Chapter 9

ROTATION OF RIGID BODIES

INTRODUCTION

The trouble began, as it usually did, with Kipanga.

Mr. Akilikubwa had brought a collection of objects to class: a metre ruler, a bicycle wheel, a hard-boiled egg, and, for reasons that would become clear shortly, an uncooked egg. He placed them on the demonstration table and announced, *"Today, we leave the comfortable world of forces and straight-line motion. Today, we enter the world of rotation."*

Kipanga, sitting at the front as usual, immediately picked up the metre ruler and balanced it horizontally on one finger at its centre. *"Like this, sir? Rotation is easy."*

"Try spinning it," said **Mr. Akilikubwa.**

Kipanga flicked one end. The ruler spun beautifully on his finger like a tiny helicopter blade.

"Now," said **Mr. Akilikubwa**, moving Kipanga's finger to one end of the ruler, *"spin it from here."*

Kipanga tried. The ruler wobbled, resisted, and barely completed half a turn before clattering onto the desk. *"What happened? It's the same ruler! Same mass! Why is it suddenly so stubborn?"*

"Excellent question. Kipute, would you like to explain?"

Kipute, who had been quietly spinning the bicycle wheel on its axle with impressive ease, looked up. *"I think it's because the mass is distributed differently relative to the point of rotation. When you spin from the centre, most of the mass is close to the pivot. When you spin from the end, more of the mass is far from the pivot, and somehow that makes it harder to spin."*

"Somehow!" **Mr. Akilikubwa** clapped his hands in delight. *"That 'somehow' is precisely what this chapter is about. In Chapters 2 and 3, we learned that a body's resistance to changes in linear motion is described by its mass. Push a* $10kg$ *box, and it resists. Push a* $100kg$ *box with the same force, and it resists ten times more. Mass is the measure of linear inertia. But Kipanga just demonstrated something fascinating: the same mass can be easy to spin or difficult to spin, depending on how it is arranged around the axis. Rotation has its own kind of inertia, and mass alone does not capture it."*

He picked up the two eggs. *"Watch carefully."* He spun both eggs on the table. One spun smoothly and continued spinning for several seconds. The other wobbled pathetically and stopped almost immediately. *"Same size, same shape, nearly the same mass. Why does one spin well and the other refuses?"*

Kipanga pointed confidently. *"The hard-boiled one spins better because it's solid inside. The raw one has liquid that sloshes around and… resists the spinning?"*

"Very close. The liquid inside the raw egg does not rotate with the shell immediately. It lags behind, and the internal friction between the liquid and the shell dissipates energy very quickly. The hard-boiled egg, being solid throughout, rotates as one rigid body. Every particle inside it moves together, in perfect coordination."

He wrote two words on the board in large letters: **RIGID BODY**.

*"**A rigid body is an object in which the distance between every pair of particles remains constant, no matter what forces act on it. It does not bend, stretch, compress, or flow.** The hard-boiled egg is approximately rigid. The raw egg is not. This chapter deals exclusively with rigid bodies, because their rotation follows clean, predictable laws."*

He paused and looked at the class with a mischievous smile. *"Now, here is the beautiful secret that will carry you through this entire chapter. You already know the physics of rotation. You just do not know that you know it."*

The students looked puzzled.

"Every concept in linear motion, every formula, every law, has an exact twin in rotational motion. Displacement becomes angular displacement. Velocity becomes angular velocity. Acceleration becomes angular acceleration. Mass becomes moment of inertia. Force becomes torque. Momentum becomes angular momentum. Even $F = ma$ *has a twin:* $\tau = I\alpha$*. Rotation is not a new subject. It is the same subject, wearing a different costume."*

Kipanga's eyes widened. *"So if I already understand $F = ma$, I already understand rotation?"*

"In principle, yes. In practice, the costume is quite convincing and the details take some getting used to. But the structure is identical. And that is why we will build this chapter as a mirror of what you already know. At every step, we will place the linear version and the rotational version side by side, so you can always see the twin."

Kipute smiled. *"A twin language."*

"Exactly. A twin language. Two dialects of the same physics."

He picked up the bicycle wheel and gave it a powerful spin. *"By the end of this chapter, you will understand why this wheel resists being tilted while it spins. You will understand why an ice skater spins faster by pulling in her arms. You will understand why a solid sphere beats a hollow cylinder in a race down a slope, even though both are released at the same time from the same height. You will calculate the energy stored in a spinning flywheel, the torque needed to stop a rotating platform, and the speed of a ball rolling down a hill. And you will do all of it using the twin of what you already know."*

He set the wheel down. It continued spinning on its axle, perfectly balanced, humming quietly.

"Welcome to Chapter 9. The world is about to start spinning."

THE BIG PICTURE: FROM LINEAR TO ROTATIONAL MOTION

Up to this point in the book, we have been remarkably successful at describing the physical world using a small set of ideas: displacement, velocity, acceleration, force, mass, momentum, energy. These ideas, forged in Chapters 2 through 8, allowed us to launch projectiles, crash vehicles, swing objects in circles, and send satellites into orbit. But all of this success was built on a quiet assumption that we are about to confront.

We treated every object as a **particle**: *a point mass with no size, no shape, and no internal structure*. When we said "a car accelerates" or "a ball is projected," we meant a single point moved in a certain way. We never asked *how* the car was oriented, or whether the ball was spinning. We never needed to.

That changes now.

Consider a simple experiment. Place a pencil on a smooth table and flick it at the centre. The pencil slides forward in a straight line without rotating. Now flick the same pencil at one end. The pencil both slides and spins. The same force, applied at different points, produces completely different motions. This is something that the particle model cannot explain, because a particle has no "centre" or "end." It is a point. Every force on a particle produces the same type of motion: translation along a straight line.

Real objects are not points. They have size, shape, and mass distributed over a volume. When forces act on such objects, two fundamentally different types of motion can occur:

Translation, *in which every particle in the body moves the same distance in the same direction in the same time.* The body shifts position without changing orientation. A box sliding across a floor is translating.

Rotation, *in which every particle in the body moves in a circle about a common axis*. The body changes orientation without necessarily changing position. A wheel spinning on a fixed axle is rotating.

Most real motions are a combination of both. A rolling football translates across the pitch while simultaneously rotating about its own axis. A thrown spanner tumbles through the air, rotating about its centre of mass while the centre of mass follows a parabolic projectile path. Even the Earth participates in both: it translates along its orbit around the Sun while rotating about its north-south axis once every 24 hours.

In Chapters 2 through 8, we mastered translation. In this chapter, we master rotation. And the beautiful surprise waiting for us is this: *rotation is not new physics*. It is the same physics, expressed in a different language.

The Twin Language

Every quantity we used to describe translation has an exact counterpart in rotation. Every equation we used to describe translational motion has an identical twin for rotational motion. The structure is perfectly parallel, and once you see this parallelism, the entire chapter becomes a translation exercise: take what you already know about linear motion, replace each quantity with its rotational twin, and the physics works.

Here is a preview of the twin pairs, without any formulas yet. We will develop each one rigorously in the sections that follow, but seeing the full map now will help you navigate the journey.

In translation, we describe *where* a body is using **displacement** (how far it has moved in a straight line). In rotation, the twin is **angular displacement** (how far the body has turned, measured in radians).

In translation, we describe *how fast* a body moves using **velocity** (rate of change of displacement). In rotation, the twin is **angular velocity** (rate of change of angular displacement, how fast the body spins).

In translation, we describe *how quickly the motion changes* using **acceleration** (rate of change of velocity). In rotation, the twin is **angular acceleration** (rate of change of angular velocity, how quickly the spin speeds up or slows down).

In translation, a body's resistance to change in motion is measured by its **mass**. Push a heavy box and it resists. Push a light box with the same force and it responds more readily. In rotation, the twin is **moment of inertia**: a body's resistance to change in rotational motion. Try to spin a metre ruler from its centre, and it spins easily. Try to spin the same ruler from one end, and it resists stubbornly. The mass has not changed, but the *distribution of mass relative to the axis* has changed. Moment of inertia captures this distribution, and it plays exactly the same role in rotation that mass plays in translation.

In translation, the cause of acceleration is **force**. In rotation, the twin is **torque**: the rotational equivalent of force. Torque is what makes things spin, just as force is what makes things accelerate. A force applied far from the pivot produces a large torque; the same force applied close to the pivot produces a small torque. This is why a long spanner is more effective than a short one, and why a door handle is placed at the edge farthest from the hinges, not next to them.

In translation, $F = ma$ is the master equation: force equals mass times acceleration. In rotation, the master equation is its twin: torque equals moment of inertia times angular acceleration. This single equation governs all rotational dynamics, just as $F = ma$ governs all translational dynamics.

In translation, the quantity of motion is **momentum** ($p = mv$). In rotation, the twin is **angular momentum**: the product of moment of inertia and angular velocity. Angular momentum is conserved when no external torque acts, just as linear momentum is conserved when no external force acts. This is why an ice skater spins faster when she pulls her arms inward: by reducing her moment of inertia, she must increase her angular velocity to keep angular momentum constant.

In translation, the energy of motion is **kinetic energy** ($\frac{1}{2}mv^2$). In rotation, the twin is **rotational kinetic energy**: $\frac{1}{2}$ times the moment of inertia times the square of the angular velocity. A spinning flywheel stores energy in exactly the same way that a moving car stores energy, but through rotation rather than translation.

In translation, the rate of doing work is **power** (Fv). In rotation, the twin is **rotational power**: torque times angular velocity.

Every pair follows the same pattern: replace displacement with angular displacement, velocity with angular velocity, mass with moment of inertia, force with torque, and the physics transfers perfectly.

Why rotation needs its own treatment

A reasonable student might ask: if rotation is just the same physics in a different costume, *why do we need a separate chapter? Why not simply apply $F = ma$ to every particle in a spinning object and add up the results?*

In principle, that approach would work. A spinning disc is nothing more than a vast number of tiny particles, each obeying $F = ma$. If we could track every particle individually, we would not need any rotational concepts at all.

In practice, this is impossible. A solid disc contains roughly 10^{25} atoms. Writing $F = ma$ for each one and solving the resulting system of equations would take longer than the age of the universe. The rotational quantities (angular velocity, moment of inertia, torque, angular momentum) are not new physics; they are powerful *shortcuts*. They compress the behaviour of 10^{25} particles into a single equation that a Form Five student can solve in two minutes.

This is the real reason rotation needs its own language: not because the physics is different, but because describing rotation in the language of translation is hopelessly inefficient. The twin language is a labour-saving device of extraordinary power.

What does "rigid" mean, and why does it matter?

In the introduction, Mr. Akilikubwa distinguished between the hard-boiled egg (rigid) and the raw egg (not rigid). This distinction is essential.

A **rigid body** *is an idealised object in which the distance between every pair of particles remains fixed, regardless of the forces applied.* A rigid body does not bend, stretch, compress, or flow. When one part of a rigid body moves, every other part moves in a perfectly predictable way, because the shape never changes.

Real objects are never perfectly rigid. A steel beam bends slightly under load. A rubber ball deforms when squeezed. Even a diamond compresses by a tiny amount when pressed. However, for most everyday objects under normal forces, the deformations are so small that treating the object as rigid introduces negligible error. Throughout this chapter, we assume all objects are perfectly rigid unless stated otherwise.

The rigidity assumption is what makes rotational physics tractable. If a body is rigid, then knowing how any one particle moves tells us how *every* particle moves. If the body rotates through an angle θ, every particle rotates through the same angle θ. If the body has angular velocity ω, every particle has the same angular velocity ω. The entire motion of the body is captured by a single variable (the angle), rather than by 10^{25} separate position vectors. This is an enormous simplification, and it is the foundation upon which the entire chapter rests.

The road ahead

With the big picture in place, here is the journey we will take through this chapter.

We begin with the **language of rotation**: angular displacement, angular velocity, angular acceleration, and the kinematic equations that connect them. These are the rotational twins of the kinematic equations you learned in Chapter 2.

Next, we identify the **centre of mass**: the special point about which a free body naturally rotates, and through which the net external force effectively acts.

Then comes the **heart of the chapter**: moment of inertia. We will derive it from first principles and calculate it for every standard shape. This is the most challenging subtopic, and also the most rewarding.

Armed with moment of inertia, we discover two powerful **shortcut theorems** that let us find moments of inertia for new axes without repeating long integrations.

We then introduce **radius of gyration**, a neat way to express how mass is distributed in a single number.

We then move from description to cause: **torque**, the thing that makes objects rotate, including the special case of a couple.

With torque established, we then develop the **energy and power** of rotation.

The crown jewel of the chapter follows: **rolling motion**, where translation and rotation combine and the twin languages merge into one.

We conclude with **angular momentum and its conservation**, one of the most profound principles in all of physics.

Finally, we lay out the full **comparison table**, followed by real-world **applications** that bring the physics to life.

The world has always been spinning. After this chapter, you will finally understand why.

ANGULAR KINEMATICS: THE LANGUAGE OF ROTATION

You already speak half of this language. In Chapter 7 (Uniform Circular Motion), we introduced the radian as the natural unit for measuring angles, defined angular displacement $\theta = \frac{s}{r}$, and established angular velocity $\omega = \frac{\Delta\theta}{\Delta t}$ as the rate of rotation. We derived the relationship $v = \omega r$ connecting linear and angular speed, and we used the period $T = \frac{2\pi}{\omega}$ and frequency $f = \frac{1}{T}$ to describe objects moving at constant speed around a circle.

But Chapter 7 dealt exclusively with **uniform** circular motion, where the angular velocity never changed. A satellite in a stable orbit, a stone on a string whirled at constant speed, a car rounding a bend at steady velocity: in all of those situations, ω was constant.

Real rotating objects are rarely so well-behaved. A fan speeds up when switched on and slows down when switched off. A car engine increases its rotational speed from idle to full throttle. A potter's wheel gradually loses speed to friction. A gymnast's somersault starts slow and becomes faster as she tucks. In every one of these situations, the angular velocity is changing.

To describe changing angular velocity, we need a new quantity: **angular acceleration**. And with angular acceleration in hand, we can build a complete set of kinematic equations for rotation, exactly mirroring the equations of Chapter 2.

Quick reference: What we already know from chapter 7

Quantity	Definition	Unit
Angular displacement, θ	Angle swept, $\theta = \frac{s}{r}$	rad
Angular velocity, ω	Rate of rotation, $\omega = \frac{\Delta\theta}{\Delta t}$	rad/s
Linear velocity from angular velocity	$v = \omega r$	m/s
Period, T	Time for one revolution, $T = \frac{2\pi}{\omega}$	s
Frequency, f	Revolutions per second, $f = \frac{\omega}{2\pi}$	Hz
Centripetal acceleration, a_c	$a_c = \omega^2 r = \frac{v^2}{r}$	m/s^2

Conversion reminder: $\omega = 2\pi f = \frac{2\pi N}{60}$, where N is the speed in rpm (revolutions per minute).

All of this remains valid. We now extend it.

Angular Acceleration

If the angular velocity of a body changes with time, the body has an **angular acceleration**, denoted by α.

The **average angular acceleration** over a time interval is:

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

The **instantaneous angular acceleration** is:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Angular acceleration is measured in **radians per second squared** (rad/s^2).

The sign convention follows naturally: if we take anticlockwise as positive, then a positive α means the rotation is speeding up in the anticlockwise direction (or slowing down in the clockwise direction). A negative α means the rotation is slowing down in the anticlockwise direction (or speeding up clockwise).

Angular acceleration is the rotational twin of linear acceleration. In Chapter 2, acceleration told us how rapidly an object's velocity was changing. Here, angular acceleration tells us how rapidly an object's angular velocity is changing. A flywheel that spins up from rest has a positive α. A grinding wheel that is switched off and gradually slows has a negative α (deceleration). A fan spinning at constant speed has $\alpha = 0$, which is exactly the uniform circular motion of Chapter 7.

Tangential Acceleration

In Chapter 7, a particle moving in a circle at constant ω had only centripetal acceleration $a_c = \omega^2 r$, directed toward the centre. This acceleration changed the *direction* of the velocity but not its magnitude.

When ω is changing, a second component of acceleration appears. Differentiating $v = \omega r$ with respect to time:

$$\frac{dv}{dt} = \frac{d(wr)}{dt} = r\frac{dw}{dt}$$

But:

$$\frac{dv}{dt} = a, \frac{dw}{dt} = \alpha$$

Thus:

$$a = \alpha r$$

The acceleration, a, acts along the **tangent** to the circular path and changes the *speed* of the particle. It is therefore known as **tangential acceleration, a_t**. Hence:

$$\mathbf{a_t = \alpha r}$$

Tangential acceleration exists only when $\alpha \neq 0$.

A particle undergoing non-uniform circular motion therefore has two perpendicular components of acceleration:

1) Centripetal: $a_c = \omega^2 r$, directed toward the centre (changes direction).

2) Tangential: $a_t = \alpha r$, directed along the tangent (changes speed).

The total linear acceleration has magnitude:

$$a = \sqrt{a_t^2 + a_c^2}$$

When $\alpha = 0$ (uniform circular motion), a_t vanishes ($a_t = 0$) and only the centripetal acceleration remains $\left(a = \sqrt{0 + a_c^2} = a_c\right)$. This is precisely the situation we studied in Chapter 7. Non-uniform circular motion is the general case, and uniform circular motion is the special case where α happens to be zero.

The Rotational Kinematic Equations

In Chapter 2, we derived four equations that describe straight-line motion under constant acceleration. Because the mathematical structure of rotation is identical, we can write down the rotational equivalents immediately by replacing every linear quantity with its angular twin.

The correspondence is:

Linear quantity	**Rotational twin**
Displacement, s	Angular displacement, θ
Initial velocity, u	Initial angular velocity, ω_0
Final velocity, v	Final angular velocity, ω
Acceleration, a	Angular acceleration, α
Time, t	Time, t (no twin needed)

The four equations, side by side:

Linear (constant a)	**Rotational (constant α)**
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s = \left(\frac{u + v}{2}\right)t$	$\theta = \left(\frac{\omega_0 + \omega}{2}\right)t$

These equations are valid only when the angular acceleration α is constant, just as the linear equations are valid only when the linear acceleration a is constant.

Notice that time t appears unchanged in both columns. Time does not have a "rotational twin" because time is the same for all observers regardless of the type of motion being described.

If you mastered Chapter 2, you already understand these equations. The physics is identical; only the symbols have changed. A flywheel starting from rest and spinning up at constant angular acceleration behaves mathematically in exactly the same way as a car starting from rest and accelerating at constant linear acceleration.

A note on angular quantities in rigid bodies:

We emphasized in Chapter 7 that all points on a rigid body share the same ω. This extends naturally: all points on a rigid body also share the same α, because if the angular velocity changes by the same amount for every point (rigidity demands this), then the rate of change α must be the same for all.

However, the linear quantities v, a_t, and a_c differ from point to point because they all depend on **r** (*the distance from the rotation axis*). A point far from the axis has a larger linear velocity, a larger tangential acceleration, and a larger centripetal acceleration than a point close to the axis, even though both share the same ω and α. This is precisely why angular quantities are more convenient than linear quantities for describing rotation: one value of ω describes the entire body, while the linear velocity varies from particle to particle.

Angular Displacement in the *n*th Second

There is one more kinematic result that appears frequently in examinations: the angular displacement turned through during a specific second of motion, say the nth second.

Derivation:

Using:

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

The total angular displacement from $t = 0$ to $t = n$ is:

$$\theta_n = \omega_0 n + \frac{1}{2}\alpha n^2$$

The total angular displacement from $t = 0$ to $t = (n-1)$ is:

$$\theta_{n-1} = \omega_0 (n-1) + \frac{1}{2}\alpha (n-1)^2$$

The angular displacement during the nth second alone is the difference:

$$\theta_{nth} = \theta_n - \theta_{n-1} = \omega_0 n + \frac{1}{2}\alpha n^2 - \omega_0 (n-1) - \frac{1}{2}\alpha (n-1)^2$$

$$= \omega_0 \big(n - (n-1)\big) + \frac{1}{2}\alpha (n^2 - (n-1)^2)$$

$$= \omega_0 (1) + \frac{1}{2}\alpha (n^2 - n^2 + 2n - 1)$$

$$= \omega_0 + \frac{1}{2}\alpha (2n-1)$$

$$\boldsymbol{\theta_{nth} = \omega_0 + \frac{1}{2}\alpha (2n-1)}$$

This is the rotational twin of $s_{nth} = u + \frac{1}{2}a(2n-1)$ for translational motion. It gives the angle turned through during one specific second, not the total angle from the start.

Key points:

1) The formula gives the angular displacement during the nth second, not at time n. For example, for $n = 3$, it gives the angle turned through between $t = 2$s and $t = 3$s.

2) If $\alpha = 0$ (uniform angular velocity), $\theta_{nth} = \omega_0$ for every second. Each second contributes the same angle, as expected.

3) If $\omega_0 = 0$ (starting from rest), $\theta_{nth} = \frac{1}{2}\alpha(2n-1)$. The angles in successive seconds are in the ratio $1:3:5:7:...$, which is the familiar odd-number sequence.

4) The unit of θ_{nth} is radians (not radians per second). Although the formula contains ω_0 (which has units rad/s), the implicit factor is $\omega_0 \times 1$s, giving radians. Similarly, $\frac{1}{2}\alpha(2n-1)$ has units rad/s^2 $\times$ s $=$ rad/s, but again the implicit $\times 1$s gives radians. This is the same dimensional subtlety as in the translational version.

Before the ideas tangle any further, let us straighten them out using a few worked examples.

BINDER Example 1

(a) Convert the following angular speeds to rad/s:

(i) 3000rpm, (ii) 8rev/s.

(b) A disc starts from rest and reaches an angular velocity of 40rad/s in 8s with constant angular acceleration. Calculate the angular acceleration, the total angle turned through, and the total number of revolutions made by disc.

Solution

(a) The solution of each part is as follows:

(i) Using $\omega = \frac{2\pi N}{60}$ and substituting N = 3000rpm:

$$\omega = \frac{2\pi \times 3000}{60}\text{rad/s} = 100\pi\text{rad/s} = 314\text{rad/s}$$

(i) Using $\omega = 2\pi f$ and substituting f = 8rev/s:

$$\omega = 2\pi \times 8\text{rad/s} = 16\pi\text{rad/s} = 50.3\text{rad/s}$$

(b) Given: $\omega_0 = 0$ (starts from rest), $\omega = 40$rad/s, t = 8s.

Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t}$$

Substituting:

$$\alpha = \frac{40\text{rad/s} - 0}{8\text{s}} = \mathbf{5rad/s^2}$$

For the angle, using $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ and substituting:

$$\theta = 0 + \frac{1}{2} \times 5\text{rad/s}^2 \times (8\text{s})^2 = \frac{1}{2} \times 5 \times 64 = \mathbf{160rad}$$

Converting to revolutions: $\frac{160\text{rad}}{2\pi\text{rad/rev}} = \mathbf{25.5rev}$

Making Sense of the Answer: *The disc completes about* 25.5 *revolutions while spinning up from rest to* 40*rad/s over* 8 *seconds. This is reasonable: it accelerates gently to a modest angular speed, so a few dozen revolutions is expected. If the answer had been* 0.5 *revolutions or* 5000 *revolutions, we would know something had gone wrong.*

Think Like a Physicist: *Notice how similar this calculation feels to a Chapter 2 problem where a car starts from rest and accelerates uniformly. The structure is identical: find the acceleration from the first equation, then find the displacement from the second. The only difference is the symbols: ω instead of* v, *θ instead of* s, *α instead of* a. *This is the twin language in action, and it will serve you throughout the entire chapter.*

BINDER Example 2

A grinding wheel rotates at 3000rpm. A brake is applied, and the wheel decelerates uniformly to rest in 15s.

(a) Calculate the angular deceleration.

(a) Calculate the total number of revolutions the wheel makes while coming to rest.

Solution

(a) First, converting the initial angular speed to rad/s:

$$\omega_0 = \frac{2\pi \times 3000}{60}\text{rad/s} = 314.2\text{rad/s}$$

The wheel comes to rest, so $\omega = 0$.

Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t}$$

Substituting:

$$\alpha = \frac{0 - 314.2\text{rad/s}}{15\text{s}} = -20.9\text{rad/s}^2$$

The negative sign confirms that the wheel is decelerating.

The magnitude of the angular deceleration is 20.9rad/s^2.

(b) Using $\theta = \frac{1}{2}(\omega_0 + \omega)t$ and substituting:

$$\theta = \frac{1}{2}(314.2 + 0)\text{rad/s} \times 15\text{s} = \frac{1}{2} \times 314.2 \times 15 = 2357\text{rad}$$

Converting to revolutions: $\frac{2357\text{rad}}{2\pi\text{rad/rev}} = 375\text{rev}$

Making Sense of the Answer: *The wheel was spinning very fast (*3000rpm *is* 50 *turns per second) and took* 15 *seconds to stop, so a few hundred revolutions during braking is reasonable. The deceleration of about* 21rad/s^2 *means the wheel loses* 21rad/s *of angular speed every second, which brings* 314rad/s *to zero in exactly* 15s. *Everything checks out.*

Think Like a Physicist: *Compare this to a car braking from* 100km/h *(*27.8m/s*) to rest. The structure is identical: initial velocity, zero final velocity, constant deceleration, find the deceleration and the stopping distance. If you can solve the car problem, you can solve the wheel problem, because the equations are the same. Only the symbols differ.*

REAL Example 3

A traditional Tanzanian potter's wheel is spun by hand. The potter gives the wheel a push that sets it spinning at 5rev/s, then begins shaping the clay. Friction gradually slows the wheel at a constant rate. After 20s, the wheel has slowed to 2rev/s, and the potter must push it again.

(a) Explain why the wheel does not stop instantly when friction begins acting on it.

(b) Calculate the angular deceleration of the wheel.

(c) If the potter does not push again, calculate the total number of revolutions the wheel would make from the initial push until it stops.

Solution

(a) The wheel has mass distributed at a distance from the rotation axis, so it possesses rotational inertia (a concept we will develop fully later). Because of this inertia, the wheel resists changes to its angular velocity, just as a heavy box on a surface resists changes to its linear velocity. Friction provides a constant decelerating torque, but the wheel's rotational inertia ensures that the angular velocity decreases gradually rather than dropping to zero instantly. The heavier the wheel and the farther its mass is from the axis, the longer it takes to stop.

(b) Converting to rad/s:

$$\omega_0 = 2\pi \times 5 = 10\pi\text{rad/s} = 31.42\text{rad/s}$$

$$\omega = 2\pi \times 2 = 4\pi\text{rad/s} = 12.57\text{rad/s}$$

Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{12.57\text{rad/s} - 31.42\text{rad/s}}{20\text{s}} = \frac{-18.85\text{rad/s}}{20\text{s}} = -0.943\text{rad/s}^2$$

The angular deceleration of the wheel is 0.943rad/s^2.

(c) From the initial push ($\omega_0 = 31.42\text{rad/s}$) to stop ($\omega = 0$), using $\omega^2 = \omega_0^2 + 2\alpha\theta$ and making θ the subject:

$$\theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{0 - (31.42\text{rad/s})^2}{2 \times (-0.943\text{rad/s}^2)} = \frac{-987.2\text{rad}^2/\text{s}^2}{-1.886\text{rad/s}^2} = 523.5\text{rad}$$

Converting to revolutions: $\frac{523.5\text{rad}}{2\pi\text{rad/rev}} = 83.3\text{rev}$

The total number of revolutions is 83.3rev.

Making Sense of the Answer: *The wheel makes about* 83 *revolutions over roughly* 33 *seconds before stopping. Most of these revolutions occur early, when the wheel is spinning fast: during the first* 20 *seconds, the wheel makes about* 70 *revolutions, but during the last* 13 *seconds, it makes only about* 13. *This pattern mirrors what we saw in Chapter 2 for a decelerating car: most of the distance is covered early, when the speed is still high.*

Think Like a Physicist: *Part (c) asks for the total angle from start to stop. Rather than splitting the motion into two stages (first* 20 *seconds, then the remaining time) and adding the angles separately, we used the equation $\omega^2 = \omega_0^2 + 2\alpha\theta$, which connects the initial state directly to the final state without needing time at all. Whenever a problem gives you initial and final angular velocities plus angular acceleration, but does not ask for time, this third kinematic equation is the most efficient tool.*

REAL Example 4

Kipanga watches a ceiling fan after the power is switched off. He counts 15 complete revolutions before the fan stops, and times the process at 12s.

(a) Calculate the initial angular velocity of the fan in rpm.

(b) Calculate the angular deceleration.

(c) How fast was the fan spinning (in rpm) after the first 4s?

Solution

(a) Total angle: $\theta = 15\text{rev} \times 2\pi\text{rad/rev} = 30\pi\text{rad} = 94.25\text{rad}$

The fan starts at ω_0 and comes to rest ($\omega = 0$) in $t = 12\text{s}$.

Using $\theta = \frac{1}{2}(\omega_0 + \omega)t$ and making ω_0 the subject:

$$\omega_0 = \frac{2\theta}{t} - \omega = \frac{2 \times 94.25\text{rad}}{12\text{s}} - 0 = 15.7\text{rad/s}$$

Converting: $\frac{15.7\text{rad/s}}{2\pi} = 2.5\text{rev/s} = 2.5 \times 60\text{rpm} = 150\text{rpm}$

The initial angular velocity is 150rpm.

(b) Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 15.7\text{rad/s}}{12\text{s}} = -1.31\text{rad/s}^2$$

The angular deceleration has magnitude 1.31rad/s^2.

(c) At $t = 4\text{s}$, using $\omega = \omega_0 + \alpha t$ and substituting:

$$\omega = 15.7\text{rad/s} + (-1.31\text{rad/s}^2) \times 4\text{s} = 15.7\text{rad/s} - 5.24\text{rad/s} = 10.5\text{rad/s}$$

Converting: $\frac{10.5\text{rad/s}}{2\pi} \times 60 = 100\text{rpm}$

After 4 seconds, the fan was spinning at 100rpm.

Making Sense of the Answer: *An initial speed of* 150rpm *(*2.5 *turns per second) is typical for a ceiling fan on medium setting. After* 4 *seconds (one-third of the total* 12*-second stopping time), the fan has lost exactly one-third of its angular speed (*50rpm *out of* 150rpm*). This is exactly what constant angular deceleration predicts: the angular velocity decreases uniformly with time, so equal fractions of time produce equal drops in speed.*

Think Like a Physicist: *We were given two pieces of information (*15 *revolutions and* 12 *seconds) and needed to find two unknowns (ω_0 and α). This is a two-equation, two-unknown problem, just like in Chapter 2. The key strategy is to choose the equation that has only one unknown first. Since both θ and* t *are known and $\omega = 0$, the fourth equation $\theta = \frac{1}{2}(\omega_0 + \omega)$t contains only ω_0 as unknown. Once ω_0 is found, any other equation gives α. Choosing the right equation first, the one with fewer unknowns, is a powerful strategy in both linear and rotational kinematics.*

BINDER Example 5

A wheel starts from rest and accelerates uniformly at 4rad/s^2.

(a) Calculate the angle turned through during the 1st, 2nd, 3rd, and 4th seconds.

(b) Show that these angles are in the ratio $1:3:5:7$.

(c) Calculate the total angle after 4s and verify that it equals the sum of the individual seconds.

Solution

(a) Using $\theta_{nth} = \omega_0 + \frac{1}{2}\alpha(2n-1)$ with $\omega_0 = 0$ and $\alpha = 4\text{rad/s}^2$:

1st second ($n = 1$): $\theta_1 = 0 + \frac{1}{2} \times 4 \times (2 \times 1 - 1) = 2 \times 1 = 2\text{rad}$

2nd second ($n = 2$): $\theta_2 = 0 + \frac{1}{2} \times 4 \times (2 \times 2 - 1) = 2 \times 3 = 6\text{rad}$

3rd second ($n = 3$): $\theta_3 = 0 + \frac{1}{2} \times 4 \times (2 \times 3 - 1) = 2 \times 5 = 10\text{rad}$

4th second ($n = 4$): $\theta_4 = 0 + \frac{1}{2} \times 4 \times (2 \times 4 - 1) = 2 \times 7 = 14\text{rad}$

The angles are 2, 6, 10, and 14 radians respectively.

(b) The ratio is $2:6:10:14 = 1:3:5:7$. *Shown.*

This is the odd-number ratio, characteristic of uniform acceleration from rest. It holds for any value of α.

(c) Total from the formula: $\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2} \times 4 \times (4)^2 = 32\text{rad}$

Sum of individual seconds: $2\text{rad} + 6\text{rad} + 10\text{rad} + 14\text{rad} = 32\text{rad}$. *Verified.*

Making Sense of the Answer: *The wheel turns only* 2rad *in the 1st second but* 14rad *in the 4th second, seven times more. This is because the wheel is faster in the 4th second than in the 1st. Uniform acceleration means the angular velocity increases steadily, so the wheel covers more angle in each successive second. The total of* 32rad *is about* 5.1 *revolutions in* 4 *seconds.*

Think Like a Physicist: *The odd-number ratio (*$1:3:5:7$*) is one of the most elegant results in kinematics, first discovered by Galileo for falling bodies. It applies equally to translational and rotational motion with uniform acceleration from rest. If an examination gives you angles in successive seconds and asks you to prove uniform acceleration, check for this ratio. Conversely, if you know the acceleration is uniform and the body starts from rest, you can predict the ratio without any calculation.*

HOT Example 6

A car engine crankshaft accelerates uniformly from 1200rpm to 4800rpm in 6s. A small pulley of radius 0.15m is attached to the crankshaft and rotates with it.

(a) Calculate the angular acceleration.

(b) Calculate the number of revolutions made during this acceleration.

(c) At the instant the crankshaft reaches 4800rpm, calculate the tangential and centripetal accelerations of a point on the rim of the pulley.

Solution

(a) Converting to rad/s:

$$\omega_0 = \frac{2\pi \times 1200}{60} = 40\pi\text{rad/s} = 125.7\text{rad/s}$$

$$\omega = \frac{2\pi \times 4800}{60} = 160\pi\text{rad/s} = 502.7\text{rad/s}$$

Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{502.7\text{rad/s} - 125.7\text{rad/s}}{6\text{s}} = 62.8\text{rad/s}^2$$

(b) Using $\theta = \frac{1}{2}(\omega_0 + \omega)t$ and substituting:

$$\theta = \frac{1}{2}(125.7 + 502.7)\text{rad/s} \times 6\text{s} = \frac{1}{2} \times 628.4 \times 6 = 1885\text{rad}$$

Converting to revolutions: $\frac{1885\text{rad}}{2\pi\text{rad/rev}} = 300\text{rev}$

It made 300 revolutions.

(c) The tangential acceleration changes the speed of the rim point.

Using $a_t = \alpha r$ and substituting:

$$a_t = 62.8\text{rad/s}^2 \times 0.15\text{m} = 9.42\text{m/s}^2$$

The tangential acceleration is 9.42m/s^2.

The centripetal acceleration changes the direction of the rim point.

Using $a_c = \omega^2 r$ at $\omega = 502.7\text{rad/s}$ and substituting:

$$a_c = (502.7\text{rad/s})^2 \times 0.15\text{m} = 37906\text{m/s}^2$$

The centripetal acceleration is 37906m/s^2.

Have you noticed this? The is centripetal acceleration is very huge, approximately 3870g!

Making Sense of the Answer: *The centripetal acceleration is nearly* 4000 *times the acceleration due to gravity. This is why engine components must be carefully balanced and manufactured from strong materials: even a tiny mass imbalance at these rotational speeds produces enormous centripetal forces. The tangential acceleration (*9.42m/s^2*, roughly equal to g) is modest by comparison because the engine takes a full* 6 *seconds to change speed. But the centripetal acceleration is huge because it depends on* ω^2*, and* ω *at* 4800rpm *is very large.*

Think Like a Physicist: *At high angular velocities, centripetal acceleration completely dominates over tangential acceleration because* $a_c \propto \omega^2$ *while* $a_t \propto \alpha$*, which is independent of* ω*. This is why high-speed rotating machinery is dangerous and why turbine blades, centrifuge rotors, and jet engine fans are engineered with extreme precision. This problem bridges Chapter 7 (centripetal acceleration) and Chapter 9 (angular acceleration), showing that non-uniform circular motion has both components acting simultaneously.*

HOT Example 7

A flywheel initially spinning at 600rpm is brought to rest by a constant braking torque. During the braking, the flywheel makes exactly 400 revolutions.

(a) Calculate the angular deceleration.

(b) Calculate the time taken to stop.

(c) Calculate the angular velocity (in rpm) of the flywheel when it has completed 300 of the 400 revolutions.

Solution

(a) Converting the given quantities:

$$\omega_0 = \frac{2\pi \times 600}{60} = 20\pi\text{rad/s} = 62.83\text{rad/s}$$

$\omega = 0$ (comes to rest)

$$\theta = 400\text{rev} \times 2\pi\text{rad/rev} = 800\pi\text{rad} = 2513\text{rad}$$

Using $\omega^2 = \omega_0^2 + 2\alpha\theta$ and making α the subject:

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta} = \frac{0 - (62.83\text{rad/s})^2}{2 \times 2513\text{rad}} = -0.786\text{rad/s}^2$$

The angular deceleration is 0.786rad/s^2.

(b) Using $\omega = \omega_0 + \alpha t$ and making t the subject:

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 62.83\text{rad/s}}{-0.786\text{rad/s}^2} = 79.9\text{s}$$

The flywheel takes approximately 80 seconds (1 minute 20 seconds) to stop.

(c) After 300 revolutions: $\theta_{300} = 300\text{rev} \times 2\pi\text{rad/rev} = 600\pi\text{rad} = 1885\text{rad}$

Using $\omega^2 = \omega_0^2 + 2\alpha\theta$ and substituting:

$$\omega^2 = (62.83\text{rad/s})^2 + 2 \times \left(-0.786\text{rad/s}^2\right) \times 1885\text{rad} = 985\text{rad}^2/\text{s}^2$$

$$\omega = \sqrt{985\text{rad}^2/\text{s}^2} = 31.4\text{rad/s}$$

Converting: $\frac{31.4\text{rad/s}}{2\pi} \times 60 = 300\text{rpm}$

The angular velocity is 300rpm.

Making Sense of the Answer: *After* 300 *out of* 400 *revolutions (*75% *of the total angle), the flywheel still has* 31.4rad/s *out of an initial* 62.83rad/s*, which is* 50% *of the original speed. This reveals an important insight: the flywheel has covered* 75% *of its total angle but lost only* 50% *of its speed. This makes perfect sense because a faster-spinning wheel covers more angle per second, so most of the* 400 *revolutions are completed while the speed is still high, and the last* 100 *revolutions take disproportionately long as the wheel crawls to a stop.*

Think Like a Physicist: *Part (c) is a "mid-journey" question: what is the velocity after a certain fraction of the total displacement? This type of question appears frequently in both linear and rotational kinematics. The third equation ($\omega^2 = \omega_0^2 + 2\alpha\theta$) is the ideal tool because it directly connects velocity to displacement without involving time. Notice how cleanly the answer* 300rpm *emerges: after* 300 *of* 400 *revolutions, the speed is exactly half the original* 600rpm*. Verifying such clean patterns is a powerful way to check your work during an examination.*

Having given the worked examples their full say, we can now welcome the next subtopic and see what fresh ideas it brings along.

CENTRE OF MASS

We have described how a thrown spanner tumbles through the air, rotating about some internal point while that point itself follows a smooth parabolic path. A thrown boomerang does the same. So does a spinning baton tossed by a cheerleader. In every case, there is *one special point in the body that moves as though the entire mass of the body were concentrated there and all external forces acted on it.* That point is the **centre of mass**.

The centre of mass is the "spokesperson" for the whole body. When we said "the ball was projected at 20m/s" in Chapter 6 (Projectile Motion), we were really saying "the centre of mass of the ball was projected at 20m/s." When we wrote F = ma in Chapter 3, we were really describing the acceleration of the centre of mass. Every time we treated an object as a particle in Chapters 2 through 8, we were unknowingly tracking its centre of mass. Now we make this explicit.

Why Centre of Mass Matters for Rotation

When a rigid body is free to move (not constrained by a pivot or axle), it naturally rotates about its centre of mass. Throw a ruler in the air and watch it spin: it rotates about a point near the middle, not about one end. This happens because the centre of mass is the point about which the mass is, in a sense, "balanced."

When a body is pivoted at a fixed point (such as a door on its hinges or a wheel on its axle), the rotation axis is determined by the pivot, and the centre of mass orbits around that pivot. But when no pivot is imposed, the body chooses its own axis, and that axis passes through the centre of mass.

Understanding where the centre of mass is located is therefore essential before we can calculate moment of inertia, because moment of inertia depends on the distance of each particle from the rotation axis, and the most natural rotation axis passes through the centre of mass.

Deriving the Centre of Mass: Two Particles

Let us begin with the simplest possible case: two particles on a line. Consider two particles of masses m_1 and m_2 placed on a light, rigid rod at positions x_1 and x_2 along the x-axis. We want to find the **balance point**, the single point at which we could support the rod and have it remain perfectly horizontal.

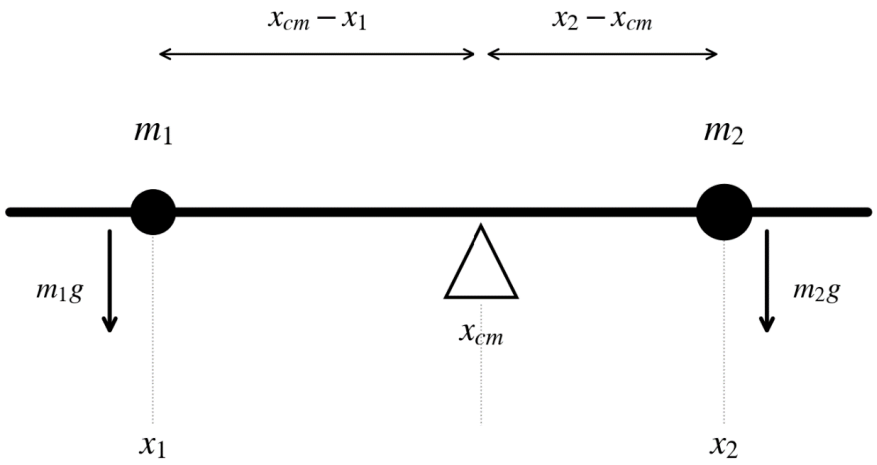


Figure: *Two particles of masses m_1 and m_2 on a light rigid rod, balanced at the centre of mass position x_{cm}. The weight $m_1 g$ acts downward at x_1 and the weight $m_2 g$ acts downward at x_2. For balance, the clockwise turning effect $m_1 g(x_{cm} - x_1)$ must equal the anticlockwise turning effect $m_2 g(x_2 - x_{cm})$.*

Suppose the balance point is at position x_{cm}. For the rod to balance, the turning effect (about x_{cm}) of the weight on one side must equal the turning effect of the weight on the other side. Since the weight of each particle is $m_i g$ and the perpendicular distance from the balance point is $|x_i - x_{cm}|$, the balance condition is:

$$m_1 g(x_{cm} - x_1) = m_2 g(x_2 - x_{cm})$$

The factor g cancels from both sides (which is why the balance point does not depend on the strength of gravity):

$$m_1(x_{cm} - x_1) = m_2(x_2 - x_{cm})$$

Expanding:

$$m_1 x_{cm} - m_1 x_1 = m_2 x_2 - m_2 x_{cm}$$

Collecting terms with x_{cm} on the left:

$$m_1 x_{cm} + m_2 x_{cm} = m_1 x_1 + m_2 x_2$$

$$(m_1 + m_2)x_{cm} = m_1 x_1 + m_2 x_2$$

Making x_{cm} the subject:

$$\mathbf{x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

This is the centre of mass of a two-particle system. It is a **weighted average** of the positions, where each position is weighted by its mass. The centre of mass lies closer to the heavier particle, which makes physical sense: a heavier particle has more "pull" on the balance point.

Notice that the derivation required only the balance of turning effects. The cancellation of g means that the centre of mass is a purely geometric-and-mass property of the system. It does not depend on gravity at all, even though we used gravity to motivate it. The same formula applies in deep space, where there is no gravity, because centre of mass can also be derived from Newton's laws applied to internal forces (the result is the same).

Generalising to Many Particles

The derivation extends naturally to any number of particles. For n particles on a line, requiring the net turning effect about x_{cm} to be zero gives:

$$\sum_{i=1}^{n} m_i\,(x_i - x_{cm}) = 0$$

Expanding:

$$\sum m_i x_i - x_{cm}\sum m_i = 0$$

Making x_{cm} the subject:

$$\mathbf{x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}}$$

For particles distributed in two or three dimensions, we apply the same formula independently in each direction:

$$\mathbf{x_{cm} = \frac{\sum m_i x_i}{M} \qquad y_{cm} = \frac{\sum m_i y_i}{M} \qquad z_{cm} = \frac{\sum m_i z_i}{M}}$$

where $M = \sum m_i$ is the total mass of the system. Each coordinate of the centre of mass is calculated independently of the others.

Centre of Mass for Continuous Bodies

For a continuous body (a solid rod, a disc, a sphere), we cannot sum over individual particles because there are infinitely many of them. Instead, we imagine dividing the body into infinitesimally small elements, each of mass dm, at position x. The sum becomes an integral:

$$\mathbf{x_{cm} = \frac{\int x\,dm}{M} = \frac{1}{M}\int x\,dm}$$

where $M = \int dm$ is the total mass of the body. Similarly for y_{cm} and z_{cm}.

To evaluate this integral, we express dm in terms of the body's geometry. For a uniform body (constant density), the mass element is proportional to the length, area, or volume element:

For a rod: $dm = \lambda\,dx$, where $\lambda = \frac{M}{L}$ is the mass per unit length.

For a plate: $dm = \sigma\,dA$, where $\sigma = \frac{M}{A}$ is the mass per unit area.

For a solid: $dm = \rho\,dV$, where $\rho = \frac{M}{V}$ is the mass per unit volume.

Derivation for a uniform rod:

Consider a uniform rod of mass M and length L, lying along the x-axis from $x = 0$ to $x = L$. The linear mass density is $\lambda = \frac{M}{L}$. A small element at position x has mass $dm = \lambda\,dx$.

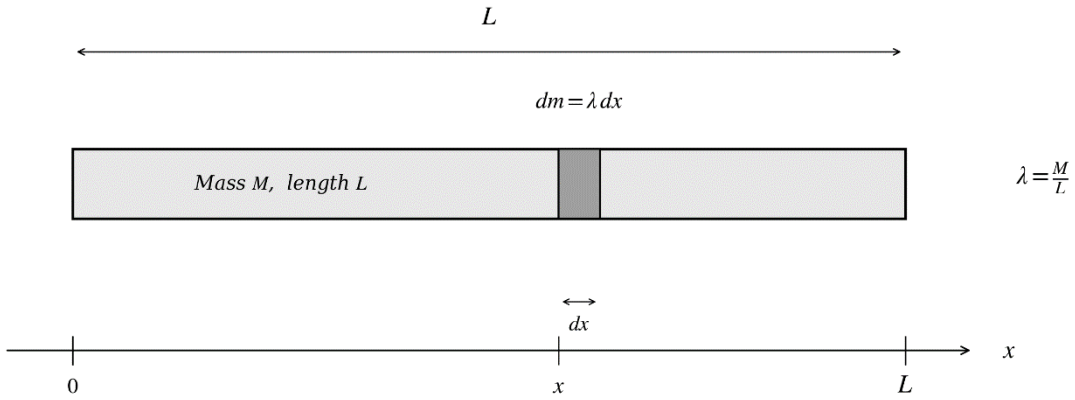


Figure: *A uniform rod of mass M and length L, lying along the x-axis from $x = 0$ to $x = L$. A small element of width dx at position x has mass $dm = \lambda dx$, where $\lambda = \frac{M}{L}$ is the linear mass density. Integrating the contribution of all such elements gives the centre of mass of the rod.*

Substituting into the integral:

$$x_{cm} = \frac{1}{M}\int_0^L x\ \lambda\, dx = \frac{\lambda}{M}\int_0^L x\ dx$$

$$= \frac{\lambda}{M}\left[\frac{x^2}{2}\right]_0^L = \frac{\lambda}{M} \times \frac{L^2}{2} = \frac{M/L}{M} \times \frac{L^2}{2} = \frac{L}{2}$$

The centre of mass of a uniform rod is at $\frac{L}{2}$, exactly at the midpoint. This confirms what symmetry tells us, but the integration method is essential for non-uniform bodies and for the moment of inertia derivations in the next subtopic.

The symmetry shortcut

For most standard uniform shapes, we do not need to perform the integration because the centre of mass can be found by **symmetry**:

If a uniform body has a geometric centre (a point, line, or plane of symmetry), the centre of mass lies at that geometric centre.

This powerful result gives us the centre of mass for all the common shapes without any calculation:

Body	**Centre of mass location**
Uniform rod	Midpoint of the rod
Uniform rectangular plate	Intersection of the diagonals
Uniform circular disc	Centre of the disc
Uniform sphere (solid or hollow)	Centre of the sphere
Uniform cylinder	Midpoint of the axis
Uniform ring (hoop)	Centre of the ring (not on the ring itself!)
Uniform triangular plate	Intersection of the medians ($\frac{1}{3}$ from base to apex)

Notice the ring: its centre of mass is at the geometric centre, which is a point in empty space. There is no material at the centre of mass! This is perfectly valid. The centre of mass is a mathematical point, not necessarily a physical location within the body.

CENTRE OF MASS OF COMPOSITE BODIES

Many real objects are not simple shapes but combinations of simple shapes. To find the centre of mass of a composite body, treat each component as a particle located at its own centre of mass, with mass equal to the component's total mass. Then apply the discrete particle formula.

For example, to find the centre of mass of an L-shaped plate, divide it into two rectangles. Find the centre of mass and mass of each rectangle. Then treat these as two particles and use the weighted average formula.

Similarly, to find the centre of mass of a body with a hole cut out (such as a disc with a smaller disc removed), treat the original body as one positive mass and the removed piece as a negative mass at the position of its own centre of mass.

Before these ideas start wandering off in search of their own centre of mass, let us pin them down with some worked examples.

BINDER Example 8

Three particles of masses 2kg, 3kg, and 5kg are placed at coordinates $(0,0)$, $(4,0)$, and $(2,3)$ respectively, where all distances are in metres.

(a) Calculate the position of the centre of mass.

(b) Explain why the centre of mass lies closest to the 5kg particle.

Solution

(a) Total mass: $\mathrm{M} = 2 + 3 + 5 = 10\mathrm{kg}$

Using $\mathrm{x_{cm}} = \frac{\sum \mathrm{m}_i \mathrm{x}_i}{\mathrm{M}}$ and substituting:

$$\mathrm{x_{cm}} = \frac{2\mathrm{kg} \times 0\mathrm{m} + 3\mathrm{kg} \times 4\mathrm{m} + 5\mathrm{kg} \times 2\mathrm{m}}{10\mathrm{kg}} = 2.2\mathrm{m}$$

Using $\mathrm{y_{cm}} = \frac{\sum \mathrm{m}_i \mathrm{y}_i}{\mathrm{M}}$ and substituting:

$$\mathrm{y_{cm}} = \frac{2\mathrm{kg} \times 0\mathrm{m} + 3\mathrm{kg} \times 0\mathrm{m} + 5\mathrm{kg} \times 3\mathrm{m}}{10\mathrm{kg}} = 1.5\mathrm{m}$$

The centre of mass is at $(2.2\mathrm{m}, 1.5\mathrm{m})$.

(b) The centre of mass formula is a weighted average, and the 5kg particle carries the largest weight in the average. Because its mass is 50% of the total, it exerts the strongest pull on the centre of mass. As a result, the point $(2.2,1.5)$ lies closest to the 5kg particle at $(2,3)$ and farthest from the lightest particle at $(0,0)$.

Making Sense of the Answer: *The centre of mass* $(2.2,1.5)$ *lies inside the triangle formed by the three particles. If it had fallen outside this triangle, we would know an error had been made, because the weighted average of positions must always lie within the region spanned by the masses.*

Think Like a Physicist: *In two dimensions, we apply the one-dimensional formula separately in each direction. The x-coordinate of the centre of mass depends only on the x-coordinates and masses, and similarly for y. The two calculations are completely independent, which makes the problem simple to set up and easy to check.*

Now that the arithmetic feels comfortable, let us see how the centre of mass behaves in the real world, where things are rarely sitting politely on coordinate axes.

REAL Example 9

A see-saw is 4m long and has a pivot at its centre. Kipanga (50kg) sits at one end, and Kipute (35kg) sits at the other end.

(a) Explain why the see-saw tilts toward Kipanga.

(b) Where should Kipute sit (measured from the pivot) so that the see-saw balances?

Solution

(a) The see-saw tilts toward Kipanga because the centre of mass of the system (Kipanga + Kipute) does not lie above the pivot. Since Kipanga is heavier than Kipute and both sit at equal distances from the pivot, the centre of mass lies on Kipanga's side. Because gravity acts through the centre of mass, the system rotates until the centre of mass is as low as possible, which means Kipanga's side goes down.

(b) For the see-saw to balance, the centre of mass of the two (Kipanga and Kipute) must lie directly above the pivot. Taking the pivot as the origin, Kipanga sits at $x_1 = -2\text{m}$ (one end) and Kipute sits at $x_2 = \text{d}$ (unknown distance on the other side).

Setting $x_{cm} = 0$ (centre of mass at the pivot):

$$0 = \frac{50\text{kg} \times (-2\text{m}) + 35\text{kg} \times \text{d}}{50\text{kg} + 35\text{kg}}$$

The denominator is non-zero, so the numerator must be zero:

$$50 \times (-2) + 35\text{d} = 0$$

$$-100 + 35\text{d} = 0$$

Making d the subject:

$$\text{d} = \frac{100}{35}\text{m} = 2.86\text{m}$$

Kipute should sit 2.86m from the pivot to balance the see-saw.

Making Sense of the Answer: *Kipute (*35kg*) is lighter than Kipanga (*50kg*), so she must sit farther from the pivot to compensate. Indeed,* $2.86\text{m} > 2.0\text{m}$*: the lighter person needs a larger distance to produce the same turning effect as the heavier person. This is the same principle used in beam balances and traditional market scales in Tanzania, where a smaller weight can balance a larger one by being placed farther from the pivot.*

Think Like a Physicist: *This is the familiar balance condition* $m_1 d_1 = m_2 d_2$ *from O-level. The centre of mass approach gives the same answer but is more general, because it works for any number of objects in any arrangement, not just two objects on a straight beam. The balance condition is simply the special case of "centre of mass lies at the pivot."*

REAL Example 10

A 12kg uniform rectangular table top measures 2.0m by 1.0m. A 4kg book is placed at one corner.

(a) Explain why the centre of mass of the table top alone is at its geometric centre.

(b) Calculate the position of the centre of mass of the system (table + book).

Solution

(a) The table top is uniform, meaning its mass is evenly distributed across its area. Because of this even distribution, the table top has two lines of symmetry (one along each pair of parallel edges). The centre of mass must lie on both lines of symmetry simultaneously, and the only point satisfying this is the intersection of the two lines: the geometric centre of the rectangle. Therefore the centre of mass of the table top alone is at its centre, at $(1.0\text{m}, 0.5\text{m})$.

(b) Setting up coordinates with one corner at the origin: the table's centre of mass is at $(1.0\text{m}, 0.5\text{m})$ with mass 12kg. The book is at the corner $(0,0)$ with mass 4kg.

Total mass: $\text{M} = 12 + 4 = 16\text{kg}$

$$x_{cm} = \frac{12\text{kg} \times 1.0\text{m} + 4\text{kg} \times 0\text{m}}{16\text{kg}} = \frac{12\text{kgm}}{16\text{kg}} = 0.75\text{m}$$

$$y_{cm} = \frac{12\text{kg} \times 0.5\text{m} + 4\text{kg} \times 0\text{m}}{16\text{kg}} = \frac{6\text{kgm}}{16\text{kg}} = 0.375\text{m}$$

The centre of mass of the system is at $(0.75\text{m}, 0.375\text{m})$.

Making Sense of the Answer: *Without the book, the centre of mass would be at* $(1.0,0.5)$. *Adding the* 4kg *book at the corner* $(0,0)$ *pulls the centre of mass toward that corner. The shift is modest because the book (*4kg*) is much lighter than the table (*12kg*). If the book were much heavier (say* 100kg*), the centre of mass would be pulled almost all the way to the corner.*

Think Like a Physicist: *We treated the entire* 12kg *table as a single particle at its centre of mass, and the* 4kg *book as a particle at its corner. This "collapse everything to its centre of mass" strategy is the power of this concept: no matter how complex a body is, once you know where its centre of mass is, you can replace it with a single particle for purposes of finding the system's centre of mass. This simplification will prove invaluable when we deal with composite bodies.*

The theory has been polite so far, dealing with well-behaved particles and tidy rectangles. Time to give it a proper workout.

HOT Example 11

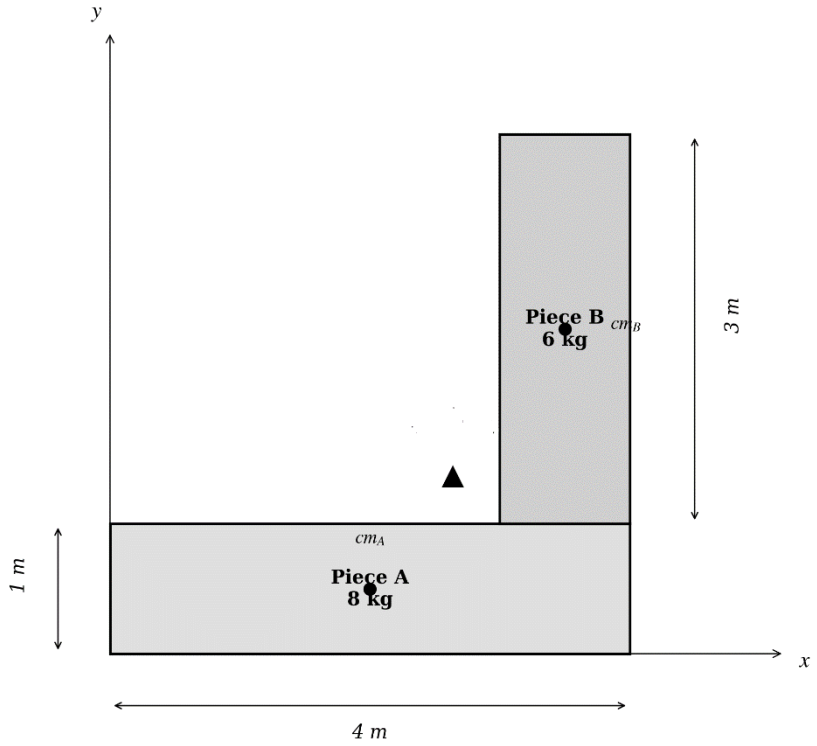
A uniform L-shaped plate is made by joining two uniform rectangular pieces:

Piece A: 4m long, 1m wide, mass 8kg (horizontal, from $x = 0$ to $x = 4$m, from $y = 0$ to $y = 1$m)

Piece B: 1m wide, 3m tall, mass 6kg (vertical, from $x = 3$m to $x = 4$m, from $y = 1$m to $y = 4$m)

Calculate the position of the centre of mass of the L-shaped plate.

Solution



The strategy is to find the centre of mass of each piece separately, then combine them as if they were two particles.

Centre of mass of Piece A (by symmetry of a rectangle): $(2.0\text{m}, 0.5\text{m})$, mass $= 8$kg

Centre of mass of Piece B (by symmetry of a rectangle): $(3.5\text{m}, 2.5\text{m})$, mass $= 6$kg

Total mass: $\text{M} = 8 + 6 = 14$kg

Using $\text{x}_{\text{cm}} = \frac{\sum \text{m}_i \text{x}_i}{\text{M}}$ and substituting:

$$\text{x}_{\text{cm}} = \frac{8\text{kg} \times 2.0\text{m} + 6\text{kg} \times 3.5\text{m}}{14\text{kg}} = 2.64\text{m}$$

Using $\text{y}_{\text{cm}} = \frac{\sum \text{m}_i \text{y}_i}{\text{M}}$ and substituting:

$$y_{cm} = \frac{8\text{kg} \times 0.5\text{m} + 6\text{kg} \times 2.5\text{m}}{14\text{kg}} = 1.36\text{m}$$

The centre of mass of the L-shaped plate is at $(2.64\text{m}, 1.36\text{m})$.

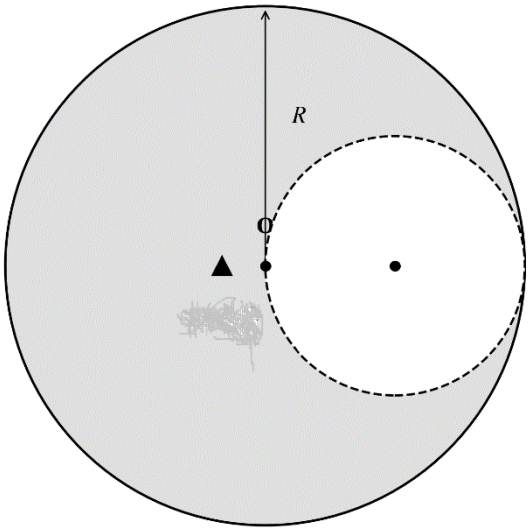
Making Sense of the Answer: *The centre of mass at* $(2.64,1.36)$ *lies near the junction of the two pieces, which is where the L-shape is "thickest." Notice that this point falls in the empty concave region of the L, not on the plate itself. This is perfectly valid: the centre of mass is a mathematical point, not necessarily a physical location within the material. A boomerang, a ring, and an L-bracket all have their centres of mass in empty space.*

Think Like a Physicist: *We solved this without any integration. We simply treated each rectangular piece as a particle at its own centre of mass, then combined them using the discrete particle formula. This "divide and conquer" strategy works for any composite body: break it into simple shapes whose centres of mass you know by symmetry, then combine. The same strategy will reappear when we use the Parallel Axis Theorem to find moments of inertia of composite bodies.*

HOT Example 12

A uniform circular disc of mass M and radius R has a smaller circular hole of radius $\frac{R}{2}$ cut out of it. The centre of the hole is at a distance $\frac{R}{2}$ from the centre of the original disc (the hole is tangent to the centre). Find the position of the centre of mass of the remaining piece.

Solution



We treat this as a subtraction problem. The key insight is:

$$(\textbf{Full disc}) = (\textbf{Remaining piece}) + (\textbf{Removed piece})$$

Therefore, the centre of mass of the full disc can be expressed in terms of the centres of mass of the remaining piece and the removed piece. We know the full disc and the removed piece; we solve for the remaining piece.

Setting up coordinates with the centre of the original disc at the origin:

Full disc: mass M, centre of mass at $x = 0$.

The removed piece is a disc of radius $\frac{R}{2}$. Since the disc is uniform, mass is proportional to area. The area of the removed piece is $\pi\left(\frac{R}{2}\right)^2 = \frac{\pi R^2}{4}$, which is $\frac{1}{4}$ of the full disc's area πR^2. Therefore the removed piece has mass $\frac{M}{4}$, and its centre of mass is at $x = \frac{R}{2}$.

The remaining piece has mass $M - \frac{M}{4} = \frac{3M}{4}$, and its centre of mass is at x_{cm} (what we want to find).

Applying the centre of mass condition for the full disc:

$$M \times 0 = \frac{3M}{4} \times x_{cm} + \frac{M}{4} \times \frac{R}{2}$$

$$0 = \frac{3M}{4} x_{cm} + \frac{MR}{8}$$

Making x_{cm} the subject:

$$\frac{3M}{4} x_{cm} = -\frac{MR}{8}$$

$$x_{cm} = -\frac{MR}{8} \times \frac{4}{3M} = -\frac{R}{6}$$

The centre of mass of the remaining piece is at a distance $\frac{R}{6}$ from the centre of the original disc, on the opposite side from the hole.

Making Sense of the Answer: *The negative sign tells us the centre of mass shifted away from the hole, which is physically correct: removing mass from one side of a symmetric body pulls the centre of mass toward the heavier (opposite) side. The shift is $\frac{R}{6}$, which is relatively small because the removed piece ($\frac{1}{4}$ of the total mass) is not dominant. If we had removed half the disc, the shift would be much larger.*

Think Like a Physicist: *The subtraction method is elegant: instead of integrating over the complicated remaining shape, we used the fact that (full disc) = (remaining piece) + (removed piece). Since we know the centre of mass of both the full disc and the removed piece, we solve for the unknown algebraically. This technique is extremely useful for any body with a hole, and it completely avoids integration. Remember this trick, because it appears frequently in examinations.*

And with that, we now know where every object's centre of mass lives, whether it is sitting politely at the geometric centre of a symmetric body, hiding between two unequal masses on a rod, lurking in the empty space inside an L-shaped bracket, or quietly shifting sideways when someone cuts a hole in a perfectly good disc. The centre of mass has been found, pinned down, and put to work. But knowing where a body rotates is only half the story. The next question is far more interesting: *how stubbornly does it resist being rotated?* Two objects of the same mass can behave very differently when you try to spin them, as Kipanga discovered with his metre ruler in the introduction. The answer lies in a quantity called the **moment of inertia**, and it is waiting for us in the next subtopic with a stack of integrals and a mischievous smile.

MOMENT OF INERTIA: THE ROTATIONAL MASS

If mass is the measure of how stubbornly an object resists being pushed, moment of inertia is the measure of how stubbornly it resists being spun. And as Kipanga discovered with his metre ruler, stubbornness in rotation depends not only on how much mass there is, but on where that mass is sitting relative to the axis of rotation.

This is the heart of the chapter. Everything before this section was preparation: we learned the language of rotation, and we found out where the natural rotation axis is. Now we answer the central question: *given a body and an axis, how do we calculate its resistance to rotation?*

The answer is the **moment of inertia**, often denoted by **I**. It plays exactly the same role in rotational dynamics that mass plays in translational dynamics. In $F = ma$, mass tells you how much force is needed to produce a given acceleration. In the rotational twin $\tau = I\alpha$, moment of inertia tells you how much torque is needed to produce a given angular acceleration. The larger the moment of inertia, the harder it is to spin the object (or to stop it once spinning).

But moment of inertia is richer than mass, and more interesting. The mass of an object is a single fixed number: a 5kg brick is 5kg whether you push it sideways, lift it upward, or throw it across the room. Moment of inertia, however, depends on the axis. The same metre ruler has a small moment of inertia about an axis through its centre (easy to spin) and a large moment of inertia about an axis through one end (hard to spin). The mass has not changed. The distribution of that mass relative to the axis has changed, and that changes everything.

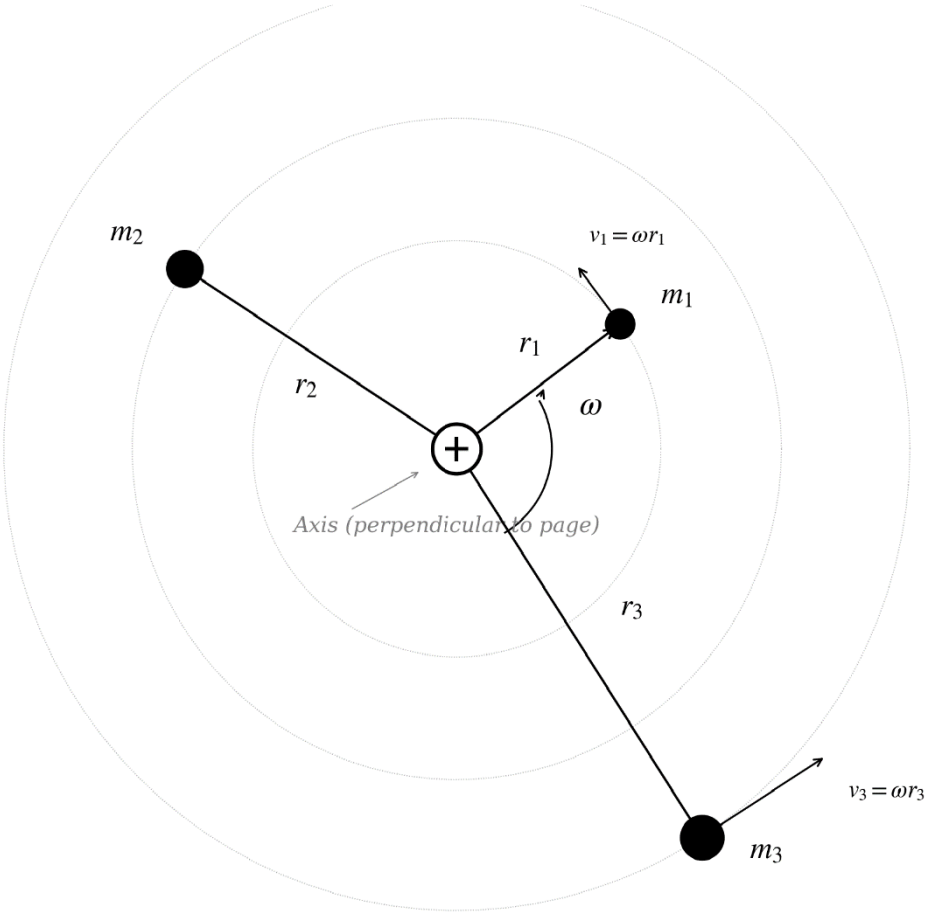
This subtopic is long. There is no way around it. We will derive the moment of inertia for eight standard shapes, each requiring a separate integration. If that sounds terrifying, take comfort in two facts. First, every derivation follows the same recipe. Once you understand the first one, the rest are variations on the same

theme. Second, at the end of this section, we will collect all eight results into a single summary table that you can use as a reference for the rest of the chapter (and the rest of your physics career). The individual derivations teach you *why* the formulas are what they are. The table lets you *use* them efficiently.

Let us begin.

What Moment of Inertia Means Physically

Imagine a collection of small masses $m_1, m_2, m_3, \ldots$ arranged at various distances $r_1, r_2, r_3, \ldots$ from a fixed rotation axis.



$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots = \sum m_i r_i^2$$

Figure: *Three particles of masses m_1, m_2, and m_3 rotate about a common axis (perpendicular to the page) with angular velocity ω. Each particle moves in a circle of radius r_i. The particle farther from the axis (m_3) has a larger tangential velocity ($v_3 = \omega r_3$) and contributes more to the moment of inertia because I depends on r^2.*

When the system rotates with angular velocity ω, each mass moves in a circle. The linear velocity of the i-th mass is $v_i = \omega r_i$ (from Chapter 7), and its kinetic energy is:

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

The total kinetic energy of the entire system is the sum over all particles:

$$K_{\text{total}} = \sum \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} \omega^2 \sum m_i r_i^2$$

Since ω is the same for all particles in a rigid body (they all rotate together), we can factor it out. What remains inside the sum depends only on the masses and their distances from the axis. We give this sum a name:

$$\mathbf{I = \sum m_i r_i^2}$$

This is the **moment of inertia** of the system about the given axis. It has units of $\text{kg} \cdot \text{m}^2$.

With this definition, the total rotational kinetic energy becomes:

$$\mathbf{KE = \frac{1}{2} I\omega^2}$$

Compare this with the translational kinetic energy $KE = \frac{1}{2}mv^2$. The structure is identical: mass m is replaced by moment of inertia I, and velocity v is replaced by angular velocity ω. The twin language strikes again.

Why r^2, not r?

You might wonder: *why does distance appear squared?* The answer comes from the kinetic energy. Each particle's kinetic energy is $\frac{1}{2}mv^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}m\omega^2 r^2$. The r^2 arises because the linear velocity itself is proportional to r (particles farther from the axis move faster), and kinetic energy depends on velocity squared. As a result, a mass at double the distance contributes **four times** as much to the moment of inertia, not twice as much. This is why the distribution of mass matters so dramatically: even a small mass placed far from the axis can dominate the moment of inertia.

This is exactly what Kipanga experienced. When he spun the ruler from the centre, most of the mass was close to the axis, and the r^2 factors were small. When he spun it from one end, half the mass was far from the axis, and the r^2 factors were large. Same mass, same ruler, very different moment of inertia.

For a continuous body, where the mass is distributed smoothly rather than concentrated in discrete particles, we replace the sum with an integral:

$$\mathbf{I = \int r^2\, dm}$$

where r is the perpendicular distance from the element dm to the rotation axis. This integral is the workhorse of this section. We will evaluate it for each of the standard shapes in the subsections that follow.

The recipe for every derivation is the same:

5) Choose a coordinate system and define the rotation axis.
6) Identify a thin element of the body at distance r from the axis.
7) Express the mass of the element dm in terms of the body's geometry and density.
8) Write $dI = r^2\, dm$.
9) Integrate over the entire body.
10) Simplify and express the result in terms of total mass M and the relevant dimension.

Once you have this recipe in your head, every derivation becomes a cooking exercise: same method, different ingredients, different result. And if cooking integrals sounds unappetizing, remember that by the time you have cooked eight of them, you will be a master chef! Or at least a master physicist, which is arguably more useful.

Before we begin the derivations, let us warm up with a simple discrete example to make sure the concept is solid.

BINDER Example 13

Four particles, each of mass 0.5kg, are connected by light rigid rods to form a square of side 0.4m. The system rotates about an axis through the centre of the square, perpendicular to the plane of the square.

(a) Calculate the moment of inertia of the system about this axis.

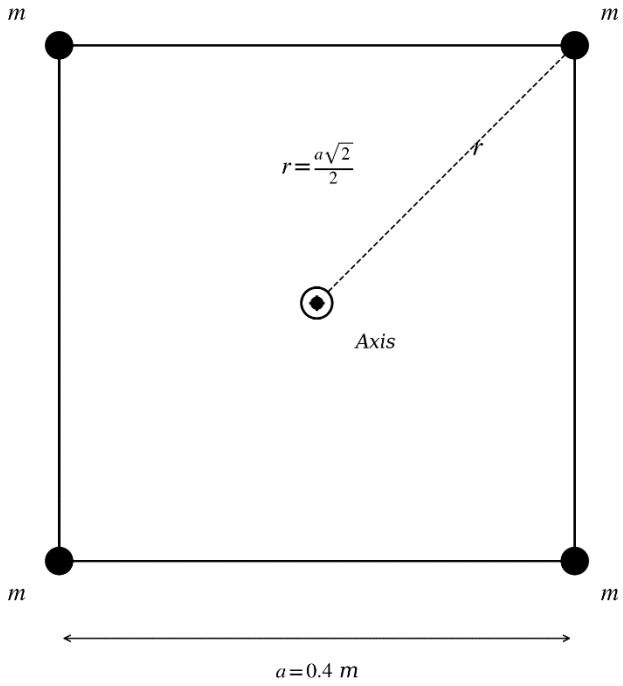
(b) If the system rotates at 10rad/s, calculate the rotational kinetic energy.

(c) Explain why moving the masses to the corners of a larger square (say, 0.8m side) would quadruple the moment of inertia, even though the total mass is unchanged.

Solution

(a) Each particle is at a corner of the square. The distance from the centre of the square to each corner is half the diagonal:

$$r = \frac{\text{diagonal}}{2} = \frac{0.4\sqrt{2}}{2}\text{m} = 0.283\text{m}$$



Since all four particles are at the same distance from the axis:

$$\mathrm{I} = \sum \mathrm{m}_i \mathrm{r}_i^2 = 4 \times 0.5\mathrm{kg} \times (0.283\mathrm{m})^2 = 4 \times 0.5 \times 0.080\mathrm{kgm}^2 = 0.16\mathrm{kgm}^2$$

The moment of inertia is $0.16\mathrm{kgm}^2$.

(b) Using $\mathrm{KE} = \frac{1}{2}\mathrm{I}\omega^2$ and substituting:

$$\mathrm{KE} = \frac{1}{2} \times 0.16\mathrm{kgm}^2 \times (10\mathrm{rad/s})^2 = 8.0\mathrm{J}$$

The rotational kinetic energy is 8.0J.

(c) If the side length doubles from 0.4m to 0.8m, the distance from the centre to each corner also doubles (0.283m to 0.566m). Because moment of inertia depends on r^2, doubling r quadruples r^2, and hence quadruples the moment of inertia. The total mass ($4 \times 0.5 = 2\mathrm{kg}$) has not changed at all. This demonstrates the key principle: moment of inertia is not just about how much mass there is, but about how far that mass is from the axis. Spreading mass outward has a dramatic effect because of the r^2 dependence.

Making Sense of the Answer: *The moment of inertia of* $0.16\mathrm{kgm}^2$ *is modest because both the masses (*0.5kg *each) and the distances from the axis (*0.28m*) are small. Doubling the square size to* 0.8m *would give* $\mathrm{I} = 0.64\mathrm{kgm}^2$*, four times larger, even though not a single gram has been added. This is the essence of the* r^2 *factor: geometry matters as much as mass.*

Think Like a Physicist: *This example used discrete masses, so we could compute I by simple addition. For continuous bodies (rods, discs, spheres), the "particles" are infinitely small elements dm, and the sum becomes an integral. But the idea is identical: add up* r^2 dm *for every bit of mass. The integral is just a sum with better manners.*

With the concept established, we now march into the derivations. Take a deep breath, keep your pen steady, and remember: each one follows the same recipe. If you can survive the first, the rest are just different flavours of the same meal.

Uniform Rod: Axis Through the Centre

This is our first derivation using integration, and it is the simplest. If you can follow this one, every derivation that follows uses the same logic with slightly different geometry. Think of it as learning to ride a bicycle: wobble through this one, and the rest become easier.

Consider a uniform rod of mass **M** and length **L**. The axis of rotation passes through the centre of the rod, perpendicular to it. We want to find the moment of inertia **I** about this axis.

Setting up the integral

We place the rod along the x-axis with the centre at the origin, so the rod extends from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$. The axis of rotation is at $x = 0$.

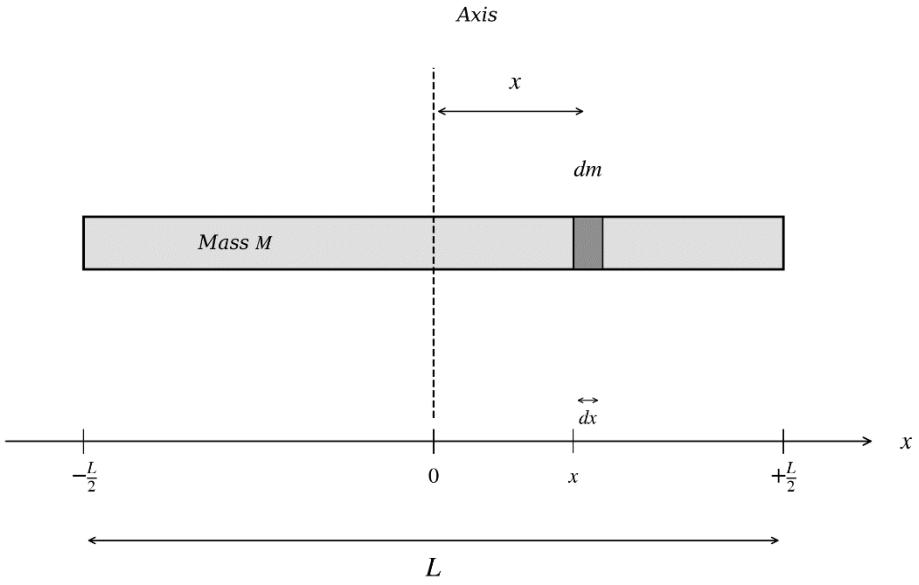


Figure: *A uniform rod of mass M and length L with the rotation axis through the centre, perpendicular to the rod. The rod extends from* $x = -\frac{L}{2}$ *to* $x = +\frac{L}{2}$. *A small element of mass dm at position x is at distance x from the axis.*

The rod is uniform, so the linear mass density (mass per unit length) is constant:

$$\lambda = \frac{M}{L}$$

We divide the rod into thin elements. A typical element is a tiny slice at position x, with width dx. Its mass is:

$$dm = \lambda\, dx = \frac{M}{L} dx$$

The distance of this element from the rotation axis is simply $|x|$, but since we square it in the moment of inertia formula ($r^2 = x^2$), the sign does not matter.
The contribution of this element to the moment of inertia is:

$$dI = x^2\, dm = x^2 \times \frac{M}{L} dx$$

Performing the integration

To find the total moment of inertia, we integrate over the entire rod from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$:

$$I = \int_{-L/2}^{+L/2} x^2 \times \frac{M}{L}\, dx$$

The constant $\frac{M}{L}$ comes out of the integral:

$$I = \frac{M}{L} \int_{-L/2}^{+L/2} x^2\, dx$$

The integral of x^2 is $\frac{x^3}{3}$:

$$I = \frac{M}{L} \left[\frac{x^3}{3}\right]_{-L/2}^{+L/2}$$

Evaluating the limits:

$$I = \frac{M}{L} \times \frac{1}{3} \left(\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3 \right)$$

$$= \frac{M}{L} \times \frac{1}{3}\left(\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right)$$

$$= \frac{M}{L} \times \frac{1}{3} \times \frac{2L^3}{8}$$

$$= \frac{M}{L} \times \frac{L^3}{12}$$

$$\mathbf{I = \frac{1}{12} ML^2}$$

This is the moment of inertia of a uniform rod about an axis through its centre, perpendicular to the rod.

Making Sense of the Result: *The result* $\frac{1}{12}ML^2$ *tells us several things. First, I is proportional to M: a heavier rod is harder to spin, which makes sense. Second, I is proportional to* L^2*: a longer rod is much harder to spin (doubling the length quadruples I), because the mass extends farther from the axis and the* r^2 *factor amplifies this effect. Third, the numerical coefficient* $\frac{1}{12}$ *is specific to this shape and this axis. Different shapes and different axes give different coefficients, as we will see in the following subsections.*

As a quick check: if L = 0 (a point mass at the axis), then I = 0. That is correct, because a point mass on the axis has zero distance from the axis and contributes nothing to the moment of inertia. If all the mass were concentrated at the ends $(\pm\frac{L}{2})$*, we would get* $I = 2 \times \frac{M}{2}\left(\frac{L}{2}\right)^2 = \frac{ML^2}{4}$*, which is larger than* $\frac{ML^2}{12}$*. This confirms that a continuous rod, with its mass spread between the ends and the centre, has a smaller moment of inertia than the same mass placed only at the ends. Mass near the axis contributes less, pulling the average down.*

Now let us cook the next dish. Same recipe, different ingredient.

Uniform Rod: Axis Through One End

This is the situation Kipanga struggled with in the introduction. The ruler is now pivoted at one end, and we expect a larger moment of inertia because more mass is farther from the axis.

Consider the same uniform rod of mass M and length L, but now the axis passes through one end, perpendicular to the rod.

Setting up the integral

This time we place the rod along the x-axis from $x = 0$ (where the axis is) to $x = L$. Everything else is the same as before.

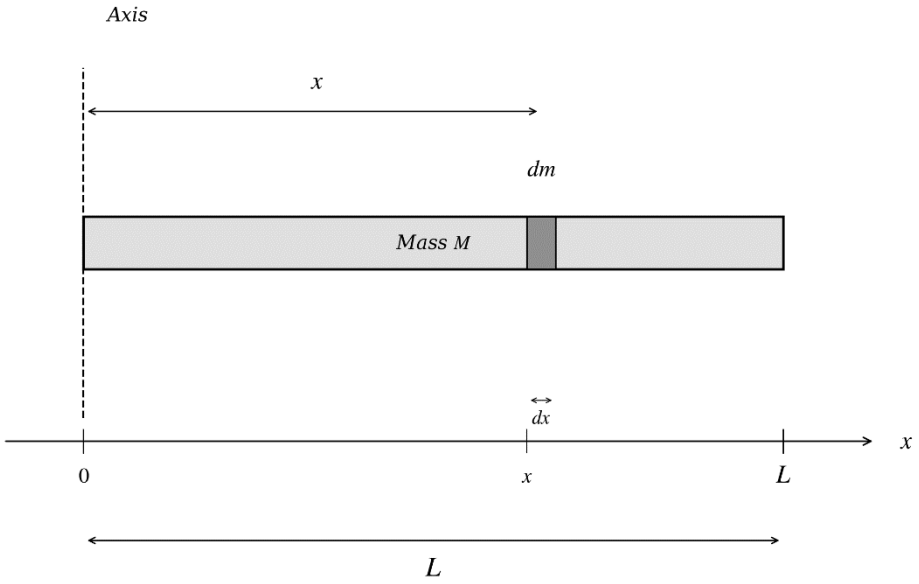


Figure: *The same uniform rod, but with the rotation axis through one end, perpendicular to the rod. The rod extends from* $x = 0$ *to* $x = L$*. A small element of mass dm at position x is at distance x from the axis. All mass is on one side, resulting in a larger moment of inertia.*

The mass element at position x is still $\mathrm{dm} = \frac{\mathrm{M}}{\mathrm{L}}\mathrm{dx}$, and its distance from the axis is x.

$$\mathrm{dI} = x^2\,\mathrm{dm} = x^2 \times \frac{\mathrm{M}}{\mathrm{L}}\mathrm{dx}$$

Performing the integration

$$\mathrm{I} = \frac{\mathrm{M}}{\mathrm{L}}\int_0^{\mathrm{L}} x^2\ \mathrm{dx} = \frac{\mathrm{M}}{\mathrm{L}}\left[\frac{x^3}{3}\right]_0^{\mathrm{L}} = \frac{\mathrm{M}}{\mathrm{L}} \times \frac{\mathrm{L}^3}{3}$$

$$\mathbf{I} = \frac{\mathbf{1}}{\mathbf{3}}\mathbf{ML^2}$$

This is the moment of inertia of a uniform rod about an axis through one end, perpendicular to the rod.

Making Sense of the Result: *Comparing the two results:* $\frac{1}{3}\mathrm{ML}^2$ *(end) versus* $\frac{1}{12}\mathrm{ML}^2$ *(centre). The end-axis moment of inertia is exactly four times larger! This makes physical sense: when the axis is at the end, all the mass is on one side, extending all the way to a distance L from the axis. When the axis is at the centre, the farthest mass is only* $\frac{\mathrm{L}}{2}$ *away. Since I depends on* r^2*, moving the axis from the centre to the end has a dramatic effect.*

This is exactly what Kipanga experienced. The same ruler, the same mass, but four times harder to spin from the end than from the centre. Physics does not care about your muscles; it cares about where the mass sits relative to the axis.

Two rods down. Let us leave the one-dimensional world and step into the plane.

Ring (Thin Hoop): Axis Through the Centre, Perpendicular to the Plane

After two rod derivations, this one will feel like a holiday. It is the simplest moment of inertia of all, and the reason is beautiful: every particle in a thin ring is at exactly the same distance from the axis.

Consider a thin ring (hoop) of mass M and radius R. The axis of rotation passes through the centre of the ring, perpendicular to its plane.

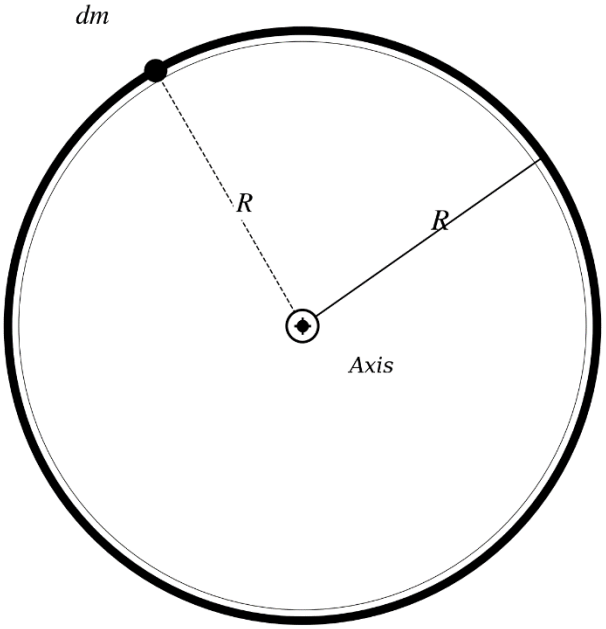


Figure: *A thin ring (hoop) of mass M and radius R, with the rotation axis through the centre, perpendicular to the plane. Every element of mass dm lies at the same distance R from the axis.*

The derivation

Every tiny piece of mass in the ring sits on the circumference, at exactly the same distance R from the axis. This makes the integral remarkably simple.

Using the general definition $\mathrm{I} = \int \mathrm{r}^2\,\mathrm{dm}$, and noting that $\mathrm{r} = \mathrm{R}$ for every element of mass in the ring, R is a constant and comes out of the integral. Integrating from zero mass to the full mass M:

$$I = \int_0^M r^2 \, dm = \int_0^M R^2 \, dm = R^2 \int_0^M dm = R^2 [m]_0^M = R^2 M$$

$$\mathbf{I = MR^2}$$

That is it. No complicated integration, no tricky limits, no algebraic gymnastics. The moment of inertia of a ring is simply MR^2.

When the distance from the axis is the same for every element, the integral collapses to a simple product. The ring is the one shape in this section where the integration is trivial, because the crucial variable (the distance from the axis) does not vary across the body.

Making Sense of the Result: *This is the largest possible moment of inertia for a body of mass M whose farthest point is at distance R from the axis. Why? Because every single gram of mass is at the maximum distance R. No mass is wasted near the axis where it would contribute less. A ring is the "most efficient" shape for resisting rotation: all its mass works at full* r^2 *leverage.*

This is also why the ring result serves as the reference against which all other shapes are compared. Every other shape of the same mass and outer radius will have $I < MR^2$*, because some of its mass will be closer to the axis than R. The disc, the sphere, the cylinder: all have coefficients less than 1 in front of* MR^2*. The ring is the champion, with a coefficient of exactly 1.*

If every derivation were this friendly, physics students would never complain. Unfortunately, the disc is next, and it requires actual work. But not much.

Disc: Axis Through the Centre, Perpendicular to the Plane

A solid disc differs from a ring in one crucial way: its mass is not all at the rim. Mass is spread from the centre ($r = 0$) to the edge ($r = R$), and the mass near the centre contributes less to the moment of inertia than the mass near the edge. We therefore expect $I < MR^2$, but we need the integration to find out by how much.

Consider a uniform solid disc of mass M, radius R, and negligible thickness. The axis of rotation passes through the centre, perpendicular to the disc.

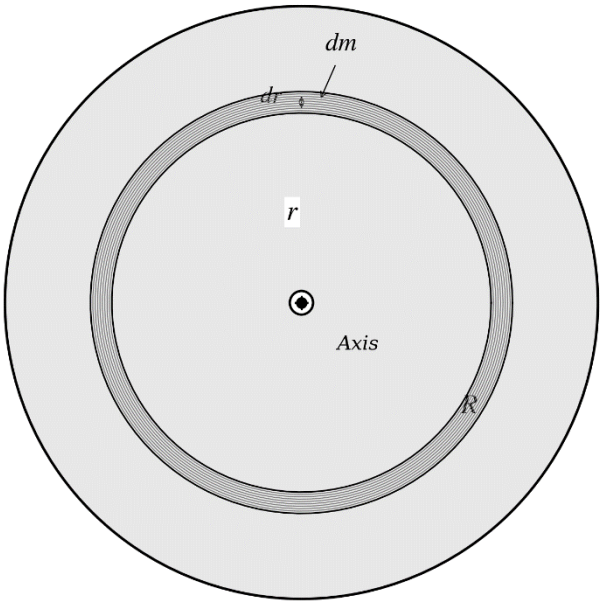


Figure: *A uniform solid disc of mass M and radius R, with the rotation axis through the centre, perpendicular to the plane. The disc is divided into concentric thin ring elements. A typical ring at radius r with width dr has area* $dA = 2\pi r \, dr$ *and mass* $dm = \sigma \, dA$*.*

Choosing the element

The key insight is to choose an element that is entirely at one distance from the axis. *What shape satisfies this?* A thin ring! If we slice the disc into concentric thin rings, each ring has all its mass at the same distance from the axis. Each ring can then use the result we just derived ($dI = r^2 \, dm$) without further thought.

Consider a thin ring of radius r and width dr within the disc. Its area is approximately the circumference times the width:

$$dA = 2\pi r\, dr$$

Since the disc is uniform, the surface-mass density (mass per unit area) is:

$$\sigma = \frac{M}{\pi R^2}$$

The mass of the thin ring element is:

$$dm = \sigma\, dA = \frac{M}{\pi R^2} \times 2\pi r\, dr = \frac{2Mr\, dr}{R^2}$$

Setting up the integral

The moment of inertia contribution from this thin ring is:

$$dI = r^2\, dm = r^2 \times \frac{2Mr\, dr}{R^2} = \frac{2M}{R^2} r^3\, dr$$

Performing the integration

Integrating from the centre ($r = 0$) to the edge ($r = R$):

$$I = \frac{2M}{R^2}\int_0^R r^3\ dr$$

$$= \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{2M}{R^2} \times \frac{R^4}{4}$$

$$= \frac{2MR^2}{4}$$

$$\mathbf{I = \frac{1}{2}MR^2}$$

The moment of inertia of a uniform solid disc about an axis through its centre, perpendicular to its plane, is $\frac{1}{2}MR^2$.

Making Sense of the Result: *The disc has half the moment of inertia of a ring of the same mass and radius. This makes excellent physical sense: the ring has all its mass at distance R, while the disc has mass spread from* $r = 0$ *to* $r = R$. *The mass near the centre contributes very little (because of the* r^2 *factor), so the average contribution is less than* R^2. *The factor of* $\frac{1}{2}$ *tells us that the "effective distance" of the disc's mass from the axis is* $\frac{R}{\sqrt{2}} \approx 0.707R$, *which is a reasonable average between 0 and R, weighted by* r^2.

Another way to see it: if you could magically redistribute the disc's mass so that all of it sits at the rim (turning it into a ring), the moment of inertia would double. This is why a bicycle wheel (essentially a ring, with most mass in the rim and tyre) has a larger moment of inertia than a solid disc of the same total mass and radius. The wheel is harder to spin up but also harder to slow down, which is why heavy rims help maintain cycling speed on flat roads.

That is four shapes down and four to go. The pattern should feel familiar by now: choose a thin element whose mass is all at the same distance from the axis, write dm, write $dI = r^2\, dm$, integrate. If you have been following the recipe, the next four are variations on the same theme, just with slightly different geometry. Let us step into three dimensions.

Solid Sphere: Axis Through the Centre

Welcome to three dimensions. The solid sphere is the most elegant derivation in this section, and the trick that makes it work is the same trick we used for the disc: choose an element whose moment of inertia you already know. For the disc, we sliced it into thin rings. For the sphere, we slice it into thin discs.

Consider a uniform solid sphere of mass M and radius R. The axis of rotation passes through the centre.

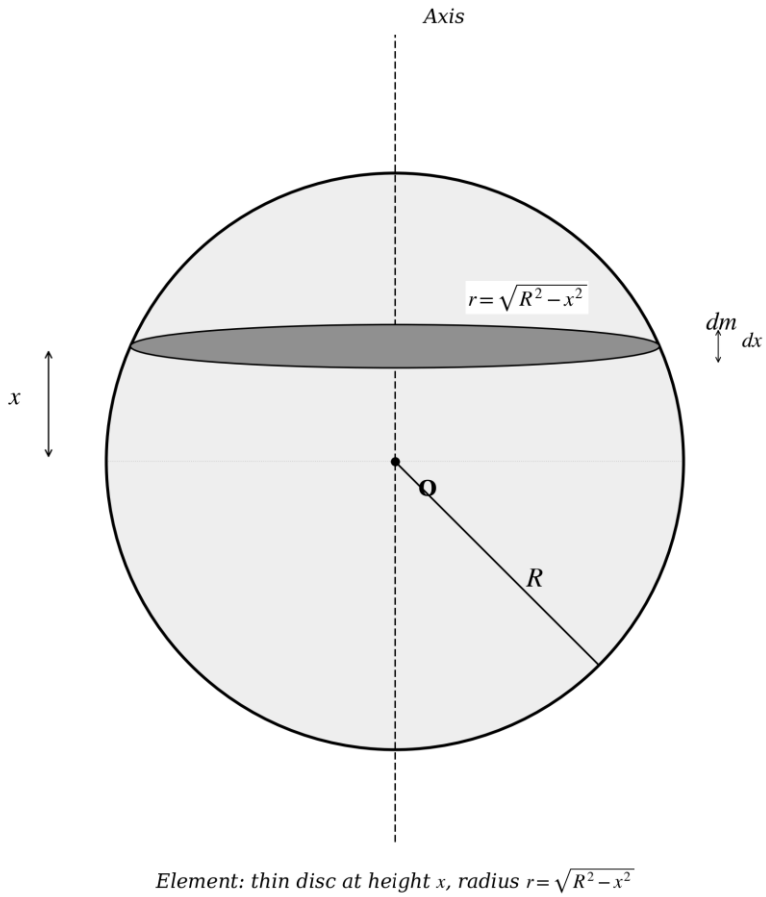


Figure: *A uniform solid sphere of mass M and radius R, with the rotation axis passing vertically through the centre. The sphere is sliced into thin disc elements perpendicular to the axis. A typical disc at height x above the centre has radius* $r = \sqrt{R^2 - x^2}$ *and thickness dx.*

Choosing the element

We slice the sphere into thin circular discs perpendicular to the rotation axis. Imagine the axis running vertically through the sphere. Each horizontal slice is a thin disc at height x above the centre, with thickness dx.

At height x, the cross-section of the sphere is a circle. Using the equation of the sphere ($x^2 + r^2 = \mathrm{R}^2$), the radius of this disc is:

$$\mathrm{r} = \sqrt{\mathrm{R}^2 - x^2}$$

The volume of the thin disc element is:

$$\mathrm{dV} = \pi \mathrm{r}^2 \, \mathrm{dx} = \pi\left(\mathrm{R}^2 - x^2\right) \mathrm{dx}$$

Since the sphere is uniform, the volume-mass density is:

$$\rho = \frac{\mathrm{M}}{\frac{4}{3}\pi \mathrm{R}^3} = \frac{3\mathrm{M}}{4\pi \mathrm{R}^3}$$

The mass of the thin disc element is:

$$\mathrm{dm} = \rho \, \mathrm{dV} = \frac{3\mathrm{M}}{4\pi \mathrm{R}^3} \times \pi\left(\mathrm{R}^2 - x^2\right) \mathrm{dx} = \frac{3\mathrm{M}}{4\mathrm{R}^3}\left(\mathrm{R}^2 - x^2\right) \mathrm{dx}$$

Moment of inertia of each disc element

Each thin disc has mass dm and radius $r = \sqrt{R^2 - x^2}$. From the previous subsection, the moment of inertia of a disc about its central axis is $\frac{1}{2}m_{disc}r_{disc}^2$. Therefore:

$$dI = \frac{1}{2}dm \times r^2 = \frac{1}{2} \times \frac{3M}{4R^3}(R^2 - x^2)\,dx \times (R^2 - x^2)$$

$$= \frac{3M}{8R^3}(R^2 - x^2)^2\,dx$$

Performing the integration

We integrate from $x = -R$ (bottom of sphere) to $x = +R$ (top of sphere):

$$I = \frac{3M}{8R^3}\int_{-R}^{+R}(R^2 - x^2)^2\,dx$$

Expanding the bracket:

$$(R^2 - x^2)^2 = R^4 - 2R^2x^2 + x^4$$

$$I = \frac{3M}{8R^3}\int_{-R}^{+R}(R^4 - 2R^2x^2 + x^4)\,dx$$

Integrating term by term:

$$\int_{-R}^{+R} R^4\,dx = R^4[x]_{-R}^{+R} = R^4 \times 2R = 2R^5$$

$$\int_{-R}^{+R} 2R^2x^2\,dx = 2R^2\left[\frac{x^3}{3}\right]_{-R}^{+R} = 2R^2 \times \frac{2R^3}{3} = \frac{4R^5}{3}$$

$$\int_{-R}^{+R} x^4\,dx = \left[\frac{x^5}{5}\right]_{-R}^{+R} = \frac{2R^5}{5}$$

Combining:

$$I = \frac{3M}{8R^3}\left(2R^5 - \frac{4R^5}{3} + \frac{2R^5}{5}\right)$$

Finding a common denominator (15):

$$2R^5 - \frac{4R^5}{3} + \frac{2R^5}{5} = R^5\left(\frac{30 - 20 + 6}{15}\right) = \frac{16R^5}{15}$$

Therefore:

$$I = \frac{3M}{8R^3} \times \frac{16R^5}{15} = \frac{48MR^2}{120} = \frac{2MR^2}{5}$$

$$\mathbf{I = \frac{2}{5}MR^2}$$

The moment of inertia of a uniform solid sphere about any axis through its centre is $\frac{2}{5}MR^2$.

Making Sense of the Result: *The coefficient $\frac{2}{5}$ is less than $\frac{1}{2}$ (the disc) and much less than 1 (the ring). This is because the sphere has mass distributed in three dimensions, and a significant fraction of that mass is close to the axis. Think of it this way: in a sphere, only the "equatorial belt" of mass is at the full distance R from the axis. Mass near the poles is very close to the axis and contributes almost nothing to I. The three-dimensional spreading of mass reduces the effective r^2 contribution compared to a flat disc.*

Notice that the result says "any axis through the centre." Because a sphere is perfectly symmetric, every axis through the centre gives the same moment of inertia. A rod has different values for different axes; a sphere does not. Symmetry is a beautiful thing.

That derivation was the longest so far, but the strategy was clean: slice the sphere into discs, use the disc formula, and add them up. The hollow sphere is next, and it is actually shorter because a hollow sphere is just a surface with no interior to worry about.

Hollow Sphere (Thin Spherical Shell): Axis Through the Centre

A thin spherical shell is the three-dimensional cousin of the ring. Just as a ring has all its mass at one distance from the centre (in a plane), a spherical shell has all its mass at one distance from the centre (in space). However, unlike the ring, the mass on a spherical shell is not all at the same distance from the rotation axis, because the axis passes through only two points on the shell (the poles), while the equator is at the maximum distance R.

Consider a thin uniform spherical shell of mass M and radius R. The axis of rotation passes through the centre.

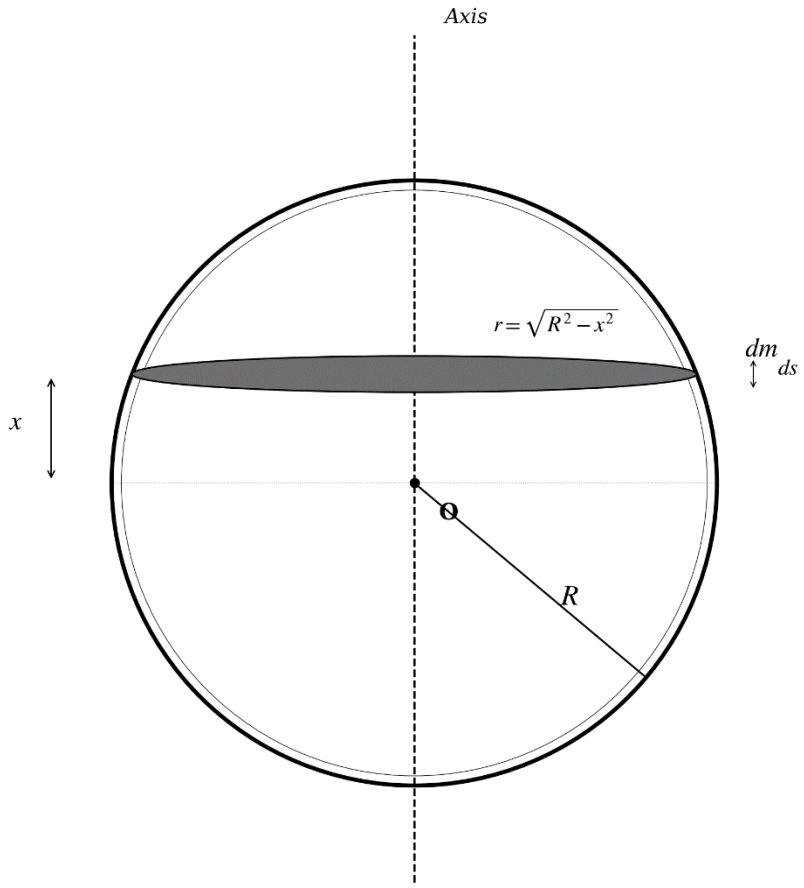


Figure: *A thin uniform spherical shell of mass M and radius R, with the rotation axis passing vertically through the centre. The shell is sliced into thin ring elements at height x. Each ring has radius* $r = \sqrt{R^2 - x^2}$ *and surface width ds. The area of each ring element is* $dA = 2\pi R\, dx$.

Choosing the element

We slice the shell into thin rings by cutting it with planes perpendicular to the rotation axis. Each ring is at height x above the centre, has radius $\mathrm{r} = \sqrt{\mathrm{R}^2 - x^2}$, and has width ds measured along the surface of the sphere.

The width along the surface is not dx but $\mathrm{ds} = \mathrm{R}\,\mathrm{d}\theta$, where θ is the polar angle measured from the top of the sphere. However, it is simpler to work with x as the variable. From the geometry: $x = \mathrm{R}\cos\theta$, so $\mathrm{dx} = -\mathrm{R}\sin\theta\,\mathrm{d}\theta$. The circumference of the ring at height x is $2\pi\mathrm{r} = 2\pi\sqrt{\mathrm{R}^2 - x^2}$ and its width along the surface is $\mathrm{ds} = \frac{\mathrm{R\,dx}}{\sqrt{\mathrm{R}^2 - x^2}}$.

The area of the ring element is:

$$dA = 2\pi\sqrt{R^2 - x^2} \times \frac{R\,dx}{\sqrt{R^2 - x^2}} = 2\pi R\,dx$$

This is a remarkably clean result: the area of each ring element depends only on dx, not on x. Every thin horizontal strip of a sphere has the same area!

The surface-mass density is:

$$\sigma = \frac{M}{4\pi R^2}$$

The mass of the ring element is:

$$dm = \sigma\,dA = \frac{M}{4\pi R^2} \times 2\pi R\,dx = \frac{M}{2R}\,dx$$

Setting up the integral

Each ring element has mass dm and is at distance $r = \sqrt{R^2 - x^2}$ from the rotation axis. Since the ring is thin (all its mass at the same distance from the axis), its moment of inertia is:

$$dI = r^2\,dm = \left(R^2 - x^2\right) \times \frac{M}{2R}\,dx$$

Performing the integration

$$I = \frac{M}{2R}\int_{-R}^{+R}\left(R^2 - x^2\right)\,dx$$

$$= \frac{M}{2R}\left[R^2 x - \frac{x^3}{3}\right]_{-R}^{+R}$$

$$= \frac{M}{2R}\left(\left(R^3 - \frac{R^3}{3}\right) - \left(-R^3 + \frac{R^3}{3}\right)\right)$$

$$= \frac{M}{2R}\left(\frac{2R^3}{3} + \frac{2R^3}{3}\right)$$

$$= \frac{M}{2R} \times \frac{4R^3}{3}$$

$$= \frac{4MR^2}{6} = \frac{2MR^2}{3}$$

$$\mathbf{I = \frac{2}{3}MR^2}$$

The moment of inertia of a thin uniform spherical shell about any axis through its centre is $\frac{2}{3}MR^2$.

Making Sense of the Result: *The hollow sphere $\left(\frac{2}{3}\right)$ has a larger coefficient than the solid sphere $\left(\frac{2}{5}\right)$. This should not be surprising: the hollow sphere has all its mass at the surface (distance R from the centre), while the solid sphere has mass distributed throughout the interior, including regions close to the axis. Mass near the axis contributes little to I, so the solid sphere's "wasted" interior mass pulls its coefficient down.*

Comparing all the round shapes so far: ring (1), hollow sphere $\left(\frac{2}{3}\right)$, disc $\left(\frac{1}{2}\right)$ solid sphere $\left(\frac{2}{5}\right)$. The trend is clear: the more mass is pushed outward (toward the maximum distance R), the larger the coefficient. The ring is the extreme case (all mass at R), and the solid sphere has the most mass near the axis (filling the entire interior).

As with the solid sphere, the result is the same for any axis through the centre, because the spherical shell is symmetric about every such axis.

Six down, two to go. The cylinders are next, and if you have survived the spheres, the cylinders will feel like a gentle stroll. They are essentially stretched versions of the disc and the ring.

Solid Cylinder: Axis Along the Cylinder

If you survived the sphere, this one will make you smile. A solid cylinder rotating about its own axis is mathematically identical to a solid disc. Not similar. Not approximately the same. Identical.

Consider a uniform solid cylinder of mass M, radius R, and length L. The axis of rotation is the central axis of the cylinder (the axis running along its length through the centre of each circular face).

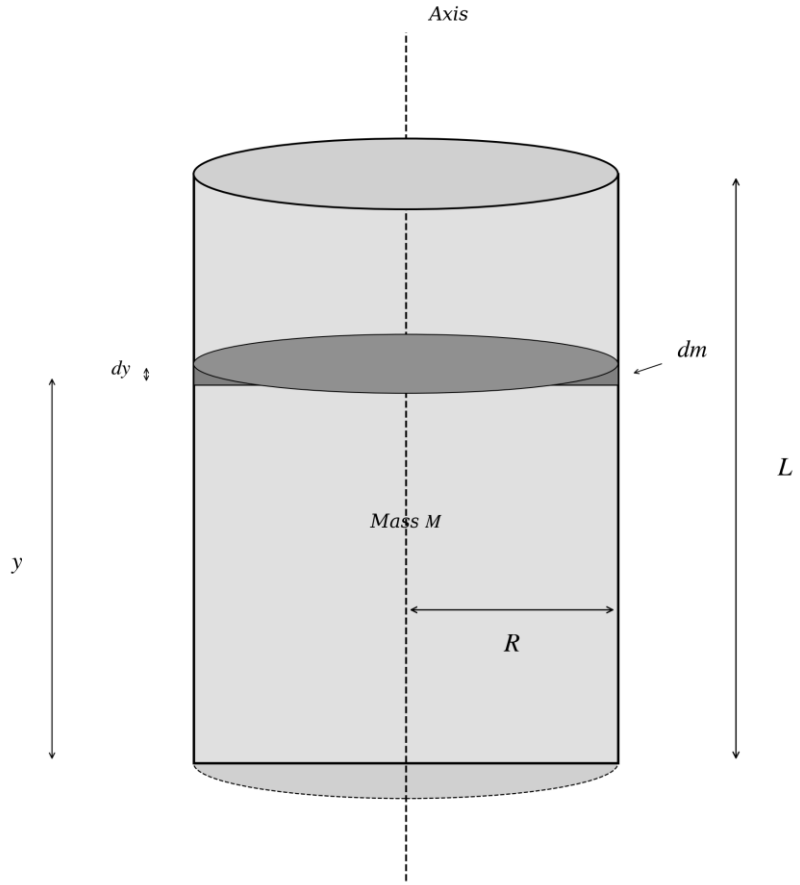


Figure: *A uniform solid cylinder of mass M, radius R, and length L, with the rotation axis along the central axis. The cylinder is sliced into thin disc elements at height y. Each disc has the same radius R, so the moment of inertia is identical to that of a disc.*

Why it is the same as a disc

Imagine slicing the cylinder into many thin circular discs, each perpendicular to the axis. Every disc has the same radius R. For each disc, the distance of any mass element from the rotation axis depends only on how far that element is from the central axis, which is the radial distance r. The position of the disc along the length of the cylinder (its height) does not affect this distance at all, because the axis runs along the length.

In other words, moving a disc up or down along the axis does not change its distance from the axis. Every disc, regardless of where it sits along the cylinder, contributes to the moment of inertia in exactly the same way as every other disc. Stacking identical discs on top of each other does not change the geometry relative to the axis.

The derivation

Each thin disc at position y along the cylinder has mass $\mathrm{dm} = \frac{\mathrm{M}}{\mathrm{L}}\mathrm{dy}$ and moment of inertia (as learned earlier in the ring subsection):

$$\mathrm{dI} = \frac{1}{2}\mathrm{dm} \times \mathrm{R}^2 = \frac{1}{2} \times \frac{\mathrm{M}}{\mathrm{L}}\mathrm{dy} \times \mathrm{R}^2$$

Integrating over the full length of the cylinder from $y = 0$ to $y = \mathrm{L}$:

$$I = \int_0^L \frac{1}{2}\frac{M}{L}R^2\,dy = \frac{MR^2}{2L}\int_0^L dy = \frac{MR^2}{2L} \times [y]_0^L = \frac{MR^2}{2L} \times L$$

$$\mathbf{I = \frac{1}{2}MR^2}$$

The moment of inertia of a uniform solid cylinder about its central axis is $\frac{1}{2}MR^2$, exactly the same as a disc.

Making Sense of the Answer: *The length L has cancelled completely. A short, fat cylinder and a long, thin cylinder of the same mass and radius have exactly the same moment of inertia about their central axis. This makes perfect sense: stretching a cylinder along the axis does not move any mass farther from the axis or closer to it. Every particle stays at the same radial distance* r, *regardless of how long the cylinder is. The moment of inertia cares only about the perpendicular distance from the axis, and the length is parallel to the axis.*

This is why a solid cylinder and a solid disc give the same formula. A disc is just a very short cylinder, and a cylinder is just a very tall disc. From the axis's point of view, they look identical.

Practical implication: a coin, a hockey puck, a tin can (solid), and a telegraph pole all have $I = \frac{1}{2}MR^2$ *about their central axis, provided they are uniform. The only things that matter are M and R.*

One more to go. The last derivation in this section. Let us finish strong.

Hollow Cylinder (Cylindrical Shell): Axis Along the Cylinder

Just as the solid cylinder is a stretched disc, the hollow cylinder is a stretched ring. And just as the ring was the simplest derivation in the flat world, the hollow cylinder will be almost as simple in the three-dimensional world.

Consider a thin uniform hollow cylinder (cylindrical shell) of mass M, radius R, and length L. The axis of rotation is the central axis of the cylinder.

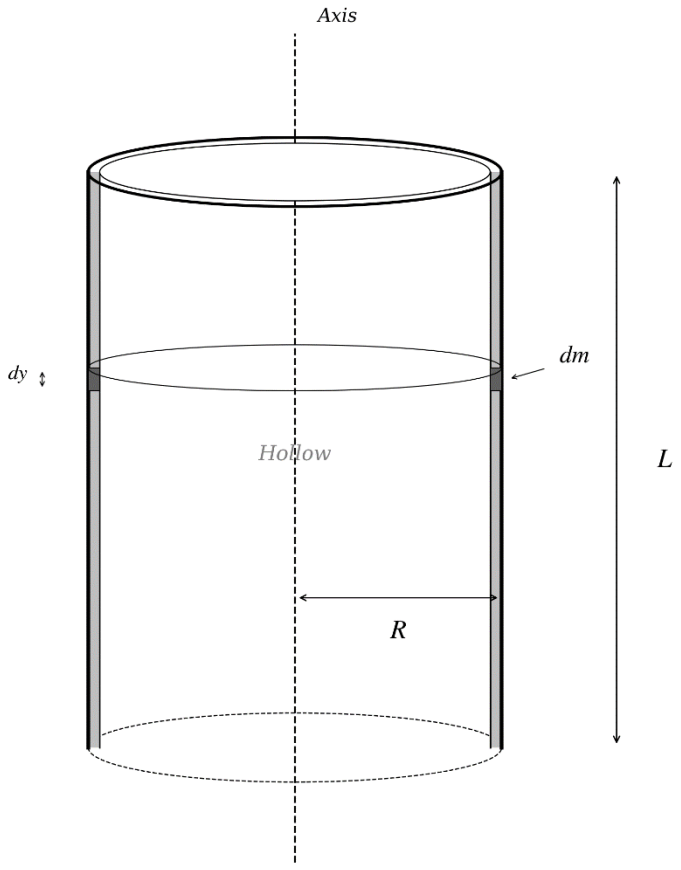


Figure: *A thin uniform hollow cylinder (cylindrical shell) of mass M, radius R, and length L, with the rotation axis along the central axis. The cylinder is sliced into thin ring elements at height y. All mass is at distance R from the axis, so the moment of inertia is identical to that of a ring.*

Why it is the same as a ring

Every element of mass in the cylindrical shell sits on the curved surface, at distance R from the central axis. Just as with the ring, the distance from the axis is the same for every element. And just as with the solid cylinder, the position along the length (the height y) does not affect this distance, because the axis runs parallel to the length.

The derivation

Each thin ring element at position y has mass $\mathrm{dm} = \frac{\mathrm{M}}{\mathrm{L}}\mathrm{dy}$ and moment of inertia (as learned earlier in the ring subsection):

$$\mathrm{dI} = \mathrm{dm} \times \mathrm{R}^2 = \frac{\mathrm{M}}{\mathrm{L}}\mathrm{dy} \times \mathrm{R}^2$$

Integrating over the full length from $y = 0$ to $y = \mathrm{L}$:

$$\mathrm{I} = \int_0^{\mathrm{L}} \frac{\mathrm{MR}^2}{\mathrm{L}}\,\mathrm{dy} = \frac{\mathrm{MR}^2}{\mathrm{L}}\int_0^{\mathrm{L}} \mathrm{dy} = \frac{\mathrm{MR}^2}{\mathrm{L}} \times [\mathrm{y}]_0^{\mathrm{L}} = \frac{\mathrm{MR}^2}{\mathrm{L}} \times \mathrm{L}$$

$$\mathbf{I = MR^2}$$

The moment of inertia of a thin uniform hollow cylinder about its central axis is MR^2, exactly the same as a ring.

Making Sense of the Answer: *Once again, the length L cancels. A short pipe and a long pipe of the same mass and radius have the same moment of inertia about their central axis. Every element of mass is at distance R from the axis, regardless of where it sits along the length. The hollow cylinder is to the ring what the solid cylinder is to the disc: the same shape, just taller.*

And just as the ring was the "champion" of flat shapes (coefficient = 1), the hollow cylinder is the champion of three-dimensional shapes rotating about their central axis. No other shape of the same mass and outer radius can have a larger moment of inertia, because no other shape has all its mass at the maximum distance from the axis.

Eight derivations complete. The integrals are done. If you are still reading, you have just accomplished something that most textbooks ask you to accept on faith. You did not accept it on faith. You derived every result from scratch, one careful integral at a time. That is not just learning physics; that is *doing* physics.

Now let us collect the fruits of our labour into one beautiful table.

Summary table: Moments of inertia of standard shapes

The table below collects all eight results, plus the discrete particle definition. This is one of the most frequently referenced tables in rotational mechanics. Commit the results to memory, but more importantly, understand *why* each coefficient is what it is.

Shape	Axis	Moment of inertia	Coefficient
Discrete particles	Any	$\mathrm{I} = \sum \mathrm{m_i r_i^2}$	depends on arrangement
Uniform rod	Through centre, ⊥ to rod	$\frac{1}{12}\mathrm{ML}^2$	$\frac{1}{12} = 0.083$
Uniform rod	Through one end, ⊥ to rod	$\frac{1}{3}\mathrm{ML}^2$	$\frac{1}{3} = 0.333$
Thin ring (hoop)	Through centre, ⊥ to plane	MR^2	1
Uniform disc	Through centre, ⊥ to plane	$\frac{1}{2}\mathrm{MR}^2$	$\frac{1}{2} = 0.5$
Solid sphere	Any axis through centre	$\frac{2}{5}\mathrm{MR}^2$	$\frac{2}{5} = 0.4$
Thin hollow sphere	Any axis through centre	$\frac{2}{3}\mathrm{MR}^2$	$\frac{2}{3} = 0.667$
Solid cylinder	Central (long) axis	$\frac{1}{2}\mathrm{MR}^2$	$\frac{1}{2} = 0.5$
Thin hollow cylinder	Central (long) axis	MR^2	1

Reading the table:

The coefficient tells you how efficiently the shape uses its mass to resist rotation. A coefficient of 1 (ring, hollow cylinder) means all mass is at the maximum distance from the axis. A coefficient of $\frac{2}{5}$ (solid sphere) means the mass is spread throughout the interior, with much of it close to the axis where it contributes less. The larger the coefficient, the harder the object is to spin for a given mass and size.

Notice the pairings: the solid cylinder has the same formula as the disc $\left(\frac{1}{2}MR^2\right)$, and the hollow cylinder has the same formula as the ring (MR^2). This is because stretching a shape along the rotation axis does not change the radial distribution of mass.

Also notice: the hollow version of any shape always has a larger coefficient than the solid version, because the hollow version pushes all its mass to the outside. Hollow sphere $\left(\frac{2}{3}\right)$ beats solid sphere $\left(\frac{2}{5}\right)$. Hollow cylinder (1) beats solid cylinder $\left(\frac{1}{2}\right)$.

This table will serve you for the rest of this chapter and beyond. Treat it as a trusted friend.

Now that the derivations are complete and the table is assembled, let us put these results to work. The following worked examples are where the formulas come alive: real objects, real numbers, real insight.

BINDER Example 14

A uniform solid sphere of mass 4kg and radius 0.1m spins about an axis through its centre at 20rad/s.

(a) Calculate the moment of inertia of the sphere.

(b) Calculate the rotational kinetic energy.

(c) A thin hollow sphere of the same mass and radius spins at the same angular velocity. Calculate its moment of inertia and explain why it is larger.

Solution

(a) From the summary table, for a solid sphere: $I = \frac{2}{5}MR^2$.

Substituting:

$$I = \frac{2}{5} \times 4\text{kg} \times (0.1\text{m})^2 = \frac{2}{5} \times 4 \times 0.01\text{kgm}^2 = 0.016\text{kgm}^2$$

The moment of inertia of the solid sphere is 0.016kgm^2.

(b) Using $KE = \frac{1}{2}I\omega^2$ and substituting:

$$KE = \frac{1}{2} \times 0.016\text{kgm}^2 \times (20\text{rad/s})^2 = 3.2\text{J}$$

The rotational kinetic energy is 3.2J.

(c) For a thin hollow sphere of the same mass and radius: $I = \frac{2}{3}MR^2$.

$$I = \frac{2}{3} \times 4\text{kg} \times (0.1\text{m})^2 = 0.0267\text{kgm}^2$$

The moment of inertia of the hollow sphere is 0.0267kgm^2.

This is larger than the solid sphere (0.016kgm^2) because the hollow sphere has all its mass on the surface, at the maximum distance R from the axis of rotation. The solid sphere has mass distributed throughout the interior, including regions close to the axis that contribute little to I. This is justified by the ratio, $\frac{2/3}{2/5} = \frac{5}{3} \approx 1.67$ which implies that the hollow sphere has about 67% more rotational inertia.

Making Sense of the Answer: *The solid sphere stores* 3.2J *of rotational kinetic energy at* 20rad/s. *The hollow sphere, with its larger moment of inertia, would store* $\frac{1}{2} \times 0.0267 \times 400 = 5.33\text{J}$ *at the same angular velocity. More inertia means more energy stored at the same spin rate, just as more mass means more kinetic energy at the same translational velocity.*

Think Like a Physicist: *Whenever a problem asks you to compare solid and hollow versions of the same shape, the answer is always the same: the hollow version has a larger moment of inertia because its mass is farther from the axis on average. This principle applies to spheres, cylinders, and any other shape.*

REAL Example 15

A bicycle wheel can be modelled as a thin ring of mass 1.5kg and radius 0.33m (most of the mass is in the rim and tyre). A solid disc of the same mass and radius is attached to a motor for comparison.

(a) Calculate the moment of inertia of each.

(b) The motor applies the same constant torque to both objects, starting from rest. Using $\tau = \text{I}\alpha$ (which we will derive formally later, but the twin of $\text{F} = \text{ma}$ should feel natural by now), explain which object reaches 10rad/s first.

(c) Both objects are spinning at 10rad/s and the motor is switched off. If friction provides the same decelerating torque to both, which one takes longer to stop?

Solution

(a) Bicycle wheel (ring): $\text{I}_{\text{ring}} = \text{MR}^2$. Substituting:

$$\text{I}_{\text{ring}} = 1.5\text{kg} \times (0.33\text{m})^2 = 0.163\text{kgm}^2$$

The moment of inertia of the bicycle wheel is 0.163kgm^2.

Disc: $\text{I}_{\text{disc}} = \frac{1}{2}\text{MR}^2$. Substituting:

$$\text{I}_{\text{disc}} = \frac{1}{2} \times 1.5\text{kg} \times (0.33\text{m})^2 = 0.0817\text{kgm}^2$$

The moment of inertia of the disc is 0.0817kgm^2.

(b) From $\tau = \text{I}\alpha$, making α the subject: $\alpha = \frac{\tau}{\text{I}}$.

Since the same torque τ is applied to both, the object with the smaller I gets the larger angular acceleration α. The disc ($\text{I} = 0.0817\text{kgm}^2$) has half the moment of inertia of the ring (0.163kgm^2), so the disc has twice the angular acceleration and reaches 10rad/s in half the time.

Hence, the disc reaches 10rad/s first.

(c) When decelerating, $\alpha = \frac{\tau}{\text{I}}$ again. The ring has the larger I, so for the same decelerating torque, the ring has the smaller angular deceleration. With a smaller deceleration, it takes longer to slow from 10rad/s to zero.

Hence, the bicycle wheel takes longer to stop.

Making Sense of the Answer: *The ring is harder to start spinning (larger I, smaller acceleration for the same torque) but also harder to stop. This is the essence of rotational inertia: it resists all changes in angular velocity, whether speeding up or slowing down. This is why heavy bicycle wheels maintain their speed well on flat roads but feel sluggish when accelerating from traffic lights, and why racing cyclists prefer lighter wheels (smaller I) for quick acceleration.*

Think Like a Physicist: *Parts (b) and (c) are the same physics seen from two directions. A large I means small α for a given torque, whether that torque is driving or braking. This is the rotational twin of Newton's second law: a heavy object (m large) accelerates slowly for a given force, and also decelerates slowly for a given braking force. The analogy is exact because $\tau = I\alpha$ has the same mathematical structure as $F = ma$.*

BINDER Example 16

A uniform rod of mass 0.3kg and length 0.8m can be pivoted either at its centre or at one end.

(a) Calculate the moment of inertia about each axis.

(b) Calculate the ratio of the two results and explain why it is exactly 4.

(c) A torque of 0.06Nm is applied. Calculate the angular acceleration for each case.

Solution

(a) Axis through the centre: using $I = \frac{1}{12}ML^2$ and substituting:

$$I_{centre} = \frac{1}{12} \times 0.3\text{kg} \times (0.8\text{m})^2 = 0.016\text{kgm}^2$$

The moment of inertia about the centre is 0.016kgm^2.

Axis through one end: using $I = \frac{1}{3}ML^2$ and substituting:

$$I_{end} = \frac{1}{3} \times 0.3\text{kg} \times (0.8\text{m})^2 = 0.064\text{kgm}^2$$

The moment of inertia about one end is 0.064kgm^2.

(b) The ratio is:

$$\frac{I_{end}}{I_{centre}} = \frac{0.064\text{kgm}^2}{0.016\text{kgm}^2} = 4$$

The end-axis moment of inertia is exactly 4 times the centre-axis moment of inertia. This is because $\frac{1/3}{1/12} = \frac{12}{3} = 4$. Physically, when the axis moves from the centre to the end, all the mass that was previously distributed symmetrically on both sides is now entirely on one side, extending to a full length L from the axis instead of $\frac{L}{2}$. Because I depends on r^2, the mass at the far end (distance L) contributes enormously compared to what it contributed when it was only at distance $\frac{L}{2}$.

(c) Using $\tau = I\alpha$ and making α the subject:

Centre axis: $\alpha = \frac{\tau}{I} = \frac{0.06\text{Nm}}{0.016\text{kgm}^2} = 3.75\text{rad/s}^2$

End axis: $\alpha = \frac{\tau}{I} = \frac{0.06\text{Nm}}{0.064\text{kgm}^2} = 0.9375\text{rad/s}^2$

The angular acceleration about the centre is 3.75rad/s^2 and about one end is 0.9375rad/s^2.

The same torque produces four times the angular acceleration when the rod is pivoted at the centre than when pivoted at the end. This is exactly what Kipanga felt when he tried to spin his ruler from different points.

Making Sense of the Answer: *The ratio of 4 is universal for any uniform rod: it does not depend on the mass or length. A toothpick, a metre ruler, and a telephone pole all show the same factor-of 4 as difference between centre-pivot and end-pivot rotation. The angular acceleration at the centre (3.75rad/s^2) is comfortably large, meaning the rod responds easily to the torque. At the end (0.94rad/s^2), the response is sluggish. Same rod, same torque, four times more stubborn.*

Think Like a Physicist: *This example is the numerical proof of what we derived earlier in different subsections. The Parallel Axis Theorem will give us a shortcut to get I_{end} from I_{centre} without re-deriving. But for now, the direct comparison confirms both derivations are correct and consistent.*

REAL Example 17

Kipanga bets Kipute that he can spin a hard-boiled egg faster than she can spin a raw egg, both starting from rest with the same push. Kipute accepts the bet.

(a) Explain, using the concept of moment of inertia, why Kipanga wins the bet.

(b) A hard-boiled egg can be modelled as a uniform solid sphere of mass 60g and radius 2.5cm. Calculate its moment of inertia.

(c) After both eggs are spinning, Kipute briefly touches the raw egg to stop it, then releases it. She notices it starts spinning again on its own. Explain this observation, and why the hard-boiled egg would behave differently.

Solution

(a) The hard-boiled egg is rigid: every particle inside rotates together as one body. Its moment of inertia is that of a solid sphere: $I = \frac{2}{5}MR^2$. On another hand, the raw egg contains liquid that does not rotate rigidly with the shell. When the egg is pushed, the shell begins to rotate but the liquid inside lags behind due to its own inertia. The effective moment of inertia of the system (shell + liquid) is larger because the internal liquid resists being dragged into rotation. Therefore, for the same applied torque and the same duration, the hard-boiled egg achieves a higher angular velocity.

(b) Converting units: $M = 60g = 0.060kg$, $R = 2.5cm = 0.025m$.

Using $I = \frac{2}{5}MR^2$ and substituting:

$$I = \frac{2}{5} \times 0.060kg \times (0.025m)^2 = 1.5 \times 10^{-5}kgm^2$$

The moment of inertia of the hard-boiled egg is $1.5 \times 10^{-5}kgm^2$.

(c) When Kipute touches the raw egg, only the shell stops. The liquid inside is still swirling because it has angular momentum that was not removed by the brief touch. When she releases the egg, the still-moving liquid exerts forces on the inside of the shell, dragging it back into rotation. The egg appears to restart on its own. A hard-boiled egg, if stopped and released, stays still because the entire body (shell + solid interior) stops together.

This is actually a classic kitchen test for distinguishing raw eggs from boiled ones.

Making Sense of the Answer: *The moment of inertia of the egg ($1.5 \times 10^{-5}kgm^2$) is tiny, which is why eggs spin so easily. The key physics is not the number but the comparison: rigid bodies spin predictably because I is well-defined, while non-rigid bodies (raw egg, liquid in a container) behave unpredictably because the internal mass does not follow the external rotation immediately.*

Think Like a Physicist: *This example connects back to the introduction of the chapter, where Mr. Akilikubwa demonstrated the difference between the hard-boiled egg and the raw egg. The moment of inertia formula works perfectly for the rigid egg, but the raw egg breaks the rigid body assumption. Recognising when a formula applies and when it does not is just as important as knowing the formula itself.*

HOT Example 18

A compound pulley consists of a solid disc of mass 6kg and radius 0.20m, with a thin ring of mass 2kg and radius 0.20m attached around its rim (both share the same axis).

(a) Calculate the total moment of inertia of the compound pulley about its central axis.

(b) If a torque of 0.5Nm is applied to the pulley, calculate the angular acceleration.

(c) Starting from rest, calculate the angular velocity after 4s and the number of revolutions completed.

Solution

(a) For the disc: $I_{disc} = \frac{1}{2}MR^2 = \frac{1}{2} \times 6kg \times (0.20m)^2 = 0.12kgm^2$

For the ring: $I_{ring} = MR^2 = 2kg \times (0.20m)^2 = 0.08kgm^2$

Since both share the same axis, the total moment of inertia is the sum:

$$I_{total} = I_{disc} + I_{ring} = 0.12kgm^2 + 0.08kgm^2 = 0.20kgm^2$$

The total moment of inertia of the compound pulley is $0.20kgm^2$.

(b) Using $\tau = I\alpha$ and making α the subject:

$$\alpha = \frac{\tau}{I} = \frac{0.5Nm}{0.20kgm^2} = 2.5rad/s^2$$

The angular acceleration is $2.5rad/s^2$.

(c) Using $\omega = \omega_0 + \alpha t$ and substituting:

$$\omega = 0 + 2.5\text{rad/s}^2 \times 4\text{s} = 10\text{rad/s}$$

The angular velocity after 4s is 10rad/s.

Using $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ and substituting:

$$\theta = 0 + \frac{1}{2} \times 2.5\text{rad/s}^2 \times (4\text{s})^2 = 20\text{rad}$$

Number of revolutions: $\frac{20\text{rad}}{2\pi\text{rad/rev}} = 3.18\text{rev}$

The pulley completes 3.18 revolutions in 4s.

Making Sense of the Answer: *The ring (2kg) contributes $0.08kgm^2$ to the moment of inertia, while the disc (6kg, three times heavier) contributes only $0.12kgm^2$. The ring contributes 40% of the total I despite being only 25% of the total mass. This is the r^2 effect: the ring has all its mass at the rim, while the disc spreads its mass from centre to rim. Position matters as much as mass.*

Think Like a Physicist: *Part (a) uses a crucial principle: when multiple bodies share the same rotation axis, their moments of inertia simply add. This is because $I = \int r^2\, dm$ is an integral over all the mass, and splitting the mass into components and integrating separately gives the same total.*

HOT Example 19

A flywheel is designed to store 10kJ of rotational kinetic energy while spinning at 3000rpm. The flywheel is a uniform solid disc.

(a) Calculate the required moment of inertia.

(b) If the flywheel is made of steel with mass 25kg, calculate the required radius.

(c) A designer proposes replacing the solid disc with a thin ring of the same mass to reduce the size. Calculate the radius of the ring that stores the same energy at the same angular velocity, and explain why it is smaller.

Solution

(a) Converting: $\omega = \frac{2\pi \times 3000}{60} = 100\pi\text{rad/s} = 314.2\text{rad/s}$

$$\text{KE} = 10\text{kJ} = 10000\text{J}$$

Using $\text{KE} = \frac{1}{2}\text{I}\omega^2$ and making I the subject:

$$\text{I} = \frac{2\text{KE}}{\omega^2} = \frac{2 \times 10000\text{J}}{(314.2\text{rad/s})^2} = 0.2026\text{kgm}^2$$

The required moment of inertia is 0.203kgm^2.

(b) For a solid disc: $\text{I} = \frac{1}{2}\text{MR}^2$. Making R the subject:

$$\text{R} = \sqrt{\frac{2\text{I}}{\text{M}}} = \sqrt{\frac{2 \times 0.2026\text{kgm}^2}{25\text{kg}}} = 0.127\text{m}$$

The required radius of the solid disc is 0.127m (about 12.7cm).

(c) For a thin ring: $\text{I} = \text{MR}^2$. Making R the subject:

$$\text{R} = \sqrt{\frac{\text{I}}{\text{M}}} = \sqrt{\frac{0.2026\text{kgm}^2}{25\text{kg}}} = 0.0900\text{m}$$

The required radius of the ring is 0.0900m (about 9.0cm).

The ring is smaller (9.0cm) than the disc (12.7cm) because the ring concentrates all its mass at radius R, while the disc wastes mass near the centre where it contributes less to I. Since every kilogram of the ring

works at full R^2 leverage, the ring achieves the same moment of inertia with a smaller radius. The ratio of radii is $\frac{0.090}{0.127} = \frac{1}{\sqrt{2}} \approx 0.707$, which follows from $MR_{ring}^2 = \frac{1}{2}MR_{disc}^2$, giving $R_{ring} = \frac{R_{disc}}{\sqrt{2}}$.

Making Sense of the Answer: *A flywheel of radius* 12.7cm *is roughly the size of a dinner plate, which is a reasonable size for a* 25kg *steel disc storing* 10kJ*. The ring alternative is about* 30% *smaller in radius, which is a significant space saving. This is why real flywheel energy storage systems (used in some hybrid vehicles and power grid stabilisation) often use ring-like designs with heavy rims rather than solid discs.*

Think Like a Physicist: *This is a design problem: given the energy requirement and the angular velocity, work backward to find the physical dimensions. The key insight from part (c) is that the ring achieves the same I with a smaller radius because it distributes mass more efficiently. In engineering, this means ring-like flywheels are more compact for the same energy storage capacity. The factor of* $\frac{1}{\sqrt{2}}$ *is exact and universal: it applies to any solid disc versus ring comparison of the same mass and moment of inertia.*

HOT Example 20

A uniform circular disc of mass 5kg and radius 0.30m has a circular hole of radius 0.10m cut from it. The centre of the hole is at the centre of the disc (a concentric hole, forming an annular disc).

(a) Calculate the mass of the removed piece.

(b) Calculate the moment of inertia of the remaining annular disc about an axis through the centre, perpendicular to the plane.

(c) Explain why the result is not simply $\frac{1}{2}MR^2$ with the remaining mass substituted for M.

Solution

(a) The disc is uniform, so mass is proportional to area.

Area of full disc: $A_{full} = \pi(0.30)^2 = 0.09\pi\text{m}^2$

Area of removed piece: $A_{hole} = \pi(0.10)^2 = 0.01\pi\text{m}^2$

Fraction removed: $\frac{A_{hole}}{A_{full}} = \frac{0.01\pi}{0.09\pi} = \frac{1}{9}$

Mass of removed piece: $m_{hole} = \frac{1}{9} \times 5\text{kg} = 0.556\text{kg}$

The mass of the removed piece is 0.556kg.

(b) We use the subtraction method: (full disc) = (remaining annular disc) + (removed small disc).

Therefore: $I_{remaining} = I_{full} - I_{hole}$

For the full disc: using $I = \frac{1}{2}MR^2$ and substituting:

$$I_{full} = \frac{1}{2} \times 5\text{kg} \times (0.30\text{m})^2 = 0.225\text{kgm}^2$$

For the removed small disc (mass 0.556kg, radius 0.10m): using $I = \frac{1}{2}mR^2$ and substituting:

$$I_{hole} = \frac{1}{2} \times 0.556\text{kg} \times (0.10\text{m})^2 = 0.00278\text{kgm}^2$$

Therefore:

$$I_{remaining} = 0.225\text{kgm}^2 - 0.00278\text{kgm}^2 = 0.222\text{kgm}^2$$

The moment of inertia of the annular disc is 0.222kgm^2.

(c) If we wrongly used $\frac{1}{2}M_{remaining}R_{outer}^2$ with $M_{remaining} = 5\text{kg} - 0.556\text{kg} = 4.444\text{kg}$:

$$\frac{1}{2} \times 4.444\text{kg} \times (0.30\text{m})^2 = 0.200\text{kgm}^2$$

This is 0.200kgm^2, which is less than the correct answer 0.222kgm^2. The wrongly used formula underestimates because it treats the annular disc as if its mass were distributed from $r = 0$ to $r = 0.30\text{m}$, just

like a full disc. But the annular disc has no mass between $r = 0$ and $r = 0.10\text{m}$. Its mass is concentrated in the region $0.10\text{m} \leq r \leq 0.30\text{m}$, farther from the axis on average than a full disc's mass. Because the mass is farther out, the true moment of inertia is larger. The subtraction method correctly accounts for the missing central region; the wrongly used formula does not.

Making Sense of the Answer: *The removed piece (*$0.556kg$*, radius* $0.10m$*) contributes only* $0.00278kgm^2$ *to the moment of inertia, which is barely* 1% *of the full disc's* $0.225kgm^2$*. This is because the removed piece is small and close to the axis, where the* r^2 *factor keeps contributions tiny. Removing the central region barely changes I, but it removes* 11% *of the mass. This is why annular (ring-like) shapes are more efficient than solid discs for rotational inertia: the central mass contributes almost nothing to I but adds to the weight.*

Think Like a Physicist: *The subtraction method (*$I_{\text{remaining}} = I_{\text{full}} - I_{\text{removed}}$*) is the rotational analogue of the subtraction method we used earlier for centre of mass in (Example 11, the disc with a hole). The principle is the same: the properties of the full body equal the sum of the properties of its parts. This technique appears frequently in examinations, and the most common error is exactly the one exposed in part (c): using the formula for a full disc with the reduced mass. Always use subtraction, never substitution.*

And with that, our interesting moment of inertia subtopic is truly complete. The hard work of this chapter is behind us. What follows builds on these results but requires no more plenty of integrations. If you survived subtopic, the rest of the chapter will feel like coasting downhill, which, as we will soon discover in a subsequent subtopic, is something that rolling objects do with surprising elegance.

THE PARALLEL AND PERPENDICULAR AXIS THEOREMS

If the previous subtopic was the hard climb, this section is the view from the top. You have just derived eight moments of inertia the honest way, integral by integral, element by element. Now we introduce two theorems that will save you from ever repeating most of that work. These theorems are shortcuts, powerful ones, and they feel like a reward for surviving the derivations.

The first theorem tells you how to find the moment of inertia about any axis parallel to one you already know. The second tells you how to relate moments of inertia about perpendicular axes in a flat body. Between them, they extend your table of eight results to an almost unlimited number of axes, without a single new integral.

The Parallel Axis Theorem (Steiner's Theorem)

The problem: You know the moment of inertia of a body about an axis through its centre of mass. You now need the moment of inertia about a different axis, parallel to the first, but displaced by some distance d. *Must you redo the entire integration?*

The answer: No. The Parallel Axis Theorem gives you the result in one line.

Statement:

If I_{cm} *is the moment of inertia of a body of mass M about an axis through its centre of mass, then the moment of inertia about any parallel axis a distance d away is:*

$$\boldsymbol{I = I_{cm} + Md^2}$$

The new moment of inertia equals the centre-of-mass value plus an extra term Md^2 that accounts for the shift. This extra term is exactly what you would get if the entire mass M were concentrated at the centre of mass and rotated about the new axis at distance d.

Derivation:

Consider a body of mass M. Let the centre of mass be at the origin, and let the CM-axis (passing through the centre of mass) be the original rotation axis. We want the moment of inertia about a new axis, parallel to the CM-axis, passing through a point at distance d from the centre of mass.

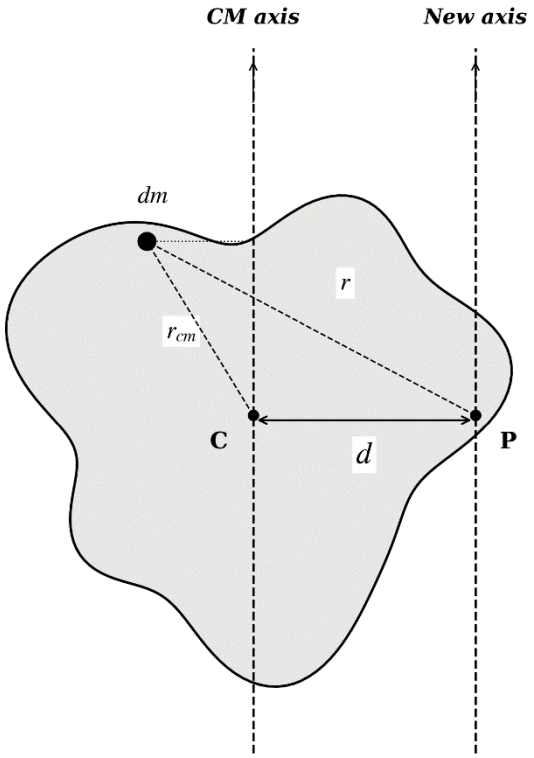


Figure: *An irregular body of mass M with two parallel axes, both perpendicular to the page. The CM axis passes through the centre of mass C. The new axis passes through point P, at distance d from C. A mass element dm at position (x, y) is at distance r_{cm} from the CM axis and distance r from the new axis.*

Set up coordinates so that the centre of mass is at the origin and the new axis is at position $(d, 0)$ in the x-y plane. For any mass element dm at position (x, y):

Distance from the cm-axis: $r_{cm}^2 = x^2 + y^2$

Distance from the new axis: $r^2 = (x - d)^2 + y^2 = x^2 - 2xd + d^2 + y^2$

The moment of inertia about the new axis is:

$$I = \int_0^M r^2\, dm = \int_0^M (x^2 - 2xd + d^2 + y^2)\, dm$$

Rearranging:

$$I = \int_0^M (x^2 + y^2)\, dm - 2d \int_0^M x\, dm + d^2 \int_0^M dm$$

The first integral is I_{cm} (moment of inertia about the cm-axis).

The second integral is $\int x\, dm = Mx_{cm} = M \times 0 = 0$, because the origin is at the centre of mass, so $x_{cm} = 0$.

The third integral is simply M.

Therefore:

$$I = I_{cm} + 0 + Md^2$$

$$\mathbf{I = I_{cm} + Md^2}$$

The middle term vanishes because we measured from the centre of mass. This is precisely why the theorem works only when one of the two axes passes through the centre of mass. If neither axis passes through the centre of mass, you cannot use the theorem directly; you must first go back to the cm-axis, then shift to the desired axis.

Verification: the rod

Earlier, we derived $I_{cm} = \frac{1}{12}ML^2$ for a rod about its centre. Then we derived $I_{end} = \frac{1}{3}ML^2$ about one end. The end is at distance $d = \frac{L}{2}$ from the centre. Applying the Parallel Axis Theorem:

$$I_{end} = I_{cm} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1+3}{12}ML^2 = \frac{4}{12}ML^2 = \frac{1}{3}ML^2$$

This matches the result we derived independently. The theorem works perfectly, and it took one line instead of a full integration. If this does not bring a smile to your face after the effort of the previous subtopic, nothing will!

Key points about the Parallel Axis Theorem:

1. One of the two axes **must** pass through the centre of mass. You cannot shift directly from one off-centre axis to another.
2. The shift term Md^2 is always **positive**, so the moment of inertia about any off-centre axis is always larger than I_{cm}. The centre-of-mass axis always gives the **minimum** moment of inertia for any set of parallel axes. This makes physical sense: the centre of mass is the point where mass is most "compactly arranged" relative to the axis.
3. The theorem works for **any** shape, not just symmetric ones. It applies to rods, discs, spheres, irregular blobs, and anything else.

The Perpendicular Axis Theorem

The problem: You have a flat (planar) body lying in the x-y plane. You know the moments of inertia about the x-axis (I_x) and the y-axis (I_y), both lying in the plane of the body. You need the moment of inertia about the z-axis, perpendicular to the body and passing through the intersection of x and y. *Must you set up a new integral?*

The answer: No. The Perpendicular Axis Theorem gives the result by simple addition.

Statement:

For a planar (flat) body lying in the x-y plane:

$$\mathbf{I_z = I_x + I_y}$$

The moment of inertia about the perpendicular axis equals the sum of the moments of inertia about any two mutually perpendicular axes in the plane of the body, all three axes meeting at the same point.

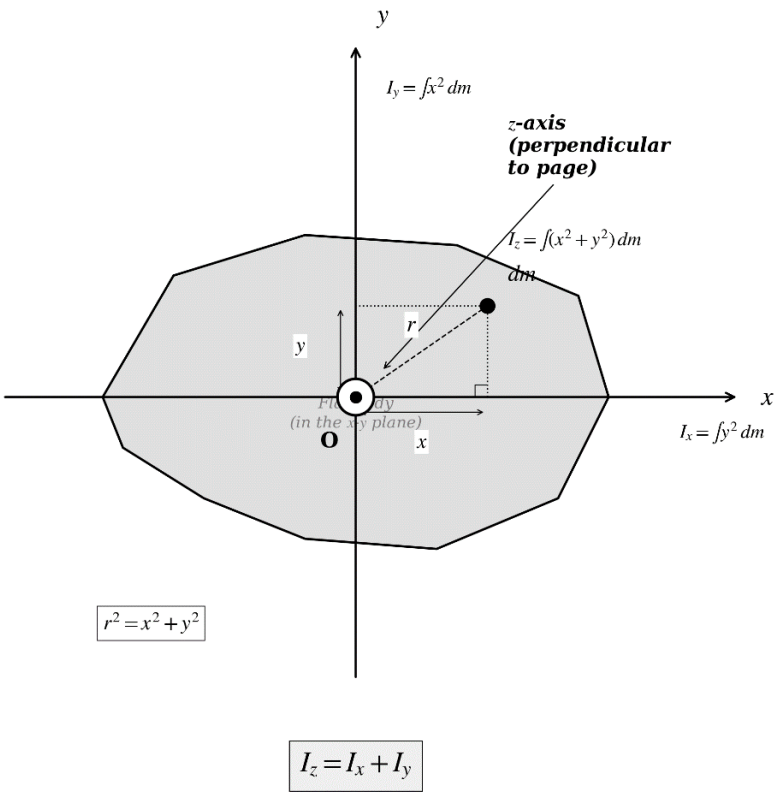


Figure: *A flat body lying in the x-y plane. The x-axis and y-axis lie in the plane of the body; the z-axis is perpendicular to the plane, passing through point O. A mass element dm at position (x, y) is at distance y from the x-axis, distance x from the y-axis, and distance* $r = \sqrt{x^2 + y^2}$ *from the z-axis. Since* $r^2 = x^2 + y^2$*, the Perpendicular Axis Theorem gives* $I_z = I_x + I_y$.

Derivation:

For any mass element dm at position (x, y) in the plane, the perpendicular distances from each axis are:

From the x-axis (which runs along x): the distance is $|y|$, so $I_x = \int_0^M y^2 \text{ dm}$

From the y-axis (which runs along y): the distance is $|x|$, so $I_y = \int_0^M x^2 \text{ dm}$

From the z-axis (perpendicular to the plane): the distance is $r = \sqrt{x^2 + y^2}$, so:

$$I_z = \int_0^M (x^2 + y^2) \text{ dm}$$

Since $x^2 + y^2$ can be split:

$$I_z = \int_0^M (x^2 + y^2) \text{ dm} = \int_0^M x^2 \text{ dm} + \int_0^M y^2 \text{ dm} = I_y + I_x$$

$$\mathbf{I_z = I_x + I_y}$$

The derivation works because $r^2 = x^2 + y^2$ splits cleanly into two independent terms. This is a consequence of Pythagoras's theorem applied to the position of dm in the plane.

Critical restriction: This theorem applies **only** to flat (planar) bodies, such as discs, rings, flat plates, and thin sheets. It does **not** apply to three-dimensional objects like spheres, solid cylinders, or cubes. The reason is that the derivation requires every mass element to lie in the x-y plane (i.e., $z = 0$ for all dm). If the body has thickness in the z-direction, the theorem fails because the distance from the x-axis is not simply $|y|$ but $\sqrt{y^2 + z^2}$.

Applications: new results without integration

The Perpendicular Axis Theorem is most powerful when combined with the symmetry of the body. Here are three important results that fall out instantly:

1) Disc about a diameter:

We know $I_z = \frac{1}{2}MR^2$ for a disc (axis through centre, perpendicular to plane). By symmetry, every diameter of a disc is equivalent, so $I_x = I_y$. Applying the theorem:

$$\frac{1}{2}MR^2 = I_x + I_x = 2I_x$$

Making I_x the subject:

$$I_{\text{diameter}} = \frac{1}{4}MR^2$$

A disc is twice as easy to spin about a diameter as about the perpendicular central axis.

2) Ring about a diameter:

We know $I_z = MR^2$ for a ring. By symmetry, $I_x = I_y$:

$$MR^2 = 2I_x$$

$$I_{\text{diameter}} = \frac{1}{2}MR^2$$

3) Rectangular plate about a perpendicular axis:

A uniform rectangular plate of mass M and sides a and b, centred at the origin. Viewed along the x-axis, the plate looks like a rod of length b; viewed along the y-axis, it looks like a rod of length a. Therefore:

$$I_x = \frac{1}{12}Mb^2 \qquad I_y = \frac{1}{12}Ma^2$$

$$\mathbf{I_z} = I_x + I_y = \frac{\mathbf{1}}{\mathbf{12}}\mathbf{M}(\boldsymbol{a^2 + b^2})$$

For a square plate ($a = b$): $\mathrm{I}_z = \frac{1}{6}\mathrm{Ma}^2$.

None of these results required a single new integral. Each was obtained by combining a known result with the Perpendicular Axis Theorem and, where applicable, symmetry. This is the theorem's gift: it turns what would be difficult integrations into one-line algebra.

Summary of new results from the Perpendicular Axis Theorem:

Body	Axis	Moment of inertia
Disc	Through centre, along a diameter	$\frac{1}{4}\mathrm{MR}^2$
Ring	Through centre, along a diameter	$\frac{1}{2}\mathrm{MR}^2$
Rectangular plate	Through centre, $\perp$ to plate	$\frac{1}{12}\mathrm{M}(\mathrm{a}^2 + \mathrm{b}^2)$
Square plate	Through centre, $\perp$ to plate	$\frac{1}{6}\mathrm{Ma}^2$

With two theorems in hand, we can now handle almost any axis without returning to the integral. Time to put them to work.

BINDER Example 21

A uniform circular disc of mass 2kg and radius 0.20m lies flat on a table.

(a) Using the Perpendicular Axis Theorem, calculate the moment of inertia of the disc about an axis along one of its diameters.

(b) Using the Parallel Axis Theorem on the result from (a), calculate the moment of inertia about an axis tangent to the disc and lying in the plane of the disc.

Solution

(a) The moment of inertia about the perpendicular central axis is:

$$\mathrm{I}_z = \frac{1}{2}\mathrm{MR}^2 = \frac{1}{2} \times 2\mathrm{kg} \times (0.20\mathrm{m})^2 = 0.04\mathrm{kgm}^2$$

By symmetry, every diameter of the disc is equivalent, so $\mathrm{I}_x = \mathrm{I}_y$. Applying the Perpendicular Axis Theorem $\mathrm{I}_z = \mathrm{I}_x + \mathrm{I}_y = 2\mathrm{I}_x$ and making I_x the subject:

$$\mathrm{I}_{\text{diameter}} = \frac{\mathrm{I}_z}{2} = \frac{0.04}{2} = 0.02\mathrm{kgm}^2$$

The moment of inertia about a diameter is $0.02\mathrm{kgm}^2$.

(b) A tangent axis in the plane of the disc is parallel to a diameter, at distance $\mathrm{d} = \mathrm{R} = 0.20\mathrm{m}$ from the centre. Applying the Parallel Axis Theorem:

$$\mathrm{I}_{\text{tangent}} = \mathrm{I}_{\text{diameter}} + \mathrm{Md}^2 = 0.02\mathrm{kgm}^2 + 2\mathrm{kg} \times (0.20\mathrm{m})^2 = 0.10\mathrm{kgm}^2$$

The moment of inertia about the tangent axis in the plane is $0.10\mathrm{kgm}^2$.

Making Sense of the Answer: *Starting from one known value ($I_z = 0.04 kgm^2$), we obtained two new values ($I_{diameter} = 0.02$ and $I_{tangent} = 0.10$) without any integration. The tangent value is five times the diameter value ($0.10/0.02 = 5$) because moving the axis from the centre to the rim adds a substantial Md^2 term. Notice the progression: diameter (0.02) < perpendicular centre (0.04) < tangent (0.10). The perpendicular axis has a larger I than the diameter because it "sees" the full radial spread of the disc, while the diameter axis has mass distributed on both sides, some of it very close.*

Think Like a Physicist: *This example uses both theorems in sequence: Perpendicular first (to get the diameter result from the perpendicular result), then Parallel (to shift from the centre to the rim). This "Perpendicular then Parallel" combination is one of the most common examination techniques. Master it, and you can find the moment of inertia of any flat body about almost any axis from a single known result.*

BINDER Example 22

A uniform disc of mass 3kg and radius 0.25m is mounted on an axle that passes through a point on its rim, perpendicular to the plane of the disc.

(a) Calculate the moment of inertia of the disc about an axis through its centre, perpendicular to its plane.

(b) Using the Parallel Axis Theorem, calculate the moment of inertia about the rim axis.

Solution

(a) From the summary table: $I_{cm} = \frac{1}{2}MR^2$.

Substituting: $I_{cm} = \frac{1}{2} \times 3\text{kg} \times (0.25\text{m})^2 = \frac{1}{2} \times 3 \times 0.0625 = 0.09375\text{kgm}^2$

The moment of inertia about the centre is 0.0938kgm^2.

(b) The rim axis is parallel to the centre axis, at distance $d = R = 0.25\text{m}$ from the centre.

Using $I = I_{cm} + Md^2$ and substituting:

$$I = 0.09375\text{kgm}^2 + 3\text{kg} \times (0.25\text{m})^2 = 0.281\text{kgm}^2$$

The moment of inertia about the rim axis is 0.281kgm^2.

Making Sense of the Answer: *The rim-axis value (0.281) is exactly three times the centre-axis value (0.09375). This is not a coincidence: $I_{rim} = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$, which is indeed three times $\frac{1}{2}MR^2$. Moving the axis from the centre to the rim always triples the moment of inertia for a disc. The Md^2 term (0.1875) is twice as large as I_{cm} (0.09375), showing that the shift contributes more than the original value.*

Think Like a Physicist: *The Parallel Axis Theorem always increases the moment of inertia when you move the axis away from the centre of mass. The increase is Md^2, which can be substantial. For the disc, shifting by one radius doubles the shift term relative to the original. This is why doors have hinges at the edge, not at the centre: the larger moment of inertia about the hinge axis makes the door swing in a stable, controlled way rather than fluttering like a butterfly.*

BINDER Example 23

A thin uniform ring of mass 0.8kg and radius 0.15m lies flat on a table. Calculate the moment of inertia about:

(a) An axis through the centre, perpendicular to the plane of the ring.

(b) An axis along a diameter of the ring (an axis in the plane of the ring through the centre).

(c) An axis tangent to the ring in its plane (touching the ring at one point, lying in the plane).

Solution

(a) From the summary table: $I_z = MR^2$.

Substituting: $I_z = 0.8\text{kg} \times (0.15\text{m})^2 = 0.018\text{kgm}^2$

The moment of inertia about the perpendicular axis through the centre is 0.018kgm^2.

(b) Using the Perpendicular Axis Theorem: $I_z = I_x + I_y$.

By symmetry, $I_x = I_y$ (every diameter of a ring is equivalent). Therefore:

$$I_z = 2I_x$$

Making I_x the subject:

$$I_x = \frac{I_z}{2} = \frac{0.018\text{kgm}^2}{2} = 0.009\text{kgm}^2$$

The moment of inertia about a diameter is 0.009kgm^2.

(c) The tangent axis is parallel to a diameter, at distance $d = R = 0.15\text{m}$ from the centre. Using the Parallel Axis Theorem (shifting from the diameter axis):

$$I_{tangent} = I_{diameter} + Md^2 = 0.009\text{kgm}^2 + 0.8\text{kg} \times (0.15\text{m})^2 = 0.027\text{kgm}^2$$

The moment of inertia about the tangent axis is 0.027kgm^2.

Making Sense of the Answer: *The three results form a satisfying progression: perpendicular centre (*0.018*), diameter (*0.009*), tangent (*0.027*). The diameter value is the smallest because the mass is distributed both above and below the axis, with much of it close to the axis. The tangent value is the largest because the axis is shifted to the rim, putting all the mass on one side. Notice how we used both theorems in sequence: Perpendicular Axis Theorem first (to get the diameter), then Parallel Axis Theorem (to shift from diameter to tangent). This combination is a common examination technique.*

Think Like a Physicist: *This example demonstrates the power of combining the two theorems. Starting from just one known result (*$I_z = MR^2$*), we obtained two more (*$I_{diameter}$ *and* $I_{tangent}$*) without any integration. The Perpendicular Axis Theorem moved us from a perpendicular axis to an in-plane axis. The Parallel Axis Theorem then moved us from the centre to the rim. Together, these two theorems can reach almost any axis from any known starting point.*

REAL Example 24

A 0.5kg uniform ruler is 1m long. A student pivots the ruler at a point 0.2m from one end.

(a) Explain why the Parallel Axis Theorem is needed here, rather than using a formula directly from the summary table.

(b) Calculate the moment of inertia about this pivot.

(c) If a torque of 0.1Nm is applied, calculate the angular acceleration.

Solution

(a) The summary table gives the moment of inertia for a rod about two specific axes: through the centre and through one end. The pivot at 0.2m from one end is neither of these. It is an off-centre axis that does not appear in the table. The Parallel Axis Theorem lets us start from the known centre-axis result and shift to any parallel axis, without deriving a new integral.

(b) The centre of the ruler is at 0.5m from either end. The pivot is at 0.2m from one end, which is $0.5\text{m} - 0.2\text{m} = 0.3\text{m}$ from the centre. So $\text{d} = 0.3\text{m}$.

From the summary table: $\text{I}_{\text{cm}} = \frac{1}{12}\text{ML}^2$.

Substituting: $\text{I}_{\text{cm}} = \frac{1}{12} \times 0.5\text{kg} \times (1\text{m})^2 = 0.04167\text{kgm}^2$

Using $\text{I} = \text{I}_{\text{cm}} + \text{Md}^2$ and substituting:

$$\text{I} = 0.04167\text{kgm}^2 + 0.5\text{kg} \times (0.3\text{m})^2 = 0.0867\text{kgm}^2$$

The moment of inertia about the pivot is 0.0867kgm^2.

(c) Using $\tau = \text{I}\alpha$ and making α the subject:

$$\alpha = \frac{\tau}{\text{I}} = \frac{0.1\text{Nm}}{0.0867\text{kgm}^2} = 1.15\text{rad/s}^2$$

The angular acceleration is 1.15rad/s^2.

Making Sense of the Answer: *The moment of inertia (*0.0867*) lies between the centre value (*0.0417*) and the end value (*$\frac{1}{3} \times 0.5 \times 1 = 0.1667$*). This is correct: the pivot is between the centre and the end, so the moment of inertia should be between the two extreme values. If it had come out larger than the end value, we would know an error had occurred.*

Think Like a Physicist: *Part (a) highlights an important point: the summary table covers only specific axes. Any other axis requires the Parallel Axis Theorem. In examinations, the pivot is often placed at an arbitrary point precisely to test whether you know the theorem. The procedure is always the same: find* I_{cm} *from the table, identify d as the distance from the centre of mass to the new axis, and apply* $I = I_{cm} + Md^2$*.*

HOT Example 25

A uniform square plate of mass M and side length a lies in the x-y plane, centred at the origin.

(a) Using the Perpendicular Axis Theorem and the known result for a rod, show that the moment of inertia about an axis through the centre, perpendicular to the plate, is $\text{I} = \frac{1}{6}\text{Ma}^2$.

(b) Using the Parallel Axis Theorem, find the moment of inertia about an axis along one edge of the plate (perpendicular to the plate).

Solution

(a) Consider the square plate lying in the x-y plane with its centre at the origin, sides parallel to the axes. The plate extends from $-\frac{a}{2}$ to $+\frac{a}{2}$ in both x and y.

The moment of inertia about the x-axis (I_x) can be found by noting that, viewed from the side, the plate looks like a uniform rod of length a in the y-direction. Every thin horizontal strip of the plate at height y is at distance $|y|$ from the x-axis. The result is the same as a rod of mass M and length a about its centre:

$$I_x = \frac{1}{12}Ma^2$$

By the same reasoning (the plate looks like a rod of length a in the x-direction when viewed along y):

$$I_y = \frac{1}{12}Ma^2$$

Applying the Perpendicular Axis Theorem:

$$I_z = I_x + I_y = \frac{1}{12}Ma^2 + \frac{1}{12}Ma^2 = \frac{2}{12}Ma^2 = \frac{1}{6}Ma^2$$

The moment of inertia about the perpendicular axis through the centre is $\frac{1}{6}Ma^2$.

(b) Taking the midpoint of one edge, the distance from the centre is $d = \frac{a}{2}$.

Using $I = I_{cm} + Md^2$ and substituting:

$$I = \frac{1}{6}Ma^2 + M\left(\frac{a}{2}\right)^2 = \frac{1}{6}Ma^2 + \frac{1}{4}Ma^2 = Ma^2\left(\frac{1}{6} + \frac{1}{4}\right) = Ma^2\left(\frac{2+3}{12}\right) = \frac{5}{12}Ma^2$$

The moment of inertia about the axis through the midpoint of one edge is $\frac{5}{12}Ma^2$.

Making Sense of the Answer: *In part (a), the coefficient $\frac{1}{6}$ for the square plate is larger than $\frac{1}{12}$ for a rod but smaller than $\frac{1}{2}$ for a disc. This is reasonable: a square plate has mass spread in two dimensions (x and y), so it contributes to I from both directions. The Perpendicular Axis Theorem captures this by adding the two in-plane contributions. In part (b), shifting the axis from the centre to the edge increases I by $\frac{1}{4}Ma^2$, nearly doubling the value. Rotation about the edge is significantly harder than rotation about the centre.*

Think Like a Physicist: *Part (a) is a beautiful example of how the Perpendicular Axis Theorem combines with an existing result (the rod formula) to produce a new result (the square plate formula) without any new integration. The argument that "the plate looks like a rod when viewed from the side" is the key insight. This reasoning generalises: the moment of inertia of any uniform rectangular plate of sides a and b about a perpendicular axis through the centre is $I = \frac{1}{12}M(a^2 + b^2)$. The square is the special case where $a = b$.*

RADIUS OF GYRATION

This is a one-idea subtopic, and the idea is elegant. After deriving moments of inertia for eight shapes, you may have noticed that every result has the form:

$$I = (\text{some fraction}) \times M \times (\text{some length})^2$$

The fraction and the length depend on the shape and the axis. For a disc, $I = \frac{1}{2}MR^2$. For a rod about its centre, $I = \frac{1}{12}ML^2$. For a solid sphere, $I = \frac{2}{5}MR^2$.

In every case, we can write:

$$I = Mk^2$$

where k is a length that absorbs both the fraction and the characteristic dimension. This length k is called the **radius of gyration**.

By definition: the **radius of gyration k** of a body about a given axis *is the distance from the axis at which the entire mass of the body would need to be concentrated (as a single point) to give the same moment of inertia as the actual distributed body.*

$$\mathbf{I = Mk^2} \qquad \text{therefore} \qquad \mathbf{k = \sqrt{\frac{I}{M}}}$$

The radius of gyration has units of metres. It is not the distance of any particular particle from the axis; it is the "effective distance" of the mass as a whole.

Physical meaning

Imagine you could take all the mass of a body and compress it into a thin ring of the same mass, centred on the rotation axis. The radius of that ring that would give the same moment of inertia as the original body is the radius of gyration.

For a thin ring: $I = MR^2 = Mk^2$, so $k = R$. The radius of gyration equals the actual radius, because all the mass is already at distance R.

For a solid disc: $I = \frac{1}{2}MR^2 = Mk^2$, so $k = R\sqrt{\frac{1}{2}} = \frac{R}{\sqrt{2}} \approx 0.707R$. The effective distance is about 71% of the actual radius, because some mass is close to the axis.

For a solid sphere: $I = \frac{2}{5}MR^2 = Mk^2$, so $k = R\sqrt{\frac{2}{5}} \approx 0.632R$. Even less, because the three-dimensional spreading puts more mass near the axis.

The radius of gyration compresses the shape information and the axis information into a single number. It tells you, at a glance, how "spread out" the mass is relative to the axis. The larger the radius of gyration, the more spread out the mass, and the harder the body is to rotate.

You have to understand that: shapes that push mass outward (ring, hollow sphere, hollow cylinder) have larger radii of gyration, closer to the actual radius R. Shapes that spread mass throughout the interior (disc, solid sphere, solid cylinder) have smaller radii of gyration.

Why radius of gyration is useful

1. **Quick comparisons:** The radius of gyration immediately tells you which of two bodies of the same mass is harder to rotate. The one with the larger k wins. You do not need to compare the full I values.
2. **Rolling motion:** Later, we will discover that the acceleration of a body rolling down an incline depends on $\frac{k^2}{R^2}$. The radius of gyration appears naturally in the rolling equation, making it the perfect tool for comparing how different shapes roll.
3. **Engineering specifications:** In structural and mechanical engineering, the radius of gyration is used to describe the stiffness of columns and the rotational properties of machine parts. It appears in the Euler buckling formula for columns and in the specifications of flywheels and rotors.

The radius of gyration is not a new physical quantity; it is a convenient repackaging of the moment of inertia. But like all good repackaging, it makes the contents easier to use.

The concept is simple enough. Let us make sure the calculations are equally comfortable.

BINDER Example 26

A uniform solid cylinder has mass 8kg and radius 0.12m. It rotates about its central axis.

(a) Calculate the moment of inertia.

(b) Calculate the radius of gyration.

(c) A thin hollow cylinder of the same mass and radius rotates about the same axis. State its radius of gyration and explain why it is larger.

Solution

(a) Using $I = \frac{1}{2}MR^2$ and substituting:

$$I = \frac{1}{2} \times 8\text{kg} \times (0.12\text{m})^2 = 0.0576\text{kgm}^2$$

The moment of inertia is 0.0576kgm^2.

(b) Using $k = \sqrt{\frac{I}{M}}$ and substituting:

$$k = \sqrt{\frac{0.0576\text{kgm}^2}{8\text{kg}}} = 0.0849\text{m}$$

The radius of gyration is 0.0849m (about 8.5cm).

Alternatively: $k = \frac{R}{\sqrt{2}} = \frac{0.12}{\sqrt{2}} = 0.0849$m. Same answer.

(c) For the hollow cylinder: $I = MR^2$, so $k = R = 0.12$m.

The hollow cylinder's radius of gyration (0.12m) is larger than the solid cylinder's (0.085m) because all the mass of the hollow cylinder is at the maximum distance R from the axis. The solid cylinder has mass distributed from $r = 0$ to $r = R$, so its effective distance (radius of gyration) is smaller. This is supported by the ratio $\frac{k_{hollow}}{k_{solid}} = \frac{R}{R/\sqrt{2}} = \sqrt{2} \approx 1.41$, implying that the hollow cylinder's effective distance is about 41% larger.

Making Sense of the Answer: *The solid cylinder's radius of gyration (*8.5cm*) is about* 71% *of the actual radius (*12cm*). This means the "average" mass in the solid cylinder behaves as if it were at* 71% *of the way from the axis to the rim. The hollow cylinder's k equals R exactly, because all mass is at the rim. These numbers will reappear later when we compare rolling speeds.*

Think Like a Physicist: *The radius of gyration is a single number that captures how a body's mass is distributed relative to the axis. Two bodies with the same k and the same mass have the same moment of inertia, even if they look completely different. A solid disc and a solid cylinder of the same mass and radius have identical radii of gyration (*$R/\sqrt{2}$*), which makes sense because they have the same moment of inertia formula.*

REAL Example 27

A traditional Tanzanian wooden club (rungu) has a heavy head and a thin handle. The total mass is 0.6kg. When swung about the end of the handle (axis at the grip), the moment of inertia is measured to be 0.15kgm^2.

(a) Calculate the radius of gyration about this axis.

(b) Explain what the radius of gyration tells us about the mass distribution of the rungu.

(c) If a uniform rod of the same mass and length had the same radius of gyration, what would its length be?

Solution

(a) Using $k = \sqrt{\frac{I}{M}}$ and substituting:

$$k = \sqrt{\frac{0.15\text{kgm}^2}{0.6\text{kg}}} = 0.50\text{m}$$

The radius of gyration is 0.50m.

(b) The radius of gyration of 0.50m tells us that the mass of the rungu behaves as though it were all concentrated at 0.50m from the grip. Since the rungu has a heavy head at the far end and a light handle near the grip, most of the mass is far from the axis. This pushes the effective distance (radius of gyration) toward the far end. If the mass were evenly distributed (like a uniform rod), the radius of gyration would be smaller because more mass would be near the grip.

(c) For a uniform rod pivoted at one end: $I = \frac{1}{3}ML^2$, so $k = \frac{L}{\sqrt{3}}$.

Setting $k = 0.50$m and making L the subject:

$$L = k\sqrt{3} = 0.50\text{m} \times \sqrt{3} = 0.866\text{m}$$

A uniform rod of the same mass and radius of gyration would be 0.866m long. If the actual rungu is shorter than this (say 0.7m), it confirms that the mass concentration at the head gives it a larger radius of gyration than a uniform rod of the same length, which is exactly the design intent: a heavy head maximizes the striking force.

Making Sense of the Answer: *A radius of gyration of* 0.50m *for a weapon designed for impact is sensible. The heavy head, placed far from the grip, maximizes the moment of inertia and therefore the angular momentum and KE at impact. This is the same principle behind a hammer, a golf club, and a baseball bat: mass at the far end makes the tool more effective.*

Think Like a Physicist: *The radius of gyration allows us to compare objects of very different shapes on equal footing. The rungu is nothing like a rod, a disc, or a sphere, yet its radius of gyration tells us exactly how its mass is distributed relative to the axis, in a single number. This is especially useful for irregular bodies where the moment of inertia cannot be calculated from a simple formula but can be measured experimentally. From the measured I and the known M, the radius of gyration follows immediately.*

HOT Example 28

Two solid spheres, A and B, have the same mass M but different radii: $\text{R}_\text{A} = 0.10\text{m}$ and $\text{R}_\text{B} = 0.20\text{m}$.

(a) Calculate the ratio of their radii of gyration.

(b) Both spheres are released from rest at the top of the same inclined plane and roll without slipping. Using the fact that the acceleration of a rolling body is $\text{a} = \frac{\text{g}\sin\theta}{1+k^2/\text{R}^2}$, determine which sphere reaches the bottom first.

(c) Explain why the result of (b) is surprising at first glance, and why it makes perfect sense when examined carefully.

Solution

(a) For a solid sphere: $\text{k} = \text{R}\sqrt{\frac{2}{5}}$.

$$\text{k}_\text{A} = 0.10\text{m}\sqrt{\frac{2}{5}} = 0.0632\text{m}$$

$$\text{k}_\text{B} = 0.20\text{m}\sqrt{\frac{2}{5}} = 0.1265\text{m}$$

$$\frac{\text{k}_\text{A}}{\text{k}_\text{B}} = \frac{0.0632}{0.1265} = \frac{1}{2}$$

The ratio of their radii of gyration equals the ratio of their radii, which is $\frac{1}{2}$.

(b) The acceleration of a rolling body is $\text{a} = \frac{\text{g}\sin\theta}{1+\text{k}^2/\text{R}^2}$.

For any solid sphere: $\text{k}^2 = \frac{2}{5}\text{R}^2$, so $\frac{\text{k}^2}{\text{R}^2} = \frac{2}{5}$.

This ratio is the same for sphere A and sphere B, regardless of their radii! Therefore:

$$\text{a} = \frac{\text{g}\sin\theta}{1+\frac{2}{5}} = \frac{\text{g}\sin\theta}{\frac{7}{5}} = \frac{5\text{g}\sin\theta}{7}$$

Both spheres have exactly the same acceleration. They reach the bottom at the same time.

(c) At first glance, it seems like the larger sphere should reach the bottom first (or latter) because it has a larger moment of inertia (or larger radius). But the rolling acceleration neither depends on I nor on R alone but on $\frac{\text{k}^2}{\text{R}^2}$, which is a pure number determined by the shape, not the size. For all solid spheres, regardless of radius, this ratio is $\frac{2}{5}$.

Understand: The physical reason is that a larger sphere has more rotational inertia (which slows it down) but also has a larger radius (which means its linear velocity is greater for the same angular velocity). These two effects cancel exactly. The result is that all solid spheres roll down an incline at the same rate, just as all objects fall at the same rate in free fall (regardless of mass). The rolling race depends on shape, not on size or mass.

Making Sense of the Answer: *The ratio* $\frac{\text{k}^2}{\text{R}^2} = \frac{2}{5}$ *is a property of the shape "solid sphere," not of any particular sphere. A marble, a bowling ball, and the Earth (if it could roll) would all have the same ratio*

and the same rolling acceleration. This is one of the most beautiful results in rotational mechanics, and the radius of gyration is what makes it transparent.

Think Like a Physicist: *The ratio* $\frac{k^2}{R^2}$ *is the key quantity for rolling motion. It is a dimensionless number that depends only on the shape and the axis, not on M or R. Shapes with larger* $\frac{k^2}{R^2}$ *roll more slowly because a greater fraction of their energy goes into rotation rather than translation. This is why a hollow cylinder* $\left(\frac{k^2}{R^2} = 1\right)$ *always loses the rolling race to a solid sphere* $\left(\frac{k^2}{R^2} = \frac{2}{5}\right)$*. We will explore this fully later.*

That wraps up the radius of gyration: one concept, one formula, and a glimpse of the rolling race that awaits. Short section, big payoff. Now let us move to the real engine of rotational dynamics: torque.

TORQUE: THE ROTATIONAL FORCE

Up to this point, we have described rotation without asking what causes it. We know *how* a body rotates, *where* it rotates, and *how stubbornly* it resists rotation. But we have said nothing about what makes a body start rotating, or stop rotating, or rotate faster. In Chapters 3, we answered the equivalent question for translation: force is the cause of acceleration (F = ma). Now we answer it for rotation.

The rotational twin of force is **torque**.

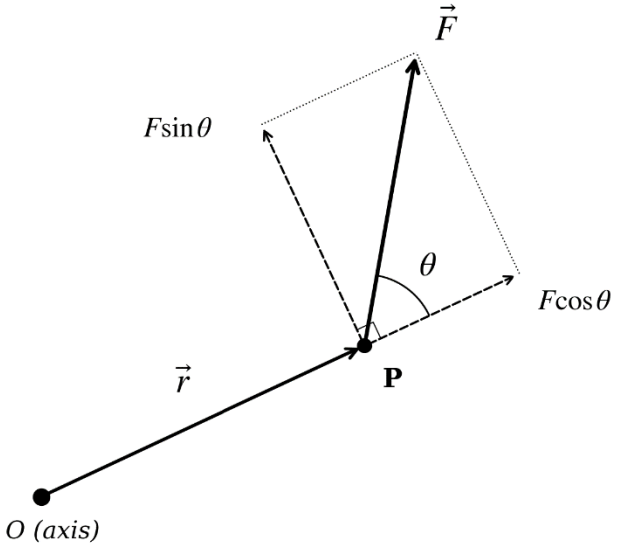
If force is a push or pull that tends to change an object's linear motion, torque *is a twist that tends to change an object's rotational motion.* You apply torque every time you open a door, turn a tap, tighten a bolt with a spanner, or stir ugali with a big wooden spoon (mwiko). In each case, a force is applied at some distance from the rotation axis, and it is this combination of force and distance that produces the twisting effect.

Definition of torque

Consider a force F applied at a point that is at position vector r from the rotation axis. The angle between r and F is θ. The **torque** $\boldsymbol{\tau}$ (also called the **moment of force**) is defined as:

$$\boldsymbol{\tau = rF\sin\theta}$$

where: - τ (Greek letter tau) is the torque, measured in newton-metres (Nm) - r is the distance from the axis to the point where the force is applied (m) - F is the magnitude of the applied force (N) - θ is the angle between r and F



$$\tau = rF\sin\theta = r \times (F\sin\theta) = F \times (r\sin\theta)$$

Figure: *A force* $\vec{F}$ *is applied at point P, located at position* $\vec{r}$ *from the axis at O. The angle between* $\vec{r}$ *and* $\vec{F}$ *is* θ*. The force is resolved into two components:* $F\sin\theta$ *perpendicular to* $\vec{r}$ *(which produces torque) and* $F\cos\theta$ *along* $\vec{r}$ *(which produces no torque). The torque is* $\tau = rF\sin\theta$*.*

The product $r\sin\theta$ is called the **perpendicular distance** (or **lever arm**) from the axis to the line of action of the force. Equivalently, $F\sin\theta$ is the component of force perpendicular to r. Either interpretation gives the same result.

Why $\sin\theta$?

Only the component of force perpendicular to the radius produces rotation. A force directed straight toward the axis (along r) produces no rotation at all, no matter how large it is. Think of pushing a door by pressing directly toward the hinge: the door does not move. But pushing perpendicular to the door (at right angles to r) produces maximum rotation. The $\sin\theta$ factor captures this: when $\theta = 90°$, $\sin\theta = 1$ and the torque is maximum. When $\theta = 0°$ or $180°$, $\sin\theta = 0$ and the torque is zero.

Maximum torque occurs when the force is perpendicular to the radius ($\theta = 90°$):

$$\boldsymbol{\tau_{max} = rF}$$

This is why door handles are placed as far as possible from the hinges (large r) and why we push perpendicular to the door (maximizing $\sin\theta$). Both choices maximise the torque.

Sign convention: We take anticlockwise torque as positive and clockwise torque as negative, consistent with our angular convention we learned earlier.

Torque as a vector

For advanced work, torque is defined as the cross product of the position vector and the force:

$$\boldsymbol{\vec{\tau} = \vec{r} \times \vec{F}}$$

The magnitude is $|\vec{\tau}| = rF\sin\theta$, which is our scalar definition. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$, given by the right-hand rule: curl the fingers of the right hand from $\vec{r}$ toward $\vec{F}$, and the thumb points in the direction of $\vec{\tau}$. For rotation in a plane, this direction is simply "into the page" (clockwise) or "out of the page" (anticlockwise).

For most problems in this chapter, the scalar form $\tau = rF\sin\theta$ with the sign convention is sufficient.

Net torque

When several forces act on a body, each produces its own torque about the axis. The **net torque** is the algebraic sum of all individual torques:

$$\boldsymbol{\tau_{net} = \sum\tau_i = \sum r_i F_i \sin\theta_i}$$

taking signs into account (positive for anticlockwise, negative for clockwise).

A body is in **rotational equilibrium** when the net torque about any point is zero:

$$\boldsymbol{\sum\tau = 0} \qquad \textit{(condition for rotational equilibrium)}$$

This is the rotational counterpart of $\sum F = 0$ for translational equilibrium.

Newton's second law for rotation

This is the master equation of rotational dynamics, the twin of $F = ma$.

Derivation:

Consider a single particle of mass m at distance r from a fixed rotation axis. A tangential force F_t acts on it. By Newton's second law for the particle:

$$F_t = ma_t$$

where $a_t = \alpha r$ is the tangential acceleration. Substituting:

$$F_t = m\alpha r$$

The torque due to this force about the axis is $\tau = F_t \times r$ (the force is tangential, so $\theta = 90°$):

$$\boldsymbol{\tau} = m\alpha r \times r = \boldsymbol{mr^2\alpha} \text{ (for one particle)}$$

For a rigid body consisting of many particles, the net torque is the sum over all particles:

$$\tau_{net} = \sum m_i r_i^2 \alpha = \alpha \sum m_i r_i^2$$

We can factor out α because it is the same for all particles in a rigid body. The remaining sum is the moment of inertia $I = \sum m_i r_i^2$:

$$\boldsymbol{\tau_{net} = I\alpha}$$

This is **Newton's second law for rotation**. It says: *the net torque on a body equals the moment of inertia times the angular acceleration.* Compare with $F_{net} = ma$: force is replaced by torque, mass by moment of inertia, linear acceleration by angular acceleration. The twin language is complete.

Linear	Rotational
$F_{net} = ma$	$\tau_{net} = I\alpha$
Force causes linear acceleration	Torque causes angular acceleration
Greater mass → less acceleration for same force	Greater I → less angular acceleration for same torque

Couples

A **couple** *consists of two forces that are equal in magnitude, opposite in direction, and parallel but not collinear (they act along different lines).* A couple produces pure rotation with no translation, because the net force is zero but the net torque is not.

Torque due to a couple:

If two forces, each of magnitude F, are separated by a perpendicular distance d, the torque of the couple is:

$$\boldsymbol{\tau_{couple} = Fd}$$

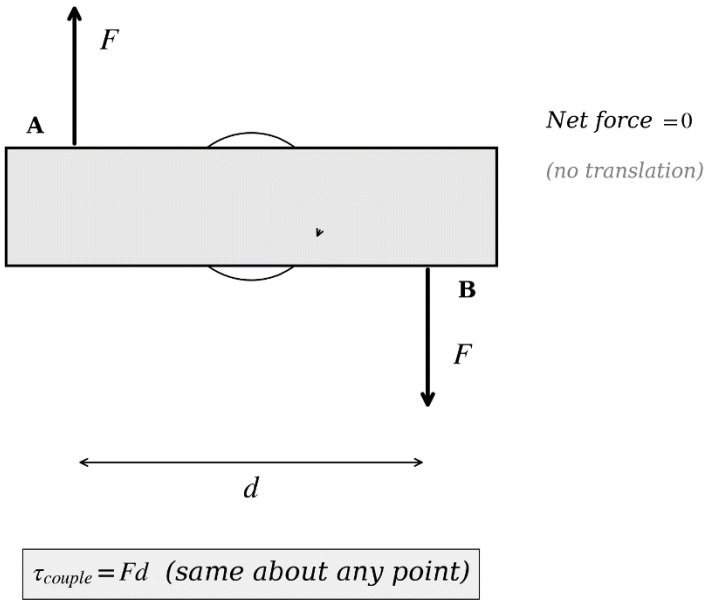


Figure: *A couple consists of two equal and opposite forces F, applied at points A and B, separated by perpendicular distance d. The net force is zero (no translation), but both forces produce torques in the same rotational direction, giving a net torque $\tau = Fd$. This torque is the same about any point.*

This torque is the same about **any** point. This is a special property of couples: unlike a single force (whose torque depends on where you measure it from), a couple produces the same torque regardless of the chosen pivot. This is because the couple produces pure rotation and no translation.

Proof: Let the two forces $+F$ and $-F$ act at points A and B, separated by distance d. Take torques about any point O at distance x from A. Then the distance from O to B is $(d - x)$ if O is between A and B, (or $(d + x)$ if O is outside).

For O between A and B: $\tau = F \times x + F \times (d - x) = Fx + Fd - Fx = Fd$

The distance x cancels, confirming that the torque of a couple is independent of the reference point.

Examples of couples in everyday life:

- Turning a steering wheel with both hands (one pushes up, the other pushes down).
- Turning a tap with thumb and finger.
- Winding a clock with a key.
- Opening a jar lid with two hands.

In each case, two opposite forces produce pure rotation without moving the axis sideways.

Difference between torque and couple:

Students often confuse these. A **torque** is produced by a single force about a chosen axis; its value depends on which axis you choose. A **couple** is produced by two equal and opposite forces; its torque is the same about any axis. A single force can produce both translation and rotation. A couple produces only rotation.

Now that we have the full machinery of rotational dynamics, from torque to $\tau = \mathrm{I}\alpha$, and from single forces to couples, let us see it all in action.

BINDER Example 29

A force of 15N is applied at the end of a 0.4m spanner, at an angle of $60°$ to the handle.

(a) Calculate the torque about the bolt at the other end.

(b) Calculate the maximum torque that could be produced by the same force at the same point on the spanner.

(c) Explain why a longer spanner makes it easier to loosen a tight bolt.

Solution

(a) Using $\tau = \mathrm{rF}\sin\theta$ and substituting:

$$\tau = 0.4\mathrm{m} \times 15\mathrm{N} \times \sin 60° = 5.20\mathrm{Nm}$$

The torque is 5.20Nm.

(b) Maximum torque occurs when $\theta = 90°$ ($\sin 90° = 1$):

$$\tau_{\max} = \mathrm{rF} = 0.4\mathrm{m} \times 15\mathrm{N} = 6.0\mathrm{Nm}$$

The maximum torque is 6.0Nm.

(c) Torque is $\tau = \mathrm{rF}\sin\theta$. A longer spanner means a larger r. For the same applied force and angle, a larger r gives a larger torque. Thus, the extra length gives the force more leverage.

Making Sense of the Answer: *At* $60°$*, the torque (*5.20Nm*) is* 87% *of the maximum (*6.0Nm*). This means applying the force at* $60°$ *instead of* $90°$ *costs only* 13% *of the torque. The* $\sin\theta$ *factor is quite forgiving near* $90°$*: even at* $45°$*, you still get* 71% *of the maximum. It is only at very small angles (pushing nearly along the handle) that the torque drops dramatically.*

Think Like a Physicist: *The torque equation has three "knobs" you can turn: r, F, and* θ*. To maximise torque, make all three as large as possible: use a long lever (large r), push hard (large F), and push perpendicular to the lever (large* $\sin\theta$*). In practice, the angle is the easiest to optimize (just push at* $90°$*), while the force is limited by human strength, and the lever length is limited by the tool available.*

BINDER Example 30

A uniform disc of mass 4kg and radius 0.25m is free to rotate about its central axis. A constant tangential force of 8N is applied at the rim.

(a) Calculate the torque about the central axis.

(b) Calculate the angular acceleration.

(c) Starting from rest, calculate the angular velocity after 3s.

Solution

(a) The force is tangential (perpendicular to the radius), so $\theta = 90°$ and $\sin\theta = 1$:

$$\tau = \mathrm{rF}\sin 90° = 0.25\mathrm{m} \times 8\mathrm{N} \times 1 = 2.0\mathrm{Nm}$$

The torque is 2.0Nm.

(b) First, the moment of inertia of the disc: $\mathrm{I} = \frac{1}{2}\mathrm{MR}^2 = \frac{1}{2} \times 4 \times (0.25)^2 = 0.125\mathrm{kgm}^2$

Using $\tau = \mathrm{I}\alpha$ and making α the subject:

$$\alpha = \frac{\tau}{\mathrm{I}} = \frac{2.0\mathrm{Nm}}{0.125\mathrm{kgm}^2} = 16\mathrm{rad/s}^2$$

The angular acceleration is $16\mathrm{rad/s}^2$.

(c) Using $\omega = \omega_0 + \alpha\mathrm{t}$ and substituting:

$$\omega = 0 + 16\mathrm{rad/s}^2 \times 3\mathrm{s} = 48\mathrm{rad/s}$$

The angular velocity after 3s is 48rad/s.

Making Sense of the Answer: *An angular acceleration of* $16\mathrm{rad/s}^2$ *is large. In* 3 *seconds, the disc reaches* $48\mathrm{rad/s}$*, which is about* 7.6 *revolutions per second. This rapid spin-up happens because the disc has a small moment of inertia (*$0.125\mathrm{kgm}^2$*) and the applied torque (*$2\mathrm{Nm}$*) is substantial relative to it. A heavier disc or one with a larger radius would accelerate more slowly for the same torque.*

Think Like a Physicist: *This example combines three subtopics: torque, moment of inertia, and angular kinematics. This is the typical structure of a rotational dynamics problem: calculate the torque, find I, use* $\tau = \mathrm{I}\alpha$ *to get* α*, then use the kinematic equations to find* ω *or* θ*. Master this chain, and you can solve any constant-torque rotation problem.*

REAL Example 31

Mr. Akilikubwa demonstrates couples by asking Kipanga and Kipute to stand on opposite sides of a laboratory turntable. Kipanga pushes the rim with a force of 20N to the left, and Kipute pushes the opposite rim with a force of 20N to the right. The turntable has a diameter of 0.6m.

(a) Explain why the turntable rotates but does not translate sideways.

(b) Calculate the torque of the couple.

(c) If the turntable has mass 5kg, calculate the angular acceleration.

Solution

(a) The two forces are equal in magnitude (20N each) and opposite in direction (left and right). Therefore the net force on the turntable is $20\mathrm{N} - 20\mathrm{N} = 0$. With zero net force, there is no translational acceleration, so the centre of the turntable does not move sideways.

However, both forces produce torques in the same rotational direction. Kipanga's force pushes the rim to the left at the top, creating a clockwise torque. Kipute's force pushes the rim to the right at the bottom, also creating a clockwise torque. The torques add rather than cancel, producing net rotation. This is the defining property of a couple: zero net force but non-zero net torque.

(b) The perpendicular distance between the two forces is the diameter: $\mathrm{d} = 0.6\mathrm{m}$.

Using $\tau_{\mathrm{couple}} = \mathrm{Fd}$ and substituting:

$$\tau = 20\mathrm{N} \times 0.6\mathrm{m} = 12\mathrm{Nm}$$

The torque of the couple is 12Nm.

(c) The turntable is a uniform disc: $\mathrm{I} = \frac{1}{2}\mathrm{MR}^2 = \frac{1}{2} \times 5 \times (0.3)^2 = 0.225\mathrm{kgm}^2$

Using $\tau = \mathrm{I}\alpha$ and making α the subject:

$$\alpha = \frac{\tau}{\mathrm{I}} = \frac{12\mathrm{Nm}}{0.225\mathrm{kgm}^2} = 53.3\mathrm{rad/s}^2$$

The angular acceleration is $53.3\mathrm{rad/s}^2$.

Making Sense of the Answer: *The angular acceleration is very large (*$53.3\mathrm{rad/s}^2$*) because the couple torque (*$12\mathrm{Nm}$*) is substantial and the disc's moment of inertia (*$0.225\mathrm{kgm}^2$*) is modest. In practice, friction would reduce this acceleration significantly. But the demonstration makes the point: a couple is an efficient way to spin something, because both forces contribute to rotation and neither is "wasted" on translation.*

Think Like a Physicist: *Note that we could have calculated the torque by taking moments about the centre:* $\tau = 20N \times 0.3m + 20N \times 0.3m = 12Nm$. *We could also have taken moments about any other point and obtained the same answer, because the torque of a couple is independent of the pivot.*

REAL Example 32

A merry-go-round at a Tanzanian school playground can be modelled as a uniform disc of mass 120kg and radius 1.5m. A student applies a constant tangential force of 60N at the rim.

(a) Calculate the torque and the angular acceleration.

(b) How long does it take to reach an angular velocity of 2rad/s (a gentle spin), starting from rest?

(c) Once spinning at 2rad/s, the student lets go. A friction torque of 15Nm opposes the motion. Calculate the time for the merry-go-round to stop.

Solution

(a) Torque: $\tau = \text{rF} = 1.5\text{m} \times 60\text{N} = 90\text{Nm}$ (force is tangential, $\theta = 90°$)

Moment of inertia: $\text{I} = \frac{1}{2}\text{MR}^2 = \frac{1}{2} \times 120\text{kg} \times (1.5\text{m})^2 = 135\text{kgm}^2$

Using $\tau = \text{I}\alpha$ and making α the subject:

$$\alpha = \frac{90\text{Nm}}{135\text{kgm}^2} = 0.667\text{rad/s}^2$$

The torque is 90Nm and the angular acceleration is 0.667rad/s^2.

(b) Using $\omega = \omega_0 + \alpha\text{t}$ and making t the subject:

$$\text{t} = \frac{\omega - \omega_0}{\alpha} = \frac{2\text{rad/s} - 0}{0.667\text{rad/s}^2} = 3.0\text{s}$$

It takes 3.0 seconds to reach 2rad/s.

(c) The friction torque decelerates the merry-go-round. Using $\tau = \text{I}\alpha$ and making α the subject:

$$\alpha_{\text{friction}} = \frac{15\text{Nm}}{135\text{kgm}^2} = 0.111\text{rad/s}^2$$

Using $\omega = \omega_0 + \alpha\text{t}$ with $\omega = 0$ (stops) and $\alpha = -0.111\text{rad/s}^2$ (decelerating):

$$0 = 2\text{rad/s} + \left(-0.111\text{rad/s}^2\right) \times \text{t}$$

$$\text{t} = \frac{2\text{rad/s}}{0.111\text{rad/s}^2} = 18\text{s}$$

The merry-go-round takes 18 seconds to stop after the student lets go.

Making Sense of the Answer: *It takes only* 3 *seconds to spin up but* 18 *seconds to stop. The asymmetry comes from the torques: the student applies* 90Nm *(strong push), but friction provides only* 15Nm *(gentle resistance). The moment of inertia of* 135kgm^2 *is large (the merry-go-round is heavy and wide), which is why even the strong push gives a modest angular acceleration and why the weak friction takes a long time to bring it to rest. This is rotational inertia in action: a massive spinning object resists changes in both directions.*

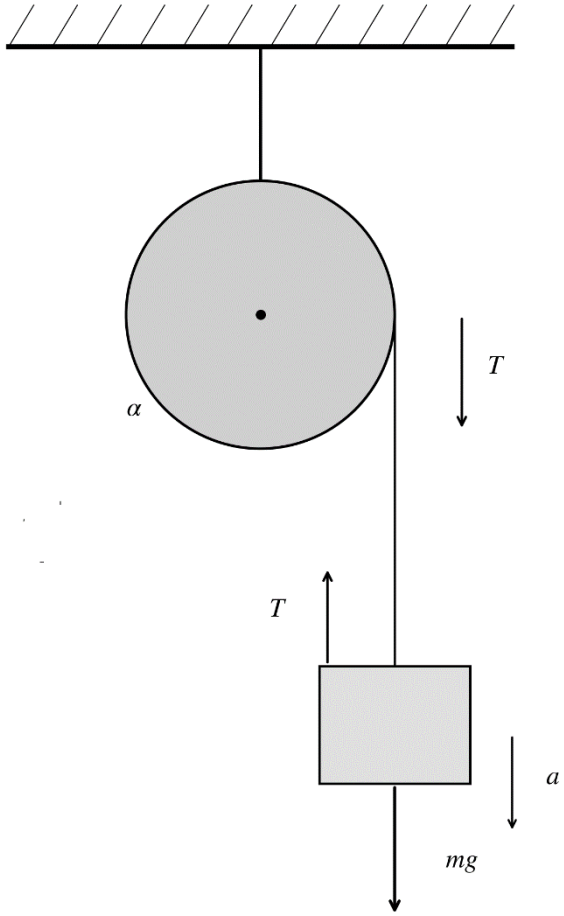
Think Like a Physicist: *This problem follows the same chain as Example 29 but with a more realistic scenario and a two-phase motion (spinning up, then coasting down). In part (c), the angular deceleration is* 6 *times smaller than the acceleration in part (a), because the friction torque is* 6 *times smaller than the student's torque. The ratio of stopping time to spin-up time* $(18/3 = 6)$ *is exactly this torque ratio. This makes the problem easy to check without recalculating everything.*

HOT Example 33

A pulley of mass 2kg and radius 0.10m is modelled as a uniform disc and is free to rotate about its central axis. A light string is wound around the pulley, and a block of mass 3kg hangs from the free end. The system is released from rest. Take $\text{g} = 9.8\text{m/s}^2$.

(a) Draw a free body diagrams for the pulley-block system.

(b) If the string does not slip on the pulley, solve for the acceleration of the block and the tension in the string.



Solution

(a)

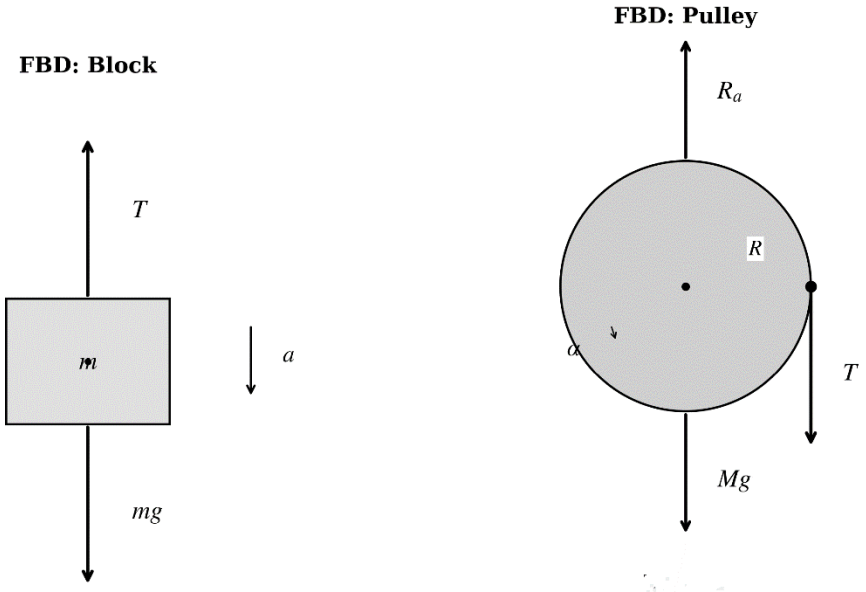


Figure: *Free body diagrams for the pulley-block system. Left: the block of mass m, with tension T upward and weight mg downward, accelerating at a downward. Right: the pulley of mass M and radius R, with tension T at the rim, axle reaction R_a, weight Mg, and angular acceleration α*

(b) **For block:** The block has two forces: weight $mg = 3kg \times 9.8m/s^2 = 29.4N$ downward, and tension T upward. Taking downward as positive (the block accelerates downward):

$$mg - T = ma$$
$$29.4 - T = 3a \quad ... (i)$$

For pulley: R_a balances with mg and both act at axis, and thus there is zero torque about the axis. The torque on the pulley is produced by the tension T in the string, acting at the rim (radius R = 0.10 m). The torque is $\tau = TR$.

The pulley's moment of inertia is $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 2 \times (0.10)^2 = 0.01kgm^2$.

Using $\tau = I\alpha$:

$$TR = I\alpha$$
$$T \times 0.10 = 0.01\alpha \quad ... (ii)$$

If the string does not slip on the pulley, then $a = \alpha R$, which gives $\alpha = \frac{a}{R} = \frac{a}{0.10} = 10a$.

Substituting into (ii):

$$0.10T = 0.01 \times 10a = 0.1a$$
$$T = a \quad ... (iii)$$

Substituting (iii) into (i):

$$29.4 - a = 3a$$
$$29.4 = 4a$$
$$a = 7.35m/s^2$$

From (iii): $T = 7.35N$

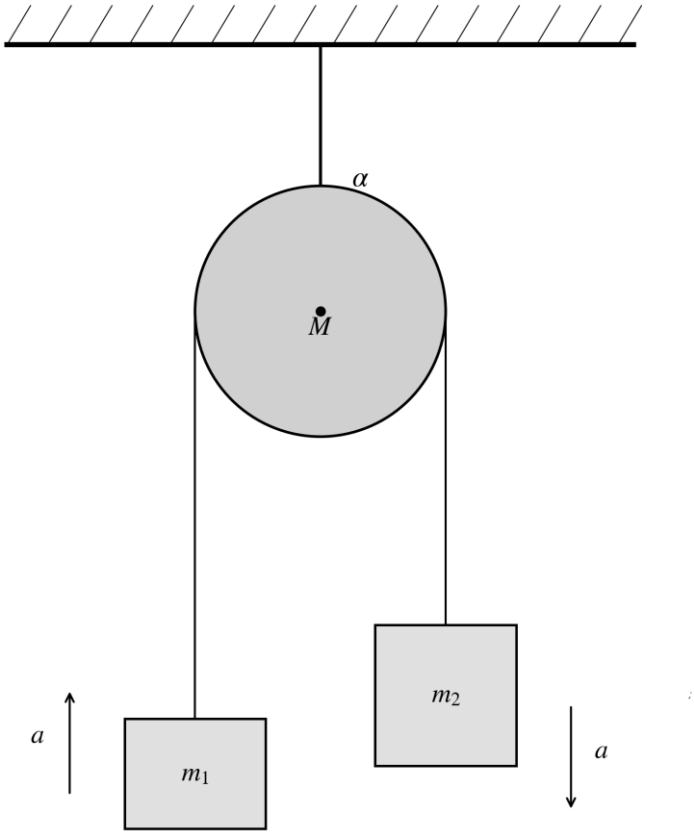
The acceleration of the block is $7.35m/s^2$ and the tension in the string is $7.35N$.

Making Sense of the Answer: *If the pulley were massless (I = 0), the block would fall freely at $9.8m/s^2$ with T = 0. With the massive pulley, the acceleration is $7.35m/s^2$, which is 75% of g. The pulley's rotational inertia slows the block down because some of the gravitational potential energy goes into spinning the pulley rather than accelerating the block. The tension ($7.35N$) is much less than the weight ($29.4N$), confirming that the block is accelerating downward. If T equalled mg, the block would not accelerate at all.*

Think Like a Physicist: *This is the classic "Atwood machine with a massive pulley" problem. The key idea is that the pulley is not just a frictionless redirector of the string; it has its own moment of inertia and must be accelerated rotationally by the string tension. The constraint $a = \alpha R$ connects the linear motion of the block to the rotational motion of the pulley, giving us the third equation needed to solve for three unknowns (a, α, T). This type of coupled linear-rotational problem is one of the most important in this chapter and appears frequently in examinations.*

HOT Example 34

Two blocks of masses $m_1 = 4kg$ and $m_2 = 6kg$ are connected by a light string that passes over a uniform disc pulley of mass $M = 2kg$ and radius $R = 0.15m$. The string does not slip on the pulley. The system is released from rest. Take $g = 9.8m/s^2$.

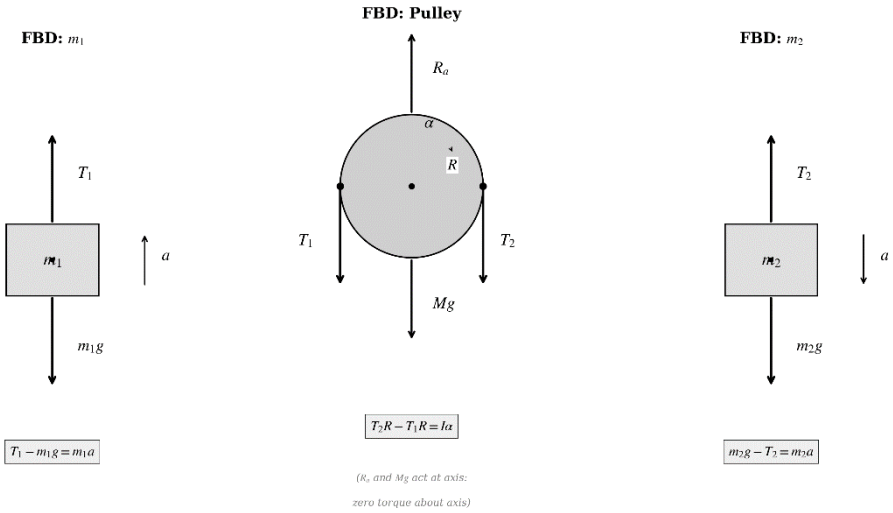


(a) Explain why the tensions on the two sides of the string are not equal.
(b) Calculate the acceleration of the system, the angular acceleration of the pulley, and both tensions.

Solution

(a) If the pulley were massless and frictionless, the string would simply transmit force without change, and the tensions would be equal on both sides. But this pulley has mass, which means it has a moment of inertia ($I = \frac{1}{2}MR^2$). To accelerate the pulley rotationally, there must be a net torque on it. The net torque is $(T_2 - T_1)R$. If $T_1 = T_2$, the net torque would be zero and the pulley would not accelerate. Therefore the tensions must be different: $T_2 > T_1$ (the tension on the heavier side is larger) to provide the torque that spins the pulley.

(b) Forces acting in each objects are clearly shown in the following free body diagrams:



Since $m_2 > m_1$, the heavier block m_2 accelerates downward and the lighter block m_1 accelerates upward, both with the same magnitude a.

For m_1 (accelerating upward, taking upward as positive):

$$T_1 - m_1 g = m_1 a$$
$$T_1 - 4 \times 9.8 = 4a$$
$$T_1 - 39.2 = 4a \quad \ldots (i)$$

For m_2 (accelerating downward, taking downward as positive):

$$m_2 g - T_2 = m_2 a$$
$$6 \times 9.8 - T_2 = 6a$$
$$58.8 - T_2 = 6a \quad \ldots (ii)$$

For the pulley (net torque produces angular acceleration):

$$T_2 R - T_1 R = (T_2 - T_1) R = I\alpha$$

The moment of inertia: $I = \frac{1}{2} MR^2 = \frac{1}{2} \times 2 \times (0.15)^2 = 0.0225 \text{kgm}^2$

$$(T_2 - T_1) \times 0.15 = 0.0225\alpha \quad \ldots (iii)$$

The no-slip constraint gives: $a = \alpha R$, so $\alpha = \frac{a}{R} = \frac{a}{0.15}$

Substituting into (iii):

$$(T_2 - T_1) \times 0.15 = 0.0225 \times \frac{a}{0.15}$$
$$(T_2 - T_1) \times 0.15 = 0.15a$$
$$T_2 - T_1 = a \quad \ldots (iv)$$

Adding equations (i) and (ii):

$$(T_1 - 39.2) + (58.8 - T_2) = 4a + 6a$$
$$19.6 + T_1 - T_2 = 10a$$

From (iv): $T_1 - T_2 = -a$

$$19.6 - a = 10a$$
$$a = \frac{19.6}{11} = 1.78 \text{m/s}^2$$

The acceleration of the system is 1.78m/s^2.

Angular acceleration: $\alpha = \frac{a}{R} = \frac{1.78}{0.15} = 11.9 \text{rad/s}^2$

The angular acceleration of the pulley is 11.9rad/s^2.

From (i): $T_1 = 4a + 39.2 = 4 \times 1.78 + 39.2 = 46.3\text{N}$

From (ii): $T_2 = 58.8 - 6a = 58.8 - 6 \times 1.78 = 48.1\text{N}$

The tension on the m_1 side is 46.3N and the tension on the m_2 side is 48.1N.

Making Sense of the Answer: *If the pulley were massless, the acceleration would be* $\frac{(m_2 - m_1)g}{m_1 + m_2} = \frac{2 \times 9.8}{10} = 1.96 m/s^2$ *and both tensions would be equal. With the massive pulley, the acceleration drops to* $1.78 m/s^2$ *(about* 9% *less) because the pulley's rotational inertia acts as extra "effective mass" in the system. The total effective mass is* $m_1 + m_2 + \frac{1}{2}M = 4 + 6 + 1 = 11kg$*, and indeed* $\frac{(m_2 - m_1)g}{11} = \frac{19.6}{11} = 1.78 m/s^2$*. The two tensions differ by only* $1.8N$*, which is the small torque needed to rotationally accelerate the relatively light pulley.*

Think Like a Physicist: *This is the full Atwood machine with a massive pulley, one of the most classic problems in rotational dynamics. The key differences from the massless-pulley version (Chapter 4) are: (1) the tensions on the two sides are different, (2) the pulley's rotational inertia adds to the system's effective mass, and (3) you need three equations (two for the blocks, one for the pulley) plus one constraint ($a = \alpha R$) to solve for four unknowns (a, α, T_1, T_2). The effective mass trick $\left(m_1 + m_2 + \frac{1}{2}M\right)$ is a powerful shortcut for checking your answer.*

ROTATIONAL WORK, ENERGY, AND POWER

We are already familiar with the energy toolkit for translational motion: work done by a force ($W = Fs\cos\theta$), the kinetic energy of a moving body ($KE = \frac{1}{2}mv^2$), and the power delivered by a force ($P = Fv$). These tools allow us to solve problems that would have been difficult or impossible using forces alone.

We now build the identical toolkit for rotation. And by "identical," we mean it: every formula is the translational formula with each quantity replaced by its rotational twin. If you understood the energy methods studied earlier, this section will feel like meeting an old friend in new clothes.

Work Done by a Torque

When a force F moves an object through a displacement s, the work done is $W = Fs$ (for a constant force in the direction of motion). The rotational twin replaces force with torque and linear displacement with angular displacement.

Consider a torque τ acting on a body that rotates through an angular displacement θ (in radians). The **work done by the torque** is:

$$\mathbf{W = \tau\theta}$$

This is easily derived from the linear definition. If a tangential force F_t acts at radius r, the torque is $\tau = F_t r$ and the arc length travelled is $s = r\theta$. Therefore:

$$W = F_t \times s = F_t \times r\theta = (F_t r)\theta = \tau\theta$$

For a variable torque, the work is:

$$W = \int_{\theta_1}^{\theta_2} \tau \, d\theta$$

The unit of rotational work is the joule (J), just as for translational work. Work is work, whether the motion is linear or rotational.

The Work-Energy Theorem for Rotation

The work-energy theorem states that the net work done on a body equals the change in its translational KE:

$$W_{net} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

The rotational twin is:

$$\mathbf{W_{net} = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2}$$

The net work done by all torques on a rotating body equals the change in its rotational KE.

Derivation:

Starting from $\tau = I\alpha$ and using $\alpha = \frac{d\omega}{dt}$:

$$W = \int \tau \, d\theta = \int I\alpha \, d\theta = \int I\frac{d\omega}{dt} \, d\theta = \int I\frac{d\omega}{dt} \times \frac{d\theta}{dt} \, dt$$

But $\frac{d\theta}{dt} = \omega$, so:

$$W = \int I\omega \, d\omega = I\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

This is the rotational work-energy theorem. It is the rotational twin of $W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$, with m replaced by I and v replaced by ω.

Rotational Kinetic Energy

We already met this earlier when we derived the moment of inertia. The **rotational KE** of a body spinning with angular velocity ω about a fixed axis is:

$$\mathbf{KE_{rot} = \frac{1}{2}I\omega^2}$$

This is the twin of $KE_{trans} = \frac{1}{2}mv^2$.

When a body both translates and rotates (as we will explore later on rolling motion), the total KE is the sum:

$$\mathbf{KE_{total} = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2}$$

The first term is the translational KE of the centre of mass. The second term is the rotational KE about the centre of mass. The two are independent and simply add.

Rotational Power

Power is the rate of doing work. In translation, $P = \frac{dW}{dt} = Fv$. In rotation:

$$P = \frac{dW}{dt} = \frac{d(\tau\theta)}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

$$\mathbf{P = \tau\omega}$$

The rotational power equals the torque times the angular velocity. This is the twin of $P = Fv$.

The unit is the watt (W), the same as for translational power.

This formula is enormously useful in engineering. The power output of an engine, a turbine, or a motor is specified in terms of the torque it delivers and the angular velocity at which it operates.

The complete twin table:

Translational	Rotational
$W = Fs$	$W = \tau\theta$
$KE = \frac{1}{2}mv^2$	$KE = \frac{1}{2}I\omega^2$
$W_{net} = \Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$	$W_{net} = \Delta KE = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$
$P = Fv$	$P = \tau\omega$

The twin language is now complete for energy. Every formula is the same structure with the rotational substitutions. If you know one column, you know both.

With the energy toolkit assembled, let us put it to work. These examples will feel familiar from what you have already know, just dressed in rotational clothes.

BINDER Example 35

A constant torque of 8Nm is applied to a wheel that is initially at rest. The wheel rotates through 20rad.

(a) Calculate the work done by the torque.

(b) If the wheel has a moment of inertia of 0.5kgm^2, calculate the angular velocity after 20rad using the work-energy theorem.

(c) Verify your answer using $\omega^2 = \omega_0^2 + 2\alpha\theta$.

Solution

(a) Using $W = \tau\theta$ and substituting:

$$W = 8\text{Nm} \times 20\text{rad} = 160\text{J}$$

The work done is 160J.

(b) Using the work-energy theorem $W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$. Since $\omega_0 = 0$:

$$160\text{J} = \frac{1}{2} \times 0.5\text{kgm}^2 \times \omega^2$$

$$\omega = \sqrt{\frac{2 \times 160\text{J}}{0.5\text{kgm}^2}} = 25.3\text{rad/s}$$

The angular velocity after 20rad is 25.3rad/s.

(c) First, find α from $\tau = I\alpha$:

$$\alpha = \frac{\tau}{I} = \frac{8\text{Nm}}{0.5\text{kgm}^2} = 16\text{rad/s}^2$$

Using $\omega^2 = \omega_0^2 + 2\alpha\theta$ and substituting:

$$\omega^2 = 0 + 2 \times 16\text{rad/s}^2 \times 20\text{rad} = 640\text{rad}^2/\text{s}^2$$

$$\omega = \sqrt{640\text{rad}^2/\text{s}^2} = 25.3\text{rad/s}$$

Both methods give the same answer. ✓

Making Sense of the Answer: *The work-energy theorem and the kinematic approach must always agree, because they describe the same physics from different perspectives. The energy method is often quicker because it bypasses the intermediate step of finding α, going directly from work to final velocity. This advantage becomes even greater when the torque varies with angle, making the kinematic approach impractical.*

Think Like a Physicist: *Compare this with a more familiar problem: a force of* 8N *pushes a* 0.5kg *box from rest through* 20m*. Work =* 160J*, final velocity =* $\sqrt{640} = 25.3$m/s*. The numbers are identical because the mathematical structure is identical. The twin language is not just a metaphor; it is an exact correspondence.*

REAL Example 36

A traditional Tanzanian grain mill uses a heavy circular grinding stone (modelled as a uniform disc) of mass 15kg and radius 0.25m, spinning at 4rev/s.

(a) Calculate the rotational KE stored in the grinding stone.

(b) A friction torque of 3Nm acts on the stone when grain is fed in. Using the work-energy theorem, calculate how many revolutions the stone makes before stopping (if not pushed again).

(c) Calculate the average power dissipated during the stopping process.

Solution

(a) Moment of inertia: $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 15\text{kg} \times (0.25\text{m})^2 = 0.469\text{kgm}^2$

Angular velocity: $\omega = 2\pi \times 4 = 8\pi\text{rad/s} = 25.13\text{rad/s}$

Using $KE = \frac{1}{2}I\omega^2$ and substituting:

$$KE = \frac{1}{2} \times 0.469\text{kgm}^2 \times (25.13\text{rad/s})^2 = 148\text{J}$$

The rotational KE is 148J.

(b) The friction torque does negative work (it opposes the motion), converting KE into heat. By the work-energy theorem:

$$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = 0 - 148\text{J} = -148\text{J}$$

Since $W = -\tau\theta$ (negative because friction opposes motion), making θ the subject:

$$\theta = \frac{148\text{J}}{3\text{Nm}} = 49.3\text{rad}$$

Converting: $\frac{49.3}{2\pi} = 7.85\text{rev}$

The stone makes about 7.9 revolutions before stopping.

(c) First, find the time to stop. Using $\omega = \omega_0 + \alpha t$ with $\alpha = -\frac{\tau}{I} = -\frac{3}{0.469} = -6.40\text{rad/s}^2$:

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 25.13}{-6.40} = 3.93\text{s}$$

Average power: $P_{avg} = \frac{W}{t} = \frac{148\text{J}}{3.93\text{s}} = 37.7\text{W}$

The average power dissipated by friction is 37.7W.

Making Sense of the Answer: *The grinding stone stores* 148J *at* 4rev/s. *This is the energy available for grinding before the stone needs another push. The stone makes about* 8 *revolutions while this energy is used up by the friction of grinding. The average power (*38W*) is modest, roughly equivalent to a dim light bulb. This explains why traditional grinding is slow and requires frequent pushing: the stone does not store enough energy for prolonged grinding at high power.*

Think Like a Physicist: *Part (b) uses the work-energy theorem instead of kinematics. Compare: using kinematics would require finding α, then using $\omega^2 = \omega_0^2 + 2\alpha\theta$, then converting to revolutions. The energy method goes directly from initial KE to the target, bypassing α. Both give the same answer, but the energy route is shorter. Energy methods are especially powerful when forces (or torques) vary, because the work-energy theorem still works while $\tau = I\alpha$ requires constant torque.*

REAL Example 37

A car engine delivers a torque of 180Nm at 4000rpm.

(a) Calculate the power output in horsepower (1hp = 746W).

(b) If the engine speed drops to 2000rpm but the torque remains the same, calculate the new power output and explain why it is halved.

(c) Explain why car manufacturers quote "peak torque" and "peak power" at different engine speeds.

Solution

(a) Converting: $\omega = \frac{2\pi \times 4000}{60} = \frac{8000\pi}{60} = 418.9\text{rad/s}$

Using $P = \tau\omega$ and substituting:

$$P = 180\text{Nm} \times 418.9\text{rad/s} = 75402\text{W} = 75.4\text{kW}$$

Converting to horsepower: $\frac{75400}{746} = 101\text{hp}$

The power output is 101hp.

(b) At 2000rpm: $\omega = \frac{2\pi \times 2000}{60} = 209.4\text{rad/s}$

$$P = 180 \times 209.4 = 37700\text{W} = 37.7\text{kW}$$

The power is halved because $P = \tau\omega$. If τ stays the same but ω halves, then P halves. Power depends on both torque and angular velocity, not torque alone. An engine producing high torque at low rpm delivers less power than the same engine producing the same torque at high rpm.

(c) Torque measures the engine's turning effect at any instant. Power measures the rate at which the engine delivers energy. Because $P = \tau\omega$, peak power occurs not at peak torque but at the product $\tau\omega$ being maximum. In most engines, torque peaks at a moderate rpm (where combustion is most efficient), while power peaks at a higher rpm (where the higher ω compensates for the slightly reduced torque). Car manufacturers quote both because they measure different things: peak torque determines how much load the engine can handle (towing, hill climbing), while peak power determines the maximum speed and acceleration.

Making Sense of the Answer: *A* 101hp *engine is a typical mid-range car engine. The numbers are realistic. The key insight is that torque alone does not determine power: a tractor engine may have more torque than a sports car engine, but the sports car produces more power because it operates at much higher rpm.*

Think Like a Physicist: *The formula* $P = \tau\omega$ *is one of the most practically useful in all of rotational mechanics. Engineers use it daily to specify motors, turbines, and engines. If you ever need to choose a motor for a machine, you will use this formula: determine the torque required by the load and the angular velocity at which it must operate, multiply, and you have the power rating of the motor you need.*

HOT Example 38

A flywheel (uniform disc, mass 6kg, radius 0.50m) is spinning at 80rad/s. A braking torque is applied that decreases linearly with angle: $\tau = 50 - 0.5\theta$, where τ is in Nm and θ is in radians (measured from the instant the brake is applied). The brake weakens as the flywheel turns because the brake pad wears down.

(a) Calculate the initial rotational KE.

(b) Using the work-energy theorem with the variable torque, calculate the angle through which the flywheel turns before stopping.

(c) Explain why the kinematic equation $\omega^2 = \omega_0^2 + 2\alpha\theta$ cannot be used for this problem.

Solution

(a) $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 6\text{kg} \times (0.50\text{m})^2 = 0.75\text{kgm}^2$

$$KE_0 = \frac{1}{2}I\omega_0^2 = \frac{1}{2} \times 0.75\text{kgm}^2 \times (80\text{rad/s})^2 = 2400\text{J}$$

The initial rotational KE is 2400J.

(b) The braking torque does negative work as it opposes the rotation. The work done by the variable braking torque as the flywheel rotates from $\theta = 0$ to $\theta = \theta_{stop}$ is:

$$W = -\int_0^{\theta_{stop}} (50 - 0.5\theta)\, d\theta = -\left[50\theta - \frac{0.5\theta^2}{2}\right]_0^{\theta_{stop}} = -\left(50\theta_{stop} - 0.25\theta_{stop}^2\right)$$

By the work-energy theorem, when the flywheel stops ($\omega = 0$):

$$W = 0 - KE_0 = -2400\text{J}$$

Therefore:

$$50\theta_{stop} - 0.25\theta_{stop}^2 = 2400$$

Rearranging:

$$0.25\theta_{stop}^2 - 50\theta_{stop} + 2400 = 0$$

Solving the quadratic equation in θ_{stop}:

$$\theta_{stop} = 80\text{rad} \quad \text{or} \quad \theta_{stop} = 120\text{rad}$$

The braking torque vanishes when $50 - 0.5\theta = 0$, i.e., at $\theta = 100$rad. The solution $\theta = 120$rad is beyond this point (it gives the negative value of $50 - 0.5\theta$, which implies that the brake has already released), so it is physically invalid.

The flywheel stops after turning through $\theta = 80$rad (12.7 revolutions).

You may check: at $\theta = 80$rad, the braking torque is $\tau = 50 - 0.5 \times 80 = 10$Nm, which is still positive. The brake is still active when the flywheel stops. ✓

(c) The kinematic equation $\omega^2 = \omega_0^2 + 2\alpha\theta$ requires **constant** angular acceleration. Since the braking torque varies with angle ($\tau = 50 - 0.5\theta$), the angular acceleration $\alpha = \frac{\tau}{I}$ also varies with angle. The kinematic equations are therefore invalid for this problem. Only the work-energy theorem, which handles variable torques through integration, can solve it correctly.

Making Sense of the Answer: *The flywheel stores* 2400J *and the brake removes this energy over* 80rad *(about* 13 *revolutions). At the start, the brake exerts* 50Nm*, but by the time the flywheel stops, the brake has weakened to* 10Nm*. The two solutions of the quadratic (*80 *and* 120*) correspond to two angles where the cumulative work equals* 2400J*: the flywheel reaches zero speed at* 80rad *(real stopping point), while* 120rad *is a mathematical solution that has no physical meaning because the brake has already disappeared at* 100rad*. Rejecting the unphysical root is an important skill in examination problems.*

Think Like a Physicist: *This is the kind of problem that makes energy methods indispensable. The variable torque makes* $\tau = I\alpha$ *useless for finding the total angle directly, but the work-energy theorem handles it*

*elegantly through integration. The quadratic with two roots is a bonus lesson: always check whether both roots are physically valid. The brake's limited range (*0 *to* 100*rad) eliminates one root, and only the student who checks the physics, not just the mathematics, gets the correct answer.*

Experimental Determination of the Moment of Inertia of a Flywheel

The derivations of moments of inertia studied earlier, give us the moment of inertia for ideal, perfectly uniform shapes. But real flywheels are not perfect: they have spokes, hubs, keyways, and manufacturing imperfections. For a real flywheel, the moment of inertia is best determined experimentally, and the method uses exactly the energy principles we have just developed.

The method

A flywheel is mounted on a horizontal axle so that it is free to rotate. A known mass m is attached to a string wound around the axle (radius r). The mass is released from rest and falls through a measured height h, unwinding the string. At the instant the string detaches (all the string has unwound), the flywheel is spinning at its maximum angular velocity ω. The flywheel then continues to spin, gradually slowing due to friction, and eventually comes to rest after n additional revolutions.

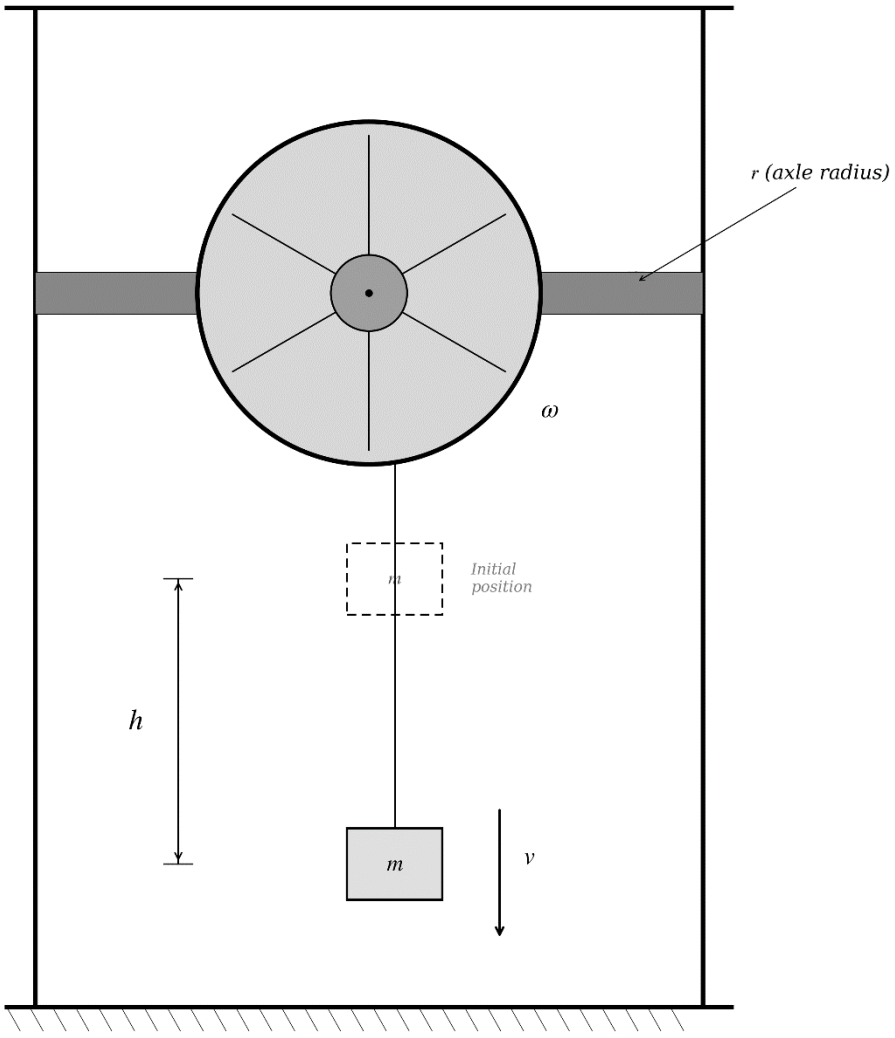


Figure: *Experimental setup for determining the moment of inertia of a flywheel. The flywheel is mounted on a horizontal axle (radius r) supported by a rigid frame. A mass m is attached to a string wound around the axle. Released from its initial position (dashed), the mass falls through height h, unwinding the string and spinning the flywheel to angular velocity ω. At the final position, the mass has downward velocity $v = \omega r$. After the string detaches, the flywheel coasts for n_2 additional revolutions before friction brings it to rest.*

The energy equation

As the mass falls through height h, gravitational potential energy is converted into three forms:

1) Translational KE of the falling mass: $\frac{1}{2}mv^2$
2) Rotational KE of the flywheel: $\frac{1}{2}I\omega^2$
3) Energy lost to friction during the winding phase

By conservation of energy:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + W_{\text{friction (winding)}}$$

Dealing with friction

The friction torque τ_f acts throughout both the winding phase (while the mass falls) and the unwinding phase (after the string detaches). If the flywheel makes n_1 revolutions during the fall and n_2 revolutions after the string detaches before stopping:

Work done by friction during the fall: $W_{f1} = \tau_f \times 2\pi n_1$

Work done by friction after detachment: $W_{f2} = \tau_f \times 2\pi n_2$

After the string detaches, the flywheel has KE $= \frac{1}{2}I\omega^2$ and this is entirely used up by friction over n_2 revolutions:

$$\frac{1}{2}I\omega^2 = \tau_f \times 2\pi n_2$$

This gives us the friction torque:

$$\tau_f = \frac{I\omega^2}{4\pi n_2}$$

The number of revolutions during the fall, n_1, is found from the geometry: the string length unwound equals the distance fallen, so $h = 2\pi n_1 r$, giving $n_1 = \frac{h}{2\pi r}$.

The no-slip condition *relates v and ω at the moment of detachment*: $v = \omega r$.

Substituting everything *into the energy equation:*

$$mgh = \frac{1}{2}m(\omega r)^2 + \frac{1}{2}I\omega^2 + \frac{I\omega^2}{4\pi n_2} \times 2\pi n_1$$

$$mgh = \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}I\omega^2 + \frac{I\omega^2 n_1}{2n_2}$$

$$mgh = \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}I\omega^2\left(1 + \frac{n_1}{n_2}\right)$$

Making I the subject:

$$\frac{1}{2}I\omega^2\left(1 + \frac{n_1}{n_2}\right) = mgh - \frac{1}{2}m\omega^2 r^2$$

$$\mathbf{I = \frac{2mgh - m\omega^2 r^2}{\omega^2\left(1 + \frac{n_1}{n_2}\right)}}$$

Finding ω:

The angular velocity at the moment of detachment can be found by measuring the time for a few revolutions immediately after detachment (using a stopwatch and counting), or from the average angular velocity during the fall**:** $\boldsymbol{\omega = \frac{2\times 2\pi n_1}{t}}$ where t is the time of fall (since the flywheel accelerates from 0 to ω, the average is $\omega/2$).

REAL Example 39

In a flywheel experiment, a mass of 2kg is attached to a string wound around the axle of radius 0.03m. The mass is released from rest and falls through 1.5m in 10s, making 8 complete turns of the flywheel. After the string detaches, the flywheel makes 40 more revolutions before stopping. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the angular velocity of the flywheel, and linear velocity of the falling mass at the instant the string detaches

(b) Calculate the moment of inertia of the flywheel.

(c) Calculate the friction torque.

Solution

(a) During the fall, the flywheel accelerates uniformly from rest. The number of revolutions is $n_1 = 8$, so the total angle is $\theta = 2\pi \times 8 = 16\pi\text{rad}$.

Using $\theta = \frac{1}{2}(\omega_0 + \omega)t$ with $\omega_0 = 0$ and making ω the subject:

$$\omega = \frac{2\theta}{t} = \frac{2 \times 16\pi \text{ rad}}{10\text{s}} = \frac{32\pi \text{ rad}}{10\text{s}} = 10.05\text{rad/s}$$

The angular velocity at detachment is 10.05rad/s.

Alternative solution

The angular velocity is directly given by the following formula:

$$\omega = \frac{2 \times 2\pi n_1}{t} = \frac{2 \times 2\pi \times 8\text{rad}}{10\text{s}} = \frac{32\pi \text{ rad}}{10\text{s}} = 10.05\text{rad/s}$$

The linear velocity of the falling mass: $v = \omega r = 10.05 \times 0.03 = \mathbf{0.302m/s}$

(b) Using the formula derived above and substituting:

$$I = \frac{2mgh - m\omega^2 r^2}{\omega^2\left(1 + \frac{n_1}{n_2}\right)}$$

Substituting:

$$I = \frac{2 \times 2\text{kg} \times 9.8\text{m/s}^2 \times 1.5\text{mkg} - 2 \times (10.05\text{rad/s})^2 \times (0.03\text{m})^2}{(10.05\text{rad/s})^2 \times \left(1 + \frac{8}{40}\right)} = 0.484\text{kgm}^2$$

The moment of inertia of the flywheel is 0.484kgm^2.

(c) Using $\frac{1}{2}I\omega^2 = \tau_f \times 2\pi n_2$ and making τ_f the subject:

$$\tau_f = \frac{I\omega^2}{4\pi n_2} = \frac{0.484\text{kgm}^2 \times (10.05\text{rad/s})^2}{4\pi \times 40} = 0.0973\text{Nm}$$

The friction torque is 0.097Nm.

Making Sense of the Answer: *The moment of inertia of $0.484kgm^2$ is a reasonable value for a laboratory flywheel. The friction torque ($0.097Nm$) is small, which is expected for a well-lubricated bearing, and it allows the flywheel to spin for 40 revolutions after detachment, consistent with low friction. The translational KE of the falling mass $\left(\frac{1}{2} \times 2 \times 0.302^2 = 0.091J\right)$ is tiny compared to the potential energy released ($58.8J$), which tells us that almost all the energy went into spinning the flywheel and overcoming friction. This is typical: the falling mass moves slowly because the axle radius is small.*

Think Like a Physicist: *This experiment is a beautiful application of conservation of energy. Three energy terms (translational KE, rotational KE, friction work) compete for the gravitational PE released by the falling mass. The clever part is using the coasting phase (after detachment) to independently determine the friction torque, which then feeds back into the main energy equation. Without this trick, we would have two unknowns (I and τ_f) in one equation. By splitting the motion into two phases (winding and coasting), we get two equations for two unknowns. This is experimental physics at its best.*

So far, every object in this chapter has either rotated in place (spinning about a fixed axis) or translated in a straight line. Real objects often do both at the same time. A ball rolling down a hill translates forward while spinning about its own centre. A wheel on a car translates along the road while rotating about its axle. In the next subtopic, we bring translation and rotation together for the first time, and discover that the shape of a rolling object determines how fast it goes. The famous rolling race is about to begin.

ROLLING MOTION

A ball rolls down a hill. A wheel rolls along a road. A cylinder rolls off a table. In each case, the object does two things at once: it translates (the centre moves forward) and it rotates (the body spins about its centre). Neither description alone captures the full motion. Translation without rotation is sliding. Rotation without translation is spinning in place. **Rolling** is the beautiful combination of both.

This subtopic brings together everything we have built in this chapter. The moment of inertia decides how stubborn the body is about spinning. The radius of gyration decides who wins the rolling race. The energy methods of the previous subtopic decide how fast the body moves. And Newton's second law, applied simultaneously to translation and rotation, reveals the role of friction in making rolling possible in the first place.

Rolling motion is where rotational dynamics meets the real world. Let us begin.

Pure Rolling: The No-slip Condition

When a wheel rolls along a road without skidding, the bottom of the wheel is momentarily at rest relative to the road. There is no sliding, no skidding, no slipping. This is called **pure rolling** (or rolling without slipping), and it imposes a precise mathematical relationship between the translational velocity of the centre and the angular velocity of the body.

The constraint:

Consider a circular body (sphere, cylinder, disc, wheel) of radius R rolling without slipping on a flat surface. In one complete revolution, the body rotates through 2π radians. During this revolution, the centre of mass advances by exactly one circumference, 2πR, because each point on the rim touches the ground exactly once.

In general, if the body rotates through angle θ, the centre of mass advances by:

$$s = R\theta$$

Differentiating with respect to time:

$$\frac{ds}{dt} = R\frac{d\theta}{dt}$$

$$\mathbf{v_{cm} = \omega R}$$

This is the **no-slip condition** for pure rolling. It connects the translational speed of the centre of mass to the angular velocity. If you know one, you know the other.

Differentiating once more:

$$\frac{dv_{cm}}{dt} = R\frac{d\omega}{dt}$$

$$\mathbf{a_{cm} = \alpha R}$$

The translational acceleration of the centre of mass equals the angular acceleration times the radius.

What pure rolling means at the contact point:

At the contact point (where the body touches the surface), two velocities combine:

1) The translational velocity of the centre: v_{cm} (forward)
2) The rotational velocity of the contact point relative to the centre: ωR (backward, because the bottom of the body moves backward relative to the centre)

The net velocity of the contact point is:

$$v_{contact} = v_{cm} - \omega R$$

For pure rolling, $v_{cm} = \omega R$, so:

$$\boldsymbol{v_{contact} = \omega R - \omega R = 0}$$

The contact point has **zero velocity**. It is instantaneously at rest. This is the defining feature of pure rolling: no sliding at the contact point.

What happens at other points on the rim:

The topmost point has the translational velocity v_{cm} (forward) plus the rotational velocity ωR (also forward at the top), giving:

$$\boldsymbol{v_{top} = v_{cm} + \omega R = 2v_{cm}}$$

The top of the rolling body moves at twice the speed of the centre. This is why mud flies backward off the top of a spinning car tyre, not off the bottom.

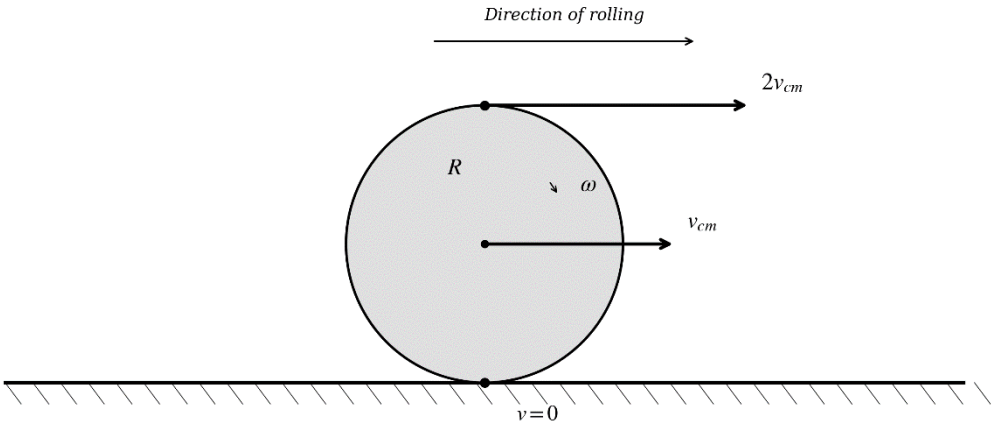


Figure: *A circular body of radius R rolling without slipping on a flat surface. At the centre of mass, the velocity is v_{cm} (forward). At the contact point (bottom), the translational velocity v_{cm} and the rotational velocity ωR cancel exactly, giving zero net velocity. At the topmost point, the two velocities add, giving $2v_{cm}$. The angular velocity ω satisfies the no-slip condition $v_{cm} = \omega R$.*

Three types of motion:

1. **Pure rolling** ($v_{cm} = \omega R$): no slipping at the contact point.
2. **Skidding** ($v_{cm} > \omega R$): the body slides forward faster than it spins. The contact point moves forward. Kinetic friction acts backward on the body, slowing the translation and speeding up the rotation until pure rolling is established.
3. **Spinning** ($v_{cm} < \omega R$): the body spins faster than it translates. The contact point moves backward. Kinetic friction acts forward on the body, speeding up the translation and slowing the rotation until pure rolling is established.

In all three cases, friction acts to bring v_{cm} and ωR into agreement. Once pure rolling is achieved, the friction at the contact point becomes **static friction** (no sliding), and it does **no work** (because the contact point is at rest). This is a crucial point that we will return to later.

Total Kinetic Energy of a Rolling Body

A rolling body has both translational and rotational KE. The total KE is their sum:

$$KE_{total} = KE_{trans} + KE_{rot} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

Using the no-slip condition $v_{cm} = \omega R$ and $I_{cm} = Mk^2$:

$$KE_{total} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}Mk^2\left(\frac{v_{cm}}{R}\right)^2$$

$$= \frac{1}{2}Mv_{cm}^2\left(1+\frac{k^2}{R^2}\right)$$

$$\mathbf{KE_{total} = \frac{1}{2}Mv_{cm}^2\left(1+\frac{k^2}{R^2}\right)}$$

This is a powerful result. The factor $\left(1+\frac{k^2}{R^2}\right)$ tells us how much extra KE the body has because of its rotation. For a solid sphere ($\frac{k^2}{R^2}=\frac{2}{5}$), the total KE is $\frac{7}{5}$ times the translational KE alone. For a hollow cylinder ($\frac{k^2}{R^2}=1$), it is exactly double.

Energy partition:

The fraction of total KE that is translational:

$$\frac{\mathbf{KE_{trans}}}{\mathbf{KE_{total}}} = \frac{\frac{1}{2}Mv_{cm}^2}{\frac{1}{2}Mv_{cm}^2\left(1+\frac{k^2}{R^2}\right)} = \frac{\mathbf{1}}{\mathbf{1+\frac{k^2}{R^2}}}$$

The fraction that is rotational:

$$\frac{\mathbf{KE_{rot}}}{\mathbf{KE_{total}}} = \frac{\frac{1}{2}Mk^2\left(\frac{v_{cm}}{R}\right)^2}{\frac{1}{2}Mv_{cm}^2\left(1+\frac{k^2}{R^2}\right)} = \frac{\mathbf{\frac{k^2}{R^2}}}{\mathbf{1+\frac{k^2}{R^2}}}$$

Rolling Down an Inclined Plane: Energy Method

A body of mass M and radius R starts from rest at the top of an inclined plane of height h and rolls without slipping to the bottom.

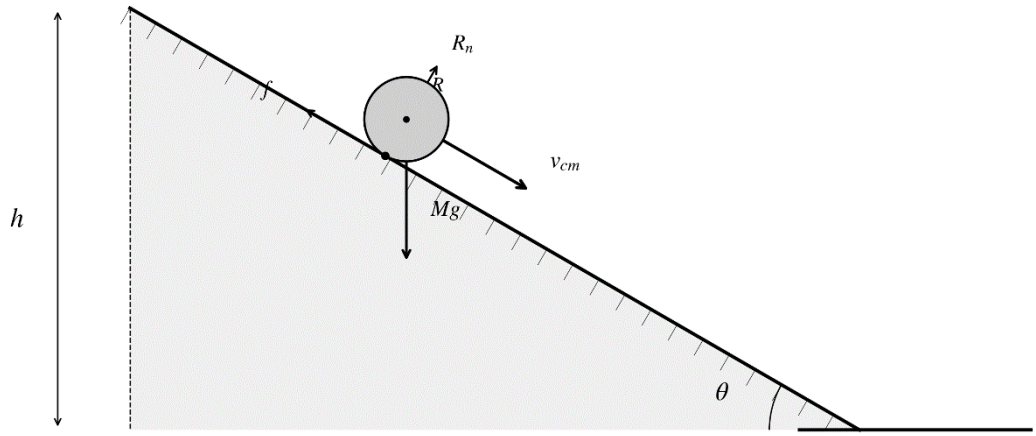


Figure: *A body of mass M and radius R rolling without slipping down an inclined plane of height h and angle θ. The forces acting on the body are: weight Mg (vertically downward from the centre of mass), normal reaction R_n (perpendicular to the surface at the contact point), and static friction f (up the slope at the contact point). The velocity of the centre of mass v_{cm} is directed down the slope. The contact point, where friction acts, is instantaneously at rest during pure rolling.*

Speed at the bottom:

By conservation of energy, the gravitational PE at the top converts entirely into KE at the bottom (no work done by friction, since static friction at the contact point does no work during pure rolling):

$$Mgh = \frac{1}{2}Mv_{cm}^2\left(1+\frac{k^2}{R^2}\right)$$

Making v_{cm} the subject:

$$v_{cm}^2 = \frac{2gh}{1+\frac{k^2}{R^2}}$$

$$\mathbf{v_{cm}} = \sqrt{\frac{\mathbf{2gh}}{\mathbf{1+\frac{k^2}{R^2}}}}$$

Compare this with a sliding (non-rolling) body, which would have $v = \sqrt{2gh}$. The rolling body is always slower, by the factor $\frac{1}{\sqrt{1+k^2/R^2}}$, because some energy goes into rotation.

Acceleration down the incline:

Using $v^2 = u^2 + 2as$ with $u = 0$ and $v^2 = \frac{2gh}{1+k^2/R^2}$, and noting that $h = s\sin\theta$ where s is the distance along the incline and θ is the angle of inclination:

$$\frac{2gs\sin\theta}{1+k^2/R^2} = 2as$$

$$\mathbf{a} = \frac{\mathbf{g\sin\theta}}{\mathbf{1+\frac{k^2}{R^2}}}$$

This is a constant acceleration (independent of position), which means the body rolls down with uniform acceleration, just like a block sliding down a frictionless incline but with the effective gravitational pull reduced by the factor $\frac{1}{1+k^2/R^2}$.

Key observations:

1. The acceleration depends on the **shape** (through $\frac{k^2}{R^2}$) but not on the **mass** or the **size**. A marble and a bowling ball (both solid spheres) have the same acceleration and reach the bottom at the same time. This is the rotational equivalent of Galileo's discovery that all objects fall at the same rate.
2. The acceleration is always less than $g\sin\theta$ (the acceleration of a sliding body on a frictionless incline). The "missing" acceleration goes into spinning the body up.
3. The smaller the $\frac{k^2}{R^2}$, the larger the acceleration and the faster the body reaches the bottom.

Now let us put numbers to these results.

BINDER Example 40

A solid sphere of mass 2kg and radius 0.08m rolls without slipping down an inclined plane of height 1.5m and angle 30°. Take $g = 9.8 m/s^2$.

(a) Calculate the speed of the centre of mass at the bottom of the incline.

(b) Calculate the translational and rotational KE at the bottom.

(c) Calculate the acceleration down the incline.

(d) Calculate the time to reach the bottom.

Solution

(a) For a solid sphere: $\frac{k^2}{R^2} = \frac{2}{5}$.

Using $v_{cm} = \sqrt{\frac{2gh}{1+k^2/R^2}}$ and substituting:

$$v_{cm} = \sqrt{\frac{2 \times 9.8\text{m/s}^2 \times 1.5\text{m}}{1 + \frac{2}{5}}} = 4.58\text{m/s}$$

The speed at the bottom is 4.58m/s.

(b) Translational KE:

$$KE_{trans} = \frac{1}{2}Mv_{cm}^2 = \frac{1}{2} \times 2\text{kg} \times (4.58\text{m/s})^2 = 21.0\text{J}$$

Total KE = Mgh = $2 \times 9.8\text{m/s}^2 \times 1.5\text{m} = 29.4\text{J}$

Rotational KE = Total − Translational = $(29.4 - 21.0)\text{J} = 8.4\text{J}$

The translational KE is 21.0J and the rotational KE is 8.4J.

(c) Using $a = \frac{g\sin\theta}{1+k^2/R^2}$ and substituting:

$$a = \frac{9.8\text{m/s}^2 \times \sin 30°}{1 + \frac{2}{5}} = 3.50\text{m/s}^2$$

The acceleration is 3.50m/s^2.

(d) The length of the incline: $s = \frac{h}{\sin\theta} = \frac{1.5\text{m}}{\sin 30°} = \frac{1.5}{0.5} = 3.0\text{m}$

Using $s = \frac{1}{2}at^2$ (starting from rest) and making t the subject:

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 3.0\text{m}}{3.50\text{m/s}^2}} = 1.31\text{s}$$

The time to reach the bottom is 1.31s.

Making Sense of the Answer: *If the sphere were sliding (not rolling) on a frictionless incline, it would reach* $v = \sqrt{2 \times 9.8 \times 1.5} = 5.42\text{m/s}$*. The rolling sphere is slower (*4.58m/s*, which is* 84% *of the sliding speed) because* 29% *of the energy goes into rotation. The acceleration (*3.50m/s^2*) is* 71% *of the sliding value (*$g\sin 30° = 4.9\text{m/s}^2$*). Notice that neither the mass (*2kg*) nor the radius (*0.08m*) appeared in the final answers for speed and acceleration. Only the shape matters.*

Think Like a Physicist: *This example uses the energy method: conservation of energy gives the speed, kinematics gives the acceleration and time. No forces or torques are needed. The energy method is elegant but has one limitation: it does not tell us the friction force. For that, we need the force method. But for finding speed, acceleration, and time, the energy method is unbeatable.*

REAL Example 41

After school, Kipanga and Kipute are cycling home. Kipanga notices that when his bicycle rides through a muddy puddle, the mud flies off the **top** of the wheel, not the bottom. He expected the mud to fly off from the bottom, where the wheel touches the mud.

Explain why the mud flies off from the top of the spinning wheel.

Solution

At the contact point (bottom of the wheel), the velocity is zero because the translational velocity v_{cm} (forward) and the rotational velocity ωR (backward) cancel exactly. Since the contact point is momentarily at rest, mud at the bottom has no tendency to fly off.

At the topmost point, the two velocities add: the translational velocity v_{cm} (forward) and the rotational velocity ωR (also forward at the top) give a total velocity of $2v_{cm}$. This is the fastest-moving point on the wheel. Mud at the top is moving at twice the bicycle's speed and is flung off tangentially when the adhesion to the tyre can no longer provide the centripetal force needed to keep it on the curved path.

Making Sense of the Answer: *The bottom of the wheel is the safest place for mud: it is momentarily at rest. The top is the most dangerous: it moves at $2v_{cm}$, the fastest point on the entire wheel. This is also why mudguards on bicycles are placed above the wheel, not below it. The mud threat comes from the top, not the bottom.*

Think Like a Physicist: *This observation is a direct consequence of the no-slip condition $v_{cm} = \omega R$. The velocity at any point on the wheel is the vector sum of the translational velocity (same for all points) and the rotational velocity (different direction at each point). At the bottom they cancel. At the top they reinforce. At the sides they are perpendicular, giving $v = \sqrt{2}\, v_{cm}$ at $45°$ to the horizontal.*

HOT Example 42

A solid cylinder and a solid sphere, both of mass M and radius R, are released from rest at the top of an inclined plane of angle θ and length L. They roll without slipping.

(a) Show that the cylinder arrives at the bottom later than the sphere, and find the ratio of their arrival times.

(b) If $\theta = 20°$ and $L = 2.5\text{m}$, calculate the arrival time for each. Take $g = 9.8\text{m/s}^2$.

(c) How much later does the cylinder arrive?

Solution

(a) The acceleration of a rolling body is $a = \frac{g\sin\theta}{1+k^2/R^2}$.

For the solid sphere: $\frac{k^2}{R^2} = \frac{2}{5}$, so $a_{sphere} = \frac{g\sin\theta}{\frac{7}{5}} = \frac{5g\sin\theta}{7}$

For the solid cylinder: $\frac{k^2}{R^2} = \frac{1}{2}$, so $a_{cylinder} = \frac{g\sin\theta}{\frac{3}{2}} = \frac{2g\sin\theta}{3}$

Since $\frac{5}{7} > \frac{2}{3}$, the sphere has the larger acceleration and arrives first.

Using $L = \frac{1}{2}at^2$ and making t the subject: $t = \sqrt{\frac{2L}{a}}$

$$\frac{\mathbf{t_{cylinder}}}{\mathbf{t_{sphere}}} = \sqrt{\frac{a_{sphere}}{a_{cylinder}}} = \sqrt{\frac{5/7}{2/3}} = \mathbf{1.0351}$$

(The cylinder takes 3.5% longer than the sphere. The ratio does not depend on θ, L, M, R, or g: it is a universal constant for the sphere versus cylinder race.)

(b) For the sphere:

$$a_{sphere} = \frac{5 \times 9.8\text{m/s}^2 \times \sin 20°}{7} = 2.394\text{m/s}^2$$

$$t_{sphere} = \sqrt{\frac{2 \times 2.5\text{m}}{2.394\text{m/s}^2}} = 1.445\text{s}$$

For the cylinder:

$$a_{cylinder} = \frac{2 \times 9.8\text{m/s}^2 \times \sin 20°}{3} = 2.235\text{m/s}^2$$

$$t_{cylinder} = \sqrt{\frac{2 \times 2.5\text{m}}{2.235\text{m/s}^2}} = 1.496\text{s}$$

The sphere arrives in 1.445s and the cylinder in 1.496s.

(c) Time difference: $t_{cylinder} - t_{sphere} = 1.496\text{s} - 1.445\text{s} = 0.051\text{s}$

The cylinder arrives 0.051 seconds (51 milliseconds) after the sphere.

Check: ratio = $\frac{1.496}{1.445} = 1.035$. Matches part (a). ✓

Making Sense of the Answer: *The difference is only* 51 *milliseconds on a* $2.5m$ *ramp. This is small but measurable with a stopwatch if you know what to look for. On a longer ramp, the gap grows (it scales as* $\sqrt{L}$*). On a* $10m$ *ramp, the gap would be about* $0.1s$*, easily visible. The fact that the ratio* $\sqrt{15/14}$ *is independent of everything except the shapes is remarkable: the same* 3.5% *time advantage holds whether the ramp is steep or gentle, short or long, on Earth or on the Moon.*

Think Like a Physicist: *Part (a) is the kind of algebraic proof that examiners love: show that the ratio depends only on shape, not on any other variable. The trick is to express both times in terms of* $L/g\sin\theta$ *and take the ratio, which cancels everything except the shape factors. Part (b) puts concrete numbers to the algebra. Part (c) shows that while the sphere always wins, the margin is slim: the cylinder is a respectable runner-up, only* 3.5% *behind.*

Rolling Down an Inclined Plane: Force and Torque method

The energy method gives us the speed and acceleration of a rolling body but tells us nothing about the forces involved. In particular, it cannot tell us the friction force. To find friction, we must apply Newton's second law for translation and Newton's second law for rotation simultaneously.

Setup: A body of mass M and radius R rolls without slipping down an inclined plane of angle θ(refer to the diagram provided in the previous subsection: energy method).

Three forces act on the body:

1) Weight Mg vertically downward (through the centre of mass)
2) Normal reaction R_n perpendicular to the surface (at the contact point)
3) Static friction f up the slope (at the contact point)

Translational equation (along the incline, taking down the slope as positive):

$$Mg\sin\theta - f = Ma_{cm} \quad \ldots (i)$$

Rotational equation (about the centre of mass, taking clockwise as positive since the body rolls down):

The weight acts through the centre of mass, so its torque about the centre of mass is zero. The normal reaction acts at the point of contact and is directed along the radius of the body, passing through the centre of mass; hence, it also produces zero torque about the centre. Only friction acts at the contact point in a direction that produces a nonzero torque.

$$fR = I_{cm}\alpha \quad \ldots (ii)$$

Constraint: $a_{cm} = \alpha R$, so $\alpha = \frac{a_{cm}}{R}$

Substituting into (ii):

$$fR = I_{cm} \times \frac{a_{cm}}{R}$$

$$f = \frac{I_{cm}a_{cm}}{R^2} \quad \ldots (iii)$$

Substituting (iii) into (i):

$$Mg\sin\theta - \frac{I_{cm}a_{cm}}{R^2} = Ma_{cm}$$

$$Mg\sin\theta = Ma_{cm} + \frac{I_{cm}a_{cm}}{R^2} = a_{cm}\left(M + \frac{I_{cm}}{R^2}\right)$$

Using $I_{cm} = Mk^2$:

$$Mg\sin\theta = a_{cm} \times M\left(1 + \frac{k^2}{R^2}\right)$$

Making a_{cm} the subject:

$$\mathbf{a_{cm} = \frac{gsin\theta}{1 + \frac{k^2}{R^2}}}$$

This is exactly the same result as the energy method. Good. The two methods must agree.

Finding the friction force:

Substituting a_{cm} back into (iii) where $I_{cm} = Mk^2$:

$$f = \frac{Mk^2}{R^2} \times \frac{gsin\theta}{1 + \frac{k^2}{R^2}} = \frac{Mgsin\theta \times \frac{k^2}{R^2}}{1 + \frac{k^2}{R^2}} = \frac{Mgsin\theta \times \frac{k^2}{R^2}}{1 + \frac{k^2}{R^2}}\left(\frac{\frac{R^2}{k^2}}{\frac{R^2}{k^2}}\right) = \frac{Mgsin\theta}{\frac{R^2}{k^2} + 1}$$

$$\mathbf{f = \frac{Mgsin\theta}{1 + \frac{R^2}{k^2}}}$$

This is the result the energy method could not give us. The friction force depends on the shape (through k^2/R^2), the mass, and the angle, but not on the radius. Let us see what it gives for specific shapes:

Body	$\frac{k^2}{R^2}$	**Friction f**
Solid sphere	$\frac{2}{5}$	$\frac{2}{7}Mgsin\theta$
Solid cylinder	$\frac{1}{2}$	$\frac{1}{3}Mgsin\theta$
Hollow sphere	$\frac{2}{3}$	$\frac{2}{5}Mgsin\theta$
Hollow cylinder	1	$\frac{1}{2}Mgsin\theta$

The hollow cylinder needs the most friction $\left(\frac{1}{2}Mgsin\theta\right)$ and the solid sphere the least $\left(\frac{2}{7}Mgsin\theta\right)$. This has practical consequences: on a surface with limited friction, the hollow cylinder is the first to start slipping, while the solid sphere is the last.

The Condition for Pure Rolling

For pure rolling to be maintained, the required friction must not exceed the maximum available static friction:

$$f \leq \mu_s R_n$$

Since $R_n = Mgcos\theta$ on an incline:

$$\frac{Mgsin\theta}{1 + \frac{R^2}{k^2}} \leq \mu_s Mgcos\theta$$

Dividing $cos\theta$ throughout:

$$\frac{tan\theta}{1 + \frac{R^2}{k^2}} \leq \mu_s$$

$$\mathbf{tan\theta \leq \mu_s\left(1 + \frac{R^2}{k^2}\right)}$$

If the incline is too steep (large θ) or the surface too smooth (small μ_s), the body cannot roll without slipping. It will slide and spin simultaneously.

For a solid sphere: $\tan\theta \leq \mu_s\left(1+\frac{5}{2}\right) = \frac{7}{2}\mu_s$, so pure rolling holds for $\tan\theta \leq 3.5\mu_s$.

For a hollow cylinder: $\tan\theta \leq \mu_s(1+1) = 2\mu_s$, so pure rolling holds for $\tan\theta \leq 2\mu_s$.

The solid sphere can roll on steeper inclines than the hollow cylinder before slipping begins, because it demands less friction.

The Role of Friction in Rolling

Students often find friction in rolling confusing. Here is a clear summary:

1. Static friction, not kinetic: During pure rolling, the contact point is at rest. Therefore, the friction is **static**, not kinetic. This is the same type of friction that prevents a book from sliding on a tilted table.

2. Friction does no work: Because the contact point (where friction acts) is instantaneously at rest, the friction force does no work. Work requires force and displacement at the same point, and the displacement of the contact point is zero. This is why the energy method works without including friction: it contributes zero to the energy equation.

3. Friction is essential for rolling: Without friction, a body on an incline would slide, not roll. Friction provides the torque about the centre of mass that spins the body up. Remove friction (a perfectly smooth surface), and the body slides without rotating, like a block on a frictionless incline. Friction is the agent that converts some of the translational acceleration into rotational acceleration.

4. Friction acts up the slope: When a body rolls down an incline, friction acts up the slope. This may seem strange (friction opposing the translational motion), but it makes sense: friction must produce a torque that spins the body in the rolling direction. The only way to produce a clockwise torque (for a body rolling down to the right) is for friction to act up the slope at the contact point.

5. Friction does not always act up the slope: If a body is rolling up an incline (decelerating), friction acts **down** the slope. It must now decelerate the spin as well as the translation. The direction of friction depends on the situation, not on a fixed rule.

Now that we understand both the energy and force methods, and the role of friction, let us tackle some challenging problems.

BINDER Example 43

A solid cylinder of mass 4kg and radius 0.10m rolls without slipping down an incline of angle $25°$. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the acceleration of the centre of mass.

(b) Calculate the friction force.

(c) Calculate the minimum coefficient of static friction required for pure rolling.

Solution

(a) For a solid cylinder: $I_{cm} = Mk^2 = \frac{1}{2}MR^2$. So:

$$\frac{k^2}{R^2} = \frac{1}{2}.$$

Using $a = \frac{g\sin\theta}{1+k^2/R^2}$ and substituting:

$$a = \frac{9.8\text{m/s}^2 \times \sin 25°}{1+\frac{1}{2}} = 2.76\text{m/s}^2$$

The acceleration is 2.76m/s^2.

(b) Using $f = \frac{Mg\sin\theta}{1+R^2/k^2}$ and substituting:

$$f = \frac{4\text{kg} \times 9.8\text{m/s}^2 \times \sin 25°}{1 + \frac{1}{1/2}} = 5.52\text{N}$$

The friction force is 5.52N, acting up the slope.

(c) For pure rolling: $f \leq \mu_s Mg\cos\theta$

Making μ_s the subject:

$$\mu_s \geq \frac{f}{Mg\cos\theta} = \frac{5.52\text{N}}{4\text{kg} \times 9.8\text{m/s}^2 \times \cos 25°} = 0.155$$

The minimum coefficient of static friction is 0.155.

Making Sense of the Answer: *The friction force ($5.52N$) is about one-third of the gravitational component along the slope ($Mg\sin 25° = 16.56N$). This is exactly the fraction $\frac{1}{3}$ from the friction table for a solid cylinder. The required μ_s of 0.155 is moderate, is easily achieved by most surfaces (rubber on concrete has $\mu_s \approx 0.6$), so the cylinder rolls comfortably on most real surfaces. Only very slippery surfaces (ice, oiled metal) would cause it to slide.*

Think Like a Physicist: *The force method gives three things the energy method cannot: the friction force, the normal reaction, and the condition for pure rolling. Whenever a problem asks for "the friction force" or "the minimum coefficient of friction," you must use the force method. The energy method is faster for speed and acceleration but blind to forces.*

REAL Example 44

On a rainy day, Kipute observes a car wheel spinning in place on a muddy patch. The car is stuck: the engine roars, the wheel spins, but the car does not move forward.

(i) Explain, using the physics of rolling, why the car cannot move.
(ii) Suggest any practical solution that will enable the car to move forward.

Solution

(i) For a wheel to propel a car forward, it must roll without slipping. During pure rolling, static friction at the contact point pushes the car forward (this is the reaction to the wheel pushing backward on the road). The wheel's rotation is converted into forward translation through friction.
On a muddy surface, the coefficient of static friction is very low. The torque from the engine spins the wheel, but the mud cannot provide enough static friction to prevent slipping. The contact point slides backward (the wheel spins faster than v_{cm}/R), and kinetic friction replaces static friction. Kinetic friction is weaker than static friction and cannot accelerate the car effectively. The wheel spins in place, the car barely moves, and the mud flies everywhere.

(ii) The solution is to increase the available friction: place sand, gravel, or a mat under the wheel. This raises μ_s, restores the no-slip condition, and allows the wheel to grip the road again.

Making Sense of the Answer: *A spinning wheel that cannot grip the road is the rolling condition in reverse: the wheel violates $v_{cm} = \omega R$ because ωR far exceeds v_{cm}. Friction tries to bring them into agreement by accelerating the car forward and decelerating the spin, but on mud the friction is too weak to do this effectively. The car is stuck because rolling requires friction, and the mud cannot provide enough.*

Think Like a Physicist: *The condition for pure rolling requires μ_s to be above a minimum value. On mud, μ_s drops below this threshold, and the wheel transitions from rolling to spinning. The physics of a stuck car is the same physics as a cylinder failing to roll on a steep, smooth incline: insufficient friction to maintain the no-slip condition.*

HOT Example 45

A solid sphere and a hollow cylinder, both of mass M and radius R, are released from rest at the top of a rough inclined plane of angle θ and height h. Both roll without slipping.

(a) Find the friction force on each body as a fraction of $Mg\sin\theta$.

(b) Show that the hollow cylinder requires 75% more friction than the solid sphere.

(c) If $\mu_s = 0.25$ and $\theta = 35°$, determine which body (if any) can roll without slipping.

Solution

(a) From the friction formula $f = \frac{Mg\sin\theta}{1+R^2/k^2}$:

Solid sphere ($k^2/R^2 = 2/5$): $\frac{R^2}{k^2} = \frac{5}{2}$

$$f_{sphere} = \frac{Mg\sin\theta}{1+\frac{5}{2}} = \frac{Mg\sin\theta}{\frac{7}{2}} = \frac{2}{7}Mg\sin\theta$$

Hollow cylinder ($k^2/R^2 = 1$): $\frac{R^2}{k^2} = 1$

$$f_{cylinder} = \frac{Mg\sin\theta}{1+1} = \frac{1}{2}Mg\sin\theta$$

The solid sphere needs $\frac{2}{7}$ of $Mg\sin\theta$ and the hollow cylinder needs $\frac{1}{2}$.

(b) The ratio:

$$\frac{f_{cylinder}}{f_{sphere}} = \frac{1/2}{2/7} = 1.75$$

$$\%\ \text{increase} = \left(\frac{1.75-1}{1}\right) \times 100 = 75\%$$

Hence, the hollow cylinder requires **75% more friction** than the solid sphere.

(c) For pure rolling, we need $f \leq \mu_s Mg\cos\theta$.

For the solid sphere:

$$\frac{2}{7}Mg\sin\theta \leq \mu_s Mg\cos\theta$$

$$\frac{2}{7}\sin\theta \leq \mu_s\cos\theta$$

$$\frac{2}{7}\tan\theta \leq \mu_s$$

$$\frac{2}{7}\tan 35° = 0.2 < \mu_s(0.25)$$

This is satisfied. The solid sphere rolls without slipping.

Performing the same for the hollow cylinder:

$$\frac{1}{2}\tan 35° = 0.35 > \mu_s(0.25)$$

This is **not** satisfied. The hollow cylinder cannot roll without slipping at $35°$ with $\mu_s = 0.25$. It will slide and spin.

The solid sphere rolls without slipping, but the hollow cylinder slides.

Hence, the body which can roll without slipping is solid sphere.

Making Sense of the Answer: *On the same incline with the same friction, the solid sphere rolls cleanly while the hollow cylinder fails. This is because the hollow cylinder demands more friction (it has more rotational inertia to spin up) but the surface cannot provide it. The solid sphere is the most "friction-efficient" roller: it demands the least friction of all standard shapes and can therefore roll on the widest range of surfaces and angles. The hollow cylinder is the least efficient, requiring the most friction and failing first on slippery surfaces.*

Think Like a Physicist: *Part (c) is exactly the kind of question that appears in examinations: "determine whether the body rolls or slides." The method is always the same: calculate the required friction from the rolling formula, compare with $\mu_s Mg\cos\theta$, and check the inequality. If satisfied, pure rolling. If not, the body slides.*

That concludes our study of rolling motion: the no-slip condition, the energy partition, the acceleration down an incline (by both energy and force methods), the friction requirement, and the condition for pure rolling. Rolling is where every idea in this chapter comes together: moment of inertia, radius of gyration, torque, energy, and Newton's laws for both translation and rotation, all working in concert.

In the next subtopic, we turn to the last great conservation law of mechanics: angular momentum, the rotational twin of linear momentum. It will complete the twin language and open the door to some of the most spectacular phenomena in physics, from pirouetting ice skaters to collapsing stars.

ANGULAR MOMENTUM AND ITS CONSERVATION

In Chapter 5, we discovered that when no external force acts on a system, its total linear momentum is conserved. This principle explained collisions, explosions, recoil, and rocket propulsion. Now we meet the rotational twin: when no external torque acts on a system, its total **angular momentum** is conserved. This principle explains why an ice skater spins faster when she pulls her arms in, why a collapsing star spins up to incredible speeds, why a cat always lands on its feet, and why a bicycle stays upright while moving.

Angular momentum completes the twin language. With it, every translational concept from Chapters 2 through 5 has a rotational counterpart:

Translational	Rotational
Force F	Torque τ
Mass m	Moment of inertia I
Velocity v	Angular velocity ω
Acceleration a	Angular acceleration α
Momentum p = mv	Angular momentum L = Iω
F = ma	τ = Iα
F = dp/dt	τ = dL/dt
If F = 0, p = constant	If τ = 0, L = constant

Definition of angular momentum

The **angular momentum L** of a rigid body rotating about a fixed axis is defined as *the product of moment of inertia and its angular velocity:*

$$\mathbf{L = I\omega}$$

where I is the moment of inertia about the rotation axis and ω is the angular velocity.

The unit of angular momentum is kgm^2s^{-1} (or equivalently, kgm^2rad/s, but since radians are dimensionless, these are the same).

Angular momentum is a vector quantity. Its direction is along the rotation axis, given by the right-hand rule: curl the fingers of the right hand in the direction of rotation, and the thumb points in the direction of L.

For a **single particle** of mass m moving with velocity v at perpendicular distance r from the axis:

$$\mathbf{L = mvr}$$

This is consistent with $L = I\omega$ because for a single particle, $I = mr^2$ and $v = \omega r$, giving $L = mr^2 \times \frac{v}{r} = mvr$.

Torque and Angular Momentum

Just as force equals the rate of change of linear momentum $\left(F = \frac{dp}{dt}\right)$, torque equals the rate of change of angular momentum:

$$\boldsymbol{\tau = \frac{dL}{dt}}$$

Derivation:

Starting from $\tau = I\alpha$ and using $\alpha = \frac{d\omega}{dt}$:

$$\tau = I\frac{d\omega}{dt}$$

If I is constant (rigid body, fixed axis): $I\frac{d\omega}{dt} = \frac{d(I\omega)}{dt}$

$$\tau = \frac{d(I\omega)}{dt} = \frac{dL}{dt}$$

This is the rotational form of Newton's second law in its most general form. It is more general than $\tau = I\alpha$ because it also applies when I changes (as in the spinning skater problem).

Conservation of Angular Momentum

Angular momentum is conserved when no external torque acts on a system. If the net external torque on a system is zero:

$$\tau_{ext} = 0 \Longrightarrow \frac{dL}{dt} = 0 \Longrightarrow L = \text{constant, and hence:}$$

$$\boldsymbol{I_1\omega_1 = I_2\omega_2}$$

This is the **law of conservation of angular momentum**: *if no external torque acts on a system, the total angular momentum remains constant, even if the moment of inertia changes.*

The key insight: When I decreases, *ω* must increase to keep L constant. When I increases, *ω* must decrease. This is the physics behind every spinning-and-pulling-in phenomenon in nature.

Applications of Conservation of Angular Momentum

1. Ice skater pulling arms in:

A skater begins a spin with arms outstretched (large I, small ω). She then pulls her arms close to her body, reducing I. Since no external torque acts about her vertical axis (friction at the ice is negligible), angular momentum is conserved: $I_1\omega_1 = I_2\omega_2$. With $I_2 < I_1$, we get $\omega_2 > \omega_1$. She spins faster. To slow down, she extends her arms again.

2. Diver performing somersaults:

A diver leaves the board with some angular momentum. In the air, no external torque acts about the centre of mass (gravity acts through the CM, producing zero torque about it). To spin faster (for somersaults), the diver pulls into a tight tuck (small I, large ω). To stop spinning before entry, the diver straightens out (large I, small ω). The angular momentum is constant throughout.

3. Collapsing star (neutron star):

A large, slowly rotating star collapses under gravity. Its radius shrinks enormously (from millions of km to about 10 km), reducing I by a factor of 10^{10} or more. Conservation of angular momentum demands a corresponding increase in ω. The resulting neutron star spins at hundreds of revolutions per second. This is the most dramatic example of angular momentum conservation in nature.

4. Rotating chair with dumbbells:

Kipute sits on a freely rotating chair holding two heavy dumbbells with arms outstretched and is given a gentle spin. She rotates slowly (large I, small ω). She pulls the dumbbells to her chest, reducing I. With no external torque (the chair rotates freely), angular momentum is conserved and she spins faster. Extending the arms again restores the original slow spin. This is the classroom version of the ice skater.

5. Planet in elliptical orbit:

A planet moves closer to the Sun at perihelion and farther at aphelion. Gravity acts along the radius, producing zero torque about the Sun. Angular momentum $L = mvr$ is conserved. At perihelion (small r), the

planet must move faster (large v). At aphelion (large r), it moves slower. This is Kepler's second law, and it is nothing more than conservation of angular momentum.

6. Spinning raw egg restarting:

In the chapter introduction, Kipute stopped a spinning raw egg and released it. The egg restarted on its own. When she touched the shell, only the shell stopped; the liquid inside kept swirling (its angular momentum was not removed). When released, the still-spinning liquid dragged the shell back into rotation. Angular momentum cannot simply disappear from the liquid without an external torque, and a brief touch on the shell is not enough.

These six examples share a single story: when no external torque acts, angular momentum is locked in. The only freedom the system has is to trade moment of inertia for angular velocity, or vice versa. The product Iω never changes.

With the principle established and the examples fresh in mind, let us see how the numbers work. And as you will discover, conservation of angular momentum produces some of the most satisfying calculations in this entire chapter.

BINDER Example 46

A disc of moment of inertia 0.4kgm^2 rotates at 10rad/s about its central axis.

(a) Calculate its angular momentum.

(b) A ring of moment of inertia 0.1kgm^2 is dropped coaxially onto the spinning disc. The ring is initially not rotating. After contact, they rotate together. Calculate the final angular velocity.

(c) Show that KE is lost in this process and explain where it goes.

Solution

(a) Using $L = I\omega$ and substituting:

$$L = 0.4\text{kgm}^2 \times 10\text{rad/s} = 4.0\text{kgm}^2\text{s}^{-1}$$

The angular momentum is $4.0\text{kgm}^2\text{s}^{-1}$.

(b) No external torque acts on the disc-ring system (the axis is fixed and frictionless at the pivot). Angular momentum is conserved:

$$I_1\omega_1 = (I_1 + I_2)\omega_f$$

Making ω_f the subject:

$$\omega_f = \frac{I_1\omega_1}{I_1 + I_2} = \frac{0.4\text{kgm}^2 \times 10\text{rad/s}}{0.4\text{kgm}^2 + 0.1\text{kgm}^2} = 8.0\text{rad/s}$$

The final angular velocity is 8.0rad/s.

(c) Initial KE:

$$KE_i = \frac{1}{2}I_1\omega_1^2 = \frac{1}{2} \times 0.4\text{kgm}^2 \times (10\text{rad/s})^2 = 20\text{J}$$

Final KE:

$$KE_f = \frac{1}{2}(I_1 + I_2)\omega_f^2 = \frac{1}{2} \times 0.5\text{kgm}^2 \times (8.0\text{rad/s})^2 = 16\text{J} < KE_i(20\text{J})$$

KE lost: $20 - 16 = 4\text{J}$

The KE decreases by 4J.

This energy is converted into heat and sound at the contact surface as the ring accelerates to match the disc's rotation through friction between the surfaces.

Relate: This is analogous to an inelastic collision in linear momentum: momentum is conserved but KE is not.

Making Sense of the Answer: *The ring was dropped onto a spinning disc, so friction between the ring and disc had to bring the ring up to speed. This friction dissipated energy as heat, just as kinetic friction always does. The angular momentum ($4.0 kgm^2 s^{-1}$) is the same before and after, but the KE dropped from $20J$ to $16J$. The fraction lost is $\frac{4}{20} = 20\%$, which equals $\frac{I_2}{I_1+I_2} = \frac{0.1}{0.5} = 20\%$. This fraction depends only on the ratio of moments of inertia, just as in linear inelastic collisions.*

Think Like a Physicist: *This problem is the rotational twin of a perfectly inelastic collision. A stationary object (ring) is struck by a moving object (disc), and they stick together. Angular momentum is conserved; KE is not. The mathematics is identical to $m_1 v_1 = (m_1 + m_2)v_f$, with I replacing m and ω replacing v. If you understood inelastic collisions in Chapter 5, you already understand this problem.*

REAL Example 47

A student sits on a freely rotating stool holding a spinning bicycle wheel horizontally above her head. The wheel spins clockwise when viewed from above. The student then flips the wheel upside down so that it now spins anticlockwise when viewed from above.

(a) Describe what happens to the student and the stool when the wheel is flipped.

(b) Explain why this happens using conservation of angular momentum.

Solution

(a) When the student flips the wheel, the stool and student begin to rotate clockwise (when viewed from above), which is the same direction as the wheel was originally spinning.

(b) Before the flip, the system (student + stool + wheel) has a total angular momentum equal to that of the spinning wheel. Call this L, directed clockwise from above. The student and stool are initially stationary, contributing zero.

Total before: $L_{total} = L_{wheel} + 0 = L$

The stool is free to rotate, so no external torque acts on the system. Therefore L_{total} must remain equal to L after the flip.

When the student flips the wheel upside down, its spin direction reverses. What was clockwise becomes anticlockwise. The wheel now carries angular momentum $-L$ (opposite to the original direction).

Total after: $L_{total} = L_{student+stool} + (-L)$

Setting total before equal to total after: $L = L_{student+stool} - L$

Making $L_{student+stool}$ the subject: $L_{student+stool} = 2L$

So the student and stool must acquire angular momentum 2L in the original (clockwise) direction. The student and stool therefore spin clockwise (from above) with angular momentum equal to twice the wheel's original angular momentum.

Making Sense of the Answer: *Before the flip, only the wheel had angular momentum. After the flip, the wheel has the same magnitude of angular momentum but in the opposite direction, and the student has twice the original angular momentum in the original direction. The total angular momentum is $2L - L = L$, which equals the original total. Nothing was created or destroyed; the angular momentum was merely redistributed within the system.*

Think Like a Physicist: *This demonstration can be performed in physics class. The surprise is that flipping a wheel makes a person spin on a stool without touching anything. But there is no mystery: the system's total angular momentum is constant. The student "borrows" angular momentum from the wheel by reversing it. If she flips the wheel back, she stops spinning. The angular momentum flows back and forth between wheel and person, always conserving the total.*

HOT Example 48

A uniform disc of mass 3kg and radius 0.20m rotates freely at 15rad/s about its central axis. A piece of putty of mass 0.5kg is dropped vertically onto the edge of the disc and sticks there.

(a) Calculate the angular momentum before the putty lands.

(b) Calculate the moment of inertia of the system after the putty lands.

(c) Calculate the final angular velocity.

(d) Calculate the percentage of KE lost.

Solution

(a) Moment of inertia of the disc: $I_{disc} = \frac{1}{2}MR^2 = \frac{1}{2} \times 3\text{kg} \times (0.20\text{m})^2 = 0.06\text{kgm}^2$

$$L_i = I_{disc} \times \omega_i = 0.06\text{kgm}^2 \times 15\text{rad/s} = 0.9\text{kgm}^2\text{s}^{-1}$$

The angular momentum before impact is $0.9\text{kgm}^2\text{s}^{-1}$.

(b) After landing, the putty sits at the rim ($r = R = 0.20\text{m}$). Its moment of inertia about the axis is $I_{putty} = mR^2 = 0.5\text{kg} \times (0.20\text{m})^2 = 0.02\text{kgm}^2$.

$$I_{total} = I_{disc} + I_{putty} = (0.06 + 0.02)\text{kgm}^2 = 0.08\text{kgm}^2$$

The total moment of inertia is 0.08kgm^2.

(c) No external torque acts on the system (the axis is fixed). Angular momentum is conserved:

$$L_i = I_{total} \times \omega_f$$

Making ω_f the subject:

$$\omega_f = \frac{L_i}{I_{total}} = \frac{0.9\text{kgm}^2\text{s}^{-1}}{0.08\text{kgm}^2} = 11.25\text{rad/s}$$

The final angular velocity is 11.25rad/s.

(d) Initial KE: $KE_i = \frac{1}{2} \times 0.06\text{kgm}^2 \times (15\text{rad/s})^2 = 6.75\text{J}$

Final KE: $KE_f = \frac{1}{2} \times 0.08\text{kgm}^2 \times (11.25\text{rad/s})^2 = 5.06\text{J}$

KE lost: $(6.75 - 5.06)\text{J} = 1.69\text{J}$

Percentage lost: $\frac{1.69\text{J}}{6.75\text{J}} \times 100\% = 25\%$

The system loses 25% of its KE.

Making Sense of the Answer: *The disc slows from* 15 *to* 11.25rad/s *(a* 25% *decrease in angular velocity). The moment of inertia increased by* 33% *(from* 0.06 *to* 0.08*), and the angular velocity decreased by* 25% *to compensate and keep L constant. Note that ω did not decrease by* 33% *because L depends on Iω, not on I alone. The* 25% *KE loss is significant and is converted deforming the putty and generating heat at the moment of impact.*

Think Like a Physicist: *This is a rotational inelastic collision. The putty "collides" with the disc and sticks (perfectly inelastic). Angular momentum is conserved, but KE is lost. The procedure is identical to a linear perfectly inelastic collision: find L before, find I after (with the new mass included), then find* ω_f *from conservation. The KE calculation is always a separate step, never assumed to be conserved.*

HOT Example 49

A merry-go-round (uniform disc, mass 100kg, radius 2.0m) is spinning freely at 1.5rad/s. A child of mass 30kg runs tangentially at 4m/s in the same direction as the merry-go-round rotates and jumps onto the edge.

(a) Calculate the angular momentum of the merry-go-round before the child jumps on.

(b) Calculate the angular momentum of the running child about the axis of the merry-go-round.

(c) Determine the final angular velocity after the child is on the merry-go-round.

(d) State whether the merry-go-round speeds up or slows down, and explain why.

Solution

(a) Moment of inertia of the merry-go-round:

$$I_m = \frac{1}{2}MR^2 = \frac{1}{2} \times 100\text{kg} \times (2.0\text{m})^2 = 200\text{kgm}^2$$

Angular momentum:

$$L_m = I_m\omega = 200\text{kgm}^2 \times 1.5\text{rad/s} = 300\text{kgm}^2\text{s}^{-1}$$

The angular momentum of the merry-go-round is $300\text{kgm}^2\text{s}^{-1}$.

(b) The child runs tangentially at speed $v = 4\text{m/s}$ at distance $r = R = 2.0\text{m}$ from the axis. The angular momentum of the child about the axis is:

$$L_{child} = mvr = 30\text{kg} \times 4\text{m/s} \times 2.0\text{m} = 240\text{kgm}^2\text{s}^{-1}$$

The angular momentum of the child is $240\text{kgm}^2\text{s}^{-1}$.

(c) After the child lands, the system (merry-go-round + child) rotates together. The child, sitting at the edge, contributes $I_{child} = mR^2 = 30\text{kg} \times (2.0\text{m})^2 = 120\text{kgm}^2$ to the moment of inertia.

$$I_{total} = I_m + I_{child} = (200 + 120)\text{kgm}^2 = 320\text{kgm}^2$$

Total angular momentum before: $L_{total} = (300 + 240)\text{kgm}^2\text{s}^{-1} = 540\text{kgm}^2\text{s}^{-1}$

Conservation of angular momentum:

$$L_{total} = I_{total} \times \omega_f$$

Making ω_f the subject:

$$\omega_f = \frac{540\text{kgm}^2\text{s}^{-1}}{320\text{kgm}^2} = 1.6875\text{rad/s}$$

The final angular velocity is 1.69rad/s.

(d) The merry-go-round speeds up slightly from 1.5rad/s to 1.69rad/s. This happens because the child brings additional angular momentum ($240\text{kgm}^2\text{s}^{-1}$) into the system by running in the same direction as the rotation. Although the child also adds moment of inertia (which tends to slow things down), the extra angular momentum outweighs the extra moment of inertia, producing a net speed increase.

Making Sense of the Answer: *The child contributes* $240\text{kgm}^2\text{s}^{-1}$ *of angular momentum, which is* 80% *of the merry-go-round's own* $300\text{kgm}^2\text{s}^{-1}$. *The combined* 540 *is then shared across* 320kgm^2 *of inertia, giving a modest increase in ω. If the child had run faster (say,* 6m/s*), the speed-up would have been more dramatic. If the child had run in the opposite direction, the merry-go-round would have slowed down.*

Think Like a Physicist: *This problem combines two sources of angular momentum: a rotating body (*$L = I\omega$*) and a moving particle (*$L = mvr$*). Both must be included in the total. The particle's angular momentum depends on how it moves relative to the axis (tangentially, radially, or at an angle). For a tangential approach,* $L = mvr$ *with full v. For a radial approach (running straight toward the centre), the angular momentum would be zero. The direction and speed of approach both matter.*

Angular momentum subtopic is complete. Angular momentum is the final piece of the twin language. With it, every concept from translational mechanics has a rotational counterpart: displacement and angle, velocity and angular velocity, acceleration and angular acceleration, mass and moment of inertia, force and torque, momentum and angular momentum, and the conservation laws that govern them all. We are now going to bring the twins face to face in a final comparison table.

COMPARISON TABLE: TRANSLATIONAL AND ROTATIONAL QUANTITIES

Throughout this chapter, we have called rotational quantities the "twins" of translational quantities. The table below places all the twins side by side. Every formula in the left column has an exact counterpart in the right column, obtained by replacing each translational quantity with its rotational twin.

This table is not decoration. It is a thinking tool. Whenever you face a rotational problem and feel lost, find the translational version you already know and translate it using this table.

Translational quantity	Symbol	Rotational quantity	Symbol
Displacement	s	Angular displacement	θ
Velocity	v	Angular velocity	ω
Acceleration	a	Angular acceleration	α
Mass (inertia)	m	Moment of inertia	I
Force	F	Torque	τ
Momentum	$p = mv$	Angular momentum	$L = I\omega$

Translational law/formula	Rotational twin
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s = \frac{1}{2}(u + v)t$	$\theta = \frac{1}{2}(\omega_0 + \omega)t$
$s_{nth} = u + \frac{1}{2}a(2n - 1)$	$\theta_{nth} = \omega_0 + \frac{1}{2}\alpha(2n - 1)$
$F = ma$	$\tau = I\alpha$
$F = \frac{dp}{dt}$	$\tau = \frac{dL}{dt}$
$W = Fs$	$W = \tau\theta$
$KE = \frac{1}{2}mv^2$	$KE = \frac{1}{2}I\omega^2$
$P = Fv$	$P = \tau\omega$
If $F = 0$, $p =$ constant	If $\tau = 0$, $L =$ constant

Translational connection	Rotational connection
$s = r\theta$	Links displacement and angle
$v = r\omega$	Links linear and angular velocity
$a_t = r\alpha$	Links tangential and angular acceleration
$v_{cm} = \omega R$	No-slip condition for rolling

How to use the table:

If a translational problem says "a force of 10N accelerates a 2kg mass," the rotational twin says "a torque of 10Nm angularly accelerates a body of moment of inertia $2kgm^2$." The mathematics is identical: $a = F/m = 5m/s^2$ becomes $\alpha = \tau/I = 5rad/s^2$. If you can solve one, you can solve the other.

This twin correspondence is not a coincidence. It arises because the mathematical structure of Newton's laws is the same for both translation and rotation. The physics is different (forces push, torques twist), but the equations have the same form. Learning one set of equations teaches you both.

APPLICATIONS OF ROTATIONAL MECHANICS

Rotational mechanics is not confined to textbooks and examination halls. It is at work in every machine, every vehicle, every spinning object in the universe. This section highlights some of the most important real-world applications.

1. Flywheels

A flywheel is a heavy rotating disc (or wheel) designed to store rotational KE. Because $KE = \frac{1}{2}I\omega^2$, a flywheel with a large moment of inertia spinning at high angular velocity stores substantial energy.

In an engine, the power strokes are intermittent (they happen only during part of each cycle). Between power strokes, the engine would slow down and jerk. A flywheel attached to the crankshaft stores energy during the power stroke and releases it between strokes, smoothing out the rotation. This is why all internal combustion engines have flywheels.

To maximize energy storage for a given mass, the mass should be placed as far from the axis as possible (maximizing I). This is why flywheels are designed with heavy rims and light centres, approaching the ring shape ($I = MR^2$) rather than the solid disc shape ($I = \frac{1}{2}MR^2$).

Modern flywheel energy storage is used in some hybrid buses, uninterruptible power supplies, and spacecraft attitude control systems. Advanced flywheels spin at tens of thousands of rpm in vacuum chambers on magnetic bearings to minimize friction losses.

2. Gyroscopes and gyroscopic stability

A gyroscope *is a spinning body mounted so that its axis of rotation can point in any direction.* Once spinning, the axis resists changes in direction. This resistance comes from the large angular momentum $L = I\omega$ along the spin axis. Changing the direction of L requires an external torque ($\tau = \frac{dL}{dt}$), and without such a torque, the axis stays fixed in space.

Gyroscopes are used in aircraft, ships, submarines, and spacecraft to maintain a reference direction independent of the vehicle's motion. The gyrocompass always points in the same direction regardless of how the vehicle turns.

Gyroscopic effects also keep a moving bicycle upright. The spinning wheels have angular momentum along a horizontal axis. Tilting the bicycle requires changing the direction of this angular momentum, which requires torque. The larger the angular momentum (faster wheel spin), the harder it is to tilt. This is why a bicycle is easier to balance at speed than at rest.

3. The spinning top and precession

A spinning top does not fall over immediately because its angular momentum resists the change in direction that falling would require. Instead, gravity applies a torque that gradually changes the direction of L, causing the spin axis to sweep out a cone. This slow sweeping motion is called **precession**. The precession rate is slower when the top spins faster (larger L means more resistance to direction change), which is why a fast-spinning top appears almost perfectly stable.

The Earth itself **precesses** (*slowly change the direction of its axis of rotation*). Its axis traces a cone with a period of about 26000 years, caused by the gravitational torques of the Sun and Moon on the Earth's equatorial bulge. This precession slowly shifts the position of the North Celestial Pole among the stars.

4. Rifling of gun barrels

Gun barrels have spiral grooves (rifling) that make the bullet spin as it leaves the barrel. The spinning bullet has angular momentum along its direction of travel. This angular momentum stabilizes the bullet's orientation, preventing it from tumbling in the air. A tumbling bullet has poor accuracy and less penetrating power. Rifling exploits gyroscopic stability to keep the bullet's nose pointing forward throughout its flight.

5. Moment of inertia in sports

Athletes instinctively use moment of inertia to control their rotation. A gymnast performing a somersault tucks tightly (small I) to spin fast, then opens out (large I) to slow the rotation before landing. A long jumper swings the arms forward during landing to transfer angular momentum from the body to the arms, preventing a backward fall. A discus thrower extends the arm fully (large r) to maximize the release speed ($v = \omega r$). In every case, the athlete is exploiting the relationship between moment of inertia, angular velocity, and angular momentum.

6. Tanzanian applications

Traditional grain mills use heavy circular grinding stones that store rotational KE and maintain steady grinding between pushes (the flywheel principle). The rungu (wooden club) concentrates mass at the far end to maximize the moment of inertia about the grip, giving more angular momentum and KE at impact. Potters' wheels use the flywheel effect to maintain smooth rotation while the potter shapes clay. Bicycle wheels on Tanzanian roads demonstrate rolling motion, gyroscopic stability, and conservation of angular momentum every day, whether the rider knows the physics or not.

This chapter has taken us from Kipanga's fumbled ruler to flywheels, from spinning eggs to collapsing stars, and from a single formula $\tau = I\alpha$ to a complete twin language that mirrors everything we ever learned about translational motion. The twin language has been spoken. Now it must be tested. In the next section, the concepts of this entire chapter: moment of inertia, torque, energy, rolling motion, and angular momentum will stop introducing themselves politely and start arriving together, uninvited, in miscellaneous worked examples. Bring your calculator, and make sure your pen does not slip, like a like a car's wheel skidding helplessly down a muddy slope!

MISCELLANEOUS WORKED EXAMPLES ON ROTATION OF RIGID BODIES

Example 50

(a) Explain why a hollow cylinder has a larger moment of inertia than a solid cylinder of the same mass and radius.

(b) A uniform solid disc of mass 5kg and radius 0.30m is mounted on a frictionless horizontal axle. A constant force of 12N is applied tangentially at the rim. Take $g = 9.8\text{m/s}^2$.

(i) Calculate the moment of inertia of the disc.

(ii) Calculate the angular acceleration.

(iii) Calculate the angular velocity after 4s, starting from rest.

(iv) Calculate the angle turned through in the 4th second.

Solution

(a) In a hollow cylinder, all the mass is concentrated at the outer radius, far from the axis of rotation. In a solid cylinder, some mass is close to the axis, where it contributes less to the moment of inertia. Since moment of inertia depends on mr^2, mass far from the axis contributes more. Therefore, the hollow cylinder, with all its mass at the maximum distance, has the larger moment of inertia.

(b) The solution of each part is as follows:

(i) $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 5\text{kg} \times (0.30\text{m})^2 = 0.225\text{kgm}^2$

The moment of inertia is 0.225kgm^2.

(ii) The tangential force at the rim produces torque $\tau = FR = 12\text{N} \times 0.30\text{m} = 3.6\text{Nm}$

$$\alpha = \frac{\tau}{I} = \frac{3.6\text{Nm}}{0.225\text{kgm}^2} = 16\text{rad/s}^2$$

The angular acceleration is 16rad/s^2.

(iii) $\omega = \omega_0 + \alpha t = 0 + 16\text{rad/s}^2 \times 4\text{s} = 64\text{rad/s}$

The angular velocity after 4s is 64rad/s.

(iv) $\theta_{nth} = \omega_0 + \frac{1}{2}\alpha(2n-1) = 0 + \frac{1}{2} \times 16 \times (2 \times 4 - 1) = 56\text{rad}$

The angle turned through in the 4th second is 56rad.

Example 51

(a) Explain why the parallel axis theorem always gives a moment of inertia that is larger than the moment of inertia about the centre of mass.
(b) A uniform rod of mass 1.2kg and length 0.80m is pivoted at a point 0.20m from one end.
(i) Calculate the moment of inertia about the centre of mass.
(ii) Calculate the moment of inertia about the pivot.

(iii) A couple of magnitude 2.4Nm is applied to the rod. Calculate the angular acceleration.

(iv) Calculate the rotational KE after 3s from rest.

Solution

(a) The parallel axis theorem states $I = I_{cm} + Md^2$. Since Md^2 is always positive (mass is positive, d^2 is positive), I about any other axis is always greater than I_{cm}. The centre of mass is the axis of minimum moment of inertia.
Physically, the moment of inertia measures how far the mass is spread from the rotation axis. When the axis passes through the centre of mass, the mass is distributed as closely as possible around the axis (some mass is near, some is far, but on average the distribution is the most compact it can be). When the axis is shifted to a distance d away from the centre of mass, every piece of mass in the body is, on average, farther from the new axis than it was from the centre of mass. Farther mass means more resistance to rotation. Therefore, the moment of inertia about any axis away from the centre of mass is always larger. The centre of mass is the "most balanced" point of the body, and rotating about it requires the least effort.

(b) The answer of each part is as follows:

(i) $I_{cm} = \frac{1}{12}ML^2 = \frac{1}{12} \times 1.2\text{kg} \times (0.80\text{m})^2 = \frac{1}{12} \times 1.2 \times 0.64 = 0.064\text{kgm}^2$

The moment of inertia about the centre is 0.064kgm^2.

(ii) The centre of mass is at the midpoint, 0.40m from each end. The pivot is 0.20m from one end, so the distance from the pivot to the centre of mass is $d = 0.40\text{m} - 0.20\text{m} = 0.20\text{m}$.

$$I_{pivot} = I_{cm} + Md^2 = 0.064\text{kgm}^2 + 1.2\text{kg} \times (0.20\text{m})^2 = (0.064 + 0.048)\text{kgm}^2 = 0.112\text{kgm}^2$$

The moment of inertia about the pivot is 0.112kgm^2.

(iii) $\alpha = \frac{\tau}{I} = \frac{2.4\text{Nm}}{0.112\text{kgm}^2} = 21.4\text{rad/s}^2$

The angular acceleration is 21.4rad/s^2.

(iv) After 3s from rest: $\omega = \alpha t = 21.4\text{rad/s}^2 \times 3\text{s} = 64.3\text{rad/s}$

$$KE = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.112\text{kgm}^2 \times (64.3\text{rad/s})^2 = 231.5\text{J}$$

The rotational KE is 232J.

Example 52

(a) **Kipanga** claims: *"A body with a larger moment of inertia always has more rotational KE."* Explain why this claim is not necessarily true.
(b) Two discs, A and B, rotate about their central axes. Disc A has $I_A = 0.5\text{kgm}^2$ and $\omega_A = 10\text{rad/s}$. Disc B has $I_B = 2.0\text{kgm}^2$ and $\omega_B = 4\text{rad/s}$.
(i) Calculate the rotational KE of each disc.

(ii) Calculate the angular momentum of each disc.

(iii) State which disc has more KE and which has more angular momentum.

Solution

(a) Rotational KE is $\frac{1}{2}I\omega^2$. It depends on both I and ω^2. A body with larger I but much smaller ω can have less KE than a body with smaller I but larger ω. The ω^2 factor means angular velocity has a greater influence on KE than moment of inertia does.

(b) The solution of each part is as follows:

(i) $KE_A = \frac{1}{2} \times 0.5\text{kgm}^2 \times (10\text{rad/s})^2 = 25\text{J}$

$KE_B = \frac{1}{2} \times 2.0\text{kgm}^2 \times (4\text{rad/s})^2 = 16\text{J}$

(ii) $L_A = I_A\omega_A = 0.5 \times 10 = 5\text{kgm}^2\text{s}^{-1}$

$L_B = I_B\omega_B = 2.0 \times 4 = 8\text{kgm}^2\text{s}^{-1}$

(iii) Disc A has more KE (25J versus 16J), but disc B has more angular momentum ($8\text{kgm}^2\text{s}^{-1}$ versus $5\text{kgm}^2\text{s}^{-1}$).

Interesting! A body can have more angular momentum but less KE than another body. This is because KE depends on ω^2 (favouring fast-spinning bodies) while L depends on ω (favouring high-inertia bodies). In this example, disc A spins faster so it wins on KE, but disc B has more inertia so it wins on angular momentum.

Example 53

(a) Explain why static friction does no work on a body that rolls without slipping on a flat surface.

(b) A solid sphere and a hollow cylinder, each of mass 2kg, are released from rest at the top of a rough incline of height 3m and angle 30°. Both roll without slipping. Take $g = 9.8\text{m/s}^2$.

(i) Calculate the speed of each body at the bottom.
(ii) Calculate the acceleration of each body.
(iii) Calculate the friction force on each body.
(iv) Determine which body arrives at the bottom first and by how much time.

Solution

(a) Static friction acts at the contact point. During pure rolling, the contact point is instantaneously at rest (zero velocity). Since work requires both force and displacement at the point of application, and the displacement of the contact point is zero, the work done by static friction is zero.

(b) The solution of each part is as follows:

(i) For the solid sphere ($k^2/R^2 = 2/5$):

$$v = \sqrt{\frac{2gh}{1 + k^2/R^2}} = \sqrt{\frac{2 \times 9.8\text{m/s}^2 \times 3\text{m}}{1 + 2/5}} = 6.48\text{m/s}$$

For the hollow cylinder ($k^2/R^2 = 1$):

$$v = \sqrt{\frac{2 \times 9.8 \times 3}{1 + 1}\text{m}^2/\text{s}^2} = 5.42\text{m/s}$$

(ii) For the solid sphere:

$$a = \frac{g\sin 30°}{1 + 2/5} = \frac{9.8 \times 0.5}{1.4}\text{m/s}^2 = 3.50\text{m/s}^2$$

For the hollow cylinder:

$$a = \frac{9.8 \times 0.5}{2}\text{m/s}^2 = 2.45\text{m/s}^2$$

(iii) For the solid sphere:

$$f = \frac{2}{7}Mg\sin 30° = \frac{2}{7} \times 2kg \times 9.8m/s^2 \times 0.5 = 2.80N$$

For the hollow cylinder:

$$f = \frac{1}{2}Mg\sin 30° = \frac{1}{2} \times 2kg \times 9.8m/s^2 \times 0.5 = 4.90N$$

(iv) Length of incline: $s = h/\sin 30° = 3/0.5 = 6m$

For the solid sphere: $t = \sqrt{2s/a} = \sqrt{2 \times 6m/3.50m/s^2} = 1.852s$

For the hollow cylinder: $t = \sqrt{2 \times 6m/2.45m/s^2} = 2.213s$

Time difference: $2.213 - 1.852 = 0.361s$

The solid sphere arrives first, beating the hollow cylinder by 0.36s.

Example 54

(a) Explain why a torque is needed to produce angular acceleration, even if the body is already rotating.

(b) A flywheel of moment of inertia $0.8kgm^2$ is rotating at 300rpm. A braking torque is applied and the flywheel comes to rest in 20s.

(i) Calculate the braking torque.

(ii) Calculate the number of revolutions before stopping.

(iii) Calculate the KE dissipated as heat.

Solution

(a) A rotating body with no net torque continues at constant angular velocity (rotational analogue of Newton's first law). To change the angular velocity (i.e. to produce angular acceleration), a net torque is required (rotational analogue of Newton's second law, $\tau = I\alpha$).

(b) $\omega_0 = 300rpm \times \frac{2\pi rad}{60s} = \frac{300 \times 2\pi}{60} = 10\pi = 31.4rad/s$

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 31.4rad/s}{20s} = -1.57rad/s^2$$

The angular deceleration is $1.57rad/s^2$.

(i) $\tau = I|\alpha| = 0.8kgm^2 \times 1.57rad/s^2 = 1.26Nm$

The braking torque is 1.26Nm.

(ii) $\theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{0 - (31.4)^2}{2 \times (-1.57)} = \frac{-986}{-3.14} = 314rad$

Number of revolutions: $N = \frac{\theta}{2\pi} = \frac{314}{2\pi} = 50$ revolutions

The flywheel makes 50 revolutions before stopping.

(iii) $KE = \frac{1}{2}I\omega_0^2 = \frac{1}{2} \times 0.8kgm^2 \times (31.4rad/s)^2 = 394J$

The KE dissipated as heat is 394J.

Be careful! Always convert rpm to rad/s before using any rotational formula. Many students forget this and substitute 300 directly, which gives wrong answers throughout the problem.

Example 55

(a) Two objects of different shapes but the same mass and radius roll down the same incline. Explain why the acceleration does not depend on mass or radius, only on shape.

(b) A uniform solid sphere of mass M and radius R rolls without slipping up an inclined plane with initial speed v_0 at the bottom.

(i) Show that the height reached is $h = \frac{7v_0^2}{10g}$.

(ii) If $v_0 = 5\text{m/s}$ and $g = 9.8\text{m/s}^2$, calculate h.

(iii) Compare this with the height a sliding block would reach with the same initial speed on a frictionless incline.

Solution

(a) The acceleration is $a = \frac{g\sin\theta}{1+k^2/R^2}$. The numerator ($g\sin\theta$) contains no mass or radius. The denominator ($1 + k^2/R^2$) depends only on the ratio k^2/R^2, which is a pure number determined by the shape of the body.

Physically, two competing effects cancel mass and radius simultaneously. On the driving side, a heavier object has a larger gravitational pull down the incline ($Mg\sin\theta$), which would accelerate it faster. But on the resisting side, a heavier object also has a larger moment of inertia ($I = Mk^2$, proportional to M), which makes it harder to spin up. The larger driving force and the larger rotational resistance both increases proportionally with M, so mass cancels. Similarly, a larger radius increases the moment of inertia (more mass farther from the axis), but it also increases the lever arm through which friction acts to produce torque. These two radius effects cancel each other exactly. What remains is the shape factor k^2/R^2, which describes how the mass is distributed relative to the outer radius.

(b) The solution of each part is as follows:

(i) At the bottom: $KE_{total} = \frac{1}{2}Mv_0^2\left(1 + \frac{k^2}{R^2}\right) = \frac{1}{2}Mv_0^2\left(1 + \frac{2}{5}\right) = \frac{1}{2}Mv_0^2 \times \frac{7}{5} = \frac{7Mv_0^2}{10}$

At the top: $PE = Mgh$, and $KE = 0$ (the sphere stops momentarily).

By conservation of mechanical energy: $\frac{7Mv_0^2}{10} = Mgh$

$$\mathbf{h = \frac{7v_0^2}{10g}}$$

(ii) $h = \frac{7\times(5\text{m/s})^2}{10\times9.8\text{m/s}^2} = 1.79\text{m}$

The height reached is 1.79m.

(ii) A sliding block:

$$\frac{1}{2}Mv_0^2 = Mgh_{slide}$$

$$\mathbf{h_{slide} = \frac{v_0^2}{2g} = \frac{25}{19.6} = 1.28m}$$

Interesting! *The rolling sphere reaches* 1.79*m, which is higher than the sliding block (*1.28*m). This seems paradoxical. The explanation is that the rolling sphere started with more total KE than the block (it had both translational and rotational KE at the bottom), so it has more energy to convert into PE. The extra height is* $\frac{7}{10g} \div \frac{1}{2g} = \frac{7}{5}$ *times the sliding height, exactly the factor* $(1 + k^2/R^2)$.

Example 56

(a) What is the difference between torque and couple? State one similarity and one difference.

(b) A grinding wheel (solid disc, mass 8kg, radius 0.25m) rotates at 120rad/s. A brake pad presses against the rim with a normal force of 40N. The coefficient of kinetic friction between the pad and the wheel is 0.6.

(i) Calculate the time to stop the wheel.

(ii) Calculate the total energy dissipated as heat.

Solution

(a) **Similarity:** Both torque and couple produce rotation.
Difference: A single torque requires a pivot point and the body may translate as well as rotate while a couple consists of two equal, opposite, parallel forces separated by a distance, and it produces pure rotation without translation.

(b) The solution of each part is as follows:

(i) *Finding the friction force*

$$f = \mu_k F_n = 0.6 \times 40\text{N} = 24\text{N}$$

Finding the braking torque

$$\tau = fR = 24\text{N} \times 0.25\text{m} = 6\text{Nm}$$

Finding moment of inertia and hence deceleration

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 8\text{kg} \times (0.25\text{m})^2 = 0.25\text{kgm}^2$$

$$\alpha = \frac{\tau}{I} = \frac{6\text{Nm}}{0.25\text{kgm}^2} = 24\text{rad/s}^2$$

Using $\omega = \omega_0 + \alpha t$ with $\omega = 0$

$$t = \frac{-\omega_0}{\alpha} = \frac{-120\text{rad/s}}{-24\text{rad/s}^2} = 5\text{s}$$

The wheel stops in 5s.

(ii) $KE = \frac{1}{2}I\omega_0^2 = \frac{1}{2} \times 0.25\text{kgm}^2 \times (120\text{rad/s})^2 = 1800\text{J}$

The energy dissipated as heat is 1800J.

Example 57

(a) Explain why a person spinning on a freely rotating chair speeds up when she pulls her arms inward.

(b) A disc of moment of inertia 0.6kgm^2 rotates at 8rad/s. A second disc of moment of inertia 0.2kgm^2, initially at rest, is dropped coaxially onto it. They rotate together.

(i) Calculate the final angular velocity.

(ii) Show that the fraction of KE lost equals $\frac{I_2}{I_1+I_2}$ where I_2 is the moment of inertia of the dropped disc.

Solution

(a) No external torque acts on the person (the chair rotates freely), so angular momentum $L = I\omega$ is conserved. When she pulls her arms inward, her moment of inertia I decreases. Since L must remain constant, ω must increase and therefore, she spins faster.

(b) The solution of each part is as follows:

(i) By conservation of angular momentum:

$$I_1\omega_1 = (I_1 + I_2)\omega_f$$

$$\omega_f = \frac{I_1\omega_1}{I_1 + I_2} = \frac{0.6\text{kgm}^2 \times 8\text{rad/s}}{(0.6 + 0.2)\text{kgm}^2} = 6\text{rad/s}$$

The final angular velocity is 6rad/s.

(ii) $KE_i = \frac{1}{2}I_1\omega_1^2$

$$KE_f = \frac{1}{2}(I_1 + I_2)\omega_f^2 = \frac{1}{2}(I_1 + I_2)\left(\frac{I_1\omega_1}{I_1 + I_2}\right)^2 = \frac{I_1^2\omega_1^2}{2(I_1 + I_2)}$$

Fraction lost:

$$\Delta KE = \frac{KE_i - KE_f}{KE_i} = 1 - \frac{KE_f}{KE_i} = 1 - \frac{\frac{I_1^2\omega_1^2}{2(I_1 + I_2)}}{\frac{1}{2}I_1\omega_1^2} = 1 - \frac{I_1}{I_1 + I_2} = \frac{I_2}{I_1 + I_2}$$

Hence:

$$\text{Fraction lost} = \frac{I_2}{I_1 + I_2}$$

Example 58

(a) A student says: *"Friction always opposes motion, so during rolling it must act in the direction opposite to the velocity of the centre of mass."* Explain why this statement is misleading.

(b) A solid cylinder of mass 3kg and radius 0.10m is given a translational velocity of 6m/s along a rough horizontal surface without any initial spin. The coefficient of kinetic friction is $\mu_k = 0.4$. Take $g = 9.8\text{m/s}^2$.

(i) Explain why the cylinder initially slides rather than rolls.
(ii) Calculate the friction force.
(iii) Calculate the translational deceleration and the angular acceleration.
(iv) Calculate the time at which pure rolling begins.
(v) Calculate the velocity when pure rolling begins.

Solution

(a) The statement is misleading because friction opposes **sliding at the contact point**, not motion of the centre of mass. During rolling down an incline, friction acts up the slope (opposing the sliding tendency at the contact point), but the centre of mass moves down the slope. Friction and the centre-of-mass velocity are in the same direction when a body accelerates rotationally.

(b) The solution of each part is as follows:

(i) Initially $v_{cm} = 6\text{m/s}$ but $\omega = 0$, so $v_{cm} \neq \omega R$. The no-slip condition is violated ($v_{cm} > \omega R$), and hence the contact point slides forward while kinetic friction acts backward.

(ii) $f = \mu_k Mg = 0.4 \times 3\text{kg} \times 9.8\text{m/s}^2 = 11.76\text{N}$

(iii) Translational: $a = -\frac{f}{M} = -\frac{11.76\text{N}}{3\text{kg}} = -3.92\text{m/s}^2$ (deceleration)

Rotational: $\alpha = \frac{fR}{I} = \frac{fR}{\frac{1}{2}MR^2} = \frac{11.76\text{N}\times 0.10\text{m}}{\frac{1}{2}\times 3\text{kg}\times(0.1\text{m})^2} = 78.4\text{rad/s}^2$ (spin-up)

(iv) Pure rolling begins when $v_{cm} = \omega R$:

Using $v = u + at$ and $\omega = \omega_0 + \alpha t$ with $\omega_0 = 0$:

$v_{cm} = 6 - 3.92t$ and $\omega = 78.4t$

Setting $v_{cm} = \omega R$ (when rolling begins):

$$6 - 3.92t = 78.4t \times 0.10$$

$$t = 0.51\text{s}$$

Pure rolling begins at $t = 0.51\text{s}$.

(v) $v = 6\text{m/s} - 3.92\text{m/s}^2 \times 0.51\text{s} = 4.0\text{m/s}$

The velocity when pure rolling begins is 4.0m/s.

Example 59

(a) Explain why the total angular momentum of a system is conserved even when internal torques act between parts of the system.

(b) A merry-go-round (mass 80kg, radius 1.5m) rotates at 2.0rad/s. A child of mass 25kg standing at the edge walks 1m inward toward the centre.

(i) Calculate the initial moment of inertia of the system.
(ii) Calculate the final moment of inertia of the system.
(iii) Calculate the final angular velocity.
(iv) Calculate the factor by which the KE changes, and explain the source of the extra KE.

Solution

(a) Internal torques come in action-reaction pairs (Newton's third law). When one part of the system exerts a torque on another, the second part exerts an equal and opposite torque back. These internal torques

cancel in the total, leaving the total angular momentum unchanged. Only external torques can change the total angular momentum.

(b) The solution of each part is as follows:

(i) $I_{disc} = \frac{1}{2}MR^2 = \frac{1}{2} \times 80\text{kg} \times (1.5\text{m})^2 = 90\text{kgm}^2$

$I_{child,i} = mR^2 = 25\text{kg} \times (1.5\text{m})^2 = 56.25\text{kgm}^2$

$I_i = (90 + 56.25)\text{kgm}^2 = 146.25\text{kgm}^2$

The initial moment of inertia of the system is 146.25kgm^2.

(ii) New position of a child from the centre: $r = 1.5\text{m} - 1\text{m} = 0.5\text{m}$

$I_{child,f} = mr^2 = 25\text{kg} \times (0.5\text{m})^2 = 6.25\text{kgm}^2$

$I_f = (90 + 6.25)\text{kgm}^2 = 96.25\text{kgm}^2$

The final moment of inertia of the system is 96.25kgm^2,

(iii) By conservation of angular momentum: $I_i\omega_i = I_f\omega_f$

$$\omega_f = \frac{I_i\omega_i}{I_f} = \frac{146.25\text{kgm}^2 \times 2.0\text{rad/s}}{96.25\text{kgm}^2} = 3.04\text{rad/s}$$

The final angular velocity is 3.04rad/s.

(iv) $KE_i = \frac{1}{2} \times 146.25\text{kgm}^2 \times (2.0\text{rad/s})^2 = 292.5\text{J}$

$KE_f = \frac{1}{2} \times 96.25\text{kgm}^2 \times (3.04\text{rad/s})^2 = 444.6\text{J}$

Factor: $\frac{KE_f}{KE_i} = \frac{444.6\text{J}}{292.5\text{J}} = 1.52$

The KE increased by a factor of 1.52 (a 52% increase).

The extra KE comes from the **work done by the child** in walking inward against the centripetal acceleration. The child must exert a radial force to move toward the centre while the system is spinning; this force does positive work, increasing the total KE of the system.

Interesting! Unlike the disc-drop problem (Example 57) where KE decreased, here the KE increased. The difference is that in Example 57 the second disc was passive (dropped on), while here the child actively does work by walking inward. Conservation of angular momentum does not require conservation of KE.

Example 60

(a) A uniform metre stick is balanced on a finger at the 50cm mark. A 200g lump of clay is then stuck to one end. Explain, without calculation, in which direction the balance point must shift and why.

(b) Three particles are positioned in space as follows:

Particle 1: mass 2kg at position $\vec{r}_1 = (3\hat{i} + 2\hat{j} + 4\hat{k})\text{m}$

Particle 2: mass 3kg at position $\vec{r}_2 = (-1\hat{i} + 4\hat{j} + 2\hat{k})\text{m}$

Particle 3: mass 5kg at position $\vec{r}_3 = (2\hat{i} - 2\hat{j} + 1\hat{k})\text{m}$

(i) Explain why the centre of mass lies closer to particle 3 than to particle 1, even without performing any calculation.
(ii) Calculate the position vector of the centre of mass.
(iii) Calculate the distance of the centre of mass from the origin.

Solution

(a) The balance point of a uniform metre stick is at its centre of mass, which is at the 50cm mark (the geometric centre). When the clay is added to one end, the system's centre of mass shifts toward the added mass, because the centre of mass is the mass-weighted average position. The balance point must

therefore shift toward the end where the clay is attached. The heavier the clay relative to the stick, the further the shift.

(b) The solution of each part is as follows:

(i) The centre of mass is the mass-weighted average position. Particle 3 has the largest mass (5kg out of a total of 10kg), contributing 50% of the weighting. The centre of mass is pulled most strongly toward the heaviest particle. Therefore, without calculating, we can predict that the centre of mass lies closer to particle 3 than to either of the other two.

(ii) The position vector of the centre of mass is:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

The total mass is:

$$M = 2kg + 3kg + 5kg = 10kg$$

Calculating each component separately:

î component:

$$x_{cm} = \frac{2kg \times 3m + 3kg \times (-1)m + 5kg \times 2m}{10kg} = 1.3m$$

ĵ component:

$$y_{cm} = \frac{2kg \times 2m + 3kg \times 4m + 5kg \times (-2)m}{10kg} = 0.6m$$

k̂ component:

$$z_{cm} = \frac{2kg \times 4m + 3kg \times 2m + 5kg \times 1m}{10kg} = 1.9m$$

Therefore:

$$\vec{r}_{cm} = (1.3\hat{i} + 0.6\hat{j} + 1.9\hat{k})m$$

Be careful! Each term in the numerator must carry the correct sign from the position vector. Particle 2 has $x = -1m$ and particle 3 has $y = -2m$. Missing a negative sign here changes the answer completely.

(iii) The distance of the centre of mass from the origin is the magnitude of $\vec{r}_{cm}$:

$$r_{cm} = |\vec{r}_{cm}| = \sqrt{x_{cm}^2 + y_{cm}^2 + z_{cm}^2}$$

Substituting:

$$r_{cm} = \sqrt{(1.3m)^2 + (0.6m)^2 + (1.9m)^2} = \sqrt{1.69 + 0.36 + 3.61}m = 2.38m$$

The centre of mass is 2.38m from the origin.

Example 61

(a) A body rotating at constant angular velocity has zero angular acceleration ($\alpha = 0$). Does this mean a point on the rim of the body has zero acceleration? Explain your answer.
(b) A grinding wheel of radius 0.25m starts from rest and accelerates uniformly at $4rad/s^2$.
(i) Calculate the tangential acceleration of a point on the rim.
(ii) Calculate the centripetal acceleration of a point on the rim at $t = 3s$.
(iii) Calculate the magnitude and direction of the total linear acceleration at $t = 3s$.

Solution

(a) No. Zero angular acceleration means the angular velocity is not changing, so the speed of the rim point is constant. However, the direction of the velocity changes continuously as the point moves around the circle. This continuous change in direction means the point has centripetal acceleration directed toward the centre, given by $a_c = \omega^2 R$. This acceleration is not zero as long as the body is rotating ($\omega \neq 0$). What is zero is the tangential acceleration ($a_t = \alpha R = 0$), which is responsible for changing the speed.

So a point on the rim of a body rotating at constant angular velocity has centripetal acceleration but no tangential acceleration. The total acceleration is purely centripetal.

(b) The solution of each part is as follows:

(i) The tangential acceleration is:

$$a_t = \alpha R = 4\text{rad/s}^2 \times 0.25\text{m} = 1.0\text{m/s}^2$$

The tangential acceleration is 1.0m/s^2.

Be careful! The tangential acceleration is constant (because α is constant), so it does not depend on time. It is the same at $t = 1\text{s}$ as at $t = 3\text{s}$.

(iii) First, finding ω at $t = 3\text{s}$:

$$\omega = \omega_0 + \alpha t = 0 + 4\text{rad/s}^2 \times 3\text{s} = 12\text{rad/s}$$

The centripetal acceleration is:

$$a_c = \omega^2 R = (12\text{rad/s})^2 \times 0.25\text{m} = 36\text{m/s}^2$$

The centripetal acceleration at $t = 3\text{s}$ is 36m/s^2.

Another point to note! Unlike the tangential acceleration, the centripetal acceleration is not constant. It grows with time because ω increases. At $t = 0$, $a_c = 0$ (the wheel is not yet turning). At $t = 3\text{s}$, $a_c = 36\text{m/s}^2$. This is already 36 times larger than the tangential acceleration!

(iv) Since a_t and a_c are perpendicular, the magnitude of the total linear acceleration is:

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{\left(1.0\text{m/s}^2\right)^2 + \left(36\text{m/s}^2\right)^2} = 36.01\text{m/s}^2$$

The angle ϕ between the total acceleration and the radius (centripetal direction) is:

$$\tan\phi = \frac{a_t}{a_c} = \frac{1.0\text{m/s}^2}{36\text{m/s}^2} = 0.0278$$

$$\phi = \tan^{-1}(0.0278) = 1.59°$$

The total acceleration has magnitude 36.0m/s^2 and is directed almost radially inward, tilted just 1.6° from the radius toward the tangent direction.

Interesting! At $t = 3\text{s}$, the centripetal acceleration (36m/s^2) completely dominates the tangential acceleration (1.0m/s^2). The total acceleration is practically centripetal. This is typical: once a body has been spinning for a few seconds, the centripetal component grows as ω^2 and quickly dwarfs the constant tangential component. Early in the motion (e.g. at $t = 0.25\text{s}$, where $\omega = 1\text{rad/s}$ and $a_c = 0.25\text{m/s}^2$), the tangential component actually dominates. The balance shifts as ω builds up.

Example 62

(a) In a simple Atwood machine with a massless pulley, the tension is the same on both sides of the string. Explain why this is no longer true when the pulley has mass.

(b) Two masses $m_1 = 5\text{kg}$ and $m_2 = 3\text{kg}$ are connected by a light, inextensible string that passes over a solid cylindrical pulley of mass $M = 2\text{kg}$ and radius $R = 0.10\text{m}$. The pulley rotates about a fixed frictionless axle. Mass m_1 hangs on the left and m_2 hangs on the right. The system is released from rest with m_1 at a height of 1.5m above m_2. Take $g = 9.8\text{m/s}^2$.

(i) Calculate the speed of the masses when m_1 has descended 0.8m.

(ii) Calculate the acceleration of the system.

(iii) Calculate the tensions T_1 and T_2.

Solution

(a) A massless pulley has zero moment of inertia ($I = 0$). By Newton's second law for rotation, $\tau = I\alpha = 0$, so the net torque on the pulley is zero. The only torques come from the tensions on each side: $(T_1 - T_2)R = 0$, which gives $T_1 = T_2$. So equal tensions follow automatically from zero mass.

When the pulley has mass, $I > 0$. The pulley must angularly accelerate as the system moves, requiring a net torque: $(T_1 - T_2)R = I\alpha \neq 0$. This demands $T_1 \neq T_2$. The tension on the heavier side must be larger than on the lighter side, and the difference provides the torque that spins the pulley.

(b) The moment of inertia of the solid cylindrical pulley:

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 2\text{kg} \times (0.10\text{m})^2 = 0.01\text{kgm}^2$$

(i) When m_1 descends by $h = 0.8\text{m}$, m_2 rises by 0.8m (the string is inextensible). The pulley rotates, gaining rotational KE.

By conservation of energy, the PE lost by m_1 equals the PE gained by m_2 plus the KE gained by both masses plus the rotational KE gained by the pulley:

$$m_1gh = m_2gh + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}I\omega^2$$

Using the no-slip condition: $v = \omega R$, so $\omega = \frac{v}{R}$. Substituting:

$$m_1gh = m_2gh + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}I\frac{v^2}{R^2}$$

Rearranging (moving m_2gh to the left side):

$$m_1gh - m_2gh = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}I\frac{v^2}{R^2}$$

$$(m_1 - m_2)gh = \frac{1}{2}v^2\left(m_1 + m_2 + \frac{I}{R^2}\right)$$

Substituting all values:

$$(5\text{kg} - 3\text{kg}) \times 9.8\text{m/s}^2 \times 0.8\text{m} = \frac{1}{2}v^2 \times \left(5\text{kg} + 3\text{kg} + \frac{0.01\text{kgm}^2}{(0.10\text{m})^2}\right)$$

$$v^2 = 3.484\text{m}^2/\text{s}^2$$

$$v = \sqrt{3.484\text{m}^2/\text{s}^2} = 1.87\text{m/s}$$

The speed of the masses when m_1 has descended 0.8m is 1.87m/s.

(ii) Since the acceleration is constant, using $v^2 = u^2 + 2as$ with $u = 0$:

$$a = \frac{v^2}{2s} = \frac{3.484\text{m}^2/\text{s}^2}{2 \times 0.8\text{m}} = 2.178\text{m/s}^2$$

The acceleration is 2.178m/s^2.

(iii) For m_1 (descending), applying Newton's second law:

$$m_1g - T_1 = m_1a$$

Making T_1 the subject:

$$T_1 = m_1(g - a) = 5\text{kg} \times \left(9.8\text{m/s}^2 - 2.178\text{m/s}^2\right) = 38.1\text{N}$$

For m_2 (ascending), applying Newton's second law:

$$T_2 - m_2g = m_2a$$

Making T_2 the subject:

$$T_2 = m_2(g + a) = 3\text{kg} \times \left(9.8\text{m/s}^2 + 2.178\text{m/s}^2\right) = 35.9\text{N}$$

The tensions are $T_1 = 38.1\text{N}$ and $T_2 = 35.9\text{N}$.

The plates are cleared, the table is wiped, and every dish has been tasted twice. If your brain feels like a flywheel that has been spinning for too long, congratulations! That is rotational KE, and it means you have been working hard. Now let us see if the Digging Deeper exercise can provide enough torque to slow you down!

DIGGING DEEPER EXERCISE 9

EXERCISE 9A: BINDER QUESTIONS

Question 1

A student claims: *"If I know how to solve $F = ma$, I already know how to solve $\tau = I\alpha$, because the mathematics is identical."* Is this claim correct? Justify your answer by identifying the translational twin of each rotational quantity in $\tau = I\alpha$.

Question 2

A solid sphere and a hollow sphere have the same mass and the same outer radius.

(a) State which one has the larger moment of inertia about an axis through the centre. Explain why.

(b) If both are acted upon by the same torque, state which one has the larger angular acceleration. Explain.

Question 3

A uniform rod can be rotated about its centre or about one end.

(a) Without using the parallel axis theorem, explain physically why the moment of inertia about the end is larger than about the centre.

(b) Using the parallel axis theorem, show that $I_{end} = 4I_{centre}$ for a uniform rod.

Question 4

By using parallel axis theorem, explain why the moment of inertia about the centre of mass is always the minimum moment of inertia for any rigid body.

Question 5

A solid disc and a ring have the same mass and the same outer radius R.

(a) Which one has the larger radius of gyration about the central axis? Explain in terms of how mass is distributed.

(b) A student says: *"If I double the mass of the disc without changing its radius, the radius of gyration doubles."* Is the student correct? Explain.

Question 6

A student tries to open a heavy door by pushing at the centre of the door instead of at the handle (far from the hinges).

(a) Explain why pushing at the centre requires more force than pushing at the handle to achieve the same angular acceleration.

(b) Explain why a couple is more effective at producing pure rotation than a single force applied at one point.

Question 7

A wheel starts from rest and undergoes uniform angular acceleration. A student measures the angular displacements in four consecutive seconds and obtains: 2rad, 6rad, 10rad, 14rad. Explain how these values confirm that the angular acceleration is uniform.

Question 8

(a) Distinguish between rotational KE and translational KE of a rolling body.

(b) Explain why a rolling body always has more KE than a sliding body of the same mass and speed.

Question 9

A solid cylinder and a hollow cylinder of the same mass and radius are released from rest at the top of an identical rough incline.

(a) Explain why the solid cylinder reaches the bottom first, even though both have the same mass and start with the same potential energy.

(b) If the incline were made perfectly frictionless, explain what would happen to both cylinders.

Question 10

During pure rolling on a flat surface, the contact point has zero velocity.

(a) Explain how the contact point can have zero velocity when the body is clearly moving forward.

(b) A motorcycle wheel picks up a small stone from the road. At what point on the wheel's rotation does the stone is more likely to fly off? Explain why.

Question 11

Friction acts up the slope when a body rolls down an incline, yet it does no work.

(a) Explain how friction can exert a force without doing any work.

(b) If friction does no work, explain where the rotational KE of the rolling body comes from.

Question 12

A disc spinning freely on a frictionless axle has a lump of clay dropped onto it.

(a) Explain why the angular velocity decreases after the clay lands.

(b) Explain why the KE of the system decreases, even though angular momentum is conserved.

Question 13

A body can have zero angular velocity but nonzero angular acceleration.

(a) Give a physical example of this situation.

(b) Can a body have nonzero angular velocity but zero angular acceleration? Give an example and explain.

Question 14

Two objects of different shapes but the same mass and radius roll without slipping down the same incline from rest.

(a) Explain why the acceleration of each object depends only on its shape and not on its mass or radius.

(b) Rank the following in order of arrival at the bottom (fastest first): solid sphere, solid cylinder, hollow sphere, hollow cylinder. Justify your ranking.

Question 15

Two students argue about a spinning raw egg that is briefly stopped and released.

Student A: *"The egg restarts because of Newton's third law."*

Student B: *"The egg restarts because of conservation of angular momentum."*

Which student is correct? Explain the physics carefully.

Question 16

The moment of inertia of a body depends on the axis of rotation.

(a) Explain, with an example, why the same body can have different moments of inertia about different axes.

(b) State the theorem that allows you to relate the moment of inertia about one axis to that about a perpendicular axis. To which shapes does this theorem apply?

Question 17

The perpendicular axis theorem applies to flat (planar) bodies but not to three-dimensional bodies like spheres.

(a) Explain physically why this restriction exists.

(b) A student tries to use the perpendicular axis theorem on a solid sphere to relate the moment of inertia about a diameter to the moment of inertia about a perpendicular diameter. Explain why this is unnecessary and what the correct relationship is.

Question 18

A flywheel and a solid ball have the same mass and the same outer radius. The flywheel is designed with a heavy rim (like a ring), while the ball has mass distributed throughout its volume.

(a) If both rotate at the same angular velocity, which stores more rotational KE? Explain in terms of moment of inertia.

(b) An engineer needs to design a rotating body that reaches high angular velocity quickly from rest under a given torque. Should the engineer choose the flywheel shape or the ball shape? Explain.

Question 19

A flywheel is designed to store rotational KE. Explain why flywheels are made with heavy rims and light centres rather than being solid discs of uniform thickness.

Question 20

A body rolling without slipping on a rough incline may begin to slip if the incline angle is increased beyond a certain value.

(a) Explain why increasing the angle can cause the body to slip.

(b) For the same coefficient of static friction and the same incline angle, which is more likely to slip: a solid sphere or a hollow cylinder? Explain.

EXERCISE 9B: REAL QUESTIONS

Question 21

A door has its handle placed at the edge farthest from the hinges, not at the centre or near the hinges. Explain, using the concept of torque, why this placement makes the door easier to open.

Question 22

A traditional Tanzanian potter shapes clay on a heavy stone wheel. The potter gives the wheel a strong push and then works the clay while the wheel gradually slows down.

(a) Explain why a heavier wheel is preferred over a lighter one for pottery.

(b) Explain why the wheel continues to spin for a long time after the push, even though no one is pushing it.

Question 23

Kipanga is helping his mother grind maize using a traditional hand-powered grinding stone. He pushes the handle at the edge of the stone. His younger brother tries to help by pushing near the centre.

Kipanga says: *"Push at the edge! You are wasting your effort near the centre."*

(a) Explain why Kipanga is correct.

(b) The grinding stone continues to rotate smoothly between pushes. Explain this observation in terms of moment of inertia and rotational KE.

Question 24

A figure skater begins a spin with both arms stretched outward. She then pulls her arms tight against her body, and her spin rate increases dramatically. No one pushes her, and the ice is nearly frictionless.

(a) Explain why pulling her arms in causes her to spin faster.

(b) Explain the change in kinetic energy as she pulls her arms tight against her body.

Question 25

A car is stuck in mud. The driver presses the accelerator hard, and the wheels spin rapidly but the car does not move.

(a) Explain, using the physics of rolling, why the car cannot move forward.

(b) A passenger places wooden planks under the spinning wheels. Explain why this helps the car to move.

Question 26

Kipute and Kipanga are watching a circus performer balance a spinning plate on a stick. The performer keeps the plate spinning fast, and it stays balanced. When the spin slows down, the plate wobbles and eventually falls.

Kipanga: *"The spinning must create some upward force that holds the plate."*

Kipute: *"No, the weight of the plate has not changed. Spinning does not reduce gravity. Something else is keeping it stable."*

Who is correct? Explain.

Question 27

A long metal rod and a short metal rod, both uniform and of the same material, are held vertically and released from the same height to fall and rotate about their lower ends (which are pinned to the ground).

(a) Explain why the longer rod takes more time to fall to the ground than the shorter rod.

(b) Explain why this result seems to contradict free fall (where all objects fall at the same rate).

Question 28

When opening a tight bottle cap, people instinctively wrap a piece of cloth around the cap before twisting.

(a) Explain how the piece of cloth helps.

(b) Some people use pliers with long handles to open very tight caps. Explain the advantage of long handles.

Question 29

In competitive diving, a diver performs three somersaults in the tuck position but straightens out before entering the water with almost no rotation.

(a) Explain how the diver controls the rate of somersaulting without any external force or torque while in the air.

(b) Explain why straightening out before entry reduces the rotation rate.

Question 30

In many Tanzanian villages, water is drawn from a well using a rope wound around a cylindrical drum. The operator turns a handle attached to the drum.

(a) Explain why a larger drum radius makes it harder to turn the handle when pulling up a full bucket, even though the bucket weight has not changed.

(b) Explain why the drum continues to rotate slightly after the operator stops turning the handle, especially when the drum is made of heavy wood.

Question 31

A rungu has most of its mass concentrated at the far end (the knob) and a thin handle.

(a) Explain why this design makes the rungu more effective as a striking weapon than a uniform stick of the same total mass and length.

(b) When thrown, the rungu rotates end over end. Explain why concentrating mass at one end affects the rotation of the thrown rungu.

Question 32

A child sits at the edge of a stationary merry-go-round. An adult gives the merry-go-round a push and it begins to spin. The child then moves to the centre.

(a) Explain what happens to the angular velocity of the merry-go-round when the child moves to the centre.

(b) The child now moves back to the edge. Explain what happens to the angular velocity.

Question 33

Mr. Akilikubwa demonstrates two identical tin cans in class. One is filled with water, the other with frozen ice (same mass of water in both). He releases both simultaneously from the top of a tilted plank.

"The ice-filled can reaches the bottom first."

Kipanga: *"But they weigh the same! How can one be faster?"*

Mr. Akilikubwa: *"The weight is the same, yes. But think about what happens inside each can as it rolls."*

Explain why the ice-filled can reaches the bottom first.

EXERCISE 9C: HOT QUESTIONS

Question 34

A wheel starts from rest and accelerates uniformly. It completes 20 revolutions in the first 5s.

(a) Calculate the angular acceleration.

(b) Calculate the angular velocity at $\mathrm{t} = 5$s.

(c) Calculate the angle turned through during the 5th second only.

(d) If the wheel has radius 0.30m, calculate the total linear acceleration of a point on the rim at $\mathrm{t} = 5$s.

Question 35

Three masses are arranged in the x-y plane as follows:

$\mathrm{m}_1 = 4$kg at $(0,0)$

$\mathrm{m}_2 = 2$kg at $(3\mathrm{m}, 0)$

$\mathrm{m}_3 = 6$kg at $(1\mathrm{m}, 4\mathrm{m})$

(a) Calculate the position of the centre of mass.
(b) Calculate the distance of the centre of mass from m_3.
(c) If m_3 is removed, calculate the new position of the centre of mass and the direction in which it shifts.

Question 36

A uniform rectangular plate of mass M, length a, and width b lies in the x-y plane.

(a) Using the result $I = \frac{1}{12}ML^2$ for a rod about its centre, and the perpendicular axis theorem, show that the moment of inertia of the plate about an axis perpendicular to its surface through its centre is $I = \frac{1}{12}M(a^2 + b^2)$.

(b) Calculate the moment of inertia of a rectangular plate of mass 2.4kg, length 0.60m, and width 0.40m about an axis perpendicular to the plate through its centre.

(c) Using the parallel axis theorem, calculate the moment of inertia about an axis perpendicular to the plate through one corner.

Question 37

A uniform disc of mass 5kg and radius 0.20m is mounted on a frictionless horizontal axle. A light cord is wound around the rim, and a constant force of 15N is applied to the free end of the cord.

(a) Calculate the moment of inertia of the disc about the axle.

(b) Calculate the torque applied to the disc.

(c) Calculate the angular acceleration.

(d) Calculate the angular velocity after 3s, starting from rest.

(e) Calculate the number of revolutions in the first 3s.

Question 38

A grinding stone (solid disc) of mass 8kg and radius 0.25m rotates at 600rpm. A brake pad is pressed against the rim with a normal force of 40N. The coefficient of kinetic friction between the pad and the stone is 0.5.

(a) Calculate the initial rotational KE of the stone.

(b) Calculate the friction force and the braking torque.

(c) Calculate the angular deceleration.

(d) Calculate the time to bring the stone to rest.

(e) Calculate the number of revolutions before stopping.

Question 39

A solid sphere of mass 3kg and radius 0.10m and a hollow cylinder of mass 3kg and radius 0.10m are both released from rest at the top of a rough incline of height 2.0m and angle 30°.

(a) Calculate the speed of each at the bottom of the incline.

(b) Calculate the time for each to reach the bottom.

(c) Calculate the time difference between their arrivals.

Question 40

A solid cylinder of mass 6kg and radius 0.15m rolls without slipping down a rough incline of angle 40°. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the acceleration of the centre of mass.

(b) Calculate the friction force required for pure rolling.

(c) Calculate the minimum coefficient of static friction needed.

(d) If the surface is oiled so that $\mu_s = 0.10$, show that the cylinder cannot roll without slipping and describe what happens instead.

Question 41

A merry-go-round (uniform disc, mass 120kg, radius 1.8m) rotates freely at 2.0rad/s. A boy of mass 40kg stands at the rim. He then walks inward to a distance of 0.6m from the centre.

(a) Calculate the moment of inertia of the system before the boy moves.

(b) Calculate the moment of inertia of the system after the boy moves.

(c) Calculate the new angular velocity.

(d) Calculate the ratio of the final KE to the initial KE, and explain where the extra energy comes from.

Question 42

Two masses $m_1 = 8$kg and $m_2 = 5$kg are connected by a light string passing over a solid cylindrical pulley of mass $M = 4$kg and radius $R = 0.12$m. The system is released from rest. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the acceleration of the system.

(b) Calculate the tensions T_1 and T_2.

(c) Calculate the speed of the masses after m_1 has descended 1.2m.

Question 43

A uniform rod of mass 1.5kg and length 0.80m is pivoted at one end and held horizontally. It is released from rest and swings downward.

(a) Calculate the moment of inertia of the rod about the pivot.

(b) Using conservation of energy, calculate the angular velocity of the rod as it passes through the vertical position.

(c) Calculate the linear speed of the free end as the rod passes through the vertical.

(d) Calculate the angular acceleration of the rod at the instant of release (when it is horizontal).

(e) Calculate the angular acceleration of the rod as it passes through the vertical.

Question 44

A flywheel (solid disc, mass 10kg, radius 0.30m) is driven by a motor that delivers a constant torque of 4.5Nm. A friction torque of 0.5Nm opposes the rotation.

(a) Calculate the net torque on the flywheel.

(b) Calculate the angular acceleration.

(c) Calculate the time to reach 120rad/s from rest.

(d) Calculate the power delivered by the motor at 120rad/s.

(e) Calculate the power dissipated by friction at 120rad/s.

Question 45

The moment of inertia of a flywheel is determined experimentally. A mass $m = 1.5$kg is attached to a string wound around the axle (radius $r = 0.025$m). The mass is released from rest and falls $h = 0.90$m, reaching a speed of 0.75m/s. After the string detaches, the flywheel makes $n_2 = 30$ additional revolutions before stopping.

(a) Calculate the angular velocity of the flywheel at the moment the string detaches.

(b) Using conservation of energy for the falling phase, derive an expression for the moment of inertia I of the flywheel in terms of m, g, h, v, r, and n_2.

(c) Calculate I.

(d) Calculate the friction torque.

Question 46

A solid sphere of mass 2kg and radius 0.05m rolls without slipping along a horizontal surface at 3m/s. It then encounters a rough incline of angle 25°.

(a) Calculate the total KE of the sphere on the horizontal surface.

(b) Calculate the maximum height the sphere reaches on the incline before momentarily stopping.

(c) Calculate the acceleration of the sphere on the incline.

(d) Calculate the friction force acting on the sphere while it rolls up the incline.

(e) In which direction does friction act on the sphere as it rolls up the incline? Explain why this is different from rolling down.

Question 47

An engineer needs to design a flywheel that stores at least 5000J of rotational KE at a maximum angular velocity of 300rad/s. The flywheel is to be a solid steel disc (density of steel $= 7800\text{kg/m}^3$).

(a) Calculate the minimum moment of inertia required.

(b) If the disc has thickness $t = 0.05\text{m}$, calculate the minimum radius of the disc.

(c) Calculate the mass of this disc.

(d) If the engineer instead uses a ring (same mass, same outer radius, inner radius $= 0.8R$), calculate the moment of inertia and the energy stored at the same angular velocity. Is the ring design better or worse for energy storage?

Question 48

A factory conveyor belt is driven by a solid cylindrical roller of mass 50kg and radius 0.20m. The belt must accelerate from rest to a linear speed of 2.0m/s in 4s. Assume the belt does not slip on the roller.

(a) Calculate the angular velocity of the roller at full speed.

(b) Calculate the angular acceleration during startup.

(c) Calculate the torque required from the motor to accelerate the roller (ignoring friction losses).

(d) If friction losses add a constant opposing torque of 3Nm, calculate the total motor torque needed and the motor power at full speed.

Question 49

A disc of moment of inertia 0.5kgm^2 rotates at 20rad/s. A second disc of moment of inertia 0.3kgm^2, initially rotating at 8rad/s in the **opposite direction**, is dropped coaxially onto the first.

(a) Taking the direction of the first disc as positive, calculate the total angular momentum before contact.

(b) Calculate the final angular velocity after they lock together.

(c) In which direction does the combined system rotate?

(d) Calculate the KE lost and express it as a percentage of the initial total KE.

Question 50

A uniform rod of mass 2kg and length 1.0m is free to rotate about a horizontal axis through one end. A bullet of mass 10g travelling horizontally at 400m/s embeds itself in the free end of the rod, which is initially hanging vertically at rest.

(a) Calculate the angular momentum of the bullet about the pivot just before impact.

(b) Calculate the moment of inertia of the system (rod + embedded bullet) about the pivot.

(c) Calculate the angular velocity of the system immediately after impact.

(d) Calculate the maximum angle through which the rod swings after impact.

Question 51

A student is asked: ***"A solid sphere rolls without slipping down an incline. What provides the centripetal acceleration for the circular motion of the sphere?"***

The student answers: *"Friction provides the centripetal acceleration, because friction is the only horizontal force."*

(a) Identify the error in the student's reasoning.

(b) Explain what friction actually does in rolling on an incline.

(c) If the sphere rolls on a flat horizontal surface at constant velocity, is friction needed? Explain.

Question 52

A solid cylinder of mass 4kg and radius 0.08m rolls without slipping at 5m/s along a horizontal surface. It then rolls up a rough incline of angle 35°.

(a) Calculate the total KE on the horizontal surface.

(b) Calculate the maximum distance along the incline the cylinder travels before stopping.

(c) Calculate the friction force while the cylinder rolls up.

(d) After momentarily stopping, the cylinder rolls back down. Calculate its speed when it returns to the horizontal surface.

Question 53

A solid sphere of mass M and radius R rolls without slipping at speed v_0 on a horizontal surface. It encounters a step of height h ($h < R$). The sphere collides with the edge of the step.

(a) Show that the angular momentum of the sphere about the edge of the step, just before the collision, is $L = Mv_0(R - h) + \frac{2}{5}MRv_0$.

(b) Assuming the sphere does not slip or bounce at the edge, and angular momentum about the edge is conserved during the very brief collision, find the angular velocity of the sphere about the edge immediately after the collision.

(c) Using energy conservation after the collision, find the minimum v_0 for the sphere to climb over the step.

Question 54

A uniform disc of mass 3kg and radius 0.20m can rotate about a fixed horizontal axis through its centre. A constant tangential force of 12N is applied at the rim for 5s, after which the force is removed.

(a) Calculate the angular velocity at the end of 5s.

(b) Calculate the rotational KE at the end of 5s.

(c) After the force is removed, a brake applies a constant friction torque of 0.8Nm. Calculate the time for the disc to stop.

(d) Calculate the total number of revolutions from start to stop.

Question 55

A solid ball of mass 0.5kg and radius 0.04m rolls without slipping at 6m/s along a flat surface toward a loop-the-loop track of radius $R_{loop} = 0.50\text{m}$.

(a) Calculate the total KE of the ball before it enters the loop.

(b) Calculate the speed of the ball at the top of the loop. Assume the ball continues to roll without slipping.

(c) For the ball to maintain contact with the track at the top of the loop, the centripetal acceleration must be at least g. Determine whether the ball completes the loop.

Question 56

A torque of $\tau = (6t - 2)\text{Nm}$ acts on a disc of moment of inertia 0.4kgm^2, where t is in seconds. The disc starts from rest at $t = 0$.

(a) At what time is the angular acceleration zero?

(b) Calculate the angular velocity at $t = 3\text{s}$.

(c) Calculate the angular momentum at $t = 3\text{s}$.

Question 57

A hollow sphere ($I = \frac{2}{3}MR^2$) of mass M and radius R rolls without slipping at speed v on a horizontal surface toward a ramp of angle θ.

(a) Show that the acceleration of the sphere on the ramp is $a = \frac{3g\sin\theta}{5}$.

(b) Show that the friction force required is $f = \frac{2}{5}Mg\sin\theta$.

(c) Show that the minimum coefficient of static friction for pure rolling is $\mu_{s,min} = \frac{2}{5}\tan\theta$.

(d) For $\theta = 45°$ and $\mu_s = 0.35$, determine whether the sphere rolls or slips. If it slips, describe qualitatively what happens.

ANSWERS

EXERCISE 9A

1. The claim is correct. The equation $\tau = I\alpha$ has exactly the same mathematical form as $F = ma$, with torque τ replacing force F, moment of inertia I replacing mass m, and angular acceleration α replacing linear acceleration a. Every algebraic manipulation that works for $F = ma$ works identically for $\tau = I\alpha$. The physics is different (forces push, torques twist), but the mathematical structure is the same.

2. (a) **The hollow sphere**. In a hollow sphere, all the mass is at the outer radius (far from the axis), while in a solid sphere, some mass is close to the axis. Since moment of inertia depends on mr^2, mass farther from the axis contributes more. The hollow sphere has $I = \frac{2}{3}MR^2$ compared with $I = \frac{2}{5}MR^2$ for the solid sphere.

(b) **The solid sphere.** From $\tau = I\alpha$, a smaller I with the same torque gives a larger α. The solid sphere has the smaller moment of inertia and therefore accelerates angularly more easily.

3. (a) When the axis is at the centre, each half of the rod extends a maximum distance of $L/2$ from the axis. When the axis is at one end, the far half of the rod extends a maximum distance of L from the axis. Since moment of inertia depends on r^2, the far half contributes much more when the axis is at the end. More mass is farther from the axis, so the moment of inertia is larger.

(b) $I_{end} = I_{centre} + Md^2$, where $d = L/2$ (distance from centre to end).

$I_{end} = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2$. Since $I_{centre} = \frac{1}{12}ML^2$, the ratio is $\frac{1/3}{1/12} = 4$. So $I_{end} = 4I_{centre}$.

4. From parallel axis theorem, $I = I_{cm} + Md^2$. Since Md^2 is always positive (M is positive, d^2 is positive), any axis at distance d from the centre of mass gives $I = I_{cm} + Md^2 > I_{cm}$. Only when $d = 0$ (the axis passes through the centre of mass) does $I = I_{cm}$. Therefore, I_{cm} is the minimum.

5. (a) The ring. The radius of gyration k represents the "effective distance" at which the entire mass would need to be concentrated to give the same moment of inertia. For the ring, all mass is at distance R from the axis, so $k_{ring} = R$. For the solid disc, mass is spread from the centre to R, so the effective distance is less: $k_{disc} = R/\sqrt{2} \approx 0.707R$. Thus the ring has the larger k because its mass is, on average, farther from the axis.

(b) The student is **incorrect**. The radius of gyration depends on shape and size, not on mass. For a solid disc of any mass, $k = R/\sqrt{2}$. Doubling the mass doubles I (from $\frac{1}{2}MR^2$ to $\frac{1}{2}(2M)R^2 = MR^2$), but $k = \sqrt{I/M} = \sqrt{MR^2/(2M)} = R/\sqrt{2}$, which is unchanged. The mass cancels.

6. (a) Torque equals force times perpendicular distance from the axis: $\tau = Fr$. The angular acceleration is determined by torque: $\alpha = \tau/I$. Pushing at the centre (small r) produces less torque for the same force than pushing at the handle (large r). To achieve the same torque (and therefore the same α), a larger force is needed at the centre.

(b) A single force produces both torque (causing rotation) and a net force (causing translation of the centre of mass). A couple consists of two equal and opposite forces, so the net force is zero (no translation), but the torques add because both forces twist in the same rotational direction. All the effort goes into rotation, with no energy wasted on translation.

7. The ratio of consecutive angular displacements is $2:6:10:14 = 1:3:5:7$, which is the odd-number sequence. This sequence is the signature of uniform acceleration from rest (Galileo's law). If the acceleration were not uniform, the ratios would deviate from $1:3:5:7$.

8. (a) Translational KE ($\frac{1}{2}Mv_{cm}^2$) is the energy due to the motion of the centre of mass while rotational KE ($\frac{1}{2}I\omega^2$) is the energy due to the spinning of the body about its centre of mass.

(b) A sliding body has only translational KE: $\frac{1}{2}Mv^2$. A rolling body has both translational KE and rotational KE: $\frac{1}{2}Mv^2\left(1 + k^2/R^2\right)$. Since $k^2/R^2 > 0$, the factor $\left(1 + k^2/R^2\right) > 1$, so the rolling body always has more KE.

9. (a) Both start with the same PE. At the bottom, this PE is shared between translational KE and rotational KE. The hollow cylinder has a larger k^2/R^2 $(= 1)$ than the solid cylinder $(= 1/2)$, so it diverts a larger fraction of its energy into rotation. With less energy going into translation, the hollow cylinder moves slower. The solid cylinder, with more translational KE, reaches the bottom first.

(b) On a frictionless incline, neither cylinder would rotate. Both would slide without rolling (like blocks), and both would reach the bottom at the same time with the same speed $v = \sqrt{2gh}$. Without friction, there is no torque to produce rotation, and the shape of the body is irrelevant.

10. (a) The contact point has two simultaneous velocities: the translational velocity v_{cm} (forward, shared by all points on the body) and the rotational velocity ωR (backward at the contact point, because the bottom of the wheel moves opposite to the direction of rolling). For pure rolling, $v_{cm} = \omega R$, so these two velocities cancel exactly at the contact point, giving zero net velocity.

(b) Near the top of the wheel. At the top, the translational and rotational velocities add to give $2v_{cm}$, which is the maximum speed on the entire wheel. The stone is flung off tangentially when the adhesion can no longer provide the centripetal force ($m\omega^2R$) needed to keep it on the curved path. This force requirement is greatest where the speed is greatest, which is at the top.

11. (a) Work requires both force and displacement at the point where the force acts. During pure rolling, the contact point (where friction acts) is instantaneously at rest. Since the displacement of the contact point is zero at every instant, the work done by friction is zero, even though the friction force itself is not zero.

(b) The rotational KE comes from the gravitational PE. As the body descends, PE converts into both translational and rotational KE. Friction does not supply the energy; it redirects it. Friction provides the torque that spins the body, but because it does no work, it does not add or remove energy from the system. It merely determines how the available PE is partitioned between translational and rotational forms.

12. (a) No external torque acts on the system, so angular momentum is conserved: $I_1\omega_1 = I_2\omega_2$. The clay adds mass at some distance from the axis, increasing the moment of inertia ($I_2 > I_1$). To keep the product $I\omega$ (angular momentum) constant, the angular velocity ω must decrease.

(b) $KE = \frac{1}{2}I\omega^2 = \frac{1}{2}I\omega^2 \times \frac{I}{I} = \frac{(I\omega)^2}{2I} = \frac{L^2}{2I}$. So, with constant momentum L, KE varies inversely proportional to the moment of inertia L. Since L is conserved (momentum is constant) and I increases, $KE = \frac{L^2}{2I}$ decreases.

13. (a) A wheel that has been brought to rest by a braking torque at the instant it stops. At that instant, $\omega = 0$, but the braking torque is still acting, producing $\alpha \neq 0$. Another example: a pendulum at its maximum displacement has $\omega = 0$ momentarily but is angularly accelerating back toward equilibrium.

(b) Yes. A wheel rotating at constant angular velocity has $\omega \neq 0$ but $\alpha = 0$. No net torque acts on it, so the angular velocity does not change. Another example is the ceiling fan at steady speed.

14. (a) The acceleration is $a = \frac{g\sin\theta}{1+k^2/R^2}$. Both the driving force ($Mg\sin\theta$, proportional to M) and the effective inertia ($M\left(1 + k^2/R^2\right)$, also proportional to M) increase with mass, so M cancels. The radius R cancels because k^2/R^2 is a pure number determined only by the shape. Only the shape factor k^2/R^2 remains.

(b) Fastest first: solid sphere ($k^2/R^2 = 2/5$), solid cylinder ($1/2$), hollow sphere ($2/3$), hollow cylinder (1).

Justification: Smaller k^2/R^2 means less energy diverted into rotation and more into translation, giving larger acceleration and faster arrival.

15. Student B is correct. When the shell is briefly stopped, the liquid inside continues to spin because no external torque has acted on it (the brief touch stops only the shell, not the liquid). The angular momentum of the liquid is not removed. When the shell is released, the still-spinning liquid exerts a torque on the shell through viscous friction, dragging it back into rotation. So the egg restarts because the liquid retained its angular momentum.

Student A's reasoning is vague. Newton's third law says the shell pushes back on the finger while being stopped, but this does not explain why the egg restarts after the finger is removed. The key physics is that the internal angular momentum of the liquid was never removed.

16. (a) Moment of inertia depends on how far each piece of mass is from the axis. For a uniform rod, rotating about the centre ($I = \frac{1}{12}ML^2$) places the mass closer to the axis on average than rotating about one end ($I = \frac{1}{3}ML^2$). The same rod has different values of I about different axes because the mass distribution relative to the axis is different.

(b) The perpendicular axis theorem: $I_z = I_x + I_y$, where x, y, z are three mutually perpendicular axes meeting at a point, with z perpendicular to the plane of the body. This theorem applies only to flat (planar) bodies such as discs, rings, and rectangular plates.

17. (a) The perpendicular axis theorem ($I_z = I_x + I_y$) is derived by considering the distance of each mass element from the z-axis in terms of its x and y coordinates: $r_z^2 = x^2 + y^2$. This decomposition works only when all mass elements lie in the x-y plane (i.e., have $z = 0$). For a three-dimensional body, mass elements are distributed in all three directions, and the simple decomposition fails.

(b) It is unnecessary because a sphere has perfect symmetry. The moment of inertia about any diameter is the same by symmetry: $I = \frac{2}{5}MR^2$. No theorem is needed to relate one diameter to another; they are all identical.

18. (a) The flywheel (ring shape). $KE = \frac{1}{2}I\omega^2$. The ring has larger moment of inertia $I = MR^2$, while the solid ball has smaller $I = \frac{2}{5}MR^2$. At the same ω, the ring stores $\frac{1}{2}MR^2\omega^2$ while the ball stores $\frac{1}{5}MR^2\omega^2$, which is 2.5 times less. So the ring having larger moment of inertia stores more energy because its mass is farther from the axis.

(b) The ball shape. From $\alpha = \tau/I$, a smaller I gives a larger angular acceleration for the same torque. The ball ($I = \frac{2}{5}MR^2$) reaches any given ω faster than the flywheel ($I = MR^2$) which has larger value of moment of inertia.

19. Rotational KE is $\frac{1}{2}I\omega^2$. For a given mass and radius, placing the mass at the rim (like a ring) maximises I to MR^2, compared with $\frac{1}{2}MR^2$ for a solid disc. A heavy rim concentrates the mass far from the axis of rotation, increasing I and therefore increasing the energy stored at any given angular velocity.

20. (a) The friction force required for pure rolling is $f = \frac{Mg\sin\theta}{1+R^2/k^2}$, which increases with θ. The maximum available static friction is $\mu_s Mg\cos\theta$, which decreases with θ (because $\cos\theta$ decreases). As θ increases, the required friction grows while the available friction shrinks. At some critical angle, the required friction exceeds the maximum available, and the body begins to slip.

(b) The hollow cylinder. It requires more friction ($\frac{1}{2}Mg\sin\theta$) than the solid sphere ($\frac{2}{7}Mg\sin\theta$) for the same mass and angle. With the same available friction, the hollow cylinder reaches the slipping threshold at a lower angle. The solid sphere can roll on steeper surfaces without slipping.

EXERCISE 9B

21. Torque equals force times perpendicular distance from the axis: $\tau = Fr$. The hinges are the axis of rotation. Placing the handle at the far edge maximizes r, so a small force produces a large torque. If the handle were near the hinges (small r), a much larger

force would be needed to produce the same torque and the same angular acceleration. Door designers place the handle at maximum r to minimize the force a person needs to apply.

22. (a) A heavier wheel has a larger moment of inertia, which means it stores more rotational KE ($\frac{1}{2}I\omega^2$) for the same angular velocity. This stored energy keeps the wheel spinning against the friction of the clay being shaped. A light wheel would lose its speed quickly because it stores less energy and the clay's resistance would slow it down rapidly.

(b) The wheel has a large moment of inertia (large rotational inertia). So a heavy wheel takes a long time to stop because of its large rotational inertia (Just as a heavy truck takes a long time to stop because of its large linear inertia). The only torque slowing it is friction at the axle and the resistance of the clay, which are small compared to the wheel's angular momentum. By $\tau = dL/dt$, a small torque takes a long time to reduce a large angular momentum to zero.

23. (a) The torque produced is $\tau = Fr$, where r is the distance from the axis. Pushing near the centre (small r) produces a small torque for the same force. Pushing at the edge (large r) produces a much larger torque for the same effort. The younger brother applies the same force but at a smaller distance, contributing less torque and therefore less angular acceleration to the stone.

(b) The grinding stone is heavy and has a large moment of inertia. Each push stores rotational KE in the stone ($\frac{1}{2}I\omega^2$). Between pushes, this stored energy maintains the rotation against the friction of grinding. The stone acts as a flywheel: it absorbs energy during the push and releases it smoothly between pushes.

24. (a) When the skater pulls her arms in, her moment of inertia decreases (mass is brought closer to the rotation axis). No external torque acts about her vertical axis (friction at the ice is negligible). By conservation of angular momentum, $I\omega =$ constant. If moment of inertia I decreases, angular momentum ω must increase to keep the product constant, and hence she spins faster.

(b) KE increases. The rotational KE is $\frac{1}{2}I\omega^2$. When I decreases and ω increases, the KE increases (because ω increases faster than I decreases, as KE $= \frac{L^2}{2I}$ and L is constant while I decreases). The extra KE comes from the work done by the skater's muscles as she pulls her arms inward against the centripetal acceleration. She does real physical work to bring her arms in, and this work becomes rotational KE.

25. (a) For a wheel to propel a car, it must roll without slipping. This requires static friction at the contact point to push the car forward. On mud, the coefficient of static friction is very low. The engine torque spins the wheel faster than $\frac{v_{cm}}{R}$(violating the no-slip condition), so the wheel spins in place. Consequently, kinetic friction replaces static friction, which is weaker and cannot accelerate the car effectively.

(b) The wooden planks provide a rough surface with a much higher coefficient of static friction than mud. The increased μ_s restores the no-slip condition: the wheel can now grip the plank, establish pure rolling ($v_{cm} = \omega R$), and static friction pushes the car forward. The planks supply the friction that the mud cannot.

26. Kipute is correct. Spinning does not create an upward force or reduce the weight of the plate. The plate's weight mg remains the same throughout. What the spin provides is angular momentum along the vertical axis. A large angular momentum resists changes in the direction of the spin axis ($\tau = dL/dt$: changing the direction of L requires a torque). Small disturbances (wobbles) try to tilt the axis, but the large L resists these changes. When the plate spins fast (large L), the resistance to tilting is strong and the plate stays balanced. When the spin slows (small L), even a small disturbance can change the axis direction significantly, causing the plate to wobble and fall.

Relate! This is gyroscopic stability, the same physics that keeps a bicycle upright and a bullet flying nose-first.

27. (a) At any angle θ from the vertical during the fall, the gravitational torque about the pivot is $\tau = Mg\frac{L}{2}\sin\theta$, and the angular acceleration is:

$$\alpha = \frac{\tau}{I} = \frac{Mg\frac{L}{2}\sin\theta}{\frac{1}{3}ML^2} = \frac{3g\sin\theta}{2L}$$

At every corresponding angle θ, the longer rod (larger L) has a smaller angular acceleration. So the longer rod is always behind throughout the entire fall. Both rods sweep through the same angle (90°), but the longer rod accelerates more slowly at every stage, so it takes more time.

Alternative Solution for (a):

Both rods rotate about their lower ends under the torque of gravity. Using conservation of energy, when the rod falls from vertical to horizontal, the centre of mass drops by $L/2$:

$$Mg\frac{L}{2} = \frac{1}{2}I\omega^2 = \frac{1}{2}\times\frac{1}{3}ML^2\times\omega^2$$

Solving: $\omega = \sqrt{\frac{3g}{L}}$. So the longer rod (larger L) reaches a smaller final angular velocity. Since both rods fall through the same angle (90°) but the longer rod rotates more slowly at the end, the longer rod takes more time to reach the ground.

(b) In free fall, all objects experience the same acceleration g regardless of mass or size, because the force (weight) and the inertia (mass) are both proportional to m. For a rotating rod, the torque is proportional to ML but the rotational inertia is proportional to ML^2. The extra factor of L in the inertia means longer rods are harder to rotate, unlike free fall where length does not matter. Rotation introduces a size dependence that free fall does not have.

28. (a) The piece of cloth increases the coefficient of friction between the hand and the cap. With higher friction, the hand can apply a larger tangential force at the rim of the cap without slipping. Since torque = force × radius, a larger tangential force at the same radius produces a larger torque. Without the piece of cloth, the hand slips before enough torque is applied to overcome the cap's resistance.

(b) Pliers extend the distance r from the axis (the centre of the cap) at which the force is applied. Since $\tau = Fr$, doubling the handle length doubles the torque for the same hand force.

Relate! Long handles provide a mechanical advantage for rotation, just as a long lever provides mechanical advantage for translation.

29. (a) After leaving the board, no external torque acts on the diver about the centre of mass (gravity acts through the CM, producing zero torque about it). Angular momentum is therefore conserved throughout the flight. The diver controls the rotation rate by changing body shape: tucking (pulling knees to chest) reduces the moment of inertia, which increases ω to conserve L.

(b) Straightening out increases the moment of inertia dramatically (arms and legs are now far from the rotation axis). Since $I\omega =$ constant, a large I requires a small ω. The diver enters the water with almost no rotation, allowing a clean, splash-free entry.

30. (a) The bucket's weight creates a torque about the drum's axis: $\tau_{bucket} = mg \times R_{drum}$. A larger drum radius R increases this opposing torque. The operator must apply a larger force at the handle to overcome this increased torque. The bucket weighs the same, but the moment arm is larger, so the torque is larger.

(b) The heavy wooden drum has a large moment of inertia, which means it stores angular momentum ($L = I\omega$) and rotational KE ($\frac{1}{2}I\omega^2$). When the operator stops, the stored angular momentum does not disappear instantly. The bucket's weight provides a torque that gradually decelerates the drum, but with large I, the deceleration is slow ($\alpha = \tau/I$ is small). The drum coasts forward briefly until friction and the bucket's torque bring it to rest.

31. (a) The concentrated mass at the far end gives the rungu a large moment of inertia about the grip point, and when swung, the knob has a large distance from the rotation axis ($r \approx L$). The KE at impact is $\frac{1}{2}I\omega^2$, and since I is large (most mass at maximum r), the KE delivered is much greater than a uniform stick where mass is spread evenly along the length. Additionally, the linear velocity of the knob is $v = \omega L$, which is the maximum possible for the given arm swing.

(b) The centre of mass of the rungu is closer to the knob (the heavy end), not at the geometric centre. When thrown, the rungu rotates about its centre of mass. Because the centre of mass is near the knob, the handle sweeps out a larger arc than the knob, giving the characteristic lopsided end-over-end tumble. If the mass were uniform, the rotation would be symmetric about the midpoint.

32. (a) When the child moves to the centre, the moment of inertia of the system (merry-go-round + child) decreases because the child's mass is now closer to the axis. No external torque acts on the system (the axle is frictionless for this argument). By conservation of angular momentum, $I\omega =$ constant, so ω increases. The merry-go-round spins faster.

(b) When the child moves back to the edge, the moment of inertia increases again. By conservation of angular momentum, ω decreases. The merry-go-round slows back down.

33. In the ice-filled can, the ice is frozen solid and rotates together with the can as a single **rigid body**. The entire mass contributes to both the translational and rotational motion.

In the water-filled can, the liquid water does not rotate with the can. As the can rolls, the shell rotates but the water inside largely slides rather than spins. The can's shell must supply all the rotational inertia, but the water still contributes to the total mass that must be translated.

The ice-filled can behaves like a solid cylinder ($k^2/R^2 = 1/2$). The water-filled can behaves somewhere between a solid cylinder and a hollow cylinder, because the non-rotating water adds translational inertia without contributing useful rotational motion. Effectively, the water-filled can has a larger "effective k^2/R^2" and therefore a smaller acceleration. Hence, the ice-filled can, with all its mass participating properly in the rolling, reaches the bottom first.

EXERCISE 9C

34. (a) 10.05rad/s^2 (b) 50.3rad/s (c) 45.2rad (7.2 revolutions) (c) 758.1m/s^2

35. (a) $(1.0\text{m}, 2.0\text{m})$ (b) 2.0m (c) $(1.0\text{m}, 0)$, 2.0m downward.

36. (b) 0.104kgm^2 (c) 0.416kgm^2

37. (a) 0.10kgm^2 (b) 3.0Nm (c) 30rad/s^2 (d) 90rad/s (e) 21.5 revolutions.

38. (a) 493.4J (b) $\text{f} = 20\text{N}$, $\tau = 5.0\text{Nm}$ (c) 20rad/s^2 (d) 3.14s (e) 15.7 revolutions.

39. (a) Solid sphere: 5.29m/s, Hollow cylinder: 4.43m/s (b) $\text{t}_{\text{sphere}} = 1.512\text{s}$, $\text{t}_{\text{cylinder}} = 1.807\text{s}$ (c) 0.295s

40. (a) 4.20m/s^2(b)12.60N (c) 0.28

(d) Since $\text{f}_{\text{max}}(4.50\text{N}) < \text{f}\,(12.60\text{N})$, the available friction is insufficient. The cylinder cannot roll without slipping. Instead, it slides and rotates simultaneously: translating faster than ωR (skidding). Kinetic friction acts up the slope, providing some torque that partially spins the cylinder, but the no-slip condition $\text{v}_{\text{cm}} = \omega\text{R}$ is never established. The cylinder accelerates faster than a pure rolling cylinder but slower than a frictionless sliding block.

41. (a) 324.0kgm^2 (b) 208.8kgm^2 (c) 3.10rad/s (d) 1.55

The final KE is 55% larger than the initial KE. The extra energy comes from the work done by the boy's muscles as he walks inward against the centripetal acceleration. Walking toward the centre on a spinning platform requires effort because the boy must overcome the tendency to be pushed outward. This muscular work is converted into rotational KE.

42. (a) 1.96m/s^2 (b) $\text{T}_1 = 62.72\text{N}$, $\text{T}_2 = 58.80\text{N}$ (c) 2.17m/s

43. (a) 0.32kgm^2 (b) 6.06rad/s (c) 4.85m/s (d) 18.4rad/s^2(e) 0rad/s^2

44. (a) 4.0Nm (b) 8.89rad/s^2 (c) 13.5s (d) 540W (e) 60W

45. (a) 30rad/s (c) 0.0239kgm^2 (d) 0.057Nm

46. (a) 12.6J (b) 0.643m (c) -2.96m/s^2 (d) 2.37N

(e) Friction acts **down the incline** when the sphere rolls up. As the sphere decelerates while rolling up, its translational speed decreases. To maintain the no-slip condition ($\text{v}_{\text{cm}} = \omega\text{R}$), the angular velocity must also decrease. Friction must provide a torque that decelerates the spin. The only way to produce a torque that slows the clockwise spin (for a sphere rolling up to the right) is for friction to act down the slope at the contact point. This is the opposite direction to rolling down, where friction acts up the slope to spin the body up.

47. (a) From $\text{KE} = \frac{1}{2}\text{I}\omega^2$:

$$\text{I}_{\text{min}} = \frac{2 \times \text{KE}}{\omega^2} = \frac{2 \times 5000\text{J}}{(300\text{rad/s})^2} = \frac{10000}{90000}\text{kgm}^2 = 0.111\text{kgm}^2$$

(b) For a solid disc: $\text{I} = \frac{1}{2}\text{MR}^2$, and $\text{M} = \rho\pi\text{R}^2\text{t}$. Substituting:

$$\text{I} = \frac{1}{2}\rho\pi\text{R}^2\text{t} \times \text{R}^2 = \frac{1}{2}\rho\pi\text{tR}^4$$

Making R the subject:

$$\text{R}^4 = \frac{2\text{I}}{\rho\pi\text{t}} = \frac{2 \times 0.111\text{kgm}^2}{7800\text{kg/m}^3 \times \pi \times 0.05\text{m}} = \frac{0.222}{1225.2}\text{m}^4 = 1.812 \times 10^{-4}\text{m}^4$$

$$\text{R} = (1.812 \times 10^{-4})^{1/4}\text{m} = 0.116\text{m}$$

The minimum radius is approximately 0.12m (12cm).

(c) $\text{M} = \rho\pi\text{R}^2\text{t} = 7800\text{kg/m}^3 \times \pi \times (0.116\text{m})^2 \times 0.05\text{m} = 7800 \times 3.1416 \times 0.01346 \times 0.05\text{kg}$

$$= 7800 \times 0.002114\text{kg} = 16.5\text{kg}$$

(d) For a ring with inner radius $\text{r} = 0.8\text{R}$ and outer radius R:

$$\text{I}_{\text{ring}} = \frac{1}{2}\text{M}(\text{R}^2 + \text{r}^2) = \frac{1}{2} \times 16.5\text{kg} \times [(0.116\text{m})^2 + (0.8 \times 0.116\text{m})^2]$$

$$= \frac{1}{2} \times 16.5 \times [0.01346 + 0.008614]\text{kgm}^2 = \frac{1}{2} \times 16.5 \times 0.02207\text{kgm}^2 = 0.182\text{kgm}^2$$

Energy stored: $\text{KE} = \frac{1}{2} \times 0.182\text{kgm}^2 \times (300\text{rad/s})^2 = 0.091 \times 90000 = 8190\text{J}$

The ring stores 8190J compared to 5000J for the disc, using the same mass and outer radius. So the ring design is better for energy storage because it places mass farther from the axis, increasing I and hence storing more energy.

48. (a) 10rad/s (b) 2.5rad/s^2 (c) 2.5Nm (d) 5.5Nm, 55W

49. (a) 7.6kgm^{2}s^{-1} (b) 9.5rad/s (c) $\omega_f = +9.5\text{rad/s}$ (positive), so the system rotates in the direction of the first disc. (d) 73.5J, 67%

50. (a) The bullet travels horizontally and hits the free end at distance $L = 1.0\text{m}$ from the pivot:

$$L_{\text{bullet}} = mvL = 0.010\text{kg} \times 400\text{m/s} \times 1.0\text{m} = 4.0\text{kgm}^2\text{s}^{-1}$$

(b) $I_{\text{rod}} = \frac{1}{3}ML^2 = \frac{1}{3} \times 2\text{kg} \times (1.0\text{m})^2 = 0.667\text{kgm}^2$

$$I_{\text{bullet}} = mL^2 = 0.010\text{kg} \times (1.0\text{m})^2 = 0.010\text{kgm}^2$$

$$I_{\text{total}} = 0.667 + 0.010 = 0.677\text{kgm}^2$$

(c) Conservation of angular momentum (the impact is very brief, so no external torque has time to act):

$$\omega = \frac{L_{\text{bullet}}}{I_{\text{total}}} = \frac{4.0\text{kgm}^2\text{s}^{-1}}{0.677\text{kgm}^2} = 5.91\text{rad/s}$$

(d) After impact, the rod swings upward. Using conservation of energy from the moment after impact to the maximum angle θ:

$$\frac{1}{2}I_{\text{total}}\omega^2 = Mg\frac{L}{2}(1 - \cos\theta) + mgL(1 - \cos\theta)$$

The left side is the initial rotational KE. The right side is the PE gained: the rod's CM rises by $\frac{L}{2}(1 - \cos\theta)$ and the bullet rises by $L(1 - \cos\theta)$.

$$\frac{1}{2} \times 0.677\text{kgm}^2 \times (5.91\text{rad/s})^2 = \left(2\text{kg} \times 9.8\text{m/s}^2 \times 0.5\text{m} + 0.010\text{kg} \times 9.8\text{m/s}^2 \times 1.0\text{m}\right)(1 - \cos\theta)$$

$$\frac{1}{2} \times 0.677 \times 34.93 = (9.8 + 0.098)(1 - \cos\theta)$$

$$11.82\text{J} = 9.898\text{N} \cdot \text{m} \times (1 - \cos\theta)$$

$$1 - \cos\theta = \frac{11.82}{9.898} = 1.194$$

$$\cos\theta = 1 - 1.194 = -0.194$$

$$\theta = \cos^{-1}(-0.194) = 101.2°$$

The rod swings 101° past the vertical (slightly beyond horizontal).

51. (a) The student confuses two different types of circular motion. The sphere rolling down an incline involves rotation about its own centre, not motion in a circular path. The "centripetal acceleration" the student refers to does not apply here. Centripetal acceleration (v^2/r) applies to an object moving along a circular path (like a car rounding a bend). A sphere rolling down a straight incline follows a straight-line path; there is no centripetal acceleration of the centre of mass. The points on the rim do move in circles about the centre and have centripetal acceleration, but this is internal to the body and is provided by internal forces within the rigid body, not by friction.

(b) Friction at the contact point provides the torque about the centre of mass that causes the sphere to spin. Without friction, the sphere would slide without rotating. Friction converts part of the gravitational pull (which would otherwise go entirely into translation) into rotation. Friction does not provide centripetal acceleration for the centre of mass.

(c) No. On a flat horizontal surface at constant velocity, both v_{cm} and ω are constant. No angular acceleration is needed (no torque required), and no linear acceleration is needed (no net force required). In the ideal case, pure rolling on a flat surface at constant velocity requires zero friction. Friction is only needed when the body must accelerate or decelerate (change v_{cm} and ω).

52. (a) 75J (b) 3.34m (c) 7.49N (down the slope) (d) 5m/s

53. (a) Just before the collision, the sphere has:

Translational angular momentum about the edge: The centre of mass moves at v_0 and is at perpendicular distance $(R-h)$ from the edge (since the edge is at height h and the centre is at height R). So $L_{trans} = Mv_0(R-h)$.

Rotational angular momentum about its own centre: $L_{rot} = I_{cm}\omega = \frac{2}{5}MR^2 \times \frac{v_0}{R} = \frac{2}{5}MRv_0$.

Both angular momenta are in the same direction (the sphere rolls forward and spins in the same sense about the edge).

$$L = Mv_0(R-h) + \frac{2}{5}MRv_0 \quad \blacksquare$$

(b) After collision, the sphere rotates about the edge with angular velocity ω'. The moment of inertia about the edge (using parallel axis theorem):

$$I_{edge} = I_{cm} + MR^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

Conservation of angular momentum about the edge:

$$Mv_0(R-h) + \frac{2}{5}MRv_0 = \frac{7}{5}MR^2\omega'$$

$$\omega' = \frac{Mv_0(R-h) + \frac{2}{5}MRv_0}{\frac{7}{5}MR^2} = \frac{5v_0(R-h) + 2Rv_0}{7R^2} = \frac{v_0(5R-5h+2R)}{7R^2} = \frac{v_0(7R-5h)}{7R^2}$$

(c) The sphere rotates about the edge. The CM starts at distance R from the edge (at ground level beside the step) and must swing up to be directly above the edge. The CM traces a circular arc of radius R about the edge.

The CM must rise by $R-(R-h) = h$.

Using energy conservation:

$$\frac{1}{2}I_{edge}\omega'^2 \geq Mgh$$

$$\frac{1}{2} \times \frac{7}{5}MR^2 \times \left(\frac{v_0(7R-5h)}{7R^2}\right)^2 \geq Mgh$$

$$\frac{7}{10}R^2 \times \frac{v_0^2(7R-5h)^2}{49R^4} \geq gh$$

$$\frac{v_0^2(7R-5h)^2}{70R^2} \geq gh$$

$$v_0 \geq \frac{\sqrt{70ghR^2}}{7R-5h}$$

The required minimum value of v_0:

$$v_0 = \frac{R\sqrt{70gh}}{7R-5h}$$

54. (a) 200rad/s (b) 1200J (c) 15s (d) 318 revolutions.

55. (a) $KE = \frac{1}{2}Mv^2(1+2/5) = \frac{1}{2} \times 0.5\text{kg} \times (6\text{m/s})^2 \times \frac{7}{5} = 12.6\text{J}$

(b) At the top of the loop, the centre of the ball is at height $2R_{loop} - r \approx 2R_{loop} = 1.0\text{m}$ (since $r \ll R_{loop}$, we approximate).

Energy conservation:

$$\frac{1}{2}Mv_0^2\left(\frac{7}{5}\right) = \frac{1}{2}Mv_{top}^2\left(\frac{7}{5}\right) + Mg(2R_{loop})$$

$$12.6\text{J} = \frac{7}{10} \times 0.5\text{kg} \times v_{top}^2 + 0.5\text{kg} \times 9.8\text{m/s}^2 \times 1.0\text{m}$$

$$v_{top} = 4.69\text{m/s}$$

(c) At the top of the loop, the centripetal acceleration must be at least g for the ball to maintain contact:

$$\frac{v_{top}^2}{R_{loop}} \geq g$$

$$\frac{22.0\text{m}^2/\text{s}^2}{0.50\text{m}} = 44.0\text{m/s}^2$$

Since $44.0\text{m/s}^2 \gg 9.8\text{m/s}^2$, the ball has more than enough speed. It completes the loop comfortably.

56. (a) $\alpha = \frac{\tau}{I} = \frac{6t-2}{0.4}\text{rad/s}^2 = (15t - 5)\text{rad/s}^2$

Setting $\alpha = 0$: $15t - 5 = 0$, so $t = \frac{1}{3}\text{s} = 0.333\text{s}$.

(b) $\omega = \int_0^t \alpha\, dt = \int_0^3 (15t - 5)\, dt = \left[\frac{15t^2}{2} - 5t\right]_0^3 = \frac{15\times 9}{2} - 5 \times 3 = 52.5\text{rad/s}$

(c) $L = I\omega = 0.4\text{kgm}^2 \times 52.5\text{rad/s} = 21.0\text{kgm}^2\text{s}^{-1}$

57. (a) $a = \frac{g\sin\theta}{1+k^2/R^2}$. For a hollow sphere, $k^2/R^2 = 2/3$:

$$a = \frac{g\sin\theta}{1 + 2/3} = \frac{g\sin\theta}{5/3} = \frac{3g\sin\theta}{5}$$

(b) $f = \frac{Mg\sin\theta}{1+R^2/k^2}$. For a hollow sphere, $R^2/k^2 = 3/2$:

$$f = \frac{Mg\sin\theta}{1 + 3/2} = \frac{Mg\sin\theta}{5/2} = \frac{2Mg\sin\theta}{5}$$

(c) For pure rolling: $f \leq \mu_s Mg\cos\theta$

$$\frac{2Mg\sin\theta}{5} \leq \mu_s Mg\cos\theta$$

$$\mu_s \geq \frac{2\sin\theta}{5\cos\theta} = \frac{2}{5}\tan\theta$$

Hence:

$$\mu_{s,min} = \frac{2}{5}\tan\theta$$

(d) Required: $\mu_s \geq \frac{2}{5}\tan 45° = \frac{2}{5} \times 1 = 0.40$

Available: $\mu_s = 0.35$

Since $0.35 < 0.40$, the sphere cannot roll without slipping. It slips while rolling up the ramp: the translational speed decreases faster than ωR (kinetic friction acts, providing less torque than needed for pure rolling). The sphere slides and rotates simultaneously, losing more energy to friction than in pure rolling. It reaches a lower maximum height than a purely rolling sphere would.