Chapter 8

GRAVITATION

INTRODUCTION

It was a quiet afternoon at Miono Secondary School. The physics class had just finished Chapter 7 on Circular Motion, and the students were packing their books. **Kipanga**, still buzzing with excitement about satellites and orbits, turned to **Mr. Akilikubwa** with a question that had been bothering him all lesson.

"Sir, you said that gravity pulls everything toward the Earth. And you said the Moon goes around the Earth in a circle. But if gravity is pulling the Moon downward, why doesn't it just fall and crash into us?"

The class went quiet. Even **Kipute**, who usually had an answer ready, paused.

Mr. Akilikubwa smiled and picked up a tennis ball from his desk. *"Kipanga, if I throw this ball gently sideways, what happens?"*

"It curves downward and hits the ground nearby," **Kipanga** answered.

"And if I throw it harder?"

"It goes further before hitting the ground."

"And if I throw it harder still?"

"Even further."

"Now," said **Mr. Akilikubwa**, his voice dropping as though sharing a secret, *"imagine I could throw it so hard that as it falls, the Earth's surface curves away beneath it at exactly the same rate. The ball keeps falling toward the Earth, but the ground keeps curving away. The ball falls forever but never lands. What would you call that?"*

Kipanga's eyes widened. *"An orbit!"*

"Exactly." **Mr. Akilikubwa** set the ball down. *"The Moon* ***is*** *falling toward the Earth! It has been falling for over four billion years. It just keeps missing. And the force that makes it fall, is the same force that pulls this tennis ball, is the force that holds you in your chair, the force that keeps the Earth circling the Sun; is the oldest, weakest, and most far-reaching force in the universe. Newton called it gravity. And he had the audacity to claim that it was the same force that made an apple fall from a tree."*

Kipute leaned forward. *"The same force? An apple and the Moon?"*

"The same law," **Mr. Akilikubwa** confirmed. *"One equation that describes both. That is the power of what you are about to learn."*

Kipanga grinned. *"So Newton figured out the Moon is basically a very fast apple?"*

Mr. Akilikubwa laughed. *"That, Kipanga, might be the best summary of gravitation anyone has ever given."*

In Chapter 7, we asked: *what real force provides the centripetal force for circular motion?* We found different answers for different situations: friction for cars on roads, tension for stones on strings, the normal reaction for banked curves. But we left one answer untouched, the most profound one of all: **gravity**.

Gravity is the force that holds the Moon in orbit around the Earth, the Earth in orbit around the Sun, and the Sun in orbit around the centre of the Milky Way galaxy. It is the force that shaped the solar system, controls the tides, and determines the fate of the universe itself. Yet it obeys a single, elegant law that Newton published in 1687, a law so simple that you will derive it yourself before this chapter is over.

In this chapter, we begin with the observations that led to our understanding of planetary motion: **Kepler's three laws**. We then meet **Newton's Law of Universal Gravitation** and use it to explain why planets move the way Kepler described. From there, we explore how gravity varies with altitude, depth, and latitude, and we introduce the concepts of **gravitational field strength** and **gravitational potential**. Finally, we apply everything to **satellites, orbits, and escape velocity**; the physics that makes space travel possible.

The mathematics builds directly on what you mastered in Chapter 7. The centripetal force equation $\frac{mv^2}{r}$ returns, but now the force supplying it is gravity: $\frac{GMm}{r^2}$. Setting these equal will unlock nearly every result in this chapter. If you understood circular motion, gravitation will feel like a natural continuation with the same physics, played on the grandest stage imaginable.

KEPLER'S LAWS OF PLANETARY MOTION

Before Newton ever thought about gravity, a German mathematician named **Johannes Kepler** spent years staring at numbers. Not just any numbers, but the most precise astronomical measurements ever collected, recorded by the Danish astronomer **Tycho Brahe**, a man so dedicated to watching the sky that he built an entire observatory on an island. Brahe gathered mountains of data on planetary positions but died in 1601 before he could make sense of it all. Kepler inherited the data and, after years of painstaking calculation (no computers, no calculators! Just pen, paper, and stubbornness), he extracted three laws that described exactly how planets move around the Sun.

Kepler did not know *why* planets obeyed these laws. He simply showed that they did. The "why" would have to wait for Newton, who arrived on the scene about 70 years later and explained all three laws using a single equation. But before we meet Newton's explanation, we must first understand what Kepler discovered.

Kepler's First Law (The Law of Orbits)

The law states that:

Each planet moves in an elliptical orbit with the Sun at one focus.

Before Kepler, everyone assumed planetary orbits were perfect circles (or combinations of circles). Kepler showed that orbits are actually **ellipses**, slightly squashed circles. An ellipse has two special points called **foci** (singular: focus). The Sun sits at one focus, not at the centre.

For most planets in our solar system, the orbits are very close to circular. The Earth's orbit, for example, deviates from a perfect circle by less than 2%. But the deviation matters: it is the reason why the Earth is slightly closer to the Sun in January than in July (this fact often surprises students who think the seasons are caused by changes in distance from the Sun. In reality, the seasons are caused by the tilt of the Earth's axis, not the distance).

However, we will often approximate planetary and satellite orbits as circular, because this simplifies the mathematics while remaining accurate for most practical calculations.

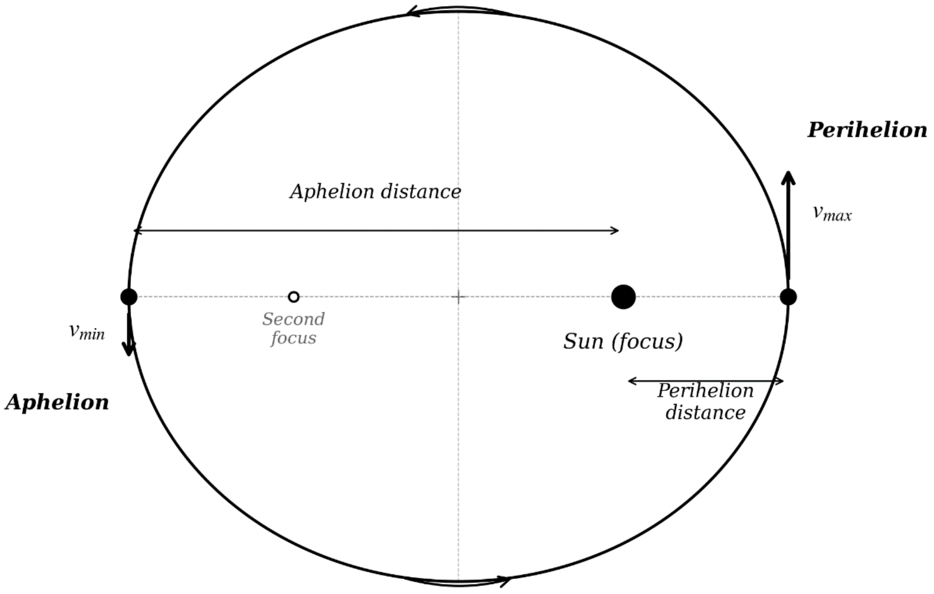


Figure*: Kepler's first law.* *A planet moves in an elliptical orbit with the Sun at one focus. The planet moves fastest at perihelion and slowest at aphelion.*

Key terms:

Perihelion: *the point in the orbit closest to the Sun.* The planet moves fastest (v_{max}) here.

Aphelion: *the point farthest from the Sun.* The planet moves slowest here (v_{min}).

For orbits around the Earth (such as satellite orbits), the equivalent terms are **perigee** (*closest*) and **apogee** (*farthest*).

Kepler's Second Law (The Law of Areas)

The law states that:

A line drawn from the Sun to a planet sweeps out equal areas in equal intervals of time.

This law tells us something profound about how the speed of a planet changes as it moves along its orbit. When the planet is close to the Sun (near perihelion), the line from the Sun to the planet is short. To sweep the same area in the same time, the planet must move faster. When the planet is far from the Sun (near aphelion), the line is longer, so the planet can sweep the same area while moving more slowly.

In short: ***planets speed up when closer to the Sun and slow down when farther away.***

This is not a coincidence or a design choice. It is a consequence of the **conservation of angular momentum** (this will be discussed in detail in **Chapter 9: Rotation of Rigid Bodies**). Because gravity acts along the line joining the planet and the Sun, it produces no torque about the Sun. With zero torque, angular momentum ($L = mvr$) is conserved. When r decreases, v must increase to keep L constant, and vice versa.

For simplified circular orbits (which we will mostly use in this topic), the speed is constant and the second law is automatically satisfied. This is because, with constant speed, the planet sweeps out equal angles in equal times, and since the radius is constant, equal angles means equal areas, agreeing with Kepler's second law.

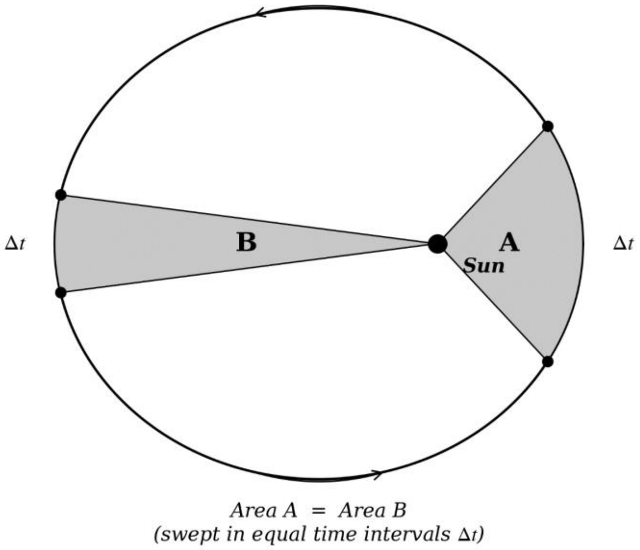


Figure: ***Kepler's second law.*** *A line drawn from the Sun to the planet sweeps out equal areas in equal time intervals Δt. Sector A (near perihelion) spans a wide arc because the planet is close to the Sun and moves fast. Sector B (near aphelion) spans a narrow arc because the planet is far from the Sun and moves slowly. Despite their different shapes, both areas are equal.*

Kepler's Third Law (The Law of Periods)

The law states that:

The square of the period of revolution of a planet is directly proportional to the cube of its mean distance from the Sun.

Mathematically:

$$T^2 \propto r^3$$

$$T^2 = kr^3$$

Where T is the orbital period, r is the mean orbital radius, and k is a constant that is the same for all planets orbiting the same central body.

If two planets orbit the same star, their periods and radii are related by:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

This is an extraordinarily powerful equation. It means that if you know the period and radius of one planet's orbit, you can find the radius of any other planet's orbit just by measuring its period. Kepler used this to map the entire solar system.

What the third law reveals

The third law tells us that planets farther from the Sun take longer to complete one orbit not just because they have a longer path, but also because they move more slowly. Both effects combine to make the period increase steeply with distance. For example, the Earth orbits the Sun in 1 year at a mean distance of 1AU (astronomical unit). Jupiter, at about 5.2AU, takes about 11.9 years. Neptune, at 30AU, takes about 165 years.

Later in this chapter, we will derive Kepler's Third Law from Newton's Law of Universal Gravitation and show that the constant $k = \frac{4\pi^2}{GM}$, where M is the mass of the central body. This derivation is one of the most beautiful results in all of physics: three empirical laws, discovered by painstaking observation, all explained by a single force law.

With Kepler's three laws now understood, let us practise applying them.

BINDER Example 1

(a) A planet moves faster at certain points in its orbit and slower at others. Using Kepler's laws, explain why this happens and identify the positions where the planet is fastest and slowest.

(b) The mean distance of the Earth from the Sun is 1.5×10^{11}m and its period of revolution is 3.156×10^7s (1 year). Calculate the constant k in Kepler's third law.

Solution

(a) Kepler's first law tells us that a planet's orbit is an ellipse with the Sun at one focus. This means the planet is not always at the same distance from the Sun; it is closer at one point (perihelion) and farther at another (aphelion).
Kepler's second law tells us that a line from the Sun to the planet sweeps equal areas in equal times. When the planet is near perihelion, the line from the Sun is short, so the planet must move through a larger arc to sweep the required area, and thus it moves faster. When the planet is near aphelion, the line is long, so a smaller arc is sufficient, and hence it moves slower.

Therefore, the planet is fastest at perihelion and slowest at aphelion.

(b) Using $T^2 = kr^3$:

$$k = \frac{T^2}{r^3} = \frac{(3.156 \times 10^7 \text{s})^2}{(1.5 \times 10^{11}\text{m})^3} = 2.95 \times 10^{-19} \text{s}^2/\text{m}^3$$

Making Sense of the Answer: *This constant is the same for every planet in the solar system. Mars, Jupiter, Saturn; they all give the same value of k. This universality is what makes Kepler's Third Law so powerful, and what hinted to Newton that a single force law governs all planetary motion.*

Think Like a Physicist: *The constant k depends on the mass of the central body (the Sun, in this case). Planets orbiting a different star would have a different k, because the star has a different mass.*

REAL Example 2

Mars has a mean orbital radius of 2.28×10^{11}m. **Kipanga** claims that since Mars is about 1.5 times farther from the Sun than Earth, its year must be about 1.5 times longer than Earth's year.

Explain why Kipanga's reasoning is incorrect, and find the actual period of Mars.

Solution

Kipanga's reasoning is wrong because the period does not increase in direct proportion to the distance. Kepler's third law states $T^2 \propto r^3$, not $T \propto r$.

Using the ratio form:

$$\frac{T_{Mars}^2}{T_{Earth}^2} = \frac{r_{Mars}^3}{r_{Earth}^3}$$

$$\frac{T_{Mars}^2}{(1\ \text{year})^2} = \left(\frac{2.28 \times 10^{11}\text{m}}{1.5 \times 10^{11}\text{m}}\right)^3 = (1.52)^3 = 3.51$$

$$T_{Mars} = \sqrt{3.51}\ \text{years} = 1.87\ \text{years}$$

Mars takes 1.87 Earth years to orbit the Sun (not 1.5 years as Kipanga predicted).

Making Sense of the Answer: *A planet 1.5 times farther out has a longer path and moves more slowly. Both effects combine through the* $T^2 \propto r^3$ *relationship, making the period nearly twice the Earth's, not just 1.5 times.*

Think Like a Physicist: *The ratio form of Kepler's third law is extremely useful because it avoids the need to know the constant k or the mass of the Sun. You only need the period and radius of one known orbit to find the other.*

HOT Example 3

A geostationary satellite orbits the Earth with a period of exactly 24 hours. The Moon orbits the Earth with a period of 27.3 days at a mean distance of 3.84×10^8m from the Earth's centre.

Calculate the orbital radius of the geostationary satellite.

Solution

Both the satellite and the Moon orbit the Earth, so the same constant k applies. Using the ratio form:

$$\frac{T_{sat}^2}{T_{Moon}^2} = \frac{r_{sat}^3}{r_{Moon}^3}$$

Making r_{sat} the subject:

$$r_{sat}^3 = r_{Moon}^3 \times \frac{T_{sat}^2}{T_{Moon}^2}$$

Where: $T_{sat} = 24\text{h} = 24 \times 3600\text{s} = 86400\text{s}$, $T_{Moon} = 27.3\ \text{days} = 27.3 \times 24 \times 3600\text{s} = 2.359 \times 10^6\text{s}$, $r_{Moon} = 3.84 \times 10^8\text{m}$

$$r_{sat}^3 = (3.84 \times 10^8\text{m})^3 \times \frac{(86400\text{s})^2}{(2.359 \times 10^6\text{s})^2} = 7.6 \times 10^{22}\text{m}^3$$

$$r_{sat} = \sqrt[3]{7.6 \times 10^{22}\text{m}^3} = 4.24 \times 10^7\text{m}$$

The orbital radius of the geostationary satellite is 4.24×10^7m (about 42400km from the Earth's centre, or about 36,000km above the surface).

Making Sense of the Answer: *The geostationary orbit is about one-ninth of the way to the Moon. Its period (1 day) is much shorter than the Moon's (27.3 days), so it must be much closer, and Kepler's third law gives us exactly how close.*

Think Like a Physicist: *This problem uses the Moon as a reference orbit to find the geostationary radius, without needing the mass of the Earth or the gravitational constant G. The ratio method is powerful precisely because the unknown constants cancel.*

Kepler told us *how* planets move. Three elegant laws, extracted from decades of data. But he could not answer the deeper question: *why* do they move this way? *What force compels a planet to follow an ellipse? Why does it sweep equal areas? Why does* T^2 *relate to* r^3 *and not to* r^2 *or* r^4*?*

The answer came from Isaac Newton, who showed that all three laws are consequences of a single universal force. In the next subtopic, we meet that force.

NEWTON'S LAW OF UNIVERSAL GRAVITATION

Kepler showed us the pattern. Three laws, elegant and precise, describing exactly how planets orbit the Sun. But Kepler could not answer the obvious question: *what force makes them do this?*

Isaac Newton answered that question with one of the boldest claims in the history of science. He proposed that the force pulling an apple to the ground and the force holding the Moon in orbit are **the same force**. Not similar forces, not analogous forces, but the *same* force, obeying the *same* law, differing only in the masses involved and the distances between them. He called it **gravitation**, and he declared it to be **universal**: every object in the universe attracts every other object. The law states that:

Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres.

Mathematically:

$$F = \frac{Gm_1m_2}{r^2}$$

Where:

F is the gravitational force of attraction between the two bodies (in N),

m_1 and m_2 are the masses of the two bodies (in kg),

r is the distance between their centres (in m),

G is the **universal gravitational constant**, with value:

$$G = 6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$$

Understanding the law

Several features of this law deserve careful attention.

1. The force is mutual

If the Earth pulls you downward with a gravitational force, you pull the Earth upward with the same force. This is Newton's third law in action. The forces are equal in magnitude and opposite in direction. The reason you accelerate more than the Earth is that your mass is vastly smaller, not that the force is different.

2. The force depends on the product of the masses

Doubling either mass doubles the force. Doubling both masses quadruples the force. If one mass is zero, the force is zero.

3. The force obeys an inverse square law

Doubling the distance reduces the force to one-quarter. Tripling the distance reduces it to one-ninth. This rapid decrease with distance is why gravitational effects from distant objects are usually negligible, even though the force technically extends to infinity.

4. The force acts along the line joining the centres

Gravity is always attractive and always acts along the straight line connecting the centres of the two masses. It never pushes; it only pulls.

The Gravitational Constant G

The constant G is extraordinarily small: $6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$. This is why you do not feel a gravitational pull toward the person sitting next to you in class, even though Newton's law says the force exists. For two people of mass 60kg each, sitting 1m apart, the gravitational force between them is about 2.4×10^{-7}N (roughly the weight of a bacterium!).

Gravity only becomes significant when at least one of the masses is enormous, like a planet or a star. The Earth's mass (6.0×10^{24}kg) compensates for the tiny value of G and produces the familiar gravitational pull we experience every day.

G was first measured experimentally by Henry Cavendish in 1798, more than a century after Newton proposed the law. Cavendish used a sensitive torsion balance to measure the weak gravitational attraction between lead spheres. His experiment is often described as "weighing the Earth," because knowing G allows the mass of the Earth to be calculated.

The Shell Theorem

Newton's law of gravitation is stated for **point particles**. But planets and stars are not points, they are extended spheres. *How do we apply the law to them?* Newton himself answered this question by proving what is now called the **Shell Theorem**, which has two parts:

Part 1: *A uniform spherical shell attracts a particle* ***outside*** *it as if the entire mass of the shell were concentrated at its centre.*

Part 2: *A uniform spherical shell exerts* ***zero*** *gravitational force on a particle* ***inside*** *it.*

Since a solid sphere can be thought of as a collection of concentric shells, Part 1 means that a uniform solid sphere (like a planet) attracts any external object as if all the sphere's mass were at its centre. This is the reason we are justified in using $F = \frac{Gm_1m_2}{r^2}$ with r measured from **centre to centre**, even though the masses are spread out over large volumes.

When you stand on the Earth's surface, the relevant distance in $F = \frac{Gm_Em}{r^2}$ is the distance from you to the **centre** of the Earth ($r_E = 6.4 \times 10^6 m$), not the distance to the ground beneath your feet.

Part 2 will become important later when we study how gravity varies with depth below the Earth's surface.

Deriving Kepler's Third Law from Newton's Law

This is where the connection between Chapters 7 and 8 becomes explicit. Consider a planet of mass **m** orbiting the Sun of mass **M** in a circular orbit of radius **r**.

From Chapter 7, the planet in circular motion requires a centripetal force:

$$F_{centripetal} = \frac{mv^2}{r}$$

This centripetal force is provided by gravity:

$$F_{gravity} = \frac{GMm}{r^2}$$

Setting them equal:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

The mass of the planet m cancels:

$$v^2 = \frac{GM}{r}$$

Now, the orbital speed is related to the period by $v = \frac{2\pi r}{T}$. Substituting:

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

$$\mathbf{T^2 = \frac{4\pi^2}{GM} r^3}$$

This is Kepler's third law, with the constant identified:

$$\mathbf{k = \frac{4\pi^2}{GM}}$$

Three observations are profound:

First: $T^2 \propto r^3$ emerges naturally from Newton's law of gravitation combined with circular motion. Kepler discovered this empirically; Newton explained *why*.

Second: The mass of the orbiting body (m) does not appear. The period depends only on the orbital radius and the mass of the central body (M). This is why *all satellites at the same altitude orbit the Earth with the same period, regardless of their mass.*

Third: the constant k depends only on G and M. If you know k (from observing any orbit), you can calculate M (the mass of the central body). This is how astronomers "weigh" the Sun, the Earth, and other planets.

Let us now put these ideas to work.

BINDER Example 4

(a) Two identical spheres, each of mass 50kg, are placed with their centres 0.5m apart. Calculate the gravitational force between them and explain why this force is not noticeable in everyday life. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) Calculate the gravitational force between the Earth and a 70kg person standing on the surface. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$.

Solution

(a) Using $F = \frac{Gm_1m_2}{r^2}$:

$$F = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 50\text{kg} \times 50\text{kg}}{(0.5\text{m})^2} = 6.67 \times 10^{-7}\text{N}$$

This force is about 0.00000067N. It is far too small to feel because the gravitational constant G is extremely small, and neither mass is large enough to compensate.

(b) Using:

$$F = \frac{GM_Em}{r_E^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg} \times 70\text{kg}}{(6.4 \times 10^6\text{m})^2} = 683\text{N}$$

The gravitational force between the Earth and a 70kg person is 683N. (This is the person's weight, close to $mg = 70 \times 9.8 = 686\text{N}$; the small difference comes from rounding).

Making Sense of the Answer: *The gravitational force between everyday objects is negligible. But when one mass is as enormous as the Earth ($6 \times 10^{24}\text{kg}$), the tiny constant G is overwhelmed and the force becomes the very weight we feel every moment of our lives.*

Think Like a Physicist: *Newton's law of gravitation and the familiar formula W = mg are not two different things. W = mg is simply the result of applying* $F = \frac{GMm}{r^2}$ *at the Earth's surface, where* $g = \frac{GM}{r_E^2}$. *We will derive this connection explicitly in the next section.*

REAL Example 5

The Earth pulls the Moon with a gravitational force, and the Moon pulls the Earth with the same force. Yet the Moon orbits the Earth, not the other way around. Explain why both bodies experience the same force but respond so differently.

Solution

The gravitational force on the Moon due to the Earth is equal in magnitude to the force on the Earth due to the Moon. However, by Newton's second law ($a = \frac{F}{m}$), the acceleration produced depends on the mass of the body. The Moon is much less massive than the Earth, so the Moon's acceleration is much greater than the Earth's. The Moon therefore moves in a large, visible orbit, while the Earth's motion is barely noticeable.

Making Sense of the Answer: *Same force, different masses, different accelerations. A mosquito and a truck experience the same collision force during impact, but only the mosquito changes its motion dramatically. The same principle applies to the Moon and the Earth.*

Think Like a Physicist: *The phrase "the Moon orbits the Earth" is a convenient simplification. More precisely, both orbit their common centre of mass. But since the Earth is so much more massive, the centre of mass is very close to the Earth's centre, making the simplification excellent.*

HOT Example 6

(a) The Earth orbits the Sun at a mean distance of $1.5 \times 10^{11}\text{m}$ with a period of $3.156 \times 10^7\text{s}$. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. Calculate the mass of the Sun.

(b) A satellite orbits the Earth at a height of 300km above the surface. Calculate its orbital speed and period in minutes. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$.

Solution

(a) From $T^2 = \frac{4\pi^2}{GM}r^3$, making M the subject:

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{4 \times (3.14)^2 \times (1.5 \times 10^{11}\mathrm{m})^3}{6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}} \times (3.156 \times 10^7\mathrm{s})^2} = 2.01 \times 10^{30}\mathrm{kg}$$

The mass of the Sun is approximately 2.0×10^{30}kg.

(b) Orbital radius: $R = r_E + h = 6.4 \times 10^6\mathrm{m} + 300 \times 10^3\mathrm{m} = 6.7 \times 10^6\mathrm{m}$

$$F_{\mathrm{centripetal}} = \frac{mv^2}{R}$$

This centripetal force is provided by gravity:

$$F_{\mathrm{gravity}} = \frac{GM_E m}{R^2}$$

Equating:

$$\frac{mv^2}{R} = \frac{GM_E m}{R^2}$$

From which:

$$v^2 = \frac{GM_E}{R}$$

$$v = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}} \times 6.0 \times 10^{24}\mathrm{kg}}{6.7 \times 10^6\mathrm{m}}} = 7726\mathrm{m/s}$$

The orbital speed is 7726m/s.

Period: $T = \frac{2\pi R}{v} = \frac{2 \times 3.14 \times 6.7 \times 10^6\mathrm{m}}{7726\mathrm{m/s}} = \frac{4.21 \times 10^7}{7726} = 5449\mathrm{s}$

$$T = \frac{5449}{60}\mathrm{min} = 90.8\mathrm{min}$$

The period is 90.8min.

Making Sense of the Answer: *A satellite at 300km altitude circles the Earth in about 91 minutes which is roughly the period of the International Space Station. Its speed of 7726m/s means it travels faster than a bullet. At lower altitudes, the speed would be even higher and the period shorter.*

Think Like a Physicist: *Notice that the satellite's mass never appeared in any calculation. Orbital speed and period depend only on the mass of the central body and the orbital radius. A feather and a truck at the same altitude would orbit at exactly the same speed, just as Galileo's insight predicts.*

With Newton's law of gravitation and its connection to circular motion now established, we have the tools to explore how gravity behaves at different locations: above, on, and below the surface of the Earth. That is the subject of the next section.

ACCELERATION DUE TO GRAVITY

In Chapter 2, we introduced $g = 9.8\mathrm{m/s^2}$ as a constant, which is the acceleration of any object falling freely near the Earth's surface. We used it in equation after equation without asking where it comes from. Now, with Newton's law of gravitation in hand, we can finally answer that question and discover that g is not truly constant at all.

The Origin of g

Consider an object of mass **m** on the surface of the Earth (mass $\mathbf{M_E}$, radius $\mathbf{r_E}$). The gravitational force on the object is its weight:

$$F = \frac{GM_E m}{r_E^2}$$

But we also know that weight equals mg:

$$mg = \frac{GM_E m}{r_E^2}$$

The mass m cancels:

$$\mathbf{g = \frac{GM_E}{r_E^2}}$$

This is a profound result. The acceleration due to gravity at the surface of a planet depends only on the **mass** and **radius** of the planet, not on the mass of the falling object. A feather and a boulder experience the same gravitational acceleration. This is a fact Galileo demonstrated centuries ago, and which Newton's law now explains mathematically.

Mass and Density of the Earth

Since $g = \frac{GM_E}{r_E^2}$, we can rearrange to find the mass of the Earth:

$$\mathbf{M_E} = \frac{\mathbf{gr_E^2}}{\mathbf{G}} = \frac{9.8\text{m/s}^2 \times (6.4 \times 10^6\text{m})^2}{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}} = 6.02 \times 10^{24}\text{kg}$$

The density of the Earth follows. Modelling the Earth as a sphere:

$$\boldsymbol{\rho} = \frac{\mathbf{M_E}}{\frac{4}{3}\boldsymbol{\pi}\mathbf{r_E^3}} = \frac{6.02 \times 10^{24}\text{kg}}{\frac{4}{3} \times 3.14 \times (6.4 \times 10^6\text{m})^3} = 5483\text{kg/m}^3$$

This is about 5500kg/m^3, roughly five and a half times the density of water. Since surface rocks have a density of only about $2500 - 3000\text{kg/m}^3$, the interior of the Earth must be much denser than the surface; evidence for a heavy iron-nickel core, deduced from nothing more than g, G, and r_E.

Variation of g with Altitude

Consider the following diagram:

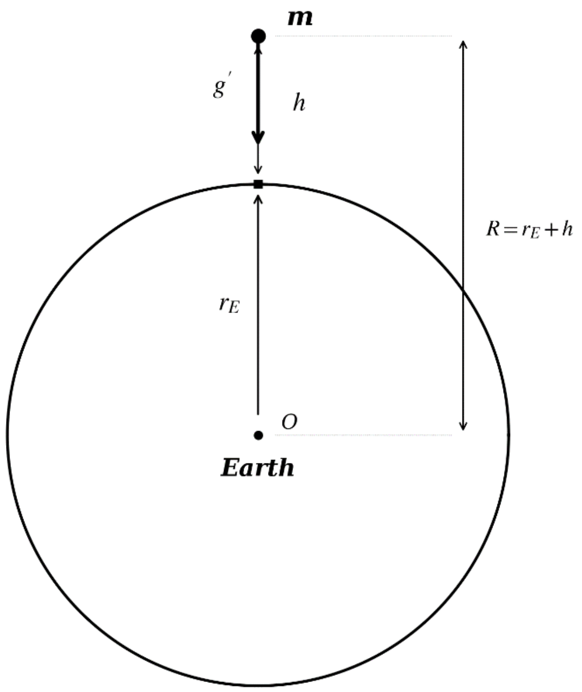


Figure: ***An object of mass m at height h above the Earth's surface.*** *The distance from the centre of the Earth is* $R = r_E + h$*. The acceleration due to gravity g' at this height is directed toward the centre and is less than g at the surface.*

At a height h above the Earth's surface, the distance from the centre of the Earth becomes:

$$R = r_E + h.$$

The acceleration due to gravity at this height is:

$$g' = \frac{GM_E}{R^2} = \frac{GM_E}{(r_E + h)^2}$$

Dividing by $g = \frac{GM_E}{r_E^2}$:

$$\frac{g'}{g} = \frac{r_E^2}{(r_E + h)^2}$$

$$\mathbf{g' = g\left(\frac{r_E}{r_E + h}\right)^2}$$

This is the **exact** expression, valid for any height. It shows that g decreases with altitude, following an inverse square relationship with the distance from the centre (not from the surface).

Approximate formula for small heights ($h \ll r_E$):

By multiplying both the numerator and the denominator inside the bracket by $\frac{1}{r_E}$, the equation can be rewritten as:

$$g' = g\left(\frac{1}{1 + \frac{h}{r_E}}\right)^2 = g\left(1 + \frac{h}{r_E}\right)^{-2}$$

If $h \ll r_E$:

$$\frac{h}{r_E} \approx 0 \text{(too small)}$$

Using the binomial approximation $(1 + x)^{-2} \approx 1 - 2x$ for small x:

$$\mathbf{g' \approx g\left(1 - \frac{2h}{r_E}\right)}$$

This approximation is useful for heights up to a few hundred kilometres, where $\frac{h}{r_E}$ is still small. For larger heights, the exact formula must be used.

Variation of g with Depth

What happens below the Earth's surface? This is where Part 2 of the Shell Theorem becomes essential.

Consider an object at depth d below the surface. Its distance from the centre is $R = r_E - d$. According to the Shell Theorem, only the mass of the Earth **below** the object (within radius R) contributes to the gravitational pull. All the mass in the shell above the object exerts zero net force on it.

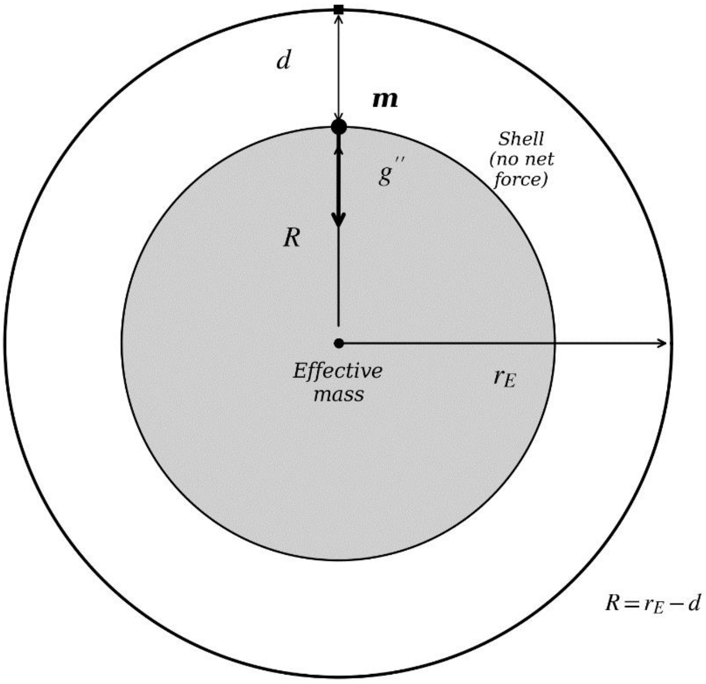


Figure: ***An object of mass m at depth d below the Earth's surface,*** *at distance $R = r_E - d$ from the centre. The shaded region represents the effective mass that exerts gravitational force on the object. The outer shell (unshaded) exerts zero net force, in accordance with the Shell Theorem.*

Assuming the Earth has uniform density ρ (an approximation, but one that reveals the essential physics):

The mass within radius R is:

$$M' = \rho \times \frac{4}{3}\pi R^3$$

The total mass of the Earth is:

$$M_E = \rho \times \frac{4}{3}\pi r_E^3$$

Dividing:

$$\frac{M'}{M_E} = \frac{R^3}{r_E^3}$$

The acceleration due to gravity at depth d is:

$$g'' = \frac{GM'}{R^2} = \frac{G}{R^2} \times \frac{M_E R^3}{r_E^3} = \frac{GM_E}{r_E^3} R = \frac{g}{r_E} R$$

Since $R = r_E - d$:

$$g'' = \frac{g}{r_E}(r_E - d)$$

Hence:

$$\mathbf{g'' = g\left(1 - \frac{d}{r_E}\right)}$$

This tells us that g decreases **linearly** with depth. At the surface ($d = 0$), $g'' = g$. At the centre of the Earth ($d = r_E$), $g'' = 0$.

An object at the centre of the Earth would be pulled equally in all directions by the surrounding mass, so the net gravitational force would be zero. This is true weightlessness with genuine zero gravitational force, not the "weightlessness" of orbit, where gravity still acts but produces no contact force.

Important assumption: The derivation above assumes uniform density throughout the Earth. In reality, the Earth's core is much denser than its mantle and crust, so the actual variation of g with depth is more complex. In fact, g initially *increases* slightly as you descend from the surface (because you move closer to the dense core), reaching a maximum at the core-mantle boundary, before decreasing to zero at the centre. However, for examination purposes, the uniform density approximation is standard.

Variation of g with Latitude

The value of g at the Earth's surface also varies with latitude, for two reasons:

Reason 1: The Earth is not a perfect sphere. The Earth is slightly flattened at the poles and bulges at the equator (an oblate spheroid). The polar radius is about 21km shorter than the equatorial radius. Since $g \propto \frac{1}{r^2}$, the smaller radius at the poles means g is slightly larger there.

Reason 2: The Earth rotates. A body on the surface of the Earth moves in a circle (due to the Earth's rotation) and therefore requires centripetal acceleration. Part of the gravitational pull goes toward providing this centripetal acceleration, leaving less to be felt as "weight."

At the equator, the centripetal acceleration is maximum because the radius of the circular path (the Earth's equatorial radius r_E) is largest. At the poles, the body is on the axis of rotation, the radius of the circular path is zero, and no centripetal acceleration is needed.

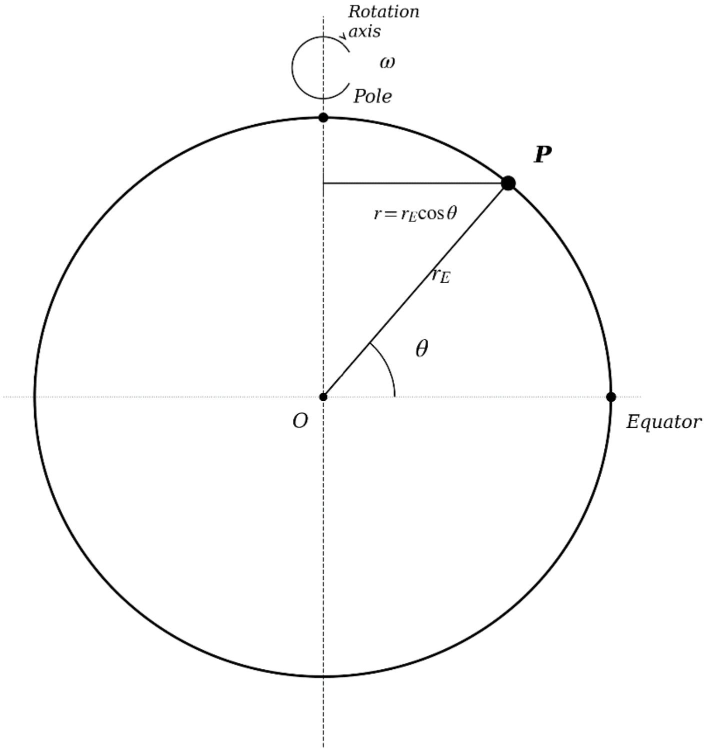


Figure: ***A point P on the Earth's surface at latitude θ.*** *The Earth rotates about its polar axis with angular velocity ω. The distance from P to the rotation axis is* $r = r_E cos\theta$*. As the Earth rotates, P moves in a horizontal circle of this radius, requiring centripetal acceleration* $\omega^2 r_E cos\theta$ *directed toward the axis.*

At latitude θ, the radius of the circular path is $r = r_E \cos\theta$, and the centripetal acceleration is $\omega^2 r_E \cos\theta$. The effective (measured) gravitational acceleration is:

$$\mathbf{g' = g - \omega^2 r_E \cos^2\theta}$$

At the **equator** ($\theta = 0°$): $g' = g - \omega^2 r_E$ (minimum)

At the **poles** ($\theta = 90°$): $g' = g$ (maximum)

The correction $\omega^2 r_E = (7.27 \times 10^{-5} \text{rad/s})^2 \times 6.4 \times 10^6 \text{m} = 0.034 \text{m/s}^2$. This is small compared to 9.8m/s^2 (about 0.3%), but it is measurable. Combined with the shape effect, the total variation in g from equator to pole is about 0.053m/s^2, giving $g \approx 9.78 \text{m/s}^2$ at the equator and $g \approx 9.83 \text{m/s}^2$ at the poles.

Summary on variation of g

As distance from the centre of the Earth **increases from zero to** r_E (moving from centre to surface): g increases linearly from 0 to g.

As distance from the centre **increases beyond** r_E (moving above the surface): g decreases as $\frac{1}{r^2}$.

The maximum value of g (under the uniform density assumption) occurs at the surface. A graph of g versus distance from the centre rises linearly to a peak at $r = r_E$ and then falls off as an inverse square curve.

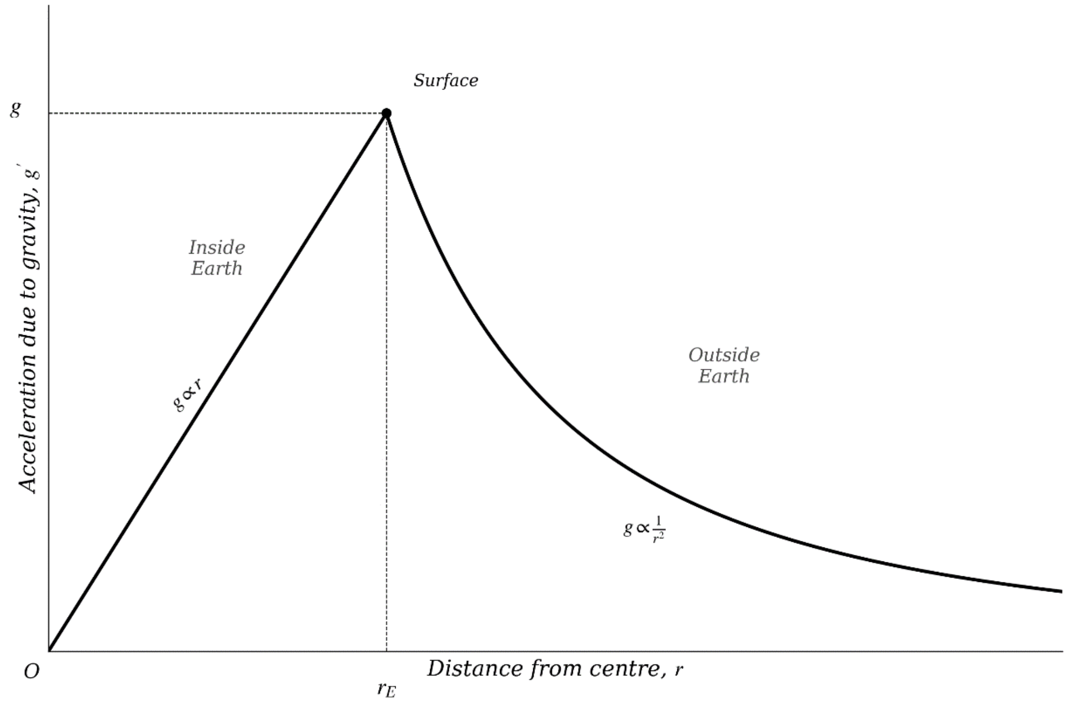


Figure: ***Variation of the acceleration due to gravity with distance from the centre of the Earth (assuming uniform density).*** *Inside the Earth, g increases linearly with distance ($g \propto r$). Outside the Earth, g decreases with the inverse square of distance ($g \propto 1/r^2$). The maximum value occurs at the surface ($r = r_E$).*

Let us now apply these ideas through worked examples.

BINDER Example 7

(a) Calculate the acceleration due to gravity on the surface of Mars. Take the mass of Mars as 6.4×10^{23}kg, the radius of Mars as 3.4×10^6m, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) A person weighs 700N on Earth. What would this person weigh on Mars?

Solution

(a) Using:

$$g_{\text{Mars}} = \frac{GM_{\text{Mars}}}{r^2_{\text{Mars}}} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.4 \times 10^{23}\text{kg}}{(3.4 \times 10^6\text{m})^2} = 3.69\text{m/s}^2$$

The acceleration due to gravity on Mars is 3.69m/s^2.

(b) The person's mass: $m = \frac{W_{\text{Earth}}}{g_{\text{Earth}}} = \frac{700\text{N}}{9.8\text{m/s}^2} = 71.4\text{kg}$

Weight on Mars: $W_{\text{Mars}} = mg_{\text{Mars}} = 71.4\text{kg} \times 3.69\text{m/s}^2 = 263\text{N}$

The person would weigh 263N on Mars.

Making Sense of the Answer: *Mars has about 38% of Earth's surface gravity. The person feels roughly a third of their Earth weight. This is why Mars rovers can make soft landings with less braking than would be needed on Earth, but enough gravity exists to keep objects firmly on the surface.*

Think Like a Physicist: *Notice that Mars is both less massive **and** smaller than Earth. Less mass reduces g, but a smaller radius increases g (because you are closer to the centre). The net effect depends on which factor wins. For Mars, the mass effect dominates, giving a lower g.*

BINDER Example 8

A body weighing 72N on the surface of the Earth is taken to a height equal to half the radius of the Earth above the surface. Calculate the new weight. Take $r_E = 6400\text{km}$.

Solution

At height $h = \frac{r_E}{2}$, the distance from the centre of the Earth becomes:

$$R = r_E + h = r_E + \frac{r_E}{2} = \frac{3}{2}r_E$$

Using the exact formula:

$$g' = g\left(\frac{r_E}{r_E + h}\right)^2 \text{ or } \frac{g'}{g} = \left(\frac{r_E}{r_E + h}\right)^2 = \left(\frac{r_E}{R}\right)^2$$

The above equation is the same as writing as:

$$\frac{mg'}{mg} = \left(\frac{r_E}{r_E + h}\right)^2 = \left(\frac{r_E}{R}\right)^2$$

Where: $mg' = W'$, $mg = W$. Thus:

$$\frac{W'}{W} = \left(\frac{r_E}{R}\right)^2 = \left(\frac{r_E}{\frac{3}{2}r_E}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Therefore:

$$W' = \frac{4}{9} \times W = \frac{4}{9} \times 72\text{N} = 32\text{N}$$

The new weight is 32N.

Making Sense of the Answer: *Moving to* $1.5r_E$ *from the centre reduces the weight to* $\frac{4}{9}$ *of the surface value which is less than half. The inverse square law is powerful: even a modest increase in distance produces a significant drop in gravitational force.*

Think Like a Physicist: *The approximate formula* $g' \approx g\left(1 - \frac{2h}{r_E}\right)$ *would give* $g' = g(1 - 1) = 0$*, which is clearly wrong. The approximation is only valid when* $h \ll r_E$*. Here* $h = 0.5r_E$*, which is not small, so the exact formula is essential.*

REAL Example 9

A gold dealer in Dar es Salaam (near sea level) uses a precision digital scale to weigh a gold bar, and reads 500.00g. The same gold bar is taken to a dealer in Mbeya (altitude approximately 1700m) and placed on an identical scale. Explain whether the reading in Mbeya will be the same, and if not, which reading will be higher.

Solution

The reading will be slightly different. A precision scale measures the gravitational force (weight) on the object and converts it to a mass reading using a calibrated value of g.

At higher altitude, g is smaller because the object is farther from the centre of the Earth. With a lower g, the gravitational force on the gold bar is slightly less, so the scale in Mbeya reads a slightly lower value than the scale in Dar es Salaam. **The Dar es Salaam reading will be higher.**

(Using the approximate formula: $\frac{\Delta g}{g} \approx \frac{2h}{r_E} = \frac{2\times1700\text{m}}{6.4\times10^6\text{m}} = 0.00053 = 0.053\%$. The difference is tiny (about 0.27g on a 500g bar), but it is real and measurable with a precision scale.)

Making Sense of the Answer: *For everyday trade this difference is negligible. But for high-precision scientific work, for gold trading in large quantities, or for pharmaceutical measurements, the variation of* g *with altitude matters. A scale calibrated in Dar es Salaam is not perfectly accurate in Mbeya.*

Think Like a Physicist: *A balance (two-pan) scale would give the same reading at both locations, because it compares masses, not weights. Both sides are affected equally by the change in* g. *A spring scale or digital scale, which measures force, is the one affected by altitude.*

HOT Example 10

At what height above the Earth's surface is the acceleration due to gravity reduced by 36% from its surface value? Take $r_E = 6400\text{km}$.

Solution

If g is reduced by 36%, the remaining value is 64% of g:

$$g' = 0.64g$$

Using the exact formula:

$$g' = g\left(\frac{r_E}{r_E + h}\right)^2$$

Substituting $g' = 0.64g$:

$$0.64g = g\left(\frac{r_E}{r_E + h}\right)^2$$

Dividing both sides by g:

$$0.64 = \left(\frac{r_E}{r_E + h}\right)^2$$

Taking the square root of both sides:

$$\sqrt{0.64} = \frac{r_E}{r_E + h}$$

$$0.8 = \frac{r_E}{r_E + h}$$

Making $(r_E + h)$ the subject:

$$r_E + h = \frac{r_E}{0.8} = 1.25r_E$$

$$h = 1.25r_E - r_E = 0.25r_E = 0.25 \times 6400\text{km} = 1600\text{km}$$

The height is 1600km above the surface.

Making Sense of the Answer: *Commercial aircraft fly at about* $10km$ *which is far too low for any noticeable change in g. Even the International Space Station at* $400km$ *experiences only about an* 11% *reduction. Gravity does not "switch off" in space; it simply weakens with distance.*

Think Like a Physicist: *Notice that* $0.64 = 0.8^2$, *so the distance from the centre need only increase by a factor of* $\frac{1}{0.8} = 1.25$. *Recognising perfect squares speeds up problem solving enormously.*

HOT Example 11

(a) A planet has the same mass as the Earth but half the radius. Calculate the acceleration due to gravity on the surface of this planet in terms of g.

(b) A planet has the same density as the Earth but twice the radius. Show that the acceleration due to gravity on its surface is 2g.

Solution

(a) Using $g_p = \frac{GM}{r_p^2}$, with $M = M_E$ and $r_p = \frac{r_E}{2}$:

$$g_p = \frac{GM_E}{\left(\frac{r_E}{2}\right)^2} = \frac{GM_E}{\frac{r_E^2}{4}} = 4 \times \frac{GM_E}{r_E^2} = 4 \times g$$

The surface gravity is 4g.

(b) For a planet with the same density ρ but radius $r_p = 2r_E$:

The mass of the planet is:

$$M_p = \rho \times \frac{4}{3}\pi r_p^3 = \rho \times \frac{4}{3}\pi(2r_E)^3 = \rho \times \frac{4}{3}\pi \times 8r_E^3 = 8\left(\rho \times \frac{4}{3}\pi r_E^3\right) = 8M_E$$

The surface gravity is:

$$g_p = \frac{GM_p}{r_p^2} = \frac{G \times 8M_E}{(2r_E)^2} = \frac{8GM_E}{4r_E^2} = 2 \times \frac{GM_E}{r_E^2} = 2g$$

The surface gravity is 2g.

Making Sense of the Answer: *Part (a): same mass squeezed into a smaller sphere means the surface is closer to the centre, dramatically increasing g. Part (b): same density but larger radius means more mass, but the surface is also farther from the centre. The mass grows as* r^3 *while g divides by* r^2*, leaving a net factor of r. Double the radius, double the g.*

Think Like a Physicist: *When comparing planets, always check what is held constant. Same mass, different radius:* $g \propto \frac{1}{r^2}$*. Same density, different radius:* $g \propto r$*. Same radius, different mass:* $g \propto M$*. Each gives a completely different scaling.*

HOT Example 12

Calculate the percentage decrease in the weight of a body when it is taken to a depth of 64km below the Earth's surface. Take $r_E = 6400\text{km}$.

Solution

Using the depth formula (assuming uniform density):

$$g'' = g\left(1 - \frac{d}{r_E}\right)$$

Substituting $d = 64\text{km}$ and $r_E = 6400\text{km}$:

$$g'' = g\left(1 - \frac{64\text{km}}{6400\text{km}}\right) = g(1 - 0.01) = 0.99g$$

The fractional decrease in g (and therefore in weight) is:

$$\frac{g - g''}{g} = \frac{g - 0.99g}{g} = 0.01 = 1\%$$

The weight decreases by 1%.

Making Sense of the Answer: 64*km is* 1% *of the Earth's radius, and the weight decreases by* 1%*. This is the beauty of the linear relationship: the percentage decrease in g at depth equals the percentage of the radius descended.*

Think Like a Physicist: *Compare altitude and depth at the same distance of* 64*km. At* 64*km altitude:* $g' \approx g\left(1 - \frac{2\times64}{6400}\right) = g(1 - 0.02) = 0.98g$*, a* 2% *decrease. At* 64*km depth: only a* 1% *decrease. Gravity decreases* ***twice as fast*** *with altitude as with depth (for small distances), because the altitude formula has a factor of* **2** *that the depth formula does not.*

HOT Example 13

Calculate the effective value of the acceleration due to gravity at the equator, taking into account the Earth's rotation. Take $g = 9.8\text{m/s}^2$, $r_E = 6.4 \times 10^6\text{m}$, and the Earth completes one rotation in 24 hours.

Solution

Calculating the angular velocity of the Earth:

$$\omega = \frac{2\pi}{\mathrm{T}} = \frac{2 \times 3.14}{24 \times 3600\mathrm{s}} = 7.27 \times 10^{-5}\mathrm{rad/s}$$

At the equator ($\theta = 0°$, so $\cos\theta = 1$), the effective gravity is:

$$\mathrm{g}' = \mathrm{g} - \omega^2 \mathrm{r_E} \cos^2\theta = \mathrm{g} - \omega^2 \mathrm{r_E}$$

Calculating $\omega^2 r_E$:

$$\omega^2 \mathrm{r_E} = (7.27 \times 10^{-5}\mathrm{rad/s})^2 \times 6.4 \times 10^6\mathrm{m} = 0.0338\mathrm{m/s^2}$$

Therefore:

$$\mathrm{g}' = 9.8\mathrm{m/s^2} - 0.034\mathrm{m/s^2} = 9.766\mathrm{m/s^2}$$

The effective g at the equator is $9.766\mathrm{m/s^2}$, (a reduction of $0.034\mathrm{m/s^2}$ (about 0.35%) due to the Earth's rotation).

Making Sense of the Answer: *The effect is tiny (about a third of one percent). A* $70kg$ *person would weigh about* $0.24N$ *less at the equator than at the pole. This is negligible for everyday purposes, but it matters for precision instruments and for satellite launches from equatorial sites.*

Think Like a Physicist: *If the Earth rotated fast enough that* $\omega^2 \mathrm{r_E} = \mathrm{g}$, *objects at the equator would be weightless. This would require a rotation period of about* 84 *minutes, which is the same as the orbital period of a low-Earth satellite. This is not a coincidence.*

The variation of g with altitude, depth, and latitude reveals that gravity is richer and more subtle than the simple constant we used in Chapter 2. In the next subtopic, we formalise this by introducing the concept of a gravitational field.

GRAVITATIONAL FIELD AND POTENTIAL

So far, we have described gravity as a force between two masses. But there is another way to think about it, one that becomes essential in more advanced physics. Instead of asking *"what force does mass A exert on mass B?"*, we ask *"what has mass A done to the space around it?"*

The answer: mass A has created a **gravitational field** which may be defined as *a region of space where any other mass placed in it will experience a gravitational force*. The field exists whether or not a second mass is present to feel it. The Earth's gravitational field fills the space around it; the Moon, satellites, and falling apples merely respond to the field that is already there.

Gravitational Field Strength

Imagine placing a small test mass **m** at some point in space near a large mass **M**. The test mass experiences a gravitational force **F** directed toward M. The **gravitational field strength** at that point is defined as *the gravitational force per unit mass experienced by a small test mass placed (at that point):*

$$\mathbf{g} = \frac{\mathbf{F}}{\mathbf{m}}$$

The SI unit of gravitational field strength is N/kg.

Since $\frac{\mathrm{F}}{\mathrm{m}}$ also equals acceleration (by Newton's second law), gravitational field strength is numerically equal to the acceleration due to gravity at that point. The units N/kg and $\mathrm{m/s^2}$ are therefore equivalent. The two quantities: field strength (N/kg) and acceleration ($\mathrm{m/s^2}$) have the same value, but they represent different physical ideas. *Field strength describes a property of the field while acceleration describes the response (resulting motion) of a mass placed in the field.*

For a point at distance **r** from the centre of a uniform sphere of mass **M**, the gravitational force on the test mass is $\mathrm{F} = \frac{\mathrm{GMm}}{\mathrm{r^2}}$ (by Newton's law). Dividing by m:

$$\mathbf{g} = \frac{\mathrm{F}}{\mathrm{m}} = \frac{\mathbf{GM}}{\mathbf{r^2}}$$

This is the same expression we derived in the previous subtopic for the acceleration due to gravity. But now we interpret it differently: it describes a property of the field created by M, independent of whatever test mass we choose to place in it.

The gravitational field strength is a **vector** quantity. It points in the direction of the force that a test mass would experience, that is, toward the mass creating the field.

Gravitational field lines

Gravitational field lines are a visual tool for representing the direction and relative strength of a gravitational field.

Rules for field lines:

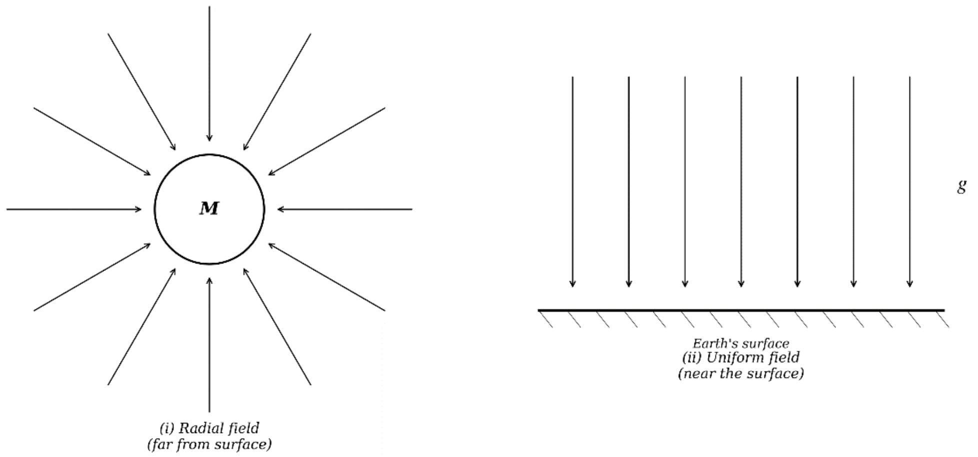
Rule 1: The direction of a field line at any point gives the direction of the gravitational field (and therefore the direction of the force on a test mass) at that point. Field lines for gravity always point **toward** the mass creating the field, because gravity is always attractive.

Rule 2: The spacing between field lines indicates the strength of the field: closer lines mean a stronger field; wider spacing means a weaker field.

How these features appear depends on the situation, as shown in the two cases below:

Radial field: Far from a spherical mass (like the Earth seen from space), the field lines point radially inward toward the centre. They spread out with distance, reflecting the inverse square decrease in field strength. This is called a **radial field**.

Uniform field: Very close to the Earth's surface (over a small area), the field lines are approximately parallel, equally spaced, and pointing vertically downward. The field is approximately **uniform** (the same strength and direction everywhere in the region). This is why we treat g as a constant in most problems near the surface.



(i) Radial field (far from surface)

(ii) Uniform field (near the surface)

Figure: ***Gravitational field lines.*** *(i) Radial field around a spherical mass M: field lines point radially inward from all directions, spreading out with distance, reflecting the inverse square decrease in field strength. (ii) Uniform field near the Earth's surface: over a small area, the field lines are parallel, equally spaced, and point vertically downward, giving an approximately constant field strength g.*

Gravitational Potential

Gravitational field strength tells us about forces. But in many problems especially those involving energy, orbits, and escape velocity, we need a quantity that describes **energy** rather than force. That quantity is **gravitational potential**.

The **gravitational potential V** at a point in a gravitational field is defined as *the work done per unit mass in bringing a small test mass from infinity to that point:*

$$V = \frac{W_{\infty \to \text{point}}}{m}$$

To understand gravitational potential, consider the work required to move a mass through a gravitational field. Suppose we want to move a small mass m from a point A to a point B in the gravitational field of a large mass M. The work done against gravity depends on the positions of A and B, not on the path taken (because gravity is a **conservative force**). This means we can define a quantity at each point in the field that determines the work done in moving between any two points.

To derive the expression for **V**, consider bringing a test mass **m** from infinity to a point at distance **r** from the centre of mass **M**.

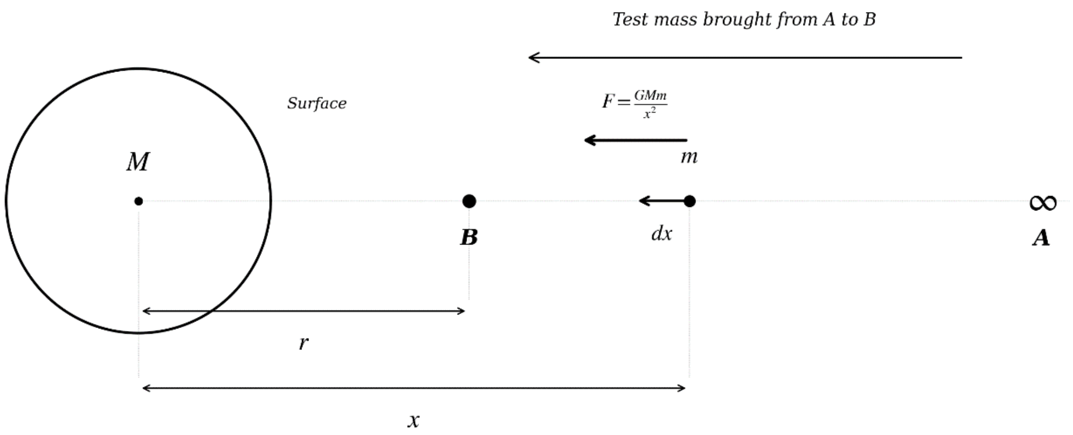


Figure: *A test mass m is brought from point A (at infinity, where V = 0) to point B (at distance r from the centre of mass M). At an intermediate position, the test mass is at distance x from the centre and experiences a gravitational force* $F = \frac{GMm}{x^2}$ *directed toward M. The small displacement dx is directed inward. The total work done by gravity as the mass moves from A to B gives the gravitational potential at B.*

At some intermediate position, at distance **x** from the centre, the gravitational force on the test mass is:

$$F = \frac{GMm}{x^2}$$

The work done by gravity in moving the test mass through a small displacement dx (inward, toward M) is dW = Fdx. The total work done by gravity in bringing the mass from infinity to distance r is:

$$W = \int_{\infty}^{r} \frac{GMm}{x^2} dx = GMm\left[-\frac{1}{x}\right]_{\infty}^{r} = GMm\left(-\frac{1}{r} + \frac{1}{\infty}\right) = -\frac{GMm}{r}$$

The work done per unit mass is:

$$V = \frac{W}{m} = -\frac{GM}{r}$$

Hence, the gravitational potential at distance r from the centre of a uniform sphere of mass M is given by:

$$\mathbf{V = -\frac{GM}{r}}$$

The SI unit of gravitational potential is J/kg (joules per kilogram).

Why is the potential negative?

The negative sign deserves careful thought, because it confuses many students.

The reference point is infinity, where the potential is defined as zero (when $r = \infty$, gravitational potential formula gives V=0). Now consider what happens as a mass moves from infinity toward a planet. Gravity pulls the mass inward, doing positive work on it. The mass speeds up, gaining kinetic energy. But this kinetic energy comes at the expense of potential energy, which decreases. Since the potential energy at infinity was zero, and the potential energy has decreased, it must now be negative.

A more negative potential means the point is deeper inside the gravitational field. The surface of the Earth has a more negative potential than a point high above it, and the centre of the Earth has the most negative potential of all.

To move a mass upward (away from the planet), you must do work against gravity. This work increases the potential energy and makes the potential less negative. At infinity, the potential returns to zero and the mass is completely free of the gravitational field.

Think of it this way: the gravitational potential tells you how much energy per kilogram you would need to supply to completely free an object from the gravitational field and send it to infinity. At the Earth's surface, this value is about 6.25×10^7J/kg, an enormous amount of energy per kilogram. This is why escaping the Earth's gravity requires such powerful rockets.

To move a mass **away** from a planet (upward), work must be done **against** gravity. This increases the potential (makes it less negative). At infinity, the potential returns to zero.

Gravitational potential energy

The gravitational potential V is a property of the field at a point (energy per unit mass). When a mass m is actually placed at that point, the gravitational potential energy of the system results.

Gravitational potential energy U *is the energy of a* ***system*** *of two masses due to their gravitational interaction*. For a mass **m** at distance **r** from the centre of mass **M**:

$$U = mV = -\frac{GMm}{r}$$

The gravitational potential energy is negative for the same reason the gravitational potential is negative: the system is bound, and energy must be supplied to separate the masses to infinity.

It is important to note that this formula replaces the familiar $U = mgh$ from earlier chapters. The formula mgh is an approximation that assumes g is constant, which is valid only near the Earth's surface over **small** height changes. The general formula $U = -\frac{GMm}{r}$ works at all distances and accounts for the variation of g with position.

The connection between the two can be seen as follows. For a given height h above the surface, the change in potential energy is:

$$\Delta U = \text{PE at h} - \text{PE at earth surface } (h = 0)$$

But:

$$\text{PE at h} = -\frac{GMm}{r_E + h}$$

$$\text{PE at earth surface} = -\frac{GMm}{r_E}$$

It follows that:

$$\Delta U = -\frac{GMm}{r_E + h} - \left(-\frac{GMm}{r_E}\right) = GMm\left(\frac{1}{r_E} - \frac{1}{r_E + h}\right) = GMm\left(\frac{h}{r_E(r_E + h)}\right)$$

For $h \ll r_E$, the denominator becomes approximately r_E^2. Thus:

$$\Delta U \approx \frac{GMmh}{r_E^2}$$

But:

$$\frac{GM}{r_E^2} = g$$

Hence, over small height changes:

$$\boldsymbol{\Delta U \approx mgh}$$

So mgh is indeed a special case of the general formula, valid when the height is small compared to the Earth's radius.

Relationship Between Field Strength and Potential

Gravitational field strength and gravitational potential are two descriptions of the same field, one based on force and the other on energy. They are connected by a precise mathematical relationship.

Gravitational field strength and gravitational potential are two descriptions of the same field, one based on force and the other on energy. They are connected by a precise mathematical relationship.

Recall that the potential is defined through work: $V = -\frac{GM}{r}$. If we differentiate this with respect to r:

$$\frac{dV}{dr} = \frac{d}{dr}\left(-\frac{GM}{r}\right) = \frac{GM}{r^2}$$

But we know that $\frac{GM}{r^2} = g$. Therefore:

$$\frac{dV}{dr} = g$$

However, there is a sign to consider. The field strength points in the direction of *decreasing* potential (toward the mass, where V becomes more negative). Since r increases **outward** (away from the mass), and g points **inward** (opposite to displacement, r), the correct relationship including direction is:

$$\mathbf{g = -\frac{dV}{dr}}$$

In words: the gravitational field strength at any point is equal to the negative of the rate of change of potential with distance at that point. The field points in the direction of steepest decrease of potential, and its magnitude equals the rate at which the potential decreases.

This is analogous to a ball on a hill. The ball accelerates in the direction of steepest descent (where height decreases fastest), and the steeper the slope, the greater the acceleration. In gravitation, the “height” is the potential, the “acceleration” is the field strength ,and the “steepness” is $\frac{dV}{dr}$.

On a graph of V against r, the gradient (slope) at any point gives the field strength at that distance. At the Earth’s surface, the slope is steep (strong field). Far from the Earth, the curve flattens (weak field). At infinity, the slope is zero and the field vanishes.

Variation of potential with distance

Outside the Earth ($r > r_E$), the potential follows:

$$V = -\frac{GM_E}{r}$$

This implies that the potential is increasing (becoming less negative) as r increases and approaching zero as $r \to \infty$.

At the surface ($r = r_E$):

$$V_{\text{surface}} = -\frac{GM_E}{r_E}$$

Inside the Earth (assuming uniform density), the potential continues to decrease toward the centre. It can be shown that the potential inside a uniform sphere is:

$$V = -\frac{GM_E}{2r_E}\left(3 - \frac{r^2}{r_E^2}\right)$$

At the centre ($r = 0$):

$$V_{\text{centre}} = -\frac{3GM_E}{2r_E}$$

This is 1.5 times more negative than at the surface. The centre of the Earth is the deepest point in the gravitational potential “well.”

At the surface ($r = r_E$):

$$V_{\text{surface}} = -\frac{GM_E}{r_E}$$

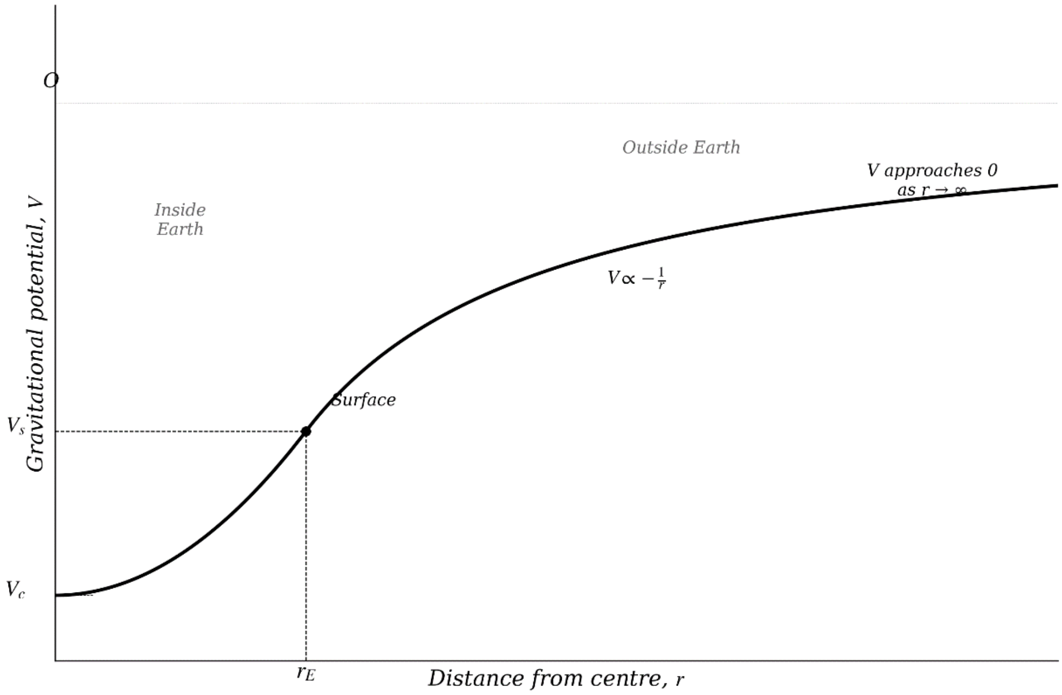


Figure: ***Variation of gravitational potential V with distance r from the centre of the Earth.*** *Inside the Earth, V decreases from V_s at the surface to $V_c = -\frac{3GM_E}{2r_E}$ at the centre. Outside the Earth, V increases (becomes less negative) as 1/r, approaching zero as r → ∞. The potential is negative everywhere, reflecting the work needed to escape the gravitational field.*

With this solid theoretical foundation in place, we are now ready to see these ideas come alive through carefully chosen worked examples.

BINDER Example 14

(a) Calculate the gravitational field strength at a height of 200km above the Earth's surface. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) Compare this with the value at the surface.

Solution

(a) The distance from the centre of the Earth is:

$$R = r_E + h = 6.4 \times 10^6\text{m} + 200 \times 10^3\text{m} = 6.6 \times 10^6\text{m}$$

$$g = \frac{GM_E}{R^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{(6.6 \times 10^6\text{m})^2} = \frac{4.0 \times 10^{14}\text{Nm/kg}}{4.356 \times 10^{13}\text{m}^2} = 9.19\text{N/kg}$$

(b) At the surface: $g_{\text{surface}} = \frac{4.0\times10^{14}}{(6.4\times10^6)^2} = \frac{4.0\times10^{14}}{4.096\times10^{13}} = 9.77\text{N/kg} > 9.19\text{N/kg}$

The field strength at 200km altitude is smaller than the value at the surface (about 94% of the surface value).

Making Sense of the Answer: 200km *seems like a great height, but it is only about* 3% *of the Earth's radius. Since* g *decreases as* $\frac{1}{r^2}$ *from the centre, a* 3% *increase in* r *produces only about a* 6% *decrease in* g.

Think Like a Physicist: *This result explains why the "weightlessness" of astronauts in orbit is not due to being far from Earth. They are in a gravitational field almost as strong as the one you are sitting in right now. Their apparent weightlessness comes from the absence of a contact force, not the absence of gravity.*

REAL Example 15

A well is dug to a depth of 3km below the Earth's surface. Explain whether the gravitational potential at the bottom of the well is higher or lower (more negative or less negative) than at the surface.

Solution

The gravitational potential at the bottom of the well is **lower** (more negative) than at the surface. Moving from the surface toward the centre of the Earth means moving deeper into the gravitational field. Gravity does positive work as the object descends, meaning the object loses potential energy. Since potential energy decreases, and $U = mV$, the potential V also decreases (becomes more negative).

To climb back out of the well, work must be done against gravity, which increases the potential back to its surface value.

Making Sense of the Answer: *Think of the gravitational field as a "well" or valley. The surface is already deep in the well (large negative potential). Going deeper makes the potential even more negative. Going upward (increasing altitude) makes it less negative, approaching zero at infinity.*

Think Like a Physicist: *The sign convention is consistent: potential decreases in the direction of the gravitational force (downward), and increases in the opposite direction (upward). This matches* $g = -\frac{dV}{dr}$*; the field points in the direction of decreasing potential.*

HOT Example 16

(a) Calculate the gravitational potential at the surface of the Earth. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) Calculate the gravitational potential at a height of 36000km above the surface (geostationary orbit altitude).

(c) Find the work done per unit mass in moving an object from the Earth's surface to the geostationary orbit.

Solution

(a) The gravitational potential is given by:

$$V_{\text{surface}} = -\frac{GM_E}{r_E} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{6.4 \times 10^6\text{m}} = -6.25 \times 10^7\text{J/kg}$$

The gravitational potential at the surface is -6.25×10^7J/kg.

(b) The distance from the centre to the geostationary orbit:

$$R = r_E + h = 6.4 \times 10^6\text{m} + 36.0 \times 10^6\text{m} = 42.4 \times 10^6\text{m}$$

$$V_{\text{geo}} = -\frac{GM_E}{R} = -\frac{4.0 \times 10^{14}\text{Nm/kg}}{42.4 \times 10^6\text{m}} = -9.43 \times 10^6\text{J/kg}$$

The gravitational potential at the geostationary orbit is -9.43×10^6J/kg.

(c) The **work done per unit mass** *equals the change in gravitational potential:*

$$\frac{\mathbf{W}}{\mathbf{m}} = \mathbf{V}_{\mathbf{geo}} - \mathbf{V}_{\mathbf{surface}} = (-9.43 \times 10^6\text{J/kg}) - (-6.25 \times 10^7\text{J/kg})$$

$$\frac{W}{m} = -9.43 \times 10^6\text{J/kg} + 6.25 \times 10^7\text{J/kg} = 5.31 \times 10^7\text{J/kg}$$

The work done is 5.31×10^7J/kg.

Making Sense of the Answer: *The positive answer confirms that work must be done to move the object upward (against gravity). For a* $1000kg$ *satellite, this is* $5.31 \times 10^{10}J$ *(*5.31×10^7 *x1000). This enormous energy requirement is why rocket launches are so expensive and why most of a rocket's mass is fuel.*

Think Like a Physicist: *The work done equals the difference in potential, not the potential itself. This is why the reference point (infinity) does not matter for practical calculations; only the difference between two points matters, and the reference cancels out.*

HOT Example 17

There is a point along the line joining the centres of the Earth and the Moon where the gravitational field strength due to both bodies is zero. Calculate the distance of this point from the centre of the Earth. Take $\mathrm{M_E} = 6.0 \times 10^{24}\mathrm{kg}$, $\mathrm{M_M} = 7.35 \times 10^{22}\mathrm{kg}$, and the Earth-Moon distance $\mathrm{D} = 3.84 \times 10^{8}\mathrm{m}$.

Solution

Let the point be at distance x from the centre of the Earth. Then it is at distance $(\mathrm{D} - \mathrm{x})$ from the centre of the Moon.

At this point, the gravitational field strength due to the Earth (pointing toward Earth) equals the field strength due to the Moon (pointing toward Moon):

$$\frac{\mathrm{GM_E}}{\mathrm{x}^2} = \frac{\mathrm{GM_M}}{(\mathrm{D} - \mathrm{x})^2}$$

G cancels:

$$\frac{\mathrm{M_E}}{\mathrm{x}^2} = \frac{\mathrm{M_M}}{(\mathrm{D} - \mathrm{x})^2}$$

Taking square roots of both sides:

$$\frac{\sqrt{\mathrm{M_E}}}{\mathrm{x}} = \frac{\sqrt{\mathrm{M_M}}}{\mathrm{D} - \mathrm{x}}$$

Cross-multiplying:

$$(\mathrm{D} - \mathrm{x})\sqrt{\mathrm{M_E}} = \mathrm{x}\sqrt{\mathrm{M_M}}$$

$$\mathrm{D}\sqrt{\mathrm{M_E}} = \mathrm{x}\sqrt{\mathrm{M_M}} + \mathrm{x}\sqrt{\mathrm{M_E}} = \mathrm{x}\left(\sqrt{\mathrm{M_M}} + \sqrt{\mathrm{M_E}}\right)$$

$$\mathrm{x} = \frac{\mathrm{D}\sqrt{\mathrm{M_E}}}{\sqrt{\mathrm{M_E}} + \sqrt{\mathrm{M_M}}}$$

Substituting:

$$\mathrm{x} = \frac{3.84 \times 10^{8}\mathrm{m}\sqrt{6.0 \times 10^{24}\mathrm{kg}}}{\sqrt{6.0 \times 10^{24}\mathrm{kg}} + \sqrt{7.35 \times 10^{22}}} = 3.46 \times 10^{8}\mathrm{m}$$

The point of zero gravitational field is $3.46 \times 10^{8}\mathrm{m}$ from the centre of the Earth (about 90% of the way to the Moon).

Making Sense of the Answer: *The neutral point is much closer to the Moon than to the Earth, because the Earth is far more massive. The Moon's weaker field can only match the Earth's strong field at a point very close to the Moon, where the Moon's inverse-square field is large enough to compensate for its smaller mass.*

Think Like a Physicist: *This is a point of zero field, not zero potential. The potential at this point is not zero, it is still negative (both the Earth and Moon contribute negative potential). A spacecraft at this point feels no net gravitational force, but it is not free from the gravitational influence of either body.*

With the concepts of gravitational field strength and potential now in place, we have two complementary descriptions of gravity: one based on force (field strength) and one based on energy (potential). Both will be essential in the next section, where we apply them to the physics of orbits and escape velocity.

SATELLITES AND ORBITAL MOTION

Everything we have built in this chapter now comes together. Newton's law of gravitation provides the force. Circular motion from Chapter 7 provides the framework. Gravitational potential from the previous section provides the energy tools. In this section, we combine all three to understand how satellites orbit, what determines their speed and period, how much energy they carry, and what makes the geostationary orbit so special.

A **satellite** *is any object that orbits another object under the influence of gravity*. The Moon is a natural satellite of the Earth. The Earth is a natural satellite of the Sun. **Artificial satellites** *are man-made objects* *placed into orbit around the Earth (or other bodies) for communication, navigation, weather monitoring, scientific research, and military purposes.*

Orbital Velocity

Consider a satellite of mass m in a circular orbit of radius R around the Earth (mass M_E). The orbital radius R is measured from the centre of the Earth, so $R = r_E + h$, where h is the height above the surface.

The satellite is in circular motion. From Chapter 7, circular motion requires a centripetal force directed toward the centre:

$$F_{centripetal} = \frac{mv^2}{R}$$

This centripetal force is provided entirely by gravity:

$$F_{gravity} = \frac{GM_Em}{R^2}$$

Setting them equal:

$$\frac{mv^2}{R} = \frac{GM_Em}{R^2}$$

The mass of the satellite m cancels from both sides. This is a result of deep significance: **the orbital velocity does not depend on the mass of the satellite.** A communication satellite of 500kg and a space station of 400000kg at the same altitude orbit at exactly the same speed.

Solving for v:

$$v^2 = \frac{GM_E}{R}$$

$$\mathbf{v = \sqrt{\frac{GM_E}{R}}}$$

Since $R = r_E + h$, the orbital velocity decreases as the height increases. A satellite in a low orbit moves faster than one in a high orbit. This makes physical sense: *a low satellite is deeper in the gravitational field, where the pull is stronger, so it needs a higher speed to avoid falling inward.*

For a satellite very close to the surface ($h \ll r_E$, so $R \approx r_E$), and using $GM_E = gr_E^2$:

$$\mathbf{v} = \sqrt{\frac{gr_E^2}{r_E}} = \sqrt{\mathbf{gr_E}} = \sqrt{9.8\text{m/s}^2 \times 6.4 \times 10^6\text{m}} = \mathbf{7920m/s \approx 7.9km/s}$$

This is the minimum orbital velocity for any satellite near the Earth's surface: about 7.9km/s, or roughly 28500km/h. Faster than a bullet, faster than any aircraft, and roughly 23 times the speed of sound.

Orbital Period

The period T is the time for one complete orbit. Since the satellite travels a distance of $2\pi R$ (the circumference) at speed v:

$$T = \frac{2\pi R}{v}$$

Substituting $v = \sqrt{\frac{GM_E}{R}}$:

$$T = \frac{2\pi R}{\sqrt{\frac{GM_E}{R}}} = 2\pi R \times \sqrt{\frac{R}{GM_E}} = 2\pi\sqrt{\frac{R^3}{GM_E}}$$

$$\mathbf{T = 2\pi\sqrt{\frac{R^3}{GM_E}}}$$

Squaring both sides gives $T^2 = \frac{4\pi^2}{GM_E}R^3$, which is Kepler's third law. We have now come full circle (no pun intended): the law Kepler discovered empirically follows directly from Newton's gravitation combined with circular motion.

The period increases with orbital radius. Low satellites orbit quickly; high satellites orbit slowly. For a satellite near the surface ($R \approx r_E$): Again using $GM_E = gr_E^2$:

$$T = 2\pi\sqrt{\frac{(r_E)^3}{gr_E^2}} = 2\pi\sqrt{\frac{r_E}{g}} = 2 \times 3.14 \times \sqrt{\frac{6.4 \times 10^6\text{m}}{9.8\text{m/s}^2}} = 5074\text{s} \approx 85\text{min}$$

The shortest possible orbital period for satellite near the Earth's surface is about 85 minutes. Any orbit with a longer period must be at a greater height.

Energy of a Satellite in Orbit

A satellite in circular orbit possesses two forms of energy: kinetic energy (because it moves) and gravitational potential energy (because it is in a gravitational field). Understanding the relationship between these energies reveals something beautiful and surprising.

Kinetic energy (for simplicity, we will use **K** to denote it and not the usual **KE):**

$$K = \frac{1}{2}mv^2$$

From the orbital velocity result,

$$v^2 = \frac{GM_E}{R}$$

Thus:

$$\mathbf{K} = \frac{1}{2}m \times \frac{GM_E}{R} = \mathbf{\frac{GM_Em}{2R}}$$

Notice that KE is **positive** (as it must be, since kinetic energy is always positive).

Gravitational potential energy:

We have derived that:

$$U = -\frac{GM_Em}{R}$$

Notice that U is **negative** (the satellite is attracted to the Earth).

Total mechanical energy:

$$E = K + U = \frac{GM_Em}{2R} - \frac{GM_Em}{R} = -\frac{GM_Em}{2R}$$

$$\mathbf{E = -\frac{GM_Em}{2R}}$$

The total energy is **negative**. This is the hallmark of a bound (attraction) system. A satellite with negative total energy cannot escape to infinity; it is gravitationally attracted (bound) to the Earth. To free the satellite completely (send it to infinity with zero residual speed), you would need to supply energy equal to:

$$|E| = \frac{GM_Em}{2R}$$

The remarkable relationships:

Looking at the three energies together:

$$\mathbf{K = \frac{GM_Em}{2R}}, \qquad \mathbf{U = -\frac{GM_Em}{R}}, \qquad \mathbf{E = -\frac{GM_Em}{2R}}$$

Several elegant patterns emerge:

1. **The kinetic energy is exactly half the magnitude of the potential energy:** $K = -\frac{1}{2}U$
2. **The total energy equals the negative of the kinetic energy:** $E = -K$
3. **The total energy equals half the potential energy:** $E = \frac{1}{2}U$

These are not coincidences. They are direct consequences of the inverse square nature of gravity combined with circular motion. They hold for any circular orbit, around any central body.

What happens when a satellite gains or loses energy?

If a satellite gains energy (for example, by firing its engines), the total energy E becomes less negative. Since $E = -\frac{GM_Em}{2R}$, a less negative E means a larger R. The satellite moves to a **higher orbit**. Counterintuitively, it also moves **slower** (since $v = \sqrt{\frac{GM_E}{R}}$ and R has increased).

If a satellite loses energy (for example, through atmospheric drag), E becomes more negative, R decreases, and the satellite drops to a **lower orbit** where it moves **faster**. This is the paradox of orbital mechanics: air resistance (drag) slows a satellite down in one sense (it loses energy), but speeds it up in another sense (it moves to a lower, faster orbit). Eventually, if enough energy is lost, the satellite spirals inward and burns up in the atmosphere.

The Geostationary Orbit

Among all possible orbits, one is particularly useful: the orbit where the satellite's period exactly matches the Earth's rotation period of 24 hours. A satellite in this orbit appears to remain stationary above a fixed point on the equator, as seen from the ground. This is called a **geostationary orbit** (also known as a **geosynchronous equatorial orbit** or a **parking orbit**).

For a satellite to be geostationary, three conditions must be met:

Condition 1: The period must be exactly 24 hours (86400s).

Condition 2: The orbit must be circular (otherwise the satellite would appear to drift back and forth during the day).

Condition 3: The orbit must lie in the equatorial plane (otherwise the satellite would appear to move north and south during the day).

The radius of the geostationary orbit can be found from Kepler's third law as follows:

$$T^2 = \frac{4\pi^2}{GM_E}R^3$$

Making R the subject:

$$R^3 = \frac{GM_ET^2}{4\pi^2}$$

$$R = \sqrt[3]{\left(\frac{GM_ET^2}{4\pi^2}\right)}$$

Substituting $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$, $M_E = 6.0 \times 10^{24}\text{kg}$, and $T = 86400\text{s}$:

$$R = \sqrt[3]{\left(\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times (86400)^2}{4 \times \pi^2}\right)} = 4.23 \times 10^7\text{m} = 42300\text{km}$$

The height above the surface:

$$h = R - r_E = 42300\text{km} - 6400\text{km} = 35900\text{km} \approx 36000\text{km}$$

Every geostationary satellite orbits at approximately 36000km above the equator. There is no choice in this; the altitude is fixed by the physics. This is why geostationary orbit positions are valuable and strictly regulated internationally.

The orbital speed in this orbit:

$$\mathrm{v} = \frac{2\pi \mathrm{R}}{\mathrm{T}} = \frac{2 \times 3.14 \times 4.23 \times 10^{7}\mathrm{m}}{86400\mathrm{s}} = 3074\mathrm{m/s} \approx 3.1\mathrm{km/s}$$

This is much slower than the 7.9km/s of a low-orbit satellite, consistent with the rule that higher orbits have lower speeds.

Why geostationary satellites are so useful:

Because a geostationary satellite appears fixed above one point on the equator, a ground antenna can be pointed at it permanently without tracking. This makes it ideal for television broadcasting, weather observation, and telecommunications. The satellite dishes you see on rooftops across Tanzania are all pointed at geostationary satellites hovering above the equator.

Energy Required to Launch a Satellite

To place a satellite into orbit, we must do two things:

First: lifting it from the surface to the orbital height (increasing its potential energy), and

Second: accelerating it to the orbital speed (giving it kinetic energy).

The total energy required *is the difference between the total energy in orbit and the total energy at rest on the surface.*

On the surface (at rest):

$$\mathrm{E_{surface}} = \mathrm{K_{surface}} + \mathrm{U_{surface}} = 0 + \left(-\frac{\mathrm{GM_E m}}{\mathrm{r_E}}\right) = -\frac{\mathrm{GM_E m}}{\mathrm{r_E}}$$

In orbit (at radius R):

$$\mathrm{E_{orbit}} = -\frac{\mathrm{GM_E m}}{2\mathrm{R}}$$

The energy that must be supplied is:

$$\Delta \mathrm{E} = \mathrm{E_{orbit}} - \mathrm{E_{surface}} = -\frac{\mathrm{GM_E m}}{2\mathrm{R}} - \left(-\frac{\mathrm{GM_E m}}{\mathrm{r_E}}\right)$$

$$\boldsymbol{\Delta \mathrm{E} = \mathrm{GM_E m}\left(\frac{1}{\mathrm{r_E}} - \frac{1}{2\mathrm{R}}\right)}$$

This is the minimum energy required, ignoring air resistance and other losses. In practice, much more energy is needed because rockets must also lift the fuel itself, overcome air resistance, and account for inefficiencies.

Let us now put all of these ideas to work.

BINDER Example 18

A satellite orbits the Earth at a height of 400km above the surface (approximately the altitude of the International Space Station).

(a) Calculate its orbital speed.

(b) Calculate its orbital period in minutes.

Take $\mathrm{M_E} = 6.0 \times 10^{24}\mathrm{kg}$, $\mathrm{r_E} = 6.4 \times 10^{6}\mathrm{m}$, and $\mathrm{G} = 6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$.

Solution

(a) The orbital radius:

$$\mathrm{R} = \mathrm{r_E} + \mathrm{h} = 6.4 \times 10^{6}\mathrm{m} + 400 \times 10^{3}\mathrm{m} = 6.8 \times 10^{6}\mathrm{m}$$

The orbital speed:

$$v = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{6.8 \times 10^6\text{m}}} = 7672\text{m/s}$$

The orbital speed is 7672m/s (about 7.67km/s).

(b) The orbital period:

$$T = \frac{2\pi R}{v} = \frac{2 \times 3.14 \times 6.8 \times 10^6\text{m}}{7672\text{m/s}} = 5567\text{s}$$

Converting to minutes:

$$T = \frac{5567\text{s}}{60} = 92.8\text{min}$$

The period is about 92.8 minutes.

The ISS completes roughly 15.5 orbits every day, experiencing 15 sunrises and 15 sunsets in 24 hours.

Making Sense of the Answer: *The speed (7.67km/s) and period (93 minutes) match the well-known values for the ISS. At this speed, the station crosses the entire width of Tanzania in about 2 minutes!*

Think Like a Physicist: *Compare with the minimum values: 7.9km/s and 85 minutes near the surface. At 400km, the speed is slightly lower and the period slightly longer, exactly as the formulas predict. The differences are small because 400km is only about 6% of the Earth's radius.*

REAL Example 19

Kipanga asks **Mr. Akilikubwa**: *"Sir, if a satellite is always falling toward the Earth, why doesn't it lose energy and crash?"*

Mr. Akilikubwa replies: *"Kipanga, it* **is** *losing energy in a sense. But look at what happens when it does."*

Explain what Mr. Akilikubwa means. Describe what happens to the speed, altitude, and period of a satellite when it gradually loses energy due to atmospheric drag.

Solution

When a satellite loses energy through atmospheric drag, its total energy $E = -\frac{GM_E m}{2R}$ becomes more negative. A more negative E corresponds to a smaller orbital radius R, so the satellite drops to a lower orbit.

At this lower orbit, the orbital speed $v = \sqrt{\frac{GM_E}{R}}$ is higher (because R is smaller), and the period $T = 2\pi\sqrt{\frac{R^3}{GM_E}}$ is shorter.

This is the orbital paradox that Mr. Akilikubwa is hinting at: drag removes energy from the satellite, yet the satellite speeds up. The satellite spirals inward into ever-lower, ever-faster orbits, until it eventually enters the dense atmosphere and burns up.

The paradox resolves when you realise that the satellite gains more kinetic energy from falling to a lower orbit than it loses to drag. The potential energy lost exceeds the kinetic energy gained plus the energy lost to drag.

Making Sense of the Answer: *A satellite slowed by drag does not simply slow down and stop, the way a car does when you take your foot off the accelerator. Instead, it trades altitude for speed, spiralling inward faster and faster until it meets its fiery end in the atmosphere.*

Think Like a Physicist: *The relationships* $K = -\frac{1}{2}U$ *and* $E = -K$ *are the key. When the satellite drops to a lower orbit,* $|U|$ *increases,* K *increases (to half of* $|U|$*), and* E *decreases (becomes more negative). The satellite has less total energy but more kinetic energy.*

HOT Example 20

(a) Calculate the kinetic energy, potential energy, and total energy of a 200kg satellite orbiting the Earth at a height of 600km.

(b) How much energy must be supplied to move this satellite from its current orbit to a new orbit at 1200km above the surface?

Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

(a) Orbital radius at 600km:

$$R_1 = r_E + h_1 = 6.4 \times 10^6\text{m} + 600 \times 10^3\text{m} = 7 \times 10^6\text{m}$$

Kinetic energy:

$$K = \frac{GM_E m}{2R_1} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6 \times 10^{24}\text{kg} \times 200\text{kg}}{2 \times 7 \times 10^6\text{m}} = 5.717 \times 10^9\text{J} = 5.72\text{GJ}$$

Potential energy:

$$U = -\frac{GM_E m}{R_1} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6 \times 10^{24}\text{kg} \times 200\text{kg}}{7 \times 10^6\text{m}} = -1.143 \times 10^{10} = -11.43\text{GJ}$$

Total energy:

$$E_1 = K + U = 5.72\text{GJ} + (-11.43\text{GJ}) = -5.72\text{GJ}$$

You can easily check the correctness of the answers:

Check: $K = -\frac{1}{2}U?\ \frac{11.43}{2} = 5.72$. Confirmed.

Check: $E = -K?\ -5.72 = -(5.72)$. Confirmed.

(b) Orbital radius at 1200km:

$$R_2 = 6.4 \times 10^6\text{m} + 1200 \times 10^3\text{m} = 7.6 \times 10^6\text{m}$$

Total energy in the new orbit:

$$E_2 = -\frac{GM_E m}{2R_2} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6 \times 10^{24}\text{kg} \times 200\text{kg}}{2 \times 7.6 \times 10^6\text{m}} = -5.266 \times 10^9\text{J} = -5.27\text{GJ}$$

Energy that must be supplied:

$$\Delta E = E_2 - E_1 = (-5.27\text{GJ}) - (-5.72\text{GJ}) = 0.45\text{GJ} = 4.5 \times 10^8\text{J}$$

The satellite needs $4.5 \times 10^8\text{J}$ (about 450MJ) of additional energy to move from the 600km orbit to the 1200km orbit.

Making Sense of the Answer: *The higher orbit has less negative total energy (it is less tightly bound). The difference, 450MJ, is the energy the satellite's engines must provide. Notice that moving 600km higher requires far less energy than the initial launch from the surface, because the satellite already has most of the energy it needs.*

Think Like a Physicist: *The explicit checks ($K = -\frac{1}{2}U$ and $E = -K$) are not just bookkeeping. They confirm that the orbit is genuinely circular. If these relationships did not hold, it would mean the orbit is elliptical or the calculation contains an error. Always verify these relationships when working with orbital energies.*

HOT Example 21

Calculate the minimum energy required to launch a 1000kg satellite from the Earth's surface into a circular orbit at a height of 300km. Ignore air resistance and the Earth's rotation.

Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

The satellite starts at rest on the surface. Its initial energy:

$$E_{\text{surface}} = 0 + \left(-\frac{GM_E m}{r_E}\right) = -\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 1000}{6.4 \times 10^6}$$

$$E_{\text{surface}} = -\frac{4.002 \times 10^{17}\text{J}}{6.4 \times 10^{6}} = -6.253 \times 10^{10}\text{J}$$

The orbital radius:

$$R = 6.4 \times 10^{6}\text{m} + 300 \times 10^{3}\text{m} = 6.7 \times 10^{6}\text{m}$$

The total energy in orbit:

$$E_{\text{orbit}} = -\frac{GM_Em}{2R} = -\frac{4.002 \times 10^{17}\text{J}}{2 \times 6.7 \times 10^{6}\text{m}} = -\frac{4.002 \times 10^{17}}{1.34 \times 10^{7}} = -2.987 \times 10^{10}\text{J}$$

Minimum energy required:

$$\Delta E = E_{\text{orbit}} - E_{\text{surface}} = (-2.987 \times 10^{10}\text{J}) - (-6.253 \times 10^{10}\text{J})$$

$$\Delta E = 3.266 \times 10^{10}\text{J} = 32.7\text{GJ}$$

The minimum energy required is 32.7GJ.

Making Sense of the Answer: 32.7*GJ for a* 1000*kg satellite is an enormous amount of energy. For comparison, a litre of rocket fuel (kerosene) releases about* 34*MJ, so this launch would need at least* 960 *litres of fuel for the payload alone. In reality, the fuel-to-payload ratio for orbital launches is typically* $20:1$ *or higher, because the rocket must also lift the fuel, the engines, and the structure.*

Think Like a Physicist: *Most of the launch energy goes into kinetic energy, not potential energy. The potential energy change is* $GM_Em\left(\frac{1}{r_E} - \frac{1}{R}\right) \approx 2.93\text{GJ}$, *while the kinetic energy needed is* $\frac{GM_Em}{2R} \approx 29.9\text{GJ}$. *About* 91% *of the energy goes into making the satellite move fast enough to stay in orbit, and only* 9% *goes into lifting it to the correct height. This is why reaching orbit is fundamentally about speed, not altitude.*

With the physics of satellites and orbits now complete, one question remains: *what if we want to escape the Earth's gravity entirely?* Not orbit, but leave. That is the subject of the next and penultimate subtopic of this topic.

Escape velocity

In the previous section, we studied satellites that orbit the Earth, bound by gravity in circular paths. Their total energy is negative, and they remain trapped in the gravitational field forever (unless something changes their energy). But what if we do not want to orbit? What if we want to *leave*?

Every child who has thrown a ball into the air has watched it slow down, stop, and fall back. Throw it harder, and it goes higher before returning. Throw it harder still, and it goes higher still. The natural question arises: *is there a speed so great that the ball never comes back? A speed at which the object rises forever, escaping the Earth's gravitational pull entirely?*

The answer is yes. That speed is called the **escape velocity**.

Derivation

Consider a body of mass m on the surface of the Earth (mass M_E, radius r_E). Suppose the body is projected vertically upward with speed v_E. We want to find the minimum speed for which the body never returns.

"Never returns" means the body reaches infinity. At infinity, the gravitational potential energy is zero (by our convention). The minimum escape condition corresponds to the body arriving at infinity with zero residual speed (just barely escaping, with no energy left over). So at infinity, both the kinetic energy and the potential energy are zero, giving a total energy of zero.

By conservation of mechanical energy, the total energy at the surface must equal the total energy at infinity:

$$K_{\text{surface}} + U_{\text{surface}} = K_{\infty} + U_{\infty}$$

$$\frac{1}{2}mv_E^2 + \left(-\frac{GM_Em}{r_E}\right) = 0 + 0$$

$$\frac{1}{2}mv_E^2 = \frac{GM_Em}{r_E}$$

The mass m cancels (escape velocity does not depend on the mass of the escaping body):

$$\frac{1}{2}v_E^2 = \frac{GM_E}{r_E}$$

$$v_E = \sqrt{\frac{2GM_E}{r_E}}$$

Using $GM_E = gr_E^2$, this can also be written as:

$$v_E = \sqrt{2gr_E}$$

Substituting values for the Earth ($g = 9.8\text{m/s}^2$, $r_E = 6.4 \times 10^6\text{m}$):

$$\mathbf{v_E} = \sqrt{2 \times 9.8\text{m/s}^2 \times 6.4 \times 10^6\text{m}} = \mathbf{11200m/s}$$

The escape velocity from the Earth's surface is approximately 11.2km/s, or about 40300km/h.

The elegant connection: Escape velocity and orbital velocity

Recall that the orbital velocity near the surface is $v_{orb} = \sqrt{gr_E}$. The escape velocity is $v_E = \sqrt{2gr_E}$. Dividing:

$$\frac{v_E}{v_{orb}} = \frac{\sqrt{2gr_E}}{\sqrt{gr_E}} = \sqrt{2}$$

$$\mathbf{v_E = \sqrt{2} \times v_{orb} \approx 1.414 \times v_{orb}}$$

The escape velocity is exactly $\sqrt{2}$ times the orbital velocity at the same radius. This relationship holds at any distance from any spherical body, not just at the Earth's surface. A satellite in circular orbit at any altitude needs only a 41.4% $\big((1.414 - 1) \times 100\%\big)$increase in speed to escape the Earth entirely. This is a remarkably small margin, and it has deep implications for space travel.

Escape velocity from any height

If a body is already at height h above the surface (at distance $R = r_E + h$ from the centre), the escape velocity from that point is:

$$\mathbf{v_E} = \sqrt{\frac{2GM_E}{R}} = \sqrt{\mathbf{\frac{2GM_E}{r_E + h}}}$$

So *the escape velocity decreases with altitude, because the body starts higher in the gravitational potential well and needs less energy to reach the top.*

Alternatively, using $GM_E = gr_E^2$:

$$\mathbf{v_E = r_E\sqrt{\frac{2g}{r_E + h}}}$$

Key features of escape velocity

Several important points are worth highlighting.

1. It does not depend on mass

A grain of sand and a spacecraft need the same escape velocity to leave the Earth. This is because both the kinetic energy and the gravitational potential energy are proportional to m, so m cancels in the energy equation. Of course, the *energy* required does depend on mass ($\frac{1}{2}mv_E^2$ is larger for a heavier object), but the *speed* does not.

2. It does not depend on the direction of projection

The derivation used energy conservation, which is a scalar equation. It does not matter whether the body is launched vertically, horizontally, or at any angle. As long as the initial speed equals the escape velocity and the body does not hit the Earth on the way out, it will escape. (In practice, launching vertically is the most common choice because it gets above the dense atmosphere quickly.)

3. It depends on the mass and radius of the planet.

A more massive planet has a higher escape velocity (stronger gravitational pull). A larger planet (same mass but greater radius r) has a lower escape velocity (the surface is farther from the centre). For a planet of mass M and radius r:

$$v_E = \sqrt{\frac{2GM}{r}}$$

Escape Velocities of Different Bodies

The escape velocity varies enormously across the solar system. The Moon, with its small mass and radius, has an escape velocity of only about 2.4km/s. Jupiter, the most massive planet, has an escape velocity of about 60km/s. The Sun's escape velocity from its surface is about 617km/s.

Why the Moon has no atmosphere

This concept directly explains one of the most striking differences between the Earth and the Moon. The Earth has a thick atmosphere; the Moon has essentially none. The reason is escape velocity.

Gas molecules in an atmosphere are in constant random motion. Their average speed depends on the temperature: hotter gases move faster. At the **high temperatures** found on the Moon's surface (which can reach over 100°C in direct sunlight), the average speed of light gas molecules such as hydrogen and helium exceeds the Moon's escape velocity of 2.4km/s. These molecules simply fly off into space and never return. Over billions of years, even heavier gas molecules gradually escape, leaving the Moon with no atmosphere.

The Earth's escape velocity (11.2km/s) is much higher than the average speed of atmospheric gas molecules at terrestrial temperatures (typically 0.3 to 0.6km/s for nitrogen and oxygen). So the Earth retains its atmosphere comfortably.

Let us now apply these ideas through worked examples.

BINDER Example 22

(a) Calculate the escape velocity from the surface of the Moon. Take the mass of the Moon as 7.35×10^{22}kg, the radius of the Moon as 1.74×10^{6}m, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) Compare this with the escape velocity from the Earth (11.2km/s) and explain why the Moon has no atmosphere while the Earth does.

Solution

(a) Using $v_E = \sqrt{\frac{2GM}{r}}$:

$$v_E = \sqrt{\frac{2 \times 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 7.35 \times 10^{22}\text{kg}}{1.74 \times 10^{6}\text{m}}} = 2374\text{m/s}$$

The escape velocity from the Moon is 2374m/s (about 2.4km/s).

(b) **Comparison:**

The Earth's escape velocity (11.2km/s) is greater (about 4.7 times greater) than the Moon's (2.4km/s).

Explanation:

Gas molecules move very fast at ordinary temperatures, but on Earth their speeds are still much lower than the escape velocity, so they remain held by gravity and the atmosphere is retained. On the Moon, however, gravity is much weaker, giving a much lower escape velocity. Because the Moon also experiences very high surface temperatures (over 100°C) in sunlight, some gas molecules gain enough speed to escape into space. Over a very long time, this gradual loss means the Moon has lost all atmospheric gases, leaving the Moon with no atmosphere.

Making Sense of the Answer: *The Moon's low escape velocity is a direct consequence of its small mass. A body with* 81 *times less mass than the Earth simply cannot hold onto fast-moving gas molecules. This single number,* 2.4*km/s, explains why astronauts on the Moon need spacesuits.*

Think Like a Physicist: *Not all molecules in a gas move at the same speed. Even if the average speed is below the escape velocity, some molecules in the high-speed tail of the distribution will exceed it and escape. Over geological time, this slow leakage is enough to strip the Moon of any atmosphere it once had.*

REAL Example 23

Kipanga watches a science fiction film where a character escapes from a planet by jumping very hard. He asks Kipute: *"Is that even possible? Could you ever jump fast enough to escape the Earth?"*

Explain whether it is theoretically possible, and discuss why rockets are needed in practice.

Solution

Theoretically, if a person could jump at 11.2km/s (40300km/h), they would escape the Earth because Escape velocity does not depend on mass. So even a person would need the same speed as a heavier rocket. However, this is completely impossible in practice as the speed of man is far below the escape velocity (the fastest a human can move their body is about 10m/s (a sprint), which is more than 1000 times too slow).

Rockets are needed because they provide a sustained thrust over many minutes, gradually building up speed. A rocket does not need to reach escape velocity all at once; it accelerates continuously until the required speed is achieved. It also needs to overcome air resistance and lift its own fuel, which is why real launches require far more energy than the theoretical minimum.

In the film, the escape might be plausible only if the planet were very small and had very low surface gravity, giving a correspondingly low escape velocity.

Making Sense of the Answer: *Escape velocity is about the speed needed, not the force of a single push. A cannon could theoretically launch a projectile at escape velocity in one shot, but the enormous acceleration would destroy any payload (and any passenger). Rockets solve this by applying a gentler acceleration over a longer time.*

Think Like a Physicist: *The derivation of escape velocity assumes no further force acts after the initial launch (pure projectile). A rocket, by contrast, burns fuel continuously, so it does not need to start at escape velocity. It can leave the Earth at any speed, as long as it keeps accelerating. The escape velocity formula gives the minimum speed for an unpowered escape.*

HOT Example 24

(a) Show that the escape velocity from the surface of a planet can be written as $\mathrm{v_E} = \sqrt{2}\mathrm{v_{orb}}$, where $\mathrm{v_{orb}}$ is the orbital velocity near the surface.

(b) A satellite orbits the Earth at a height of 500km. Calculate the orbital velocity and the escape velocity at this altitude. By what percentage must the satellite's speed increase to escape the Earth?

Take $\mathrm{M_E} = 6.0 \times 10^{24}\mathrm{kg}$, $\mathrm{r_E} = 6.4 \times 10^{6}\mathrm{m}$, and $\mathrm{G} = 6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$.

Solution

(a) The orbital velocity near the surface is:

$$\mathrm{v_{orb}} = \sqrt{\frac{\mathrm{GM}}{\mathrm{r}}}$$

The escape velocity from the surface is:

$$\mathrm{v_E} = \sqrt{\frac{2\mathrm{GM}}{\mathrm{r}}}$$

Dividing $\mathrm{v_E}$ by $\mathrm{v_{orb}}$:

$$\frac{v_E}{v_{orb}} = \frac{\sqrt{\frac{2GM}{r}}}{\sqrt{\frac{GM}{r}}} = \sqrt{\frac{2GM/r}{GM/r}} = \sqrt{2}$$

Therefore $v_E = \sqrt{2}v_{orb}$.

(b) Orbital radius at 500km:

$$R = r_E + h = 6.4 \times 10^6\text{m} + 500 \times 10^3\text{m} = 6.9 \times 10^6\text{m}$$

Orbital velocity:

$$v_{orb} = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{6.9 \times 10^6\text{m}}} = 7616\text{m/s}$$

Escape velocity at this altitude:

$$v_E = \sqrt{2} \times v_{orb} = 1.414 \times 7616\text{m/s} = 10{,}769\text{m/s}$$

Percentage increase needed:

$$\frac{v_E - v_{orb}}{v_{orb}} \times 100\% = \frac{10769\text{m/s} - 7616\text{m/s}}{7616\text{m/s}} \times 100\% = 41.4\%$$

The satellite needs a 41.4% increase in speed to escape.

Note: This is exactly $\left(\sqrt{2} - 1\right) \times 100\% = 41.4\%$, and it is the same at every altitude. A satellite anywhere in a circular orbit is always just 41.4% of extra speed away from freedom.

Making Sense of the Answer: 41.4% *may seem like a small margin, but in energy terms it is not. Since kinetic energy scales as* v^2, *a* 41.4% *increase in speed requires a* 100% *increase in kinetic energy (since* $\left(\sqrt{2}\right)^2 = 2$*). The satellite must exactly double its kinetic energy to escape.*

Think Like a Physicist: *The result* $v_E = \sqrt{2}v_{orb}$ *connects beautifully to the energy relationships learned earlier. For a circular orbit,* $E = -K$, *so the total energy is the negative of the kinetic energy. To escape, the total energy must become zero, which requires adding energy equal to* $|E| = K$. *This doubles the kinetic energy, which increases the speed by a factor of* $\sqrt{2}$. *The physics is perfectly self-consistent.*

We have now completed the core theory of gravitation: from Kepler's empirical laws, through Newton's universal force, the variation of gravity with position, the concepts of field and potential, the energy of orbits, and finally escape velocity. In the next section, we gather the applications of these ideas into a single, coherent picture.

APPLICATIONS OF GRAVITATION

The equations we have derived in this chapter were not built for textbooks. They were built for the universe. From the motion of planets to the design of satellite networks, from measuring the mass of distant stars to explaining why the ocean tides rise and fall, gravitation touches nearly every branch of physics and engineering. In this section, we step outside the classroom and see how the physics we have mastered finds its way into the real world.

1. Weighing the Earth and other Celestial bodies

One of the most remarkable applications of Newton's law of gravitation is that it allows us to "weigh" objects we can never place on a scale. If we can observe any orbit around a body (its period and radius), Kepler's third law immediately gives us the mass of the central body:

$$\mathbf{M = \frac{4\pi^2R^3}{GT^2}}$$

This is how the mass of the Earth was first determined (using the Moon's orbit), how the mass of the Sun is known (using the Earth's orbit), and how the masses of distant planets are measured (using their moons or spacecraft flybys). We do not need to touch the object; we only need to watch something orbit it.

Even the mass of a distant star can be estimated if it has a companion star or planet whose orbital parameters can be measured. Gravitation turns every orbit into a weighing scale.

2. Communication satellites and the geostationary belt

Modern telecommunications depend heavily on geostationary satellites. As we showed earlier, these satellites orbit at approximately 36000km above the equator with a period of exactly 24 hours, appearing stationary above a fixed point on the ground.

A single geostationary satellite can “see” about one-third of the Earth’s surface. Three satellites, spaced 120° apart around the equator, can cover almost the entire globe (except for small regions near the poles). This is the basis of global satellite communication, television broadcasting, and weather monitoring.

The geostationary orbit is sometimes called the **Clarke orbit**, after the science fiction writer Arthur C. Clarke, who first proposed the idea of geostationary communication satellites in 1945, more than a decade before the first satellite (Sputnik) was launched. Clarke’s prediction, based entirely on the physics of this chapter, turned out to be one of the most commercially valuable ideas in the history of technology.

Because all geostationary satellites must orbit at the same altitude and in the same plane, the available “slots” are limited. The International Telecommunication Union (ITU) regulates the allocation of these orbital positions to prevent interference between neighbouring satellites.

3. Low Earth Orbit (LEO) satellites

Not all satellites need to be geostationary. Satellites in **low Earth orbit** (typically 200 to 2000km altitude) move much faster and have much shorter periods (about 90 to 130 minutes). They pass over different parts of the Earth’s surface on each orbit, making them ideal for applications that require global coverage over time rather than a fixed view.

Earth observation satellites photograph the surface with high resolution, monitoring deforestation, urban growth, agricultural productivity, and natural disasters. **Weather satellites** in polar orbits scan the entire Earth twice a day. The **International Space Station** orbits at about 400km and serves as a laboratory for research in microgravity.

Global navigation systems such as GPS (United States), GLONASS (Russia), Galileo (European Union), and BeiDou (China) use constellations of satellites at medium altitudes (about 20000km). By receiving signals from at least four satellites simultaneously, a GPS receiver on the ground can calculate its position to within a few metres. The physics that makes this possible is precisely the orbital mechanics of this chapter.

4. Tides

The tides are one of the oldest known effects of gravitation. The Moon’s gravitational pull on the Earth is not uniform: it is stronger on the side of the Earth closest to the Moon and weaker on the far side. This difference in gravitational pull (called the **tidal force**) stretches the Earth slightly along the Earth-Moon line, creating two bulges in the oceans. As the Earth rotates beneath these bulges, coastal areas experience two high tides and two low tides each day.

The Sun also contributes to tides, but because it is much farther away, its tidal effect is about half that of the Moon. When the Sun and Moon are aligned (during new moon and full moon), their tidal effects add together, producing exceptionally high tides called **spring tides**. When they are at right angles (during quarter moons), the tides are weaker, called **neap tides**.

Tides affect not only the oceans. The Earth’s solid crust also deforms slightly (by about 25cm) under tidal forces, and tidal forces from Jupiter cause intense volcanic activity on its moon Io, the most volcanically active body in the solar system.

5. Determining the density of the Earth

Earlier, we showed that $g = \frac{GM_E}{r_E^2}$, which gives $M_E = \frac{gr_E^2}{G}$. Once the mass is known, the density follows from:

$$\boldsymbol{\rho} = \frac{M_E}{\frac{4}{3}\pi r_E^3} = \frac{\mathbf{3g}}{\mathbf{4\pi G r_E}}$$

This result is significant because it reveals that the Earth’s average density (about 5500kg/m^3) is much higher than the density of surface rocks (2500 to 3000kg/m^3). Without ever drilling to the centre, physicists deduced that the Earth must have a very dense core, most likely composed of iron and nickel. This conclusion, drawn from nothing more than g, G, and r_E, was later confirmed by seismic studies.

6. Space travel and interplanetary missions

Every interplanetary mission uses the gravitational physics of this chapter. Launching a spacecraft from Earth requires overcoming the Earth's gravitational potential well, which demands enormous energy. Once in space, the spacecraft can use the gravity of other planets to change its speed and direction, a technique called a **gravitational slingshot** (or gravity assist).

In a gravity assist, a spacecraft approaches a planet, is deflected by the planet's gravity, and leaves with a different speed and direction. No fuel is burned during the manoeuvre; the energy comes from the planet's orbital motion. The Voyager spacecraft, launched in 1977, used gravity assists from Jupiter and Saturn to reach speeds high enough to leave the solar system entirely. Voyager 1 is currently the most distant human-made object, over 24 billion kilometres from Earth, still sending data back using a 23-watt radio transmitter (about the power of a refrigerator light bulb).

7. Apparent weightlessness in orbit

We have mentioned weightlessness several times in this chapter, but it deserves a clear, consolidated explanation because it is one of the most widely misunderstood concepts in physics.

An astronaut in orbit is not beyond the reach of gravity. As we calculated in Example 14, the gravitational field at the altitude of the ISS is still about 94% of its surface value. The astronaut floats not because gravity is absent, but because both the astronaut and the spacecraft are falling toward the Earth at exactly the same rate. There is no relative acceleration between the astronaut and the spacecraft, so there is no contact force (no normal reaction from the floor, no tension from a support). With zero contact force, the astronaut feels weightless.

This is exactly the same physics as the momentary weightlessness at the top of a hill (in Chapter 7), but sustained continuously. In orbit, the object is always at the "top of the hill" because the circular path ensures that gravitational acceleration exactly equals centripetal acceleration at every point.

True weightlessness (zero gravitational force) would require being infinitely far from all masses, which is physically impossible. What astronauts experience in orbit is more accurately called **free fall** or **microgravity**.

From satellites silently circling the Earth to the tides rising along the coast of Dar es Salaam, from GPS guiding a driver through Dodoma to the Voyager spacecraft sailing beyond the edge of the solar system, gravitation is the invisible thread connecting all of it. The equations we derived in this chapter are not abstract mathematics; they are the operating instructions for the largest structures in the universe.

And with that, the core of gravitation is complete. In the next section, the concepts of this entire chapter will come together in miscellaneous worked examples, where the physics will stop being polite and start working together. If the individual sections were the instruments, the miscellaneous examples are the orchestra. Tune up, and let us play.

MISCELLANEOUS WORKED EXAMPLES ON GRAVITATION

Example 25

(a) Explain why all objects, regardless of their mass, fall with the same acceleration in a gravitational field (in the absence of air resistance).

(b) Two stars orbit each other with a period of 2.0 years. The mean distance between their centres is 3.0×10^{11}m. Calculate the combined mass of the two-star system. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

(a) The gravitational force on an object of mass m near a planet of mass M is $F = \frac{GMm}{r^2}$. By Newton's second law, the acceleration is $a = \frac{F}{m} = \frac{GM}{r^2}$. The mass m of the falling object appears in both the gravitational force (which pulls it) and the inertia (which resists acceleration), and these two effects cancel exactly. Thus the acceleration depends only on M and r, not on m.

(b) For a binary star system, Kepler's third law gives:

$$T^2 = \frac{4\pi^2 R^3}{G(M_1 + M_2)}$$

Making the combined mass the subject:

$$M_1 + M_2 = \frac{4\pi^2 R^3}{GT^2}$$

Converting the period: $T = 2.0\text{ years} = 2.0 \times 365.25 \times 24 \times 3600\text{s} = 6.312 \times 10^7\text{s}$

Substituting:

$$M_1 + M_2 = \frac{4 \times (3.14)^2 \times (3.0 \times 10^{11}\text{m})^3}{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times (6.312 \times 10^7\text{s})^2} = 4.01 \times 10^{30}\text{kg}$$

The combined mass of the two stars is approximately 4.0×10^{30}kg.

Example 26

(a) Explain why the gravitational potential at any point near a planet is always negative.

(b) Calculate the gravitational potential at a point 1000km above the Earth's surface and the work done in moving a 50kg object from the surface to this point. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

(a) Gravitational potential is defined with the reference point at infinity, where $V = 0$. To reach any point near a planet from infinity, gravity does positive work on the object (it pulls the object inward). This means the object loses potential energy as it approaches the planet. Since the potential energy at infinity is zero and the object has less potential energy at any point near the planet, the potential must be negative. The closer to the planet, the more negative the potential becomes.

(b) At height $h = 1000\text{km} = 1.0 \times 10^6\text{m}$:

$$R = r_E + h = 6.4 \times 10^6\text{m} + 1.0 \times 10^6\text{m} = 7.4 \times 10^6\text{m}$$

$$\mathbf{V_{1000}} = -\frac{GM_E}{R} = -\frac{4.002 \times 10^{14}\text{J}}{7.4 \times 10^6\text{m}} = \mathbf{-5.408 \times 10^7 J/kg}$$

At the surface:

$$V_{\text{surface}} = -\frac{GM_E}{r_E} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{6.4 \times 10^6\text{m}} = -6.253 \times 10^7\text{J/kg}$$

The work done in moving the 50kg object from the surface to this height:

$$W = m(V_{1000} - V_{\text{surface}}) = 50\text{kg} \times \left((-5.408 \times 10^7) - (-6.253 \times 10^7)\right)\text{J/kg}$$

$$\mathbf{W} = 50\text{kg} \times 8.45 \times 10^6\text{J/kg} = \mathbf{4.225 \times 10^8 J = 422.5 MJ}$$

Example 27

(a) **Kipanga** declares confidently to the class: *"If the Earth suddenly stopped rotating, we would all fly off into space!"* **Kipute** shakes her head, showing disagreement. **Mr. Akilikubwa** hides a smile. Who is correct? Explain why.

(b) Calculate the angular velocity at which the Earth would need to rotate for objects at the equator to become weightless. Express this as a multiple of the current angular velocity of the Earth, and find the corresponding length of a day. Take $g = 9.8\text{m/s}^2$ and $r_E = 6.4 \times 10^6\text{m}$.

Solution

(a) Kipute is correct. Kipanga has the physics backwards. The Earth's rotation currently **reduces** the effective gravity at the equator (because part of the gravitational pull goes toward providing centripetal acceleration for the rotation). If the Earth stopped rotating, no centripetal acceleration would be needed, and the effective gravity would actually **increase** slightly. Everyone would feel heavier, not lighter. There would be no tendency to fly off at all.

Flying off into space would require the Earth to rotate much **faster**, not to stop. Objects would leave the surface only if the required centripetal acceleration exceeded g, so that gravity could no longer hold them down.

(b) At the equator, objects become weightless when the centripetal acceleration equals g:

$$\omega^2 r_E = g$$

Making ω the subject:

$$\omega = \sqrt{\frac{g}{r_E}} = \sqrt{\frac{9.8\text{m/s}^2}{6.4\times10^6\text{m}}} = \sqrt{1.531\times10^{-6}\text{s}^{-2}} = 1.237\times10^{-3}\text{rad/s}$$

The required angular velocity for weightless is 1.237×10^{-3}rad/s.

The current angular velocity of the Earth:

$$\omega_{\text{current}} = \frac{2\pi}{24\times3600\text{s}} = \frac{6.283}{86400\text{s}} = 7.272\times10^{-5}\text{rad/s}$$

The ratio:

$$\frac{\omega}{\omega_{\text{current}}} = \frac{1.237\times10^{-3}}{7.272\times10^{-5}} = 17.0$$

The Earth would need to rotate 17 times faster than it currently does.

The corresponding length of a day:

$$T = \frac{2\pi}{\omega} = \frac{6.283}{1.237\times10^{-3}\text{s}^{-1}} = 5079\text{s} = \frac{5079}{60}\text{min} = 84.7\text{min}$$

The length of a day would be about 84.7 minutes.

This is the same as the orbital period of a satellite near the Earth's surface, which is not a coincidence: at this rotation speed, the surface itself moves at orbital velocity, and objects sitting on the equator are effectively in orbit.

Example 28

(a) A satellite in a higher orbit moves more slowly than one in a lower orbit, even though more energy was needed to place it there. Explain why this is not a contradiction.

(b) Two satellites A and B orbit the Earth in circular orbits. Satellite A has orbital radius 2R and satellite B has orbital radius 8R. Find the ratio of:

(i) their orbital speeds,

(ii) their orbital periods,

(iii) their total mechanical energies.

Solution

(a) The orbital speed is determined by the balance between gravity and centripetal force: $v = \sqrt{\frac{GM}{R}}$. A higher orbit has a larger R, which gives a smaller v. The satellite moves more slowly because gravity is weaker at greater distances, so less speed is needed to maintain the circular path.

The total energy, however, is $E = -\frac{GMm}{2R}$. A higher orbit has a larger R, giving a less negative (larger) total energy. More energy was indeed needed to place it there. The apparent contradiction dissolves when you realise that kinetic energy and total energy are different quantities. A higher satellite has less kinetic energy but much more potential energy, and the increase in potential energy outweighs the decrease the kinetic energy, giving a higher total energy overall.

(b) Let the orbital radius of A be $R_A = 2R$ and that of B be $R_B = 8R$.

(i) Orbital speed ratio:

$$\frac{v_A}{v_B} = \frac{\sqrt{\frac{GM}{2R}}}{\sqrt{\frac{GM}{8R}}} = \sqrt{\frac{8R}{2R}} = \sqrt{4} = 2$$

Satellite A moves 2 times faster than satellite B.

(ii) Period ratio, using $T = 2\pi\sqrt{\frac{R^3}{GM}}$:

$$\frac{T_A}{T_B} = \sqrt{\frac{R_A^3}{R_B^3}} = \sqrt{\frac{(2R)^3}{(8R)^3}} = \sqrt{\frac{8}{512}} = \sqrt{\frac{1}{64}} = \frac{1}{8}$$

Satellite A has a period 8 times shorter than satellite B.

(iii) Total energy ratio (assuming both satellites have the same mass m):

$$\frac{E_A}{E_B} = \frac{-\frac{GMm}{2 \times 2R}}{-\frac{GMm}{2 \times 8R}} = \frac{\frac{1}{4R}}{\frac{1}{16R}} = \frac{16}{4} = 4$$

The total energy of satellite A is 4 times the total energy of satellite B.

Understand this: Since both energies are negative, this means A has a more negative energy (it is more tightly bound to the Earth). This is consistent with A being in a lower orbit.

Example 29

(a) Explain why the approximate formula $g' \approx g\left(1 - \frac{2h}{r_E}\right)$ works well for small heights but fails for large heights.

(b) At what height above the Earth's surface is the gravitational field strength exactly half the surface value? Express your answer in terms of r_E and in kilometres. Take $r_E = 6400\text{km}$.

Solution

(a) The exact formula is: $g' = g\left(\frac{r_E}{r_E+h}\right)^2$. The approximate formula is obtained by writing this as $g\left(1 + \frac{h}{r_E}\right)^{-2}$ and applying the binomial expansion, which gives $g\left(1 - \frac{2h}{r_E} + \cdots\right)$. The higher-order terms (containing $\frac{h^2}{r_E^2}$ and beyond) are dropped. When h is much smaller than r_E, these dropped terms are negligible and the approximation is accurate. When h is comparable to r_E, the dropped terms become significant and the approximation gives increasingly wrong results. For example, at $h = \frac{r_E}{2}$, the approximate formula gives $g' = 0$, while the exact formula gives $g' = 0.44g$.

(b) Setting $g' = \frac{1}{2}g$ in the exact formula:

$$\frac{1}{2}g = g\left(\frac{r_E}{r_E + h}\right)^2$$

Dividing both sides by g:

$$\frac{1}{2} = \left(\frac{r_E}{r_E + h}\right)^2$$

Taking the square root:

$$\frac{1}{\sqrt{2}} = \frac{r_E}{r_E + h}$$

Making $(r_E + h)$ the subject:

$$r_E + h = r_E\sqrt{2}$$

$$\mathbf{h = r_E\left(\sqrt{2}-1\right) = 0.4142r_E}$$

In kilometres:

$$\mathbf{h} = 0.4142 \times 6400\text{km} = \mathbf{2651km}$$

The gravitational field strength is halved at a height of $0.4142r_E$, or approximately 2651km above the surface.

Example 30

(a) A student claims: *"Since the gravitational field strength is zero at the centre of the Earth, the gravitational potential must also be zero there."* Identify and correct the error in this reasoning.

(b) Calculate the gravitational potential at the surface and at the centre of the Earth, assuming uniform density. Show that the potential at the centre is 1.5 times more negative than at the surface. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

(a) The student is confusing two different quantities. Gravitational field strength (g) is the force per unit mass, while gravitational potential (V) is the work done per unit mass in bringing a mass from infinity to that point. They are related by $g = -\frac{dV}{dr}$, which means that g is zero when the *rate of change* of V is zero, not when V itself is zero. At the centre of the Earth, V is at its minimum (most negative) value, and the gradient of V is zero (like the bottom of a valley, where the slope is flat but the height is at its lowest). Zero field strength means the potential has reached a turning point, not that the potential is zero.

(b) At the surface:

$$\mathbf{V_{surface}} = -\frac{GM_E}{r_E} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{6.4 \times 10^6\text{m}} = \mathbf{-6.25 \times 10^7 J/kg}$$

To find the potential at the centre, we derive it from first principles. Inside the Earth (assuming uniform density), the gravitational field strength at distance r from the centre is $g_{\text{inside}} = \frac{GM_E}{r_E^3}r$. Since $g = -\frac{dV}{dr}$, we can find the potential difference between the centre and the surface by integrating from the centre ($r = 0$) to the surface ($r = r_E$):

$$V_{\text{surface}} - V_{\text{centre}} = \int_0^{r_E} g_{\text{inside}}\ dr = \int_0^{r_E} \frac{GM_E}{r_E^3} r\ dr$$

$$= \left|\frac{GM_E}{r_E^3} \times \frac{r^2}{2}\right|_0^{r_E} = \frac{GM_E}{r_E^3} \times \frac{r_E^2}{2} = \frac{GM_E}{2r_E}$$

Making V_{centre} the subject:

$$\mathbf{V_{centre}} = V_{\text{surface}} - \frac{GM_E}{2r_E} = -\frac{GM_E}{r_E} - \frac{GM_E}{2r_E} = -\frac{2GM_E + GM_E}{2r_E} = -\mathbf{\frac{3GM_E}{2r_E}}$$

Numerically:

$$\mathbf{V_{centre}} = -\frac{3}{2} \times \frac{GM_E}{r_E} = \frac{3}{2} \times (-6.25 \times 10^7\text{J/kg}) = \mathbf{-9.375 \times 10^7 J/kg}$$

The ratio:

$$\frac{\mathbf{V_{centre}}}{\mathbf{V_{surface}}} = \frac{-9.375 \times 10^7\text{J/kg}}{-6.25 \times 10^7\text{J/kg}} = \mathbf{1.5}$$

The potential at the centre is 1.5 times more negative than at the surface.

Understand: The factor $\frac{3}{2}$ arises naturally from the integration: the potential difference between centre and surface is $\frac{GM_E}{2r_E}$, which when subtracted from $V_{surface} = -\frac{GM_E}{r_E}$ gives $-\frac{3GM_E}{2r_E}$. So the potential at the centre is indeed 1.5 times more negative than at the surface, confirming that the centre is the deepest point in the gravitational potential well.

Example 31

(a) Explain why the escape velocity from a planet does not depend on either the mass or the direction of projection of the escaping object.

(b) A planet has a mass 4 times the Earth's mass and a radius 2 times the Earth's radius. Calculate the escape velocity from this planet in terms of the Earth's escape velocity and hence express your answer in km/h.

Solution

(a) **Independence from mass:** The escape velocity is derived from conservation of energy: $\frac{1}{2}mv_E^2 = \frac{GMm}{r}$. The mass m of the escaping object appears on both sides and cancels. The kinetic energy needed to escape is proportional to m, but so is the gravitational potential energy that must be overcome. These two effects scale identically, leaving the required speed independent of mass.

Independence from direction: The energy equation is a scalar equation. It involves v^2 (which is the square of the speed, not the velocity), so only the magnitude of the speed matters, not its direction. Whether the object is launched vertically, horizontally, or at any angle, the energy balance is the same. As long as the initial speed equals v_E and the object does not collide with the surface, it will escape.

(b) The escape velocity from a planet of mass M and radius r is $v_E = \sqrt{\frac{2GM}{r}}$.

For the given planet: $M_p = 4M_E$ and $r_p = 2r_E$.

$$\mathbf{v_p} = \sqrt{\frac{2G \times 4M_E}{2r_E}} = \sqrt{\frac{4}{2} \times \frac{2GM_E}{r_E}} = \sqrt{2} \times \sqrt{\frac{2GM_E}{r_E}} = \mathbf{\sqrt{2} \times v_{E,Earth}}$$

The escape velocity from this planet is $\sqrt{2}$ times the Earth's escape velocity:

$$v_p = \sqrt{2} \times 11.2\text{km/s} = 1.414 \times 11.2\text{km/s} = \mathbf{15.8km/s}$$

Example 32

(a) Explain two advantages of placing a communication satellite in a geostationary orbit rather than a low Earth orbit.

(b) A telecommunications company plans to place an 800kg satellite in geostationary orbit. Take $M_E = 6.0 \times 10^{24}$kg, $r_E = 6.4 \times 10^6$m, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Calculate:

(i) the orbital radius,

(ii) the orbital speed,

(iii) the total energy of the satellite in orbit,

(iv) the minimum energy required to launch the satellite from the Earth's surface to this orbit.

Solution

(a) **First advantage:** A geostationary satellite remains fixed above one point on the equator, so ground antennas can be permanently pointed at it without tracking. This greatly simplifies the ground equipment needed for communication and broadcasting.

Second advantage: A single geostationary satellite can see about one-third of the Earth's surface continuously, providing uninterrupted coverage over that region. A low-orbit satellite passes overhead for only a few minutes at a time, requiring a constellation of many satellites and complex handover systems to maintain continuous coverage.

(b) The solution of each part is as follows:

(i) Using $T = 24h = 86400s$:

$$R^3 = \frac{GM_ET^2}{4\pi^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg} \times (86400\text{s})^2}{4 \times (3.14)^2} = 7.567 \times 10^{22}\text{m}^3$$

$$\mathbf{R} = \sqrt[3]{(7.567 \times 10^{22}\text{m}^3)} = 4.23 \times 10^7\text{m} = \mathbf{42300km}$$

(ii) Orbital speed:

$$\mathbf{v} = \frac{2\pi R}{T} = \frac{2 \times 3.14 \times 4.23 \times 10^7\text{m}}{86400\text{s}} = 3075\text{m/s} = \mathbf{3.08km/s}$$

(iii) Total energy in orbit:

$$E_{\text{orbit}} = -\frac{GM_Em}{2R} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg} \times 800\text{kg}}{2 \times 4.23 \times 10^7\text{m}}$$

$$\mathbf{E_{orbit}} = -\frac{3.202 \times 10^{17}\text{J}}{8.46 \times 10^7\text{m}} = \mathbf{-3.784 \times 10^9 J = -3.78GJ}$$

(iv) Total energy at the surface (at rest):

$$E_{\text{surface}} = -\frac{GM_Em}{r_E} = -\frac{3.202 \times 10^{17}\text{J}}{6.4 \times 10^6\text{m}} = -5.003 \times 10^{10}\text{J} = -50.03\text{GJ}$$

Minimum energy required:

$$\varDelta E = E_{\text{orbit}} - E_{\text{surface}} = (-3.78\text{GJ}) - (-50.03\text{GJ}) = 46.25\text{GJ}$$

The minimum energy required to launch the 800kg satellite to geostationary orbit is approximately 46.3GJ.

Example 33

(a) Explain why the gravitational field strength is zero at the neutral point between the Earth and the Moon, but the gravitational potential is not zero at that point.

(b) Two masses $M_1 = 4.0 \times 10^{24}$kg and $M_2 = 1.0 \times 10^{24}$kg have their centres separated by a distance of 5.0×10^8m. Find the position of the neutral point measured from M_1. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

(a) At the neutral point, the gravitational field due to the Earth (pointing toward Earth) is equal in magnitude to the field due to the Moon (pointing toward Moon). Since these fields point in opposite directions and are equal, they cancel, giving zero net field.

However, gravitational potential is a scalar, not a vector. It has no direction and cannot cancel by opposition. Both the Earth and the Moon contribute negative potential at every point in the space between them. At the neutral point, the total potential is the sum of two negative values, which is itself negative. Zero field means the forces balance; it does not mean the energies cancel.

(b) Let the neutral point be at distance x from M_1. Then it is at distance $(5.0 \times 10^8\text{m} - x)$ from M_2.

At the neutral point, the field strengths are equal:

$$\frac{GM_1}{x^2} = \frac{GM_2}{(5 \times 10^8 - x)^2}$$

G cancels:

$$\frac{M_1}{x^2} = \frac{M_2}{(5 \times 10^8 - x)^2}$$

Taking the square root of both sides:

$$\frac{\sqrt{M_1}}{x} = \frac{\sqrt{M_2}}{5 \times 10^8 - x}$$

Cross-multiplying:

$$(5\times10^8-\mathrm{x})\sqrt{\mathrm{M_1}}=\mathrm{x}\sqrt{\mathrm{M_2}}$$

$$5\times10^8\sqrt{\mathrm{M_1}}=\mathrm{x}\left(\sqrt{\mathrm{M_1}}+\sqrt{\mathrm{M_2}}\right)$$

$$\mathrm{x}=\frac{5\times10^8\mathrm{m}\times\sqrt{\mathrm{M_1}}}{\sqrt{\mathrm{M_1}}+\sqrt{\mathrm{M_2}}}$$

Substituting:

$$\mathrm{x}=\frac{5\times10^8\mathrm{m}\times\sqrt{4\times10^{24}\mathrm{kg}}}{\sqrt{4\times10^{24}\mathrm{kg}}+\sqrt{1\times10^{24}\mathrm{kg}}}=3.33\times10^8\mathrm{m}$$

The neutral point is $3.33\times10^8\mathrm{m}$ from $\mathrm{M_1}$.

Example 34

(a) Explain why a ball thrown upward from the Moon's surface would reach a much greater height than the same ball thrown with the same speed from the Earth's surface.

(b) A ball is thrown vertically upward from the surface of the Moon with a speed of 20m/s. Take $\mathrm{g_{Moon}}=1.6\mathrm{m/s^2}$.

(i) Calculate the maximum height reached.

(ii) If the same ball is instead thrown horizontally at 20m/s from the edge of a crater 50m deep on the Moon, calculate the time to reach the bottom and the horizontal distance covered.

Solution

(a) The maximum height reached by a ball thrown upward with speed u is $\mathrm{H}=\frac{\mathrm{u^2}}{2\mathrm{g}}$. Since the Moon's gravitational acceleration ($1.6\mathrm{m/s^2}$) is about 6 times weaker than the Earth's ($9.8\mathrm{m/s^2}$), the same initial speed produces about 6 times greater height on the Moon. The weaker gravity decelerates the ball more slowly, allowing it to travel much farther before stopping.

(b)

(i) Using $\mathrm{v^2}=\mathrm{u^2}-2\mathrm{g_{Moon}H}$, and setting $\mathrm{v}=0$ at maximum height:

$$0=(20\mathrm{m/s})^2-2\times1.6\mathrm{m/s^2}\times\mathrm{H}$$

$$\mathbf{H}=\frac{(20\mathrm{m/s})^2}{2\times1.6\mathrm{m/s^2}}=\mathbf{125m}$$

Make sense…! On Earth, the same throw would reach $\frac{(20\mathrm{m/s})^2}{2\times9.8\mathrm{m/s^2}}=20.4\mathrm{m}$. The Moon height is about 6 times greater, as expected.

(ii) The ball is thrown horizontally, so the initial vertical velocity is zero. On the Moon, the vertical acceleration is $\mathrm{g_{Moon}}=1.6\mathrm{m/s^2}$.

Time to fall 50m:

$$\mathrm{h}=\frac{1}{2}\mathrm{g_{Moon}t^2}$$

$$\mathrm{t}=\sqrt{\frac{2\mathrm{h}}{\mathrm{g_{Moon}}}}=\sqrt{\frac{2\times50\mathrm{m}}{1.6\mathrm{m/s^2}}}=\sqrt{62.5\mathrm{s^2}}=7.91\mathrm{s}$$

The time to reach the bottom is 7.91s.

Horizontal distance:

$$\mathrm{x}=\mathrm{u}\times\mathrm{t}=20\mathrm{m/s}\times7.91\mathrm{s}=158.1\mathrm{m}$$

The horizontal distance covered is 158.1m.

Do you see….? On the Moon, the ball takes 7.91s to reach the bottom (compared to 3.19s on Earth) and covers 158.1m horizontally (compared to 63.9m on Earth). The weaker gravity gives the ball much more time in the air, and therefore a much greater range.

Example 36

(a) Explain why a planet with the same density as Earth but twice the radius would have a surface gravity of 2g, not 4g.

(b) Planet X has a radius 3 times that of Earth and a surface gravity of 27m/s^2. Calculate the density of Planet X in terms of the Earth's density. Take $g_{\text{Earth}} = 9.8\text{m/s}^2$.

Solution

(a) For a planet with the same density ρ but radius $2r_E$, the mass is:

$$M = \rho \times \frac{4}{3}\pi(2r_E)^3 = 8\rho \times \frac{4}{3}\pi r_E^3 = 8M_E$$

The mass increases by a factor of 8 (because volume scales as r^3). The surface gravity is:

$$g = \frac{GM}{r^2} = \frac{G \times 8M_E}{(2r_E)^2} = \frac{8GM_E}{4r_E^2} = 2 \times \frac{GM_E}{r_E^2} = 2g_{\text{Earth}}$$

The mass increases 8-fold but the radius squared increases 4-fold, so the net effect is $\frac{8}{4} = 2$. Hence, for same density, g varies linearly with radius ($g \propto r$), not quadratically.

(b) Using $g = \frac{GM}{r^2}$ and $M = \rho \times \frac{4}{3}\pi r^3$:

$$g = \frac{G\rho \times \frac{4}{3}\pi r^3}{r^2} = \frac{4}{3}\pi G\rho r$$

For Earth: $g_E = \frac{4}{3}\pi G\rho_E r_E$

For Planet X: $g_X = \frac{4}{3}\pi G\rho_X \times 3r_E$

Dividing:

$$\frac{g_X}{g_E} = \frac{\rho_X \times 3r_E}{\rho_E \times r_E} = \frac{3\rho_X}{\rho_E}$$

$$\frac{\rho_X}{\rho_E} = \frac{g_X}{3g_E} = \frac{27\text{m/s}^2}{3 \times 9.8\text{m/s}^2} = 0.918$$

Hence $\rho_X = 0.918\rho_E$

The density of Planet X is approximately 0.92 times the Earth's density.

Example 36

(a) Explain the difference between gravitational potential and gravitational potential energy.

(b) A 500g object is at a point in the Earth's gravitational field where the gravitational potential is $-4.0 \times 10^7\text{J/kg}$. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(i) Calculate the gravitational potential energy of the object at this point.

(ii) Calculate the distance of this point from the centre of the Earth and the gravitational field strength at this point.

Solution

(a) Gravitational potential (V) is a property of the gravitational field at a particular point. It is defined as the work done per unit mass in bringing a test mass from infinity to that point. Its unit is J/kg, and it does not depend on the mass of any object placed there.

Gravitational potential energy (U) is a property of the system of two masses. It is the actual energy stored in the gravitational interaction when a specific mass m is placed at that point: $U = mV$. Its unit is J. Unlike potential, it depends on the mass of the object.

In short: potential describes the field; potential energy describes what happens when a specific mass enters that field.

(b) The solution of each part is as follows:

(i) Converting mass: $m = 500\text{g} = 0.5\text{kg}$

$$\mathbf{U} = \text{mV} = 0.5\text{kg} \times (-4.0 \times 10^7\text{J/kg}) = \mathbf{-2.0 \times 10^7 J = -20MJ}$$

(ii) From $V = -\frac{GM_E}{r}$, making r the subject:

$$r = -\frac{GM_E}{V} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{-4.0 \times 10^7\text{J/kg}}$$

$$r = \frac{4.002 \times 10^{14}\text{Nm/kg}}{4.0 \times 10^7\text{J/kg}} = 1.0 \times 10^7\text{m} = 10000\text{km}$$

The height above the surface:

$$\mathbf{h} = r - r_E = 10000\text{km} - 6400\text{km} = \mathbf{3600km}$$

The gravitational field strength at this point:

$$\mathbf{g} = \frac{GM_E}{r^2} = \frac{4.002 \times 10^{14}\text{Nm/kg}}{(1 \times 10^7\text{m})^2} = \frac{4.002 \times 10^{14}}{1 \times 10^{14}} = \mathbf{4.0N/kg}$$

Alternatively, using $g = -\frac{V}{r} = \frac{4.0\times10^7\text{J/kg}}{1.0\times10^7\text{m}} = \mathbf{4.0N/kg}$. Both methods agree.

Example 37

(a) Explain why a geostationary satellite must orbit in the equatorial plane and cannot be placed in a polar orbit with a 24-hour period.

(b) A satellite is observed to have an orbital period of 8 hours. Calculate its orbital radius and height above the Earth's surface. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

(a) A geostationary satellite must appear stationary above a fixed point on the Earth's surface. For this to happen, the satellite must rotate with the Earth at exactly the same angular velocity and in the same direction. If the satellite were in a polar orbit (passing over the poles), it would orbit in a plane that does not rotate with the Earth. Even if its period were 24 hours, it would appear to trace a figure-of-eight path as seen from the ground, moving north and south during the day, and would not remain fixed above one point. Only an orbit in the equatorial plane keeps the satellite directly above the same point on the equator at all times.

(b) Converting the period: $T = 8\text{h} = 8 \times 3600\text{s} = 28800\text{s}$

Using Kepler's third law:

$$R^3 = \frac{GM_E T^2}{4\pi^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg} \times (28800\text{s})^2}{4 \times (3.14)^2}$$

$$R^3 = \frac{4.002 \times 10^{14}\text{Nm/kg} \times 8.294 \times 10^8\text{s}^2}{39.48} = \frac{3.319 \times 10^{23}}{39.48} = 8.407 \times 10^{21}\text{m}^3$$

$$R = \sqrt[3]{(8.407 \times 10^{21}\text{m}^3)} = 2.032 \times 10^7\text{m} = 20320\text{km}$$

The orbital radius is 20320km.

Height above the surface:

$$h = R - r_E = 20320\text{km} - 6400\text{km} = 13920\text{km}$$

The satellite orbits at a height of 13920km above the surface.

Example 38

(a) A satellite in a circular orbit fires its engines briefly to increase its speed. Explain what happens to the shape of the orbit, the altitude, and the period immediately after the burn.

(b) A 300kg satellite is in a circular orbit at 600km altitude. It needs to be moved to a circular orbit at 1000km altitude. Calculate the total energy in each orbit and the energy that must be supplied by the engines. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^{6}\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

(a) Immediately after a brief speed increase, the satellite has more kinetic energy than required for its current circular orbit. The orbit is no longer circular; it becomes **elliptical**, with the burn point as the lowest point (perigee) of the new ellipse. The satellite climbs to a **higher altitude** on the opposite side of the orbit (apogee). The **period increases** because the elliptical orbit has a larger semi-major axis. If a second burn is performed at the apogee to increase speed again, the orbit can be circularised at the higher altitude.

(b) Orbital radius at 600km:

$$R_1 = r_E + 600 \times 10^3\text{m} = 6.4 \times 10^6\text{m} + 6.0 \times 10^5\text{m} = 7.0 \times 10^6\text{m}$$

Total energy in the lower orbit:

$$E_1 = -\frac{GM_Em}{2R_1} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg} \times 300\text{kg}}{2 \times 7.0 \times 10^6\text{m}}$$

$$\mathbf{E_1} = -\frac{1.201 \times 10^{17}\text{J}}{1.4 \times 10^7\text{m}} = \mathbf{-8.576 \times 10^9 J = -8.58GJ}$$

Orbital radius at 1000km:

$$R_2 = r_E + 1000 \times 10^3\text{m} = 6.4 \times 10^6\text{m} + 1.0 \times 10^6\text{m} = 7.4 \times 10^6\text{m}$$

Total energy in the higher orbit:

$$\mathbf{E_2} = -\frac{GM_Em}{2R_2} = -\frac{1.201 \times 10^{17}\text{J}}{2 \times 7.4 \times 10^6\text{m}} = \mathbf{-8.115 \times 10^9 J = -8.12GJ}$$

Energy supplied by the engines:

$$\Delta E = E_2 - E_1 = (-8.12\text{GJ}) - (-8.58\text{GJ}) = 0.46\text{GJ} = \mathbf{4.6 \times 10^8 J = 460MJ}$$

The engines must supply approximately 460MJ of energy.

Make sense…! This is a modest amount compared to the total orbital energy, because the satellite is only being raised by 400km.

You have survived all subtopics of gravitation. You have derived orbital velocities, escaped planets, weighed the Sun, and sent a spacecraft to Mars (on paper, at least). The miscellaneous examples gave you a taste of what happens when the ideas combine. Now the Digging Deeper Exercise takes over in the next page, and it has no worked solutions to hold your hand. Just you, the physics, and the knowledge that G is a very small number. Good luck. You are ready.

DIGGING DEEPER EXERCISE 8

EXERCISE 8A: BINDER QUESTIONS

Question 1

A heavy iron ball and a light wooden ball are released from the same height above the ground in a vacuum. Explain why both balls reach the ground at the same time.

Question 2

Newton’s law of gravitation states that every mass attracts every other mass. Explain why you do not feel a gravitational pull toward the person sitting next to you in class, even though the force exists.

Question 3

The Earth exerts a gravitational force on the Moon, and the Moon exerts an equal force on the Earth (Newton’s third law). Explain why the Moon orbits the Earth rather than both bodies moving equally.

Question 4

Explain why the gravitational potential at every point near a planet is negative, and explain what it means physically for the potential to become “more negative” as you move closer to the planet.

Question 5

Explain why the acceleration due to gravity decreases with altitude above the Earth’s surface but also decreases with depth below the surface, even though the reasons are different in each case.

Question 6

Explain why all satellites at the same altitude orbit the Earth with the same speed and the same period, regardless of their mass.

Question 7

The total mechanical energy of a satellite in circular orbit is negative. Explain what this means physically, and explain what would happen if the total energy became zero, and if it became positive.

Question 8

Explain why the escape velocity from a planet does not depend on the direction in which the object is launched.

Question 9

Explain why the Moon has no atmosphere while the Earth retains a thick one, using the concept of escape velocity.

Question 10

A satellite in circular orbit gradually loses energy due to atmospheric drag. Explain why the satellite speeds up rather than slows down as it loses energy.

EXERCISE 8B: REAL QUESTIONS

Question 11

A precision digital scale in a gold shop in Dar es Salaam reads 100.000g for a gold sample. The same sample is taken to a shop in Iringa (altitude about 1600m) and placed on an identical scale. Explain whether the reading in Iringa will be the same, higher, or lower, and whether the actual mass of the gold has changed.

Question 12

Astronauts aboard the International Space Station (orbiting at about 400km altitude) are often shown floating inside the spacecraft. Many people believe this is because there is no gravity in space. Explain why this belief is incorrect and describe the real reason for the apparent weightlessness.

Question 13

Tanzania's coastline experiences two high tides and two low tides each day. Explain why there are two high tides per day rather than just one.

Question 14

Kipanga says: *"If I dug a tunnel straight through the centre of the Earth and jumped in, I would fall faster and faster until I reached the centre, and then I would be crushed by the enormous gravity there."*

Kipute replies: *"You have at least two things wrong, Kipanga."*

Identify and explain the two errors in Kipanga's reasoning.

Question 15

Explain why rockets are launched from sites as close to the equator as possible, and why they are usually launched in the eastward direction.

Question 16

A geostationary satellite and a low-orbit satellite both orbit the Earth. The geostationary satellite is much higher and moves much more slowly. Explain why the geostationary satellite has more total energy (less negative) than the low-orbit satellite, even though it moves more slowly.

Question 17

Explain why the value of g at the Earth's surface varies from about 9.78m/s^2 at the equator to about 9.83m/s^2 at the poles, giving both reasons for this variation.

Question 18

Kipanga looks at the night sky and asks Mr. Akilikubwa: *"Sir, if the Sun suddenly disappeared, would the Earth immediately fly off in a straight line, or would it keep orbiting for a while?"*

Mr. Akilikubwa replies: *"That is a much deeper question than you realise, Kipanga."*

Explain what would happen to the Earth's motion if the Sun's gravitational field suddenly vanished, and explain why the Earth would move in a straight line rather than a curve.

Question 19

The Voyager 1 spacecraft, launched in 1977, used gravity assists from Jupiter and Saturn to gain enough speed to leave the solar system. Explain how a spacecraft can gain speed by flying past a planet, even though no fuel is burned during the encounter.

Question 20

Kepler's third law states that $T^2 \propto r^3$ for planets orbiting the Sun. Explain why this relationship only holds for objects orbiting the same central body, and explain what would change if the central body had a different mass like Earth.

EXERCISE 8C: HOT QUESTIONS

Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$, $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6400\text{km}$, *and* $g = 9.8\text{m/s}^2$ *unless otherwise stated.*

Question 21

The Moon orbits the Earth at a mean distance of 3.84×10^8m with a period of 27.3 days.

(a) Use this data to calculate the mass of the Earth.

(b) Calculate the orbital speed of the Moon.

(c) Using your value from (a) and the known radius of the Earth, calculate g at the Earth's surface and compare with the accepted value of 9.8m/s^2.

Question 22

(a) Calculate the gravitational potential at the surface of the Earth.

(b) A 2.0kg object is lifted from the surface to a height where the potential is -4.0×10^7J/kg. Calculate the work done against gravity.

(c) Calculate the gravitational field strength at the height found in (b).

Question 23

A student calculates the height at which g is halved using the approximate formula:

$$\frac{1}{2}g = g\left(1 - \frac{2h}{r_E}\right),$$

and gets:

$$h = \frac{r_E}{4} = 1600\text{km}$$

(a) Use the exact formula to find the correct height.

(b) Calculate the percentage error introduced by the student's use of the approximation.

(c) Why the approximation fails here.

Question 24

(a) Calculate the escape velocity from the surface of a planet whose mass is 3.2×10^{23}kg and whose radius is 2.4×10^6m.

(b) Show that the escape velocity from this planet's surface is $\sqrt{2}$ times the orbital velocity near the surface.

(c) A probe is launched from the surface of this planet at 5000m/s. Determine whether the probe will escape the planet, orbit it, or fall back.

Question 25

Two masses of 5.0×10^{24}kg and 2.0×10^{24}kg are separated by a distance of 6.0×10^8m.

(a) Find the position of the neutral point (where the net gravitational field is zero) measured from the larger mass.

(b) Calculate the gravitational potential at the neutral point.

Question 26

A student calculates the energy required to launch a satellite into orbit by computing only the kinetic energy in orbit: $E_{\text{launch}} = \frac{1}{2}mv^2$. The satellite has mass 500kg and orbits at 400km altitude.

(a) Calculate the student's answer in GJ.

(b) Calculate the correct minimum launch energy.

(c) Explain why the student's answer is wrong, and determine whether it overestimates or underestimates the true value.

Question 27

The planet Mars has mass 6.4×10^{23}kg and radius 3.4×10^{6}m.

(a) Calculate the acceleration due to gravity on the surface of Mars.

(b) Calculate the escape velocity from the surface of Mars.

(c) A ball is thrown vertically upward on Mars with a speed of 15m/s. Calculate the maximum height reached and the total time of flight.

Question 28

An engineer designs a satellite system for weather monitoring. The satellite must orbit at an altitude that gives it a period of exactly 6 hours.

(a) Calculate the orbital radius and altitude of this satellite.

(b) Calculate the orbital speed.

(c) Calculate the minimum energy required to launch a 400kg satellite into this orbit from the Earth's surface.

(d) If the satellite costs Tsh 50 billion and the launch fuel costs Tsh 200,000 per megajoule, estimate the fuel cost and comment on whether it is a significant fraction of the total cost.

Question 29

A planet has uniform density $\rho = 5000\text{kg/m}^3$ and radius $\text{r} = 5.0 \times 10^{6}$m.

(a) Calculate the mass of the planet.

(b) Calculate g at the surface.

(c) Calculate g at a depth of 2.0×10^{6}m below the surface.

(d) Calculate the gravitational potential at the surface and at the centre.

Question 30

A satellite is in a circular orbit at 500km altitude.

(a) Calculate its orbital speed and escape velocity at this altitude.

(b) By what factor must the satellite's speed be increased to escape the Earth from this orbit?

(c) By what factor must the satellite's kinetic energy be increased to escape?

(d) If the satellite has mass 800kg, calculate the energy that must be supplied to escape from this orbit.

Question 31

A space agency plans to place a 1200kg satellite in geostationary orbit.

(a) Calculate the orbital radius, speed, and period of the geostationary orbit.

(b) Calculate the total energy of the satellite in the geostationary orbit.

(c) Calculate the minimum energy required to launch the satellite from the Earth's surface.

(d) If the rocket's overall efficiency (fuel energy to useful orbital energy) is 5%, and rocket fuel has an energy density of 3.0MJ/kg, calculate the mass of fuel required.

Question 32

The Earth orbits the Sun at 1.5×10^{11}m with a period of 1 year. Jupiter orbits at 7.78×10^{11}m.

(a) Calculate the orbital period of Jupiter in years.

(b) A spacecraft is sent from Earth to Jupiter along a Hohmann transfer orbit. Calculate the transfer time.

Question 33

At what angular velocity would the Earth need to rotate so that a person at the equator would feel 75% of their normal weight?

(a) Calculate this angular velocity.

(b) Express it as a multiple of the Earth's current angular velocity.

(c) Find the length of a "day" at this rotation speed.

Question 34

A body is projected vertically upward from the Earth's surface with a speed equal to half the escape velocity.

(a) Show that the maximum height reached is $\frac{r_E}{3}$.

Calculate this height in kilometres.

ANSWERS

EXERCISE 8A

1. The gravitational force on each ball is $F = \frac{GMm}{r^2}$, where m is the mass of the ball. By Newton's second law, the acceleration is $a = \frac{F}{m} = \frac{GM}{r^2}$. Because the mass m appears in both the gravitational force (which pulls the ball down) and the inertia (which resists acceleration), these two effects cancel exactly. As a result, the acceleration depends only on M (the Earth's mass) and r (the distance from the centre), not on m. Therefore, both balls fall with the same acceleration g and, since they start from the same height with zero initial velocity, they reach the ground at the same time.

2. The gravitational force between two people is $F = \frac{Gm_1m_2}{r^2}$. For two people the product of mass is so small that the gravitational pull is negligible. This force is because the gravitational constant G is extraordinarily tiny (6.67×10^{-11}). Gravity only becomes noticeable when at least one of the masses is extremely large (like a planet), because a large mass compensates for the smallness of G. Since neither person has a mass anywhere close to a planet, the force between them is far too weak to feel.

3. By Newton's third law, the gravitational forces are equal in magnitude. However, by Newton's second law, the acceleration produced is $a = \frac{F}{m}$, which depends on the mass of the body being accelerated. Because the Earth is about 81 times more massive than the Moon, the Earth's acceleration is 81 times smaller than the Moon's for the same force. As a result, the Moon moves in a large, visible orbit, while the Earth merely wobbles slightly.

4. Gravitational potential is defined relative to infinity, where $V = 0$. As a mass moves from infinity toward a planet, gravity does positive work on it (pulling it inward), so the mass loses potential energy. Because the potential energy at infinity is zero and the mass has less energy at any point near the planet, the potential must be negative. The closer to the planet, the more work gravity has done, and therefore the more potential energy has been lost. Thus the potential becomes more negative with decreasing distance. Physically, a more negative potential means the object is more tightly bound (attracted) to the planet, and more energy would be needed to free it and return it to infinity.

5. Above the surface, g decreases because the distance from the centre of the Earth increases. Since $g = \frac{GM_E}{r^2}$, a larger r gives a smaller g. The entire mass of the Earth still contributes to the gravitational pull, but the inverse square law weakens it with distance.

Below the surface, g decreases for a completely different reason. As you descend, the shell of mass above you exerts zero net gravitational force on you (by the Shell Theorem). Therefore, only the mass below you contribute to g. Because this effective mass decreases as you go deeper, g decreases as well. At the centre, all the mass is distributed symmetrically around you, giving zero net force and hence $g = 0$.

Thus above the surface, g decreases because distance increases. Below the surface, g decreases because the effective mass decreases.

6. For a satellite in circular orbit, gravity provides the centripetal force: $\frac{mv^2}{R} = \frac{GM_Em}{R^2}$. The satellite's mass m appears on both sides and cancels, giving $v = \sqrt{\frac{GM_E}{R}}$. Because v depends only on M_E and R, all satellites at the same orbital radius R have the same speed. Since the period is $T = \frac{2\pi R}{v}$, and both R and v are independent of m, the period is also the same for all satellites at the same altitude.

8. A negative total energy means the satellite is gravitationally bound to the Earth. The satellite has kinetic energy (positive) and gravitational potential energy (negative), but the magnitude of the potential energy exceeds the kinetic energy, so the sum is negative. As a result, the satellite does not have enough energy to escape to infinity; it remains trapped in orbit.

If the total energy became exactly zero, the satellite would have just enough energy to reach infinity with zero residual speed. This is the threshold of escape. The satellite would follow a parabolic trajectory, never returning but barely escaping. If the total energy became positive, the satellite would escape with speed to spare, following a hyperbolic trajectory. Therefore, negative total energy means a bound orbit, zero means the boundary of escape, and positive means the object is free.

8. The escape velocity is derived from conservation of energy: $\frac{1}{2}mv_E^2 - \frac{GMm}{r} = 0$. This is a scalar equation involving only the magnitude of the speed, not its direction. Because energy has no direction (it is a scalar quantity), the total kinetic energy is the same whether the object moves vertically, horizontally, or at any angle. Therefore, as long as the speed equals v_E and the object does not collide with the planet's surface on its way out, it will escape regardless of the direction of launch.

9. The escape velocity from the Moon (2.4km/s) is much lower than the escape velocity from the Earth (11.2km/s), because the Moon has much less mass. Gas molecules in any atmosphere are in constant random motion, and their speeds depend on the temperature. At the temperatures found on the Moon's surface (which can exceed 100°C in direct sunlight), the fastest gas molecules reach speeds above 2.4km/s and escape into space. Over billions of years, this gradual loss has stripped the Moon of all its atmospheric gases.

On the Earth, the average molecular speed at typical surface temperatures is about 0.3 to 0.6km/s, which is far below the escape velocity of 11.2km/s. As a result, virtually no atmospheric molecules have enough speed to escape, and the Earth retains its atmosphere comfortably.

10. When a satellite loses energy, its total energy $E = -\frac{GM_Em}{2R}$ becomes more negative. Because E is inversely proportional to R, a more negative E corresponds to a smaller orbital radius. The satellite therefore drops to a lower orbit.

At this lower orbit, the orbital speed is $v = \sqrt{\frac{GM_E}{R}}$. Since R has decreased, v increases. Thus the satellite moves faster despite having lost total energy. This happens because the fall to a lower orbit converts gravitational potential energy into kinetic energy. The gain in kinetic energy exceeds the energy lost to drag, so the net effect is an increase in speed.

EXERCISE 8B

11. The reading in Iringa will be slightly **lower**. A digital scale measures the gravitational force (weight) on the object and converts it to a mass reading using a calibrated value of g. Because Iringa is at a higher altitude than Dar es Salaam, the gravitational field strength g is slightly weaker there (since g decreases with altitude). As a result, the gravitational force on the gold is slightly less, and the scale displays a slightly lower reading.

However, the actual mass of the gold has not changed at all. Mass is an intrinsic property of the object and does not depend on location. Only the weight (the gravitational force) changes with altitude.

12. This belief is incorrect because the gravitational field at 400km altitude is still about 94% as strong as on the ground. Gravity is very much present at the altitude of the ISS.

The astronauts float because both they and the spacecraft are in **free fall** toward the Earth at exactly the same rate. Since gravity accelerates everything equally, the astronaut and the floor of the spacecraft fall together. There is no relative acceleration between the astronaut and the spacecraft, and therefore no contact force (no normal reaction from the floor, no tension from any support). With zero contact force, the astronaut feels weightless.

13. The Moon's gravity pulls on the Earth, but the pull is not uniform. It is strongest on the side of the Earth closest to the Moon (because that side is nearer) and weakest on the side farthest from the Moon (because that side is more distant). This difference in gravitational pull, called the tidal force, stretches the Earth's oceans along the Earth-Moon line.

On the near side, the Moon's pull is stronger than the average, so water is pulled toward the Moon, creating a bulge (high tide). On the far side, the Moon's pull is weaker than the average, so the water is "left behind" relative to the solid Earth, creating a second bulge on the opposite side. As the Earth rotates on its axis once per day, any coastal location passes through both bulges, experiencing two high tides and two low tides approximately every 24 hours.

14. First error: Kipanga says gravity is enormous at the centre. This is wrong. As you descend into the Earth, the shell of mass above you exerts zero net force on you (Shell Theorem, Part 2). Only the mass below you pulls you downward. Because this effective mass decreases as you approach the centre, g decreases as well. At the centre, all the mass is distributed symmetrically around you, so the net gravitational force is zero. Therefore, far from being crushed, Kipanga would experience zero gravity at the centre.

Second error: Kipanga says he would fall faster and faster all the way to the centre. This is also wrong. He would accelerate as he falls (because g still points toward the centre), but g is decreasing as he descends. He would reach maximum speed at the centre (where $g = 0$ and the acceleration changes direction). After passing the centre, gravity would pull him back, decelerating him. He would oscillate back and forth through the centre, like a mass on a spring, never stopping at the centre unless resistive forces like air resistance slowed him down.

15. The Earth rotates from west to east (in the eastward direction). A point on the equator moves (at about 465m/s) due to the Earth's rotation. Because the rocket is already moving eastward (with Earth) at this speed before launch, launching in the eastward direction adds this speed to the rocket's velocity for free, reducing the amount of fuel needed to reach orbital speed.

The equator provides the maximum benefit because the rotational speed is greatest there (the equator has the largest distance from the rotation axis). At higher latitudes, the rotational speed is smaller (since $v = \omega r_E \cos\theta$), so the "free" speed boost is reduced. Therefore, launching from an equatorial site in the eastward direction gives the most efficient use of fuel.

16. The total energy of a satellite in circular orbit is $E = -\frac{GM_E m}{2R}$. The geostationary satellite has a much larger orbital radius R, so its total energy is less negative (closer to zero). This means it is less tightly bound to the Earth.

Although the geostationary satellite has less kinetic energy (because it moves more slowly), it has much more gravitational potential energy (because it is much farther from the Earth). The increase in potential energy far exceeds the decrease in kinetic energy. As a result, the total energy of the geostationary satellite is higher (less negative) than that of the low-orbit satellite. This is consistent with the fact that more energy must be supplied to place a satellite in a higher orbit.

17. Reason 1 (Shape): The Earth is not a perfect sphere; it is slightly flattened at the poles and bulges at the equator (an oblate spheroid). The polar radius is shorter (about 21km shorter) than the equatorial radius. Since $g \propto \frac{1}{r^2}$, the smaller radius at the poles means the surface is closer to the centre, and therefore g is slightly larger there.

Reason 2 (Rotation): The Earth rotates, and a body on the surface requires centripetal acceleration to move in a circle. At the equator, the centripetal acceleration is $\omega^2 r_E$ (0.034m/s^2), which is directed inward (toward the rotation axis). Because part of the gravitational pull goes toward providing this centripetal acceleration, the effective g measured at the equator is reduced. At the poles, the body is on the rotation axis, so no centripetal acceleration is needed and the full gravitational pull is felt.

Both effects work together: they make g smaller at the equator and larger at the poles.

18. If the Sun's gravitational field suddenly vanished, the Earth would immediately move in a straight line tangent to its orbit at the instant of disappearance, at its current orbital speed (about 30km/s). It would not continue to curve, and it would not "orbit for a while."

The reason is Newton's first law. An object continues in a straight line at constant speed unless acted upon by a net force. The only force keeping the Earth in a curved orbit is the Sun's gravity. The moment that force disappears, there is nothing to deflect the Earth from a straight path. Therefore, the Earth would fly off along the tangent to its orbit at that instant.

This also shows that circular (or elliptical) motion requires a continuous centripetal force. The orbit is not a natural state; it is a constant forced deflection from straight-line motion. Remove the force, and the natural straight-line motion resumes immediately.

19. In a gravity assist, the spacecraft approaches a planet, is deflected by the planet's gravity, and leaves with a different speed and direction. From the planet's reference frame, the spacecraft's speed relative to the planet is the same before and after the encounter (it is like an elastic collision). However, from the Sun's reference frame, the spacecraft gains speed because the planet itself is moving.

The key is that the planet is orbiting the Sun at high speed. As the spacecraft swings around the planet, it is briefly "dragged along" by the planet's gravity in the direction of the planet's motion. This transfers a tiny amount of the planet's orbital kinetic energy to the spacecraft. Because the planet is enormously more massive than the spacecraft, the planet's speed decreases by an unmeasurably small amount, while the spacecraft gains a significant speed boost. Hence the spacecraft accelerates without burning fuel, at the expense of a negligible slowing of the planet.

20. Kepler's Third Law, derived from Newton's gravitation, gives $T^2 = \frac{4\pi^2}{GM} r^3$, where M is the mass of the central body. The constant of proportionality $\frac{4\pi^2}{GM}$ depends on M. For all planets orbiting the Sun, M is the same (the Sun's mass), so they all share the same constant, and $T^2 \propto r^3$ holds with a single constant.

However, if we compare an object orbiting the Sun with an object orbiting the Earth, the two central masses are different. Because $M_{Sun} \gg M_{Earth}$, the constant $\frac{4\pi^2}{GM}$ is much smaller for the Sun, meaning that for the same orbital radius, the period around the Sun would be much shorter. Therefore the T^2-r^3 relationship holds within each system (planets around the Sun, moons around Jupiter, satellites around the Earth), but with a different constant for each central body. A more massive central body gives a smaller constant, hence shorter periods for the same radius.

EXERCISE 8C

21. (a) 6.02×10^{24}kg (b) 1023m/s (or 1.02km/s) (c) 9.80m/s^2. This agrees with the accepted value of 9.8m/s^2, confirming the consistency of the data.

22. (a) -6.25×10^7J/kg (b) 4.5×10^7J (or 45MJ) (c) 4.0N/kg

23. (a) 2651km (b) 39.6% (c) The height, h was significantly large compared to the radius of the earth.

24. (a) 4218m/s (b) $v_{orb} = 2982$m/s, Ratio: $\frac{v_E}{v_{orb}} = \frac{4218}{2982} = 1.414 = \sqrt{2}$. Confirmed.

(c) The probe's speed (5000m/s) exceeds the escape velocity (4218m/s). Because the probe has more than enough kinetic energy to overcome the gravitational potential well, it will escape the planet entirely and fly off into space. It will not orbit, because orbiting requires a specific speed less than escape velocity.

25. (a) 3.68×10^8m (b) -1.481×10^6J/kg

26. (a) 14.7GJ (b) 16.6GJ

(c) The student's answer (14.7GJ) **underestimates** the true value (16.6GJ). The student accounted for the kinetic energy needed in orbit but forgot that the satellite must also be lifted from the surface to the orbital height. Lifting requires increasing the gravitational potential energy, which costs additional energy ($\Delta U = 1.9$GJ in this case). The correct calculation includes both the kinetic energy and the change in potential energy automatically, because it uses the total energy difference between the surface and the orbit.

27. (a) 3.69m/s^2 (b) 5.0km/s (c) 30.5m, 8.13s

28. (a) R = 16780km, h = 10380km (b) 4.88km/s (c) 20240MJ or (2.024×10^{10}J)

(d) Fuel cost: 20240MJ × Tsh 200,000/MJ = Tsh 4.048×10^9 ≈ Tsh 4 billion.

As a fraction of total cost: $\frac{4}{50} = 8\%$. The fuel cost is a relatively small fraction of the total satellite cost. The satellite itself (with its instruments, solar panels, and communication equipment) is far more expensive than the fuel needed to launch it. This is why making satellites lighter is less about fuel savings and more about reducing the size (and cost) of the rocket.

29. (a) 2.618×10^{24}kg (b) 6.98m/s^2 (c) 4.19m/s^2 (d)Surface: -3.49×10^7J/kg, centre: -5.24×10^7J/kg

30. (a) $v_{orb} = 7616$m/s, $v_E = 10770$m/s (b) Speed factor $= \sqrt{2} = 1.414$ (c) Energy factor = 2 (d) 2.320×10^{10}J or 23.2GJ

31. (a) R = 42300km, v = 3.08km/s,T = 24 hours (by design) (b) −5.68GJ (c) 69.3GJ

(d) Fuel energy needed (at 5% efficiency): $\frac{69.3\text{GJ}}{0.05} = 1386\text{GJ} = 1.386 \times 10^{12}\text{J}$

Fuel mass: $\frac{1.386\times10^{12}\text{J}}{3.0\times10^{6}\text{J/kg}} = 4.62 \times 10^5\text{kg} = 462$ tonnes

The fuel mass (462 tonnes) is about 385 times the satellite mass (1200kg). This enormous fuel-to-payload ratio illustrates why reaching geostationary orbit is one of the most expensive feats in engineering.

32. (a) 11.8 years

(b) Semi-major axis of transfer orbit:

$$a = \frac{r_E + r_J}{2} = \frac{1.5 \times 10^{11} + 7.78 \times 10^{11}}{2} = \frac{9.28 \times 10^{11}}{2} = 4.64 \times 10^{11}\text{m}$$

Period of the transfer orbit:

$$\frac{T_{transfer}^2}{T_E^2} = \frac{a^3}{r_E^3} = \frac{(4.64 \times 10^{11})^3}{(1.5 \times 10^{11})^3} = \frac{9.987 \times 10^{34}}{3.375 \times 10^{33}} = 29.59$$

$$T_{transfer} = \sqrt{29.59} \times 1 \text{ year} = 5.44 \text{ years}$$

Transfer time (half the orbit): $\frac{5.44}{2} = 2.72$ years

A Hohmann transfer to Jupiter takes approximately 2.7 years.

33. (a) 6.19×10^{-4}rad/s (b) 8.52 (c) 2.82h (2 hours, 49 minutes).

34(a) The launch speed is $v = \frac{1}{2}v_E = \frac{1}{2}\sqrt{\frac{2GM_E}{r_E}}$

Using conservation of energy between the surface and the maximum height h (where $v = 0$):

$$\frac{1}{2}mv^2 - \frac{GM_Em}{r_E} = 0 - \frac{GM_Em}{r_E + h}$$

Substituting $v^2 = \frac{1}{4} \times \frac{2GM_E}{r_E} = \frac{GM_E}{2r_E}$:

$$\frac{1}{2}m \times \frac{GM_E}{2r_E} - \frac{GM_Em}{r_E} = -\frac{GM_Em}{r_E + h}$$

$$\frac{GM_Em}{4r_E} - \frac{GM_Em}{r_E} = -\frac{GM_Em}{r_E + h}$$

Dividing through by GM_Em:

$$\frac{1}{4r_E} - \frac{1}{r_E} = -\frac{1}{r_E + h}$$

$$\frac{1 - 4}{4r_E} = -\frac{1}{r_E + h}$$

$$-\frac{3}{4r_E} = -\frac{1}{r_E + h}$$

$$\frac{3}{4r_E} = \frac{1}{r_E + h}$$

$$r_E + h = \frac{4r_E}{3}$$

$$h = \frac{4r_E}{3} - r_E = \frac{4r_E - 3r_E}{3} = \frac{r_E}{3}$$

(b) $h = \frac{6400\text{km}}{3} = 2133\text{km}$