressure difference drives the flow in both cases.

Question 50

A flat circular disc of radius $10cm$ is placed on a thin film of oil ($\eta = 0.5Nsm^{-2}$, film thickness $0.8mm$) on a flat surface. The disc is rotated at 90 revolutions per minute. Calculate the torque required to maintain the rotation.

Question 51

A ball of mass $0.2kg$ is projected horizontally at $30ms^{-1}$ from a height of $25m$. Ignoring air resistance, calculate where it lands. Now explain qualitatively how viscous drag from the air would change:

(i) the horizontal distance, and

(ii) the time of flight, compared to the no-drag case.

Question 52

A patient's artery has an internal radius of $2.5mm$. Due to plaque buildup, the radius at one section is reduced to $1.5mm$. Blood ($\rho = 1060kgm^{-3}$) flows at $0.3ms^{-1}$ in the healthy section where the pressure is 1.2×10^4Pa. Calculate:

(a) the blood speed in the narrowed section,

(b) the pressure in the narrowed section, and

(c) the percentage reduction in volume flow rate through a long section of the narrowed artery compared to the healthy artery. Comment on the medical significance of your answers.

ANSWERS

EXERCISE 10A

1. A molecule deep inside a liquid is surrounded by neighbouring molecules on all sides. The attractive forces from these neighbours act equally in all directions, so the net force on the interior molecule is zero and its potential energy is at a minimum.

A molecule at the surface has neighbours only below and to the sides. There are very few molecules above (only the sparse molecules of the vapour or air). Consequently, the surface molecule experiences a net inward pull, directed into the bulk of the liquid. To bring a molecule from the interior to the surface, work must be done against this inward pull. Therefore, surface molecules have higher potential energy than interior molecules.

Since the surface has higher energy, the liquid surface tends to contract to the smallest possible area, because this minimises the total surface energy.

2. Needle floats: When the needle is placed flat on the water surface, its weight is distributed along the length of the needle. The water surface behaves like a stretched elastic membrane due to surface tension. The vertical component of the surface tension force acting along the contact line around the needle supports the needle's weight. The surface is depressed slightly but not broken.

Needle sinks after soap is added: Liquid soap is a surfactant that dramatically reduces the surface tension of water (from about 0.073Nm^{-1} to about 0.025Nm^{-1}). With reduced surface tension, the upward surface tension force along the contact line is no longer sufficient to support the weight of the steel needle. Consequently, the surface breaks and the needle sinks.

3. By the capillary rise formula:

$$h = \frac{2\gamma\cos\theta}{\rho g r}$$

For the same liquid in the same type of tube, γ, θ, ρ, and g are all constant. Therefore:

$$h \propto \frac{1}{r}$$

The height of rise is inversely proportional to the radius of the tube. A narrower tube (smaller r) gives a larger height h, and a wider tube gives a smaller height. This is because in a narrower tube, the same surface tension force acts along a smaller circumference but supports a taller, thinner column of liquid.

4. Concave meniscus: The liquid surface curves upward near the wall of the container. This occurs when the adhesive forces (between the liquid and the solid) are stronger than the cohesive forces (between the liquid molecules). The liquid is attracted to the wall and climbs upward along it. The angle of contact is less than $90°$. Example: water in a glass tube.

Convex meniscus: The liquid surface curves downward near the wall. This occurs when the cohesive forces are stronger than the adhesive forces. The liquid molecules are pulled away from the wall and toward each other. The angle of contact is greater than $90°$. Example: mercury in a glass tube.

5. By the equation of continuity for an incompressible fluid:

$$A_1 v_1 = A_2 v_2$$

where A is the cross-sectional area and v is the fluid velocity. This equation is a statement of conservation of mass: the volume of fluid entering any section per unit time must equal the volume leaving per unit time.

At the narrow section, $A_2 < A_1$. For the product Av to remain constant, v_2 must be greater than v_1. The fluid speeds up because the same volume of fluid must pass through a smaller area in the same time. No additional force is needed; the speed increase is an automatic consequence of mass conservation.

6. Bernoulli's equation for horizontal flow is:

$$P + \frac{1}{2}\rho v^2 = \text{constant}$$

The term P represents the pressure energy per unit volume, and $\frac{1}{2}\rho v^2$ represents the kinetic energy per unit volume. The sum of these two is constant along a streamline.

This can be understood as an energy budget: the fluid has a fixed total energy per unit volume. If the fluid speeds up (kinetic energy increases), the pressure energy must decrease to keep the total constant. Conversely, if the fluid slows down, the pressure increases.

Therefore, in a horizontal pipe, where the fluid moves faster (such as at a constriction), the pressure is lower, and where it moves slower (such as in a wider section), the pressure is higher.

7. When a ball spins about a vertical axis while moving forward through the air, it drags air around it in the direction of spin. On one side of the ball, the dragged air adds to the oncoming airflow, increasing the air speed on that side. On the opposite side, the dragged air opposes the oncoming flow, decreasing the air speed.

By Bernoulli's principle, the side with faster air has lower pressure, and the side with slower air has higher pressure. This pressure difference creates a net sideways force that deflects the ball toward the low-pressure side.

This is called the **Magnus effect**.

8. When a viscous fluid flows through a pipe, the layer in direct contact with the pipe wall adheres to the wall and has zero velocity. This is called the **no-slip condition**, and it applies to all real fluids on all solid surfaces.

The layer immediately above the wall is slowed by viscous drag from the stationary layer. The next layer is slowed less, and so on. Each layer is retarded by the one below it and pushed forward by the one above it. Consequently, the velocity increases smoothly from zero at the wall to a maximum at the centre, where no wall is nearby to exert drag.

Sketch: side view of pipe with horizontal arrows increasing in length from the wall (zero) to the centre (maximum), with a dashed parabolic envelope connecting the arrow tips.

9. Liquids: Viscosity in liquids arises from intermolecular attractive forces that resist the sliding of adjacent layers. As temperature increases, the molecules gain kinetic energy. The increased molecular motion weakens the intermolecular forces, making it easier for layers to slide over one another. Consequently, viscosity decreases.

Gases: Viscosity in gases does not arise from intermolecular bonds (which are negligible in gases). Instead, it arises from the transfer of momentum between layers by molecules moving randomly back and forth between them. At higher temperatures, gas molecules move faster and more vigorously, transferring more momentum between layers. As a result, the friction between layers increases and viscosity rises.

The two mechanisms are fundamentally different: intermolecular forces (liquids) versus molecular momentum transfer (gases).

10. At the moment of release ($t = 0$): The ball is at rest ($v = 0$). The only forces are the weight W acting downward and the upthrust U acting upward. Since the ball is denser than glycerine, $W > U$, so there is a net downward force. The ball accelerates downward.

As the ball speeds up: The viscous drag $F = 6\pi\eta rv$ increases with velocity. The net downward force becomes $W - U - F$, which decreases as v increases. Therefore, the acceleration decreases.

At terminal velocity: The drag has grown until $F = W - U$. The net force is zero and the acceleration is zero. The ball falls at a constant terminal velocity v_t.

For the remainder of the fall: The ball continues at constant velocity v_t until it reaches the bottom.

[**Sketch:** velocity-time graph. Curve starts at origin ($v = 0$ at $t = 0$), rises steeply at first, then the gradient decreases, and the curve levels off asymptotically at a horizontal dashed line labelled v_t]

EXERCISE 10B

11. The pond skater's weight is very small (typically less than $0.01N$), and it is distributed across six legs, each of which presses on the water surface along a contact line. The surface tension force acts upward along the contact perimeter around each leg. Because the insect is light and the total contact perimeter is large relative to its weight, the upward surface tension force is sufficient to support it. The water surface bends slightly but does not break.

A coin has a much greater weight concentrated over a small area. The contact perimeter is short relative to the coin's weight. Consequently, the surface tension force along this perimeter is far too small to support the weight. The surface breaks immediately and the coin sinks.

The key factor is not density alone but the ratio of weight to contact perimeter. The pond skater has a favourable ratio; the coin does not.

12. On a freshly waxed car, the waxy coating is hydrophobic. The cohesive forces between water molecules are much stronger than the adhesive forces between water and wax. Consequently, the angle of contact is greater than $90°$, and the water molecules are pulled together into compact, nearly spherical beads that sit on the surface.

On an unwaxed car, the paint surface is more hydrophilic. The adhesive forces between water and the paint are comparable to or stronger than the cohesive forces within the water. The angle of contact is small (well below $90°$), so the water spreads out and wets the surface, forming flat patches.

The difference is entirely determined by the angle of contact, which depends on the relative strengths of adhesive and cohesive forces at the liquid-solid interface.

13. In untilled soil, the tiny spaces between soil particles act as capillary tubes. Water from the moist subsoil rises upward through these narrow channels by capillary action and reaches the surface, where it evaporates into the air. This continuous capillary rise followed by evaporation steadily dries the soil.

When the farmer tills the top layer, the soil structure is broken up into larger, irregular clumps with much wider air gaps. Capillary rise is inversely proportional to the gap width ($h \propto 1/r$). The wider gaps in the tilled layer have very weak capillary rise, so water from below cannot reach the surface easily. Consequently, the capillary connection between the moist subsoil and the dry air is broken, and the moisture is retained in the soil below the tilled layer.

14. When a strong wind blows horizontally across the top of a roof, the air above the roof moves at high speed. The air inside the house (below the roof) is stationary or nearly so.

By Bernoulli's equation, the fast-moving air above the roof has lower pressure than the still air below the roof. This pressure difference creates a net upward force on the roof:

$$F = (P_{below} - P_{above}) \times A_{roof} = \frac{1}{2}\rho v^2 \times A_{roof}$$

If this upward force exceeds the weight of the roof and the strength of its fastenings, the roof is lifted off.

15. Dar es Salaam is significantly hotter than Moshi in the early morning. The viscosity of a liquid decreases with increasing temperature because the higher kinetic energy of molecules weakens the intermolecular forces that resist the sliding of adjacent layers. Consequently, the oil in Dar es Salaam has a lower viscosity than the same oil in cold Moshi.

By Poiseuille's formula, $Q = \frac{\pi \Delta P R^4}{8\eta L}$, the volume flow rate is inversely proportional to the viscosity η. Since the funnel dimensions, the oil level (which determines ΔP), and the geometry are identical in both cases, the lower viscosity in hot Dar es Salaam produces a higher flow rate. Hence, the oil flows faster in Dar es Salaam and slower in Moshi.

16. For an aircraft to maintain level flight, the lift force must equal the weight:

$$\frac{1}{2}\rho A(v_1^2 - v_2^2) = Mg$$

At high altitude, the air density ρ is lower than at sea level. For the lift force to remain equal to the weight (which has not changed), the speed difference $(v_1^2 - v_2^2)$ must increase to compensate for the lower density. This requires higher airspeed.

Therefore, aircraft fly faster at high altitude because the thinner air generates less lift per unit speed. The aircraft must fly faster to produce the same lift force needed to support its weight.

17. In a healthy artery, blood flows in a laminar (smooth, orderly) pattern. So the Reynolds number is typically below the turbulent threshold of 2000-3000.

When an artery is partially blocked by plaque (atherosclerosis), the internal radius at the blockage is reduced. By the equation of continuity, the blood speed increases at the narrowed section. This increased speed raises the Reynolds number. If the Reynolds number exceeds approximately 2000-3000, the flow transitions from laminar to turbulent.

Turbulent flow produces chaotic, fluctuating pressure variations that create audible vibrations in the artery wall. These vibrations are the "murmur" the doctor hears through the stethoscope. Therefore, the murmur indicates that the artery is narrowed enough to cause turbulent blood flow, which is a sign of significant atherosclerosis.

18. Kipanga's claim (gravity pulls large drops harder) is partially correct but incomplete. The gravitational force on a drop is $W = \frac{4}{3}\pi r^3 \rho g$, which is indeed larger for a larger drop. However, this alone does not explain why large drops hit the ground harder, because larger drops also experience greater drag.

Kipute's claim (the difference is about terminal velocity) is the correct and complete explanation. By the terminal velocity formula:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

The terminal velocity is proportional to r^2. A drop with twice the radius has four times the terminal velocity. This is because the weight grows as r^3 while the Stokes drag grows only as r (since $F = 6\pi\eta r v$). For larger drops, the weight increases faster than the drag, so a higher velocity is needed before the drag catches up. Hence, large drops reach a much higher terminal velocity and hit the ground harder.

Kipute is correct. The key physics is the r^2 dependence of terminal velocity, not simply the gravitational force.

19. When a high-speed train passes, the air between the person and the train is dragged along by the train and moves at high speed. By Bernoulli's equation, this fast-moving air has lower pressure than the still air on the far side of the person (away from the track).

The pressure difference across the person creates a net force directed toward the train:

$$F = (P_{\text{still}} - P_{\text{moving}}) \times A_{\text{body}} \approx \frac{1}{2}\rho v^2 \times A_{\text{body}}$$

The faster the train, the larger the pressure difference and the stronger the force. At very high speeds (such as modern express trains), this force can be large enough to pull a person off balance and into the path of the train.

Safety lines are painted well back from the platform edge so that passengers stand far enough from the train that the high-speed air has slowed significantly before reaching them, reducing the pressure difference and the inward force to a safe level.

20. The saline solution flows from the IV bag through the thin tube into the patient's vein. The flow is driven by the pressure difference between the liquid surface in the bag and the entry point in the vein. This pressure difference is the hydrostatic pressure:

$$\Delta P = \rho g h$$

where h is the vertical height of the liquid surface above the entry point. Raising the bag increases h, which increases ΔP.

The flow through the thin tube is governed by Poiseuille's formula:

$$Q = \frac{\pi \Delta P R^4}{8\eta L}$$

Since Q is directly proportional to ΔP, a larger pressure difference produces a larger flow rate. Therefore, raising the IV bag increases the flow rate of saline into the patient.

21. A soap bubble has two surfaces (inner and outer), each with surface energy γ per unit area. The total surface energy of the bubble is proportional to its total surface area.

Among all closed shapes enclosing a given volume of air, a sphere has the smallest surface area. Since surface tension drives the film toward the configuration of minimum surface energy, the bubble adopts the shape that minimises its total surface area for the volume of air trapped inside. That shape is a sphere.

Therefore, regardless of how the bubble was blown (through a circular ring, an irregular frame, or by hand), it always relaxes into a spherical shape because this is the state of lowest surface energy.

22. By Poiseuille's formula, the flow rate through a cylindrical tube is:

$$Q = \frac{\pi \Delta P R^4}{8\eta L}$$

The flow rate depends on the **fourth power** of the radius. If the student switches from a narrow straw of radius R to a wider straw of radius 2R (with the same length and the same sucking pressure), the flow rate increases by a factor of:

$$\frac{Q_2}{Q_1} = \left(\frac{2R}{R}\right)^4 = 16$$

The wider straw gives 16 times the flow rate. This dramatic increase comes entirely from the R^4 dependence. Doubling the radius does not merely double the flow; it increases it sixteenfold. This is why even a small increase in straw width makes a thick milkshake much easier to drink.

23. The ocean surface is a water-air interface with high surface tension ($\gamma_{\text{water}} \approx 0.073\text{Nm}^{-1}$). When oil is placed on this surface, three interfaces are involved: oil-air, oil-water, and water-air.

The surface energy of the oil-air interface ($\gamma_{\text{oil}} \approx 0.03\text{Nm}^{-1}$) plus the oil-water interface is less than the surface energy of the original water-air interface. Consequently, replacing the high-energy water-air surface with the lower-energy combination of oil-air and oil-water surfaces reduces the total surface energy of the system.

Since physical systems tend to minimise their energy, the oil spreads spontaneously to cover as much of the water surface as possible. The spreading continues until the oil forms a very thin film, at which point the energy reduction from further spreading becomes negligible.

24. By Stokes' law, a particle falling through air reaches a terminal velocity:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

The terminal velocity is proportional to r^2. Fine dust particles have very small radii (of the order of micrometres), so their terminal velocity is extremely small. At such low speeds, even the slightest air current keeps the particles suspended. They take hours to settle.

Coarse sand grains have larger radii. Since $v_t \propto r^2$, the terminal velocity of sand is many times larger than that of fine dust. Consequently, sand grains fall rapidly and settle within seconds.

So the r^2 dependence of terminal velocity is the reason why particle size determines settling time so dramatically.

EXERCISE 10C

25. (a) 146Pa (b) $1.99 \times 10^5\text{Pa}$

Hint for 25(b): The total pressure inside the bubble is the sum of atmospheric pressure, hydrostatic pressure at depth $h = 10\text{m}$, and excess pressure:

$$P_{\text{total}} = P_{\text{atm}} + \rho g h + \Delta P$$

26. (a) 12cm (b) 1.33Pa

27. (a) 1mm (b) $1.13 \times 10^{-4}\text{J}$ (c) $5.24 \times 10^{-4}\text{K}$

28. (a) 3.67cm (b) Since the tube is only 3cm long (shorter than 3.67cm), the water rises to the top of the tube but does **not overflow**. Instead, the meniscus at the top adjusts its curvature (becomes flatter, increasing the effective radius of curvature) until the excess pressure across the meniscus is just sufficient to support a liquid column equal to the tube length (3cm). The angle of contact increases from $0°$ to a value satisfying:

$$\cos\theta' = \frac{\rho g h_{\text{tube}}}{2\gamma} = \frac{1000 \times 9.8 \times 4 \times 10^{-4} \times 0.03}{2 \times 0.072} = 0.8167$$

$$\theta' = \cos^{-1}(0.8167) \approx 35.2°$$

29. $1.86 \times 10^{-3}\text{m} \approx 1.9\text{mm}$

30. (a) 8ms^{-1} (b) $2.20 \times 10^{5}\text{Pa}$ (c) $2.26 \times 10^{-2}\text{m}^{3}\text{s}^{-1}$

31. (a) 9.90ms^{-1} (b) 1.278s (c) 12.65m

32. 16.0 litres per second

33. Error 1: Bernoulli's equation does not apply at supersonic speeds. Bernoulli's equation is valid only for incompressible flow. At supersonic speeds ($v > 340\text{ms}^{-1}$ in air), the air is compressed significantly by shock waves. The density is no longer constant, and Bernoulli's equation breaks down completely. The actual pressure distribution around a supersonic wing must be calculated using compressible flow theory, not Bernoulli.

Error 2: The student has confused dynamic pressure with total pressure. Even within the valid range of Bernoulli's equation, the pressure around a wing does not drop to zero. The dynamic pressure $\frac{1}{2}\rho v^2$ is the kinetic energy contribution, not the actual static pressure at the wing. The static pressure can decrease but it cannot physically reach zero or become negative in a real fluid. The total pressure (static + dynamic) remains constant along a streamline; the static pressure decreases only by the amount that the dynamic pressure increases.

Therefore, the student's conclusion that the wings have near-zero pressure is wrong on both counts: the equation does not apply, and even if it did, the reasoning misinterprets what the terms mean.

34. (a) $6.28 \times 10^{-3}\text{m}^{3}\text{s}^{-1}$ (b) $0.1\text{m}^{3}\text{s}^{-1}$

35. 16

36. (a) 0.0481ms^{-1} (b) 0.703Nsm^{-2} (c) 0.185

Since $R_e = 0.185 < 1$, the flow around the sphere is laminar and Stokes' law is valid.

37. Student A (air bubble in water, one surface):

$$\Delta P_A = \frac{2\gamma}{r} = \frac{2 \times 0.073\text{Nm}^{-1}}{2 \times 10^{-3}\text{m}} = 73\text{Pa}$$

Student B (soap bubble in air, two surfaces):

$$\Delta P_B = \frac{4\gamma}{r} = \frac{4 \times 0.025\text{Nm}^{-1}}{2 \times 10^{-3}\text{m}} = 50\text{Pa}$$

Student A gets 73Pa and Student B gets 50Pa. Student A's claim that her bubble has "more surface tension" is **incorrect**. In fact, $\gamma_{\text{water}} = 0.073\text{Nm}^{-1} > \gamma_{\text{soap}} = 0.025\text{Nm}^{-1}$, so water does have higher surface tension. However, the real reason Student A gets a higher excess pressure is a combination of the higher surface tension of water **and** the fact that the air bubble has the formula $\frac{2\gamma}{r}$ while the soap bubble has $\frac{4\gamma}{r}$. Despite the factor of 4 (which should favour the soap bubble), the much higher surface tension of water wins. The student confused the result (higher excess pressure) with a single cause (surface tension) without considering that the two types of bubble use different formulas.

38. 62.6ms^{-1}

39. (a) 1.316 (b) Tsh $158{,}000$

40. (a) $4.8 \times 10^{-3}\text{N}$ (b) 0.49g

41. $2.78 \times 10^{5}\text{Pa}$

42. $7225\text{s} \approx 2$ hours

43. 71.4ms^{-1}

44. 56.6N

45. Under isothermal conditions, total surface area is conserved (for soap bubbles):

$$3 \times 2 \times 4\pi r^2 = 2 \times 4\pi R^2 \Longrightarrow R^2 = 3r^2 \Longrightarrow R = r\sqrt{3}$$

Both methods give the same result, confirming $R = r\sqrt{3}$.

Ratio of excess pressures:

$$\frac{\Delta P_{\text{original}}}{\Delta P_{\text{new}}} = \frac{4\gamma/r}{4\gamma/R} = \frac{R}{r} = \frac{r\sqrt{3}}{r} = \sqrt{3} \approx 1.73$$

The excess pressure in the original bubbles is $\sqrt{3}$ times that in the new bubble.

46. (a) $1.853 \times 10^{-4}\text{m}$ (b) 0.0408Nm^{-1}

47. (a) Sphere density $= 1.5\rho$, fluid density $= \rho$:

$$v_t = \frac{2r^2(1.5\rho - \rho)g}{9\eta} = \frac{2r^2 \times 0.5\rho \times g}{9\eta} = \frac{r^2\rho g}{9\eta}$$

(b) Second sphere: radius $2r$, density 3ρ:

$$v_t{}' = \frac{2(2r)^2(3\rho - \rho)g}{9\eta} = \frac{2 \times 4r^2 \times 2\rho \times g}{9\eta} = \frac{16r^2\rho g}{9\eta}$$

$$\frac{v_t}{v_t{}'} = \frac{\frac{r^2\rho g}{9\eta}}{\frac{16r^2\rho g}{9\eta}} = \frac{1}{16}$$

48. 14.4ms^{-1}, $0.0432\text{m}^3 = 43.2\,\text{litres per minute}$

49. By Poiseuille's formula, each small pipe (radius $\frac{R}{2}$, length L) carries:

$$Q_{\text{small}} = \frac{\pi\Delta P(R/2)^4}{8\eta L} = \frac{\pi\Delta PR^4}{8\eta L \times 16} = \frac{Q}{16}$$

Four such pipes in parallel carry a total of:

$$Q_{\text{total}} = 4 \times \frac{Q}{16} = \frac{Q}{4}$$

The parallel system carries only **one quarter** of the flow rate of the single pipe. Replacing one pipe with four pipes of half the radius is a very poor trade, because the R^4 law amplifies the effect of the radius reduction far more than the four-fold increase in number compensates.

50. This is the rotating disc problem. Consider an annular ring at radius r, width dr. Area $dA = 2\pi r\,dr$. Velocity at radius r: $v = \omega r$. Velocity gradient: $dv/dy = \omega r/d$.

Viscous force on ring: $dF = \eta \times 2\pi r\,dr \times \omega r/d$

Torque from ring: $d\tau = r\,dF = 2\pi\eta\omega r^3\,dr/d$

Total torque:

$$\tau = \frac{2\pi\eta\omega}{d}\int_0^R r^3\,dr = \frac{2\pi\eta\omega}{d} \times \frac{R^4}{4} = \frac{\pi\eta\omega R^4}{2d}$$

Given: $R = 0.10\text{m}$, $\eta = 0.5\text{Nsm}^{-2}$, $d = 0.8 \times 10^{-3}\text{m}$, $\omega = 2\pi \times 90/60 = 3\pi\,\text{rad}\,\text{s}^{-1}$

$$\tau = \frac{\pi \times 0.5\text{Nsm}^{-2} \times 3\pi\,\text{rad}\,\text{s}^{-1} \times (0.10\text{m})^4}{2 \times 8 \times 10^{-4}\text{m}} = 0.925\text{Nm}$$

51. Without air resistance:

Time of flight: $t = \sqrt{\frac{2H}{g}} = \sqrt{2 \times \frac{25\text{m}}{9.8\text{ms}^{-2}}} = 2.26\text{s}$

Horizontal distance: $x = v_0 t = 30\text{ms}^{-1} \times 2.26\text{s} = 67.8\text{m}$

Effect of viscous drag:

(i) Horizontal distance decreases. Air resistance opposes the horizontal motion, decelerating the ball throughout its flight. The horizontal velocity decreases continuously instead of remaining constant. Consequently, the ball lands closer to the launch point.

(ii) Time of flight increases slightly. During the downward fall, air resistance acts upward (opposing the downward motion), reducing the net downward acceleration. The ball takes slightly longer to reach the ground than in the no-drag case.

52. (a) By the equation of continuity:

$$\mathrm{v_2 = v_1 \times \frac{A_1}{A_2} = v_1 \times \frac{r_1^2}{r_2^2} = 0.3ms^{-1} \times \frac{(2.5mm)^2}{(1.5mm)^2} = 0.3ms^{-1} \times \frac{6.25}{2.25} = 0.833ms^{-1}}$$

(b) By Bernoulli's equation (horizontal flow):

$$\mathrm{P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) = 1.2 \times 10^4Pa + \frac{1}{2} \times 1060kgm^{-3} \times ((0.3ms^{-1})^2 - (0.833ms^{-1})^2)}$$

$$\mathrm{P_2 = 1.17 \times 10^4Pa}$$

The pressure drops in the narrowed section, which can cause the artery wall to collapse further (a dangerous positive feedback).

(c) By Poiseuille's formula, $\mathrm{Q \propto R^4}$:

$$\frac{\mathrm{Q_{narrowed}}}{\mathrm{Q_{healthy}}} = \left(\frac{1.5\mathrm{mm}}{2.5\mathrm{mm}}\right)^4 = (0.6)^4 = 0.1296$$

Percentage reduction:

$$(1 - 0.1296) \times 100\% = 87\%$$

Medical significance: A 40% $\left(\frac{2.5-1.5}{2.5} \times 100\%\right)$ reduction in radius causes an 87% reduction in blood flow through a long narrowed segment. The blood speed nearly triples at the constriction, and the pressure drops. The reduced pressure can cause further collapse of the artery wall. The combination of severely reduced flow and potential wall collapse makes atherosclerosis life-threatening even when the visible narrowing appears moderate on imaging.