Chapter 7

UNIFORM CIRCULAR MOTION

INTRODUCTION

The Miono Secondary School bus was returning from a geography field trip when disaster struck while negotiating a sharp bend at Kilimani Hill. The driver, perhaps distracted by Kipanga's loud singing from the back seats, entered the curve at excessive speed. Physics cared nothing for good intentions or desperate braking. The tires screeched, lost their grip on the asphalt, and the bus began its inevitable journey off the road; sliding sideways like a reluctant dancer, crashing through the wooden barrier, and finally coming to rest at a precarious angle in the roadside ditch.

Silence. Then chaos. Students screamed, bags tumbled, and someone's food container burst open, sending mandazi flying through the air in parabolic arcs that would have made Chapter 6 proud!

Miraculously, no one was seriously hurt. As Mr. Akilikubwa helped the shaken students get out of the bus, **Kipanga** stood there dusty and untidy. He asked in confusion, *"Sir, the driver was turning the steering wheel. I saw him. Why didn't the bus turn?"*

Mr. Akilikubwa pointed at the dramatic skid marks etched into the road surface. The marks told an unambiguous story: they curved initially, obediently following the road's bend, then suddenly straightened into a ruler-straight line heading directly toward the ditch. *"Kipanga, the driver did turn the steering wheel. But steering alone doesn't bend a vehicle's path. Look at those marks. See how they curve, then suddenly go straight! That's the moment the tires lost their grip."*

Kipute, brushing dirt from her uniform, studied the marks with scientific curiosity. *"So the bus wanted to go straight, but the road wanted it to curve?"*

"Exactly!" **Mr. Akilikubwa's** eyes lit up despite the circumstances. *"Every object moving in a straight line wants to continue in a straight line, remember Newton's first law? To force something into a curved path requires a continuous inward force pulling it toward the center of the curve. For a car on a road, that force comes from friction between tires and asphalt."*

He walked to where the skid marks transitioned from curved to straight. *"But friction has limits. At low speeds on gentle curves, no problem. But this curve is sharp, and we were going too fast. The required inward force exceeded what friction could provide. The instant that happened..."* he snapped his fingers *"...the tires slid instead of gripped, the inward force vanished, and the bus immediately obeyed Newton's first law. Straight line. Straight into the ditch."*

Kipanga looked nervously at the bent barrier. *"So... if we'd gone even faster?"*

"We'd have slid off even earlier in the curve," **Mr. Akilikubwa** said seriously. *"This is why speed limits exist on curves, Kipanga. This is why mountain roads have those warning signs with curved arrows and recommended speeds. Engineers calculate the maximum safe speed based on the curve's radius and the expected friction. Exceed that speed, and physics becomes unforgiving."*

As they waited for the rescue vehicle, **Mr. Akilikubwa** sketched a diagram in his notebook. ***"Circular motion,"*** he announced, *"is nature's most deceptive phenomenon. It looks simple! Just going around in circles. But beneath that apparent simplicity lies a profound truth: maintaining a circular path requires constant work, constant force! Satellites circle Earth at precisely calculated speeds because gravity provides relentless inward pull. Clothes stick to the drum of a spinning washing machine because the drum wall provides inward force."*

Kipanga nodded slowly, understanding dawning. *"So circular motion isn't the natural state. Straight-line motion is natural. Curves require force."*

"Precisely! You've just grasped the central insight of this entire chapter." **Mr. Akilikubwa** smiled. *"And you learned it from skid marks and a ditch instead of a textbook. Perhaps this accident was the universe's way of teaching physics."*

"I would have preferred the textbook, sir," **Kipanga** muttered, picking mandazi crumbs from hair.

Welcome to circular motion, where every curve is a battle against inertia, where every turn demands its tribute of force, and where the difference between a safe journey and a roadside ditch can be calculated with precision. In this chapter, we will discover why the moon does not fall despite Earth's gravity constantly pulling it inward, why racing cars lean dramatically into turns, why your wet clothes cling to the washing machine drum during spin cycle, and why that innocent-looking curve at Kilimani Hill requires respect, attention, and most importantly, the right speed. The bus has already taught us the consequences of getting it wrong. Now let us learn the mathematics of getting it right.

UNDERSTANDING UNIFORM CIRCULAR MOTION

Circular motion *is the movement of an object along a circular path*. You observe it everywhere, for example: a car turning a corner, a satellite orbiting Earth, clothes spinning in a washing machine, a stone whirled on a string, the moon circling our planet, or a ball rolling around the inside of a bowl.

The most important insight about circular motion is this: it is not natural. Left undisturbed, every object moves in a straight line at constant velocity accordance with Newton's first law. To force an object into a curved path requires continuous application of force. Remove that force, and the object immediately returns to straight-line motion, as our bus at Kilimani Hill dramatically demonstrated.

When an object moves along a circular path at constant speed, we call this **uniform circular motion.** The word "uniform" refers to the speed remaining constant; the object covers equal distances in equal times around the circle. A satellite orbiting Earth at constant altitude, a car maintaining steady speed around a circular track, or a DVD spinning at constant rotation rate all exhibit uniform circular motion.

But here lies a subtle trap that catches many students: *constant speed does **not** mean constant velocity*. This distinction is crucial to understanding circular motion.

Velocity in Circular Motion

Velocity is a vector quantity possessing both magnitude (speed) and direction. In circular motion, even when speed remains constant, the direction of motion continuously changes as the object moves around the circle. The velocity vector always points **tangent to the circle** at the object's current position, that is, perpendicular to the radius at that point.

Imagine a stone tied to a string being whirled in a horizontal circle. At every instant, the stone's velocity points along the tangent to its circular path. As the stone moves around the circle, this direction continuously rotates. The speed might be constant, but the direction changes every fraction of a second, which means the velocity vector is constantly changing.

This is why, when the string breaks, the stone does not continue moving in a circle or fall straight down; it flies off tangentially, following the direction of its velocity at the instant the string snapped. The bus at Kilimani Hill did exactly the same thing: when friction failed to provide the necessary inward force, the bus continued along the tangent to the curve, which happened to point straight into the ditch.

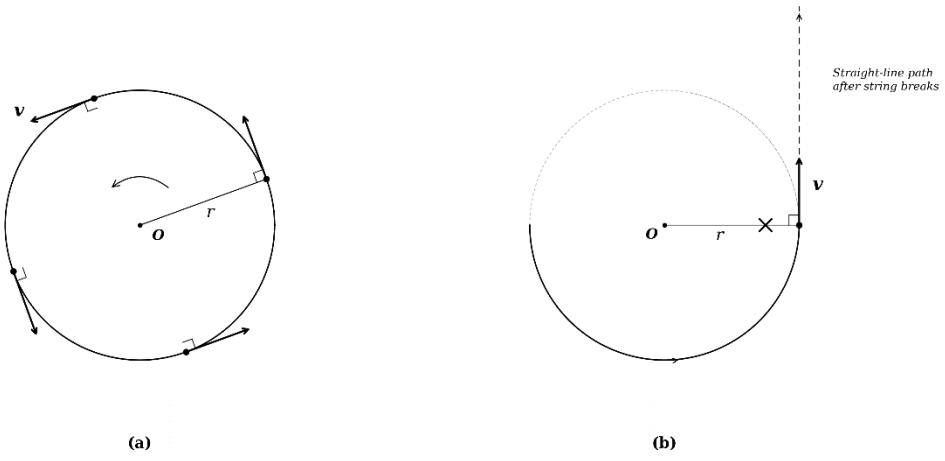


Figure: ***Velocity in circular motion:*** *(a) Velocity is always tangent to the circular path and perpendicular to radius; speed is constant but direction changes continuously. (b) When string breaks, centripetal force vanishes and the object continues along the tangent (Newton's first law).*

Acceleration in Circular Motion

If velocity is changing, acceleration must exist. This conclusion is inescapable. Acceleration is defined as the rate of change of velocity, and since velocity in circular motion continuously changes direction, there must be continuous acceleration.

This acceleration is always directed toward the centre of the circle. We call it **centripetal acceleration** (from Latin "centrum" meaning center and "petere" meaning to seek). The word literally means "center-seeking." Even though the object moves tangentially, the acceleration points radially inward.

This might seem strange. We usually think of acceleration as changing an object's speed, that is making it faster or slower. But acceleration can also change an object's direction while leaving speed unchanged. In uniform circular motion, the centripetal acceleration continuously deflects the velocity vector toward the center, bending the object's path into a circle while maintaining constant speed.

Centripetal Force

Newton's second law tells us that acceleration requires force: F = ma. If an object in circular motion experiences centripetal acceleration toward the center, there must be a force causing that acceleration. This force is called **centripetal force** which is *the inward force that maintains circular motion.*

Centripetal force is not a new type of force. It is simply whatever force happens to be pulling the object toward the center:

- **For a satellite orbiting Earth:** ***gravity*** *provides centripetal force.*
- **For a car turning a corner:** ***friction*** *between tires and road provides centripetal force.*
- **For a stone on a string:** ***tension*** *in the string provides centripetal force.*
- **For clothes in a spinning washing machine drum:** *the* ***normal reaction*** *from the drum wall provides centripetal force.*
- For our bus at Kilimani Hill: friction should have provided centripetal force, but the required force exceeded friction's capacity, so the bus slid off

The key insight: *circular motion requires continuous inward force. Remove or reduce that force, and circular motion immediately becomes straight-line motion along the tangent. This is not a flaw in the object or the motion; it is Newton's first law in action. Objects naturally move in straight lines. Only continuous inward force can maintain a curved path.*

With these foundational concepts established, let us explore some worked examples to deepen our understanding before we introduce the mathematical relationships in the next subtopic.

BINDER Example 1

A car moves at constant speed of 20m/s around a circular track. Explain whether the car is accelerating, and if so, in which direction.

Solution

Yes, the car is accelerating even though its speed is constant.

Acceleration is the rate of change of velocity, and velocity is a vector (magnitude and direction). Although the car's speed (magnitude of velocity) remains constant at 20m/s, the direction of the velocity continuously changes as the car moves around the circular track. At every instant, the velocity points tangent to the circle at the car's position. As the car moves, this tangent direction rotates, meaning the velocity vector is constantly changing.

Since velocity is changing, acceleration must exist. This acceleration points toward the centre of the circle; it is centripetal acceleration. The car is therefore continuously accelerating toward the centre of the circular track.

Making Sense of the Answer: *Acceleration does not only mean speeding up or slowing down. It also means changing direction. The car changes direction continuously as it goes around the circle, so it must be accelerating even though its speed stays constant.*

Think Like a Physicist: *Whenever you analyze circular motion, separate speed (scalar) from velocity (vector) in your mind. Constant speed in a circle always means changing velocity, which always means acceleration exists.*

BINDER Example 2

A stone tied to a string is whirled in a horizontal circle at constant speed. When the string suddenly breaks, the stone flies off in a straight line tangent to the circle rather than continuing to move in a circle or falling straight down. Explain why this happens.

Solution

While the string was intact, it provided tension force pulling the stone inward toward the center of the circle. This centripetal force continuously deflected the stone's velocity toward the centre, maintaining the circular path.

The moment the string broke, the centripetal force vanished. With no centripetal force acting on the stone, Newton's first law applies: an object in motion continues in a straight line at constant velocity unless acted upon by a force. The stone's velocity at the instant of breaking pointed tangent to the circle. With no force to deflect it from this direction, the stone continued moving straight along this tangent line.

Hence, the stone does not continue in a circle because circular motion requires continuous centripetal force which disappeared when the string broke. It does not fall straight down because it has horizontal velocity that persists in the absence of horizontal forces.

Making Sense of the Answer: *The tangent is the direction the stone was moving at the instant the string broke. Without the string pulling it inward, nothing deflects it from this straight-line path, so it continues moving straight ahead.*

Think Like a Physicist: *When analyzing what happens when centripetal force disappears, always identify the velocity direction at that instant. The object will continue along that tangent line because that is the direction it was already moving.*

REAL Example 3

A passenger sits in a car taking a sharp bend at speed. Many people feel as if they are being "thrown outward" against the door. Explain what is actually happening using correct physics.

Solution

The passenger is not thrown outward. Instead, the car pushes the passenger inward. This inward push provides the centripetal force needed for circular motion. However, the passenger's body wants to continue moving in a straight line (Newton's first law). So the "outward" sensation is just the passenger's inertia resisting the inward force. There is no real outward force; the real force always points toward the centre of the curve.

Making Sense of the Answer: *We feel "thrown outward" in circular motion because our bodies want to go straight while something (car door, drum wall, etc.) pushes us inward to force us into the curve. The inward push feels like we are being thrown the opposite direction, but the real force is always inward.*

Think Like a Physicist: *When you feel pushed to the outside during circular motion, recognize this as evidence of the inward centripetal force acting on you. Your sensation of being thrown outward is your inertia resisting the inward force and not an actual outward force.*

HOT Example 4

A car travels around a circular roundabout. At point A, the car is heading north. At point B, one-quarter of the way around the circle, the car is heading east. The car maintains constant speed throughout.

(a) Explain why the car's velocity has changed between points A and B even though its speed has not changed.
(b) If the car's velocity at point A is 15m/s north and at point B is 15m/s east, determine the change in velocity between these two points (magnitude and direction).
(c) Explain why this change in velocity requires a force, and identify the force that provides it for a car on a roundabout.
(d) Explain in which direction this force must act.

Solution

(a) Velocity is a vector quantity with both magnitude (speed) and direction. Although the car's speed remains constant at 15 m/s, the direction changes from north at point A to east at point B. Since the direction component of velocity has changed, the velocity vector has changed, even though the magnitude (speed) stayed the same.
(b) *Resolving velocities*

For point A: $(v_A)_x = 0, (v_A)_y = 15\text{m/s}$ (north taken as the positive y-direction)

For point B: $(v_B)_x = 15\text{m/s}, (v_B)_y = 0$ (east taken as the positive x-direction)

Finding horizontal and vertical change in velocity

$\Delta v_x = (v_B)_x - (v_A)_x = 15\text{m/s} - 0 = 15\text{m/s}$

$\Delta v_y = (v_B)_y - (v_A)_y = 0 - 15\text{m/s} = -15\text{m/s}$

Finding magnitude and direction of change in velocity

Magnitude: $\Delta v = \sqrt{(\Delta v_x)^2 + (\Delta v_y)^2} = \sqrt{(15\text{m/s})^2 + (-15\text{m/s})^2} = 21.2\text{m/s}$

Direction: $\theta = \tan^{-1}\left(\frac{\Delta v_y}{\Delta v_x}\right) = \tan^{-1}\left(\frac{-15}{15}\right) = -45°$(The negative sign indicates the direction is **below the positive x-axis**).

Hence, the change in velocity is 21.2m/s directed southeast (45° south of east), pointing toward the center of the roundabout.

(c) Newton's second law states that acceleration requires a net force to act on an object: F = ma. Since the car's velocity changes direction as it moves around the roundabout, the car has centripetal acceleration. Therefore, a force must act on the car. For a car moving around a roundabout, this force is provided by static friction between the tyres and the road. As the car turns, the tyres tend to move sideways relative to the road surface. Static friction prevents this sideways motion and acts toward the centre of the circular path. This frictional force provides the centripetal force required to keep the car moving along the curved path.

(d) The force must act toward the centre of the roundabout. This is because the required acceleration (centripetal acceleration) points toward the centre, and force must act in the same direction as the acceleration it produces (F = ma).

Making Sense of the Answer: *The change in velocity points toward the centre because the car's path is being continuously bent inward. Friction must act in that same inward direction to produce this change. Without sufficient friction (wet road, icy conditions), the car cannot generate enough inward force and slides off the curve tangentially; just like our bus at Kilimani Hill.*

Think Like a Physicist: *In circular motion problems, always remember that force and acceleration point toward the centre, while velocity points along the tangent. These are perpendicular to each other at every instant. The centripetal force continuously changes the direction of the tangent velocity vector, bending the path into a circle without changing the speed.*

With these fundamental concepts established, that circular motion requires continuous inward force, that velocity is always tangential while acceleration is always radial, and that constant speed does not mean constant velocity; we are ready to develop the mathematical relationships that allow us to calculate these quantities precisely. The next subtopic introduces angular motion and the quantitative tools we need to solve circular motion problems.

ANGULAR MOTION

In the previous topic, we established the conceptual foundation of circular motion: velocity is tangential, acceleration is radial, and continuous inward force is required. Now we develop the mathematical tools to describe and calculate circular motion quantitatively. To do this, we introduce angular quantities which is a powerful alternative way of describing circular motion that often proves more convenient than linear quantities like distance and speed.

Radian Measure

Before discussing angular motion, we must understand the proper unit for measuring angles in physics: the radian.

You are already familiar with degrees: a full circle contains 360°, a right angle is 90°, and so on. Degrees are arbitrary; there is no fundamental reason why a circle should be divided into 360 parts rather than 100 or 1000. The radian, by contrast, emerges naturally from the geometry of the circle itself. By definition:

One radian is the angle subtended at the center of a circle by an arc length equal to the radius of the circle.

To understand it properly, imagine a circle of radius r. If you measure an arc along the circumference with length exactly equal to r, the angle at the center corresponding to that arc is one radian. This definition creates a natural relationship between arc length s, radius r, and angle θ (measured in radians):

$$\boldsymbol{\theta = \frac{s}{r}}$$

From this, we can derive the conversion between radians and degrees. The full circumference of a circle is $2\pi r$. This circumference subtends a complete revolution, which is 360° or, in radians:

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians}$$

Therefore: $\boldsymbol{2\pi \text{ radians} = 360°}$

In physics, we almost always use radians rather than degrees when working with circular motion. The mathematics becomes cleaner, the formulas simpler, and the relationships more transparent. Unless explicitly stated otherwise, assume **angles in circular motion are measured in radians.**

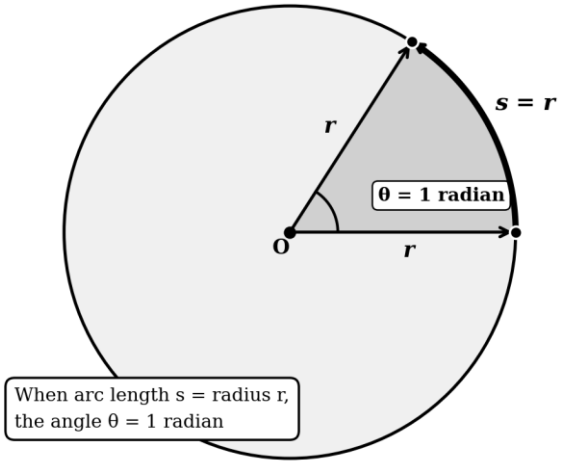


Figure: *A circle of radius r with an arc of length s = r. The angle θ subtended at the centre O is exactly 1 radian. Since the full circumference equals 2πr, a complete revolution equals 2π radians.*

Angular displacement

When an object moves along a circular path from one position to another, we can describe its motion using either linear displacement (arc length travelled) or angular displacement (angle swept out).

Angular displacement (θ) *is the angle through which an object has rotated about the centre of the circle.* It is measured in radians.

If an object moves along an arc of length s on a circle of radius r, the angular displacement is:

$$\theta = \frac{s}{r}$$

Angular displacement is the same for all points on a rotating rigid object, regardless of how far they are from the centre. A point near the rim and a point near the hub of a wheel rotate through the same angle θ, even though the point on the rim travels a much longer arc length. This is one advantage of angular quantities as they describe the rotation of the entire object with a single number.

Angular velocity

Just as linear velocity describes how quickly linear position changes with time, angular velocity describes how quickly angular position changes with time. **Angular velocity (ω)** *is the rate of change of angular displacement:*

$$\boldsymbol{\omega = \frac{\theta}{t}} \text{ (for motion starting from rest)}$$

Or more generally:

$$\boldsymbol{\omega = \frac{\Delta\theta}{\Delta t}}$$

Angular velocity is measured in radians per second (rad/s). It tells us how many radians the object rotates through per unit time. A wheel rotating at $\omega = 10$ rad/s sweeps out 10 radians every second. Since **2π radians equals one complete revolution**, this wheel completes $10/(2\pi) \approx 1.59$ revolutions every second.

Angular velocity is a vector quantity, with direction along the axis of rotation, but for our purposes in uniform circular motion, we focus on its magnitude, which tells us the rate of rotation.

Relationship between linear and angular velocity

Here is the crucial connection between the linear and angular velocity of circular motion:

$$\mathbf{v = r\omega}$$

This beautifully simple equation relates:

- v: linear velocity (in m/s)
- r: radius of the circular path (in m)
- ω: angular velocity (in rad/s)

Derivation:
If an object moves along an arc of length s on a circle of radius r, the angular displacement is:

$$\theta = \frac{s}{r}$$

From which:

$$s = r\theta$$

The linear velocity (v) is displacement divided by time:

$$v = \frac{s}{t} = \frac{r\theta}{t}$$

But $\frac{\theta}{t} = \omega$
Hence:

$$\mathbf{v = r\omega}$$

This relationship reveals a profound truth: *all points on a rotating rigid object have the same angular velocity ω, but points farther from the centre (with greater value of r) have greater linear speed v.* A point on the outer edge of a spinning disk moves faster (covers more distance per second) than a point near the center, but both rotate through the same angle per second.

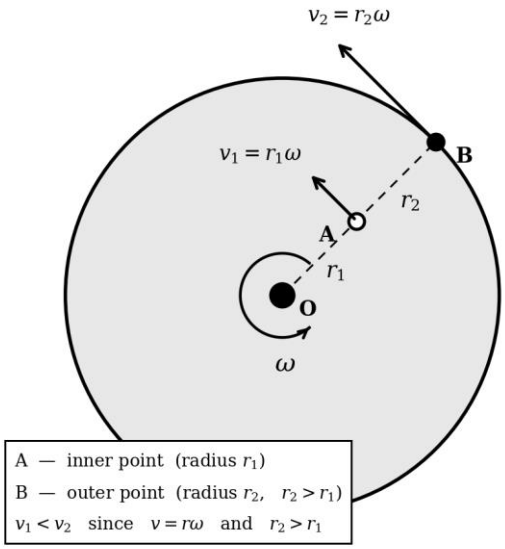


Figure: *Two points A and B on a rotating rigid disk. Both rotate with the same angular velocity ω, but since v = rω, the outer point B (radius r_2) moves with greater linear velocity than the inner point A (radius r_1). The length of each velocity arrow reflects this difference.*

Period and Frequency

For objects in uniform circular motion that complete regular revolutions, we can describe their motion using period and frequency.

Period (T) *is the time required for one complete revolution.* It is measured in seconds.

$$T = \frac{\text{Total time taken, t}}{\text{Number of revolutions, n}}$$

Frequency (f) *is the number of complete revolutions per unit time.* It is measured in hertz (Hz), where 1 Hz = 1 revolution per second = 1 s^{-1}.

$$f = \frac{\text{Number of revolutions, n}}{\text{Total time taken, t}}$$

Period and frequency are reciprocals of each other:

$$\mathbf{T = \frac{1}{f}} \text{ and } \mathbf{f = \frac{1}{T}}$$

Relating angular velocity to period and frequency

In one complete revolution, an object rotates through 2π radians. This takes time T (the period), and the angular velocity is:

$$\boldsymbol{\omega = \frac{2\pi}{T}}$$

Alternatively, since $f = \frac{1}{T}$:

$$\boldsymbol{\omega = 2\pi f}$$

These relationships allow us to convert easily between angular velocity, period, and frequency:

- *Know the period?* Calculate $\omega = \frac{2\pi}{T}$
- *Know the frequency?* Calculate $\omega = 2\pi f$
- *Know the angular velocity?* Calculate $T = \frac{2\pi}{\omega}$ or $f = \frac{\omega}{2\pi}$

Combined with $v = r\omega$, we can now express linear velocity in terms of period or frequency:

$$\mathbf{v = \omega r = r\left(\frac{2\pi}{T}\right) = \frac{2\pi r}{T}}$$

Or $$\mathbf{v = r\omega = r(2\pi f) = 2\pi rf}$$

These formulas are powerful tools for analyzing circular motion problems.

Summary of key relationships

Let us consolidate the mathematical framework we have developed:

Arc length and angle:

$$s = r\theta \ (\theta \text{ in radians})$$

Angular velocity:

- $\omega = \frac{\Delta\theta}{\Delta t}$ (rad/s)
- $\omega = \frac{2\pi}{T}$
- $\omega = 2\pi f$

Linear and angular velocity:

$$v = \omega r$$

Linear velocity from period/frequency:

- $v = \frac{2\pi r}{T}$
- $v = 2\pi rf$

Period and frequency:

- $\mathrm{T} = \frac{1}{\mathrm{f}}$
- $\mathrm{f} = \frac{1}{\mathrm{T}}$

With these tools, we can solve a wide variety of circular motion problems. Let us practice applying them through worked examples.

BINDER Example 5

A wheel of radius 0.4m rotates at 5 revolutions per second. Calculate:

(a) The period of rotation. (b) The angular velocity. (c) The linear velocity of a point on the rim.

Solution

$\mathrm{f} = 5\,\mathrm{rev/s} = 5\mathrm{Hz}$

(a) Period:

$$\mathrm{T} = \frac{1}{\mathrm{f}} = \frac{1}{5\mathrm{s}^{-1}} = 0.2\mathrm{s}$$

The period is 0.2s.

(b) Angular velocity:

$$\omega = 2\pi\mathrm{f} = 2\pi\,\mathrm{rad} \times 5\mathrm{s}^{-1} = 10\pi = 31.4\,\mathrm{rads}^{-1}$$

The angular velocity is 31.4rad/s

(c) Linear velocity:

$$\mathrm{v} = \omega\mathrm{r} = 0.4\mathrm{m} \times 31.4\mathrm{rads}^{-1} = 12.6\mathrm{m/s}$$

The linear velocity at the rim is 12.6 m/s.

Making Sense of the Answer: *The wheel spins 5 times per second, so each revolution takes* $0.2s$*. A point on the rim travels the circumference (*$2\pi r = 2.51m$*) in 0.2s, giving speed* $v = \frac{2.51}{0.2} = 12.6ms^{-1}$*, which matches our calculation.*

Think Like a Physicist: *Always check that your units work out correctly. Angular velocity must be in rad/s (not rev/s), and when you multiply ω (rad/s) by r (m), you get v (m/s). The "rad" unit effectively cancels because* ***radians are dimensionless*** *(they are a ratio of lengths).*

BINDER Example 6

A point on the equator of the Earth moves in a circle as the Earth rotates. Taking the Earth's radius as 6400km and the period of rotation as 24 hours, calculate the linear velocity of a point on the equator.

Solution

Using $\mathrm{v} = \frac{2\pi\mathrm{r}}{\mathrm{T}}$;

Where: $\mathrm{r} = 6400\,\mathrm{km} = 6.4 \times 10^6\,\mathrm{m}$, $\mathrm{T} = 24\,\mathrm{hours} = 24 \times 3600\mathrm{s} = 86400\mathrm{s}$

Substituting values:

$$\mathrm{v} = \frac{2\pi \times 6.4 \times 10^6\mathrm{m}}{86400\mathrm{s}} = 465\mathrm{m/s}$$

The linear velocity at the equator is 465m/s.

Making Sense of the Answer: *This is about 1670 km/h, quite fast! Yet we do not feel this motion because we are moving with the Earth. This velocity is why launching rockets eastward from equatorial locations provides a "boost"; the rocket already has this velocity before its engines fire.*

Think Like a Physicist: *Always convert to standard SI units (metres and seconds) before calculating. Period in hours must become seconds:* $24\,h \times 3600\,s/h = 86{,}400\,s$*. Failing to convert units properly is a common source of errors in circular motion problems.*

REAL Example 7

Two children sit on a merry-go-round: Amina sits 1m from the center, while Baraka sits 2m from the centre (at the outer edge). The merry-go-round rotates at constant angular velocity.

(a) Compare the angular velocities of the two children. Explain.
(b) Compare their linear velocities.

Solution

(a) **Both children have the same angular velocity ω.** They are on the same rigid platform, so they both rotate through the same angle in the same time. When the platform completes one revolution, both Amina and Baraka complete one revolution together, even though they travel different distances.
(b) Using $v = r\omega$:
Amina ($r = 1\text{m}$): $v_A = 1\omega$
Baraka ($r = 2.0\text{ m}$): $v_B = 2\omega$

Hence, **Baraka's linear velocity is twice Amina's linear velocity** (Baraka moves twice as fast as Amina in linear terms).

Making Sense of the Answer: *Think of a spinning disk with dots painted on it. All dots complete one rotation together (same ω), but outer dots trace larger circles and therefore move faster (larger v = rω).*

Think Like a Physicist: *When analyzing rotating rigid objects, angular quantities (ω, θ) are the same for all points, but linear quantities (v, s) increase with radius. This is why ω is often more convenient than v for describing rotation; one number describes the entire object.*

HOT Example 8

A DVD rotates in a player at 1800 revolutions per minute. The DVD has inner radius 2cm (where data begins) and outer radius 6cm (edge of data region). Take $\pi = 3.14$.

(a) Calculate the linear velocity at the inner radius.
(b) Calculate the linear velocity at the outer radius.
(c) Calculate how much faster (in percentage) the outer edge moves compared to the inner radius.
(d) Explain why DVD players must vary their rotation speed when reading from different parts of the disk to maintain constant linear read speed.

Solution

(a) $f = 1800\ \frac{\text{rev}}{\text{min}} \times \frac{1\text{min}}{60\text{s}} = 30\text{rev/s} = 30\text{Hz}$

Angular velocity:

$\omega = 2\pi f = 2 \times 3.14 \times 30 = 188.4\ \text{rad/s}$

Linear velocity at inner radius:

$v_{\text{inner}} = r_{\text{inner}} \times \omega = 0.02\text{m} \times 188.4\text{rad/s} = 3.77\text{m/s}$

The linear velocity at the inner radius is 3.77m/s.

(b) Linear velocity at outer radius:

$v_{\text{outer}} = r_{\text{outer}} \times \omega = 0.06\text{m} \times 188.4\text{rad/s} = 11.3\text{m/s}$

The linear velocity at the outer radius is 11.3m/s.

(c) Speed difference $= v_{\text{outer}} - v_{\text{inner}} = (11.3 - 3.77)\text{m/s} = 7.53\text{m/s}$

$$\text{Percentage increase} = \frac{\text{Speed difference}}{v_{\text{inner}}} \times 100\% = \frac{7.53}{3.77} \times 100\% = 200\%$$

The outer edge moves **200% faster** than the inner radius.

(d) If the angular velocity ω were constant, the linear speed $v = r\omega$ would vary dramatically from inner to outer radius. To maintain constant linear read speed:
- When reading inner tracks (small r), the player must spin the disk faster (larger ω) to achieve $v = r\omega = \text{constant}$.
- When reading outer tracks (large r), the player must spin slower (smaller ω).

Therefore, since $v = r\omega$ must stay constant, and r increases as we read outward, ω must decrease proportionally: $\omega \propto \frac{1}{r}$ for constant v.

Making Sense of the Answer: *Old vinyl records played at constant angular velocity, which meant outer grooves moved faster past the needle than inner grooves. This limited sound quality. CDs and DVDs solved this by varying angular velocity to maintain constant linear velocity, improving performance and allowing more efficient use of disk space.*

Think Like a Physicist: *The relationship $v = r\omega$ creates an inverse relationship between r and ω when v must remain constant: $v = \frac{v}{r}$. Doubling the radius requires halving the angular velocity to maintain the same linear velocity. This principle applies anywhere constant linear speed is needed despite varying radius: car transmissions, conveyor systems, and optical disk drives all use this concept.*

These angular quantities and relationships provide powerful tools for analyzing circular motion. In the next subtopic, we will use these tools to develop formulas for centripetal acceleration and centripetal force, allowing us to calculate the forces required to maintain circular motion and understand why certain speeds or radii make circular motion impossible without sufficient inward force.

CENTRIPETAL ACCELERATION AND FORCE

In the previous subtopics, we established that objects in circular motion experience acceleration toward the centre, and we developed the mathematical tools (angular quantities) to describe rotation. We have not yet answered the deeper question: *exactly how large is this centripetal acceleration? And what force is required to produce it?*

These are not merely academic questions. They are life-and-death questions.

When engineers design curved highways, they must calculate exactly how much friction is needed to keep cars on the road at various speeds. If the required force exceeds what friction can deliver, vehicles will slide off the road, just as our bus did at Kilimani Hill. When satellites are launched into orbit, mission controllers must determine precisely what velocity produces the right centripetal acceleration for a stable orbit. When you whirl a stone on a string, the tension you feel in your hand is the centripetal force, and if that tension exceeds what the string can handle, it snaps, sending the stone flying tangentially just as we saw earlier.

Understanding centripetal acceleration quantitatively transforms circular motion from an interesting concept into a powerful tool for solving real problems. To accomplish this, we need formulas. And those formulas begin with one central derivation: the magnitude of centripetal acceleration.

Deriving the Magnitude of Centripetal Acceleration

Earlier in this chapter, we established that an object in uniform circular motion is always accelerating toward the centre of the circle, even though its speed remains constant. We called this centripetal acceleration and understood its direction. Now we determine its magnitude.

The key question is: *If an object moves at constant speed v around a circle of radius r, how large is the centripetal acceleration?*

Setting up the problem

Consider an object moving at constant speed v along a circular path of radius r. At some instant, the object is at point A, moving with velocity $\mathbf{v_A}$. A short time Δt later, the object has moved to point B, where its velocity is $\mathbf{v_B}$. Both velocities have the same magnitude v (because speed is constant), but they differ in direction.

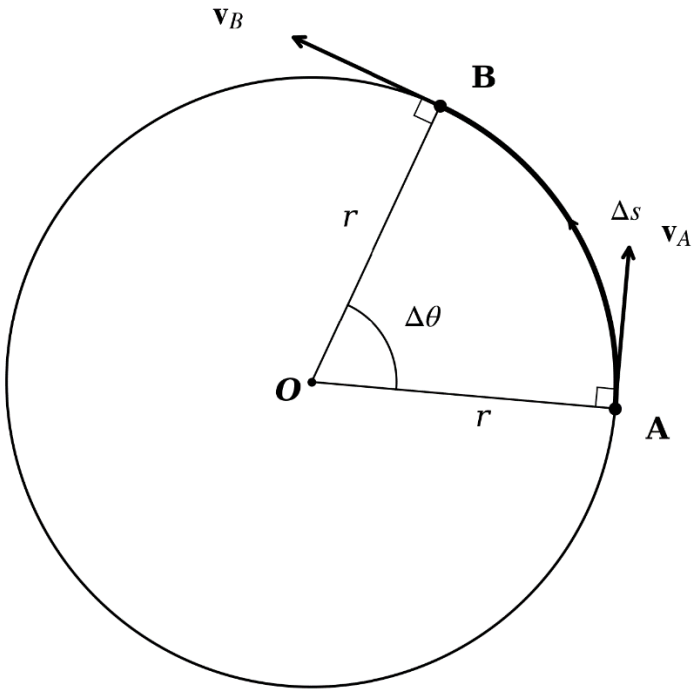


Figure (a): *An object moves at constant speed along a circular path of radius r from point A to point B, sweeping through angle Δθ and covering arc length Δs. The velocity at each point is tangent to the circle and perpendicular to the radius.*

Finding the change in velocity

The change in velocity is:

$$\mathbf{\Delta v} = \mathbf{v_B} - \mathbf{v_A}.$$

This difference **Δv** points toward the center of the circle.

To find **Δv**, we use a velocity triangle. Place $\mathbf{v_A}$ and $\mathbf{v_B}$ tail-to-tail. Because both have magnitude **v** and the angle between them equals the angular displacement **Δθ** (the same angle the radius sweeps through), the velocity triangle is an isosceles triangle with two sides of length v and an included angle Δθ.

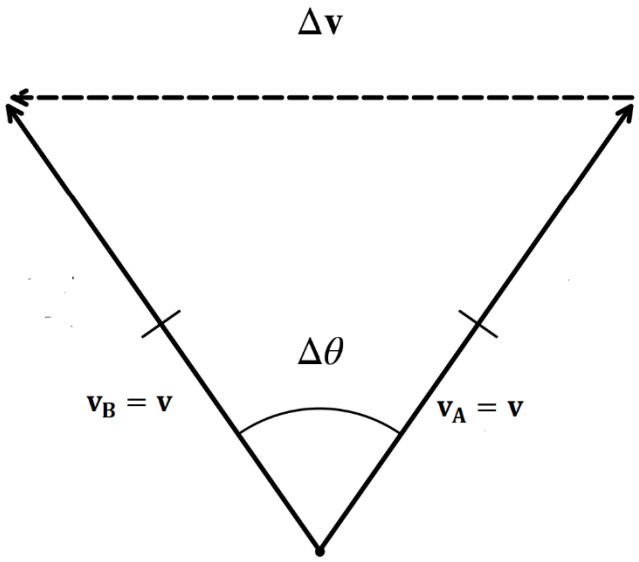


Figure (b): *The velocity vectors from (a) placed tail-to-tail form an isosceles triangle. Both sides have magnitude v, and the angle between them equals* $\boldsymbol{\Delta\theta}$*. The change in velocity,* $\boldsymbol{\Delta v = v_B - v_A}$*, is shown by the dashed arrow.*

For small angles $\Delta\theta$ (measured in radians), the length of the arc connecting the tips of $\mathbf{v_A}$ and $\mathbf{v_B}$ approximately equals the straight-line distance, $\mathbf{\Delta v}$.

But; arc length, s = radius, r × angle, $\Delta\theta$

Where: $s = \Delta v$, $r = v$.

Thus:

$$\Delta v \approx v\Delta\theta \text{ (for small } \Delta\theta)$$

Now, dividing both sides by Δt:

$$\frac{\Delta v}{\Delta t} \approx v\left(\frac{\Delta\theta}{\Delta t}\right)$$

As Δt approaches zero, $\frac{\Delta v}{\Delta t}$ becomes the instantaneous acceleration, and $\frac{\Delta\theta}{\Delta t}$ becomes angular velocity ω. Therefore:

$$a = v\omega$$

But we know that $v = r\omega$, so we can substitute:

$$a = (\omega r)\omega = \omega^2 r$$

This gives us one formula for centripetal acceleration:

$$\mathbf{a = \omega^2 r}$$

Alternatively, we can eliminate ω using $\omega = \frac{v}{r}$:

$$a = r\left(\frac{v}{r}\right)^2 = r\left(\frac{v^2}{r^2}\right) = \frac{v^2}{r}$$

This gives us the more commonly used formula:

$$\mathbf{a = \frac{v^2}{r}}$$

Let us pause and appreciate what we have discovered. The centripetal acceleration depends on two factors:

1. **Speed squared (v^2):** Doubling your speed quadruples the required centripetal acceleration. This is why *curves that feel comfortable at 30 km/h become terrifying at 60 km/h*; the required acceleration increases by a factor of four.
2. **Inverse of radius:** Tighter curves (smaller r) require greater acceleration. This is why *a gentle highway curve can be navigated at high speed, but a sharp corner demands slowing down.*

These formulas explain why the bus at Kilimani Hill could not make the turn. The curve had a certain radius r, and the bus was traveling at speed v. The required centripetal acceleration was $a = \frac{v^2}{r}$. But friction could only provide a limited acceleration. When $\frac{v^2}{r}$ exceeded what friction could deliver, the tires lost grip and the bus slid off tangentially.

Direction of centripetal acceleration

The magnitude formulas $a = \frac{v^2}{r}$ and $a = \omega^2 r$ tell us how large the acceleration is, but they do not specify direction. We established earlier that centripetal acceleration points toward the centre of the circle, but let us see why this must be true.

The change in velocity $\mathbf{\Delta v = v_B - v_A}$ connects the tip of $\mathbf{v_A}$ to the tip of $\mathbf{v_B}$ when both are drawn from a common origin. As the time interval Δt becomes infinitesimally small, the angle $\Delta\theta$ approaches zero, and the chord connecting v_A and v_B becomes perpendicular to both velocity vectors. Since the velocity vectors are themselves tangent to the circle, $\mathbf{\Delta v}$ must be perpendicular to the tangent, which means it points along the radius.

Furthermore, since $\mathbf{\Delta v = v_B - v_A}$ represents the change needed to go from $\mathbf{v_A}$ to $\mathbf{v_B}$ as the object moves counterclockwise around the circle, and since v_B has rotated counterclockwise from v_A, the difference vector points inward, toward the centre.

Therefore, *centripetal acceleration always points radially inward, toward the centre of the circle.*

This radial direction is crucial. It means centripetal acceleration is always perpendicular to velocity (which points tangentially). *A force perpendicular to velocity changes the direction of motion without changing speed* which is exactly what we need for uniform circular motion.

Centripetal force

Now we bring in Newton's second law: $\mathbf{F = ma.}$

If an object of mass m experiences centripetal acceleration a toward the centre, there must be a net force F toward the centre producing that acceleration.

The centripetal force is:

$$\mathbf{F_c = ma_c = m\left(\frac{v^2}{r}\right) = m\omega^2 r}$$

Let us be absolutely clear about what "centripetal force" means. *It is not a new type of force like gravity, friction, or tension. Centripetal force is simply* ***whatever*** *net force happens to be directed toward the centre of the circular path.* It is a descriptive term, not a fundamental force.

As mentioned earlier, different situations provide centripetal force in different ways:

Planetary orbits: Gravity provides centripetal force.

Car on curve: Friction between tires and road provides centripetal force.

Stone on string: Tension in the string provides centripetal force.

Satellite in orbit: Earth's gravity provides centripetal force.

Clothes in washing machine: Normal reaction force from drum wall provides centripetal force.

Electron around nucleus (classical model): Electric attraction provides centripetal force.

In each case, we simply identify what physical force acts toward the centre, then set it equal to the required centripetal force $\frac{mv^2}{r}$. This gives us an equation we can solve for unknowns like speed, radius, or the magnitude of the providing force.

What happens when centripetal force is insufficient?

The formula $F_c = \frac{mv^2}{r}$ tells us what force is **required** to maintain circular motion at speed v and radius r. *But what if the available force is less than this required amount?*

Consider our bus at Kilimani Hill. The maximum static friction force available between tires and road might be $F_{max} = \mu_s R$, where μ_s is the coefficient of static friction and R is the normal reaction. For the bus to successfully navigate a curve of radius r, the required centripetal force $\frac{mv^2}{r}$ must not exceed this maximum:

$$\frac{mv^2}{r} \leq F_{max}$$

If v is too large (excessive speed), or r is too small (sharp curve), or μ is too small (wet or icy road), this inequality is violated. When that happens, the available force cannot provide the necessary centripetal acceleration. The tires slip, the inward force decreases, and the vehicle slides outward; though "outward" is slightly misleading! What actually happens is that the vehicle continues along the tangent to the curve, which appears to be "outward" relative to the curve's centre.

This is why speed limits on curves exist. Engineers calculate the curve radius r and the typical friction coefficient μ for various road conditions, then determine the maximum safe speed as follows:

$$F_{max} = \frac{mv_{max}^2}{r}$$

For a car on a level road: $F_{max} = \mu_s mg$

So $\mu_s mg = \frac{mv_{max}^2}{r}$ or $\mu_s g = \frac{v_{max}^2}{r}$

Making v_{max} the subject:

$$\mathbf{v_{max} = \sqrt{\mu_s gr}}$$

If you exceed this speed $\sqrt{\mu_s gr}$, you are demanding more centripetal force than friction can supply. Once this limit is crossed, the tyres can no longer maintain the necessary grip, and the vehicle will begin to slide outward. Eventually, you face the same consequence as the Miono Bus Kilimani Hill accident; a stark reminder that physics does not negotiate!

Connecting to angular velocity

Since we have two expressions for centripetal acceleration ($\frac{v^2}{r}$ and $\omega^2 r$), we similarly have two expressions for centripetal force:

$$F_c = \frac{mv^2}{r} = m\omega^2 r$$

Use whichever is more convenient for the problem at hand. Often, when dealing with rotating rigid objects (wheels, turntables, planets), angular velocity ω is given or easier to work with, making $F_c = m\omega^2 r$ more useful.

For example, every point on a rotating disk has the same ω but different v (since $v = r\omega$ increases with r). The centripetal force required increases with radius: $F_c = m\omega^2 r$. So *points farther from the centre need stronger centripetal force to maintain circular motion at the same angular velocity.* This is why *standing at the edge of a spinning merry-go-round is more dangerous than standing near the centre:* ***the required centripetal force is much greater at larger radii.***

Summary of key formulas

Let us consolidate what we have learned:

Centripetal acceleration:

- $a_c = \frac{v^2}{r}$ (when linear velocity v is known).
- $a_c = \omega^2 r$ (when angular velocity ω is known).
- Direction: always toward the centre (radially inward).

Centripetal force:

- $F_c = \frac{mv^2}{r}$ (when linear velocity v is known).
- $F_c = m\omega^2 r$ (when angular velocity ω is known).
- Direction: always toward the centre (radially inward).
- Not a new type of force; it is whatever net force acts toward the centre.

These formulas are the foundation for analyzing all circular motion problems. In the following subtopics, we will apply them to specific situations: objects on strings, conical pendulums, cars on curves, and objects in vertical circles. But the physics remains the same: identify what provides the centripetal force, set it equal to $\frac{mv^2}{r}$ or $m\omega^2 r$, and solve.

With this mathematical framework established, let us practice applying it through worked examples.

BINDER Example 9

A car of mass 1200kg travels around a circular track of radius 50m at constant speed 20m/s. Calculate: (a) The centripetal acceleration. (b) The centripetal force required.

Solution

(a) Centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{(20\text{m/s})^2}{50\text{m}} = 8\text{m/s}^2$$

The centripetal acceleration is 8m/s^2 toward the center of the track.

(b) Centripetal force:

$$F_c = ma_c = 1200\text{kg} \times 8\text{m/s}^2 = 9600\text{N}$$

The required centripetal force is 9600 N toward the centre.

Making Sense of the Answer: *The acceleration 8m/s² is less than g = 9.8m/s², so it is a comfortable turn; not too extreme. The force 9600N must be provided by friction between the tires and track. If friction cannot provide this (wet track, bald tires), the car will* ***slide off tangentially****.*

Think Like a Physicist: *It is better to specify direction when stating acceleration or force. "8m/s²" is incomplete; "8 m/s² toward the centre" is complete. In circular motion, the inward direction is as important as the magnitude.*

BINDER Example 10

A stone of mass 0.5kg is tied to a string and whirled in a horizontal circle of radius 0.8m. The stone completes 4 revolutions per second. Calculate:

(a) The angular velocity.

(b) The centripetal acceleration.

(c) The tension in the string.

Solution

(a) Using:

$$\omega = 2\pi f$$

Where: f = 4 Hz

Substituting:

$$\omega = 2\pi \times 4\text{s}^{-1} = 8\pi = 25.1\text{rad/s}$$

The angular velocity is 25.1 rad/s.

(b) Using:

$$a = \omega^2 r$$

Where: ω = 25.1 rad/s, r = 0.8m

Substituting:

$$a = (25.1\text{rad/s})^2 \times 0.8\text{m} = 504\text{m/s}^2$$

The centripetal acceleration is 504m/s².

(c) The tension in the string provides the centripetal force. Therefore:

$$T = F_c = ma$$

Where: m = 0.5 kg, a = 504m/s²

Substituting:

$$T = 0.5\text{kg} \times 504\text{m/s}^2 = 252\text{N}$$

The tension in the string is 252N.

Making Sense of the Answer: *The centripetal acceleration is about 51 times the acceleration due to gravity! This shows how rapidly the required acceleration grows with angular velocity. Even a light stone (0.5 kg) demands 252N of tension, which is equivalent to supporting a mass of about 25.7 kg. This is why strings break when you whirl objects too fast.*

Think Like a Physicist: *When frequency or angular velocity is given instead of linear velocity, use a = ω²r directly. Converting to linear velocity first and then using a = v²/r gives the same answer but takes more steps.*

BINDER Example 11

The moon orbits Earth at approximately constant distance $3.84 \times 10^8\text{m}$ from Earth's centre, completing one orbit in 27.3 days. Calculate:

(a) The moon's orbital speed (b) The centripetal acceleration of the moon.

Solution

(a) The moon travels a distance equal to the circumference 2πr in time T:

$$v = \frac{2\pi r}{T}$$

Where: $T = 27.3\ \text{days} = 27.3\text{days} \times 24\text{min/day} \times 3600\text{sec/min} = 2.36 \times 10^6\text{s}$

$r = 3.84 \times 10^8\ \text{m}$

Substituting values:

$$T = \frac{2\pi \times 3.84 \times 10^8\text{m}}{2.36 \times 10^6\text{s}} = 1020\ \text{m/s} \approx 1.02\ \text{km/s}$$

The moon's orbital speed is approximately 1020 m/s or 1.02 km/s.

(b) Centripetal acceleration is given by:

$$a_c = \frac{v^2}{r} = \frac{(1020\text{m/s})^2}{3.84 \times 10^8\text{m}} = 0.0027\ \text{m/s}^2$$

The moon's centripetal acceleration is 0.0027m/s² toward Earth's centre.

Making Sense of the Answer: *This tiny acceleration (about 1/3600 of g) is Earth's gravitational pull on the moon at that distance. It is small because the moon is far away, but it's enough to continuously deflect the moon's path into a circle instead of a straight line. Without this acceleration, the moon would fly off tangent to its orbit.*

Think Like a Physicist: *This example shows that circular motion does not require large accelerations; it just requires continuous acceleration toward the centre. Even tiny centripetal acceleration can maintain circular motion if applied persistently. The moon has been held in orbit by this gentle 0.0027 m/s² for billions of years!*

REAL Example 12

During a school trip to the Dar es Salaam amusement park, **Kipanga** sits on a spinning ride that rotates faster and faster. At first he enjoys it. But as the ride speeds up, he feels increasing pressure from the seat pushing against his back. He shouts to Kipute, *"Something is pushing me outward!"*

Kipute, sitting next to him, shouts back, *"Nothing is pushing you outward, Kipanga. Think about what is really happening!"*

Explain, why Kipanga feels pushed outward even though no outward force acts on him.

Solution

When the ride rotates, Kipanga must move in a circular path. This requires a centripetal force directed toward the centre, which is provided by the seat pushing against his back. However, due to inertia (Newton's first law), Kipanga's body tends to continue in a straight line. The seat must continuously push him inward to overcome this tendency. What he feels as being "pushed outward" is actually his inertia resisting the inward push of the seat. As the ride speeds up, the required centripetal force increases (since $F_c = m\omega^2 r$), so the seat pushes harder, making the sensation feel stronger. But no outward force exists; the real force is always directed inward toward the centre.

Making Sense of the Answer: *What we call "feeling pushed outward" is actually feeling the inward force that maintains circular motion. It is the same sensation you feel when a car accelerates forward and the seat pushes your back. You feel "pushed backward," but the real force is forward. In both cases, the sensation is caused by inertia resisting the applied force.*

Think Like a Physicist: *Never write "centrifugal force" as a real force in your solutions. The outward sensation is a consequence of inertia, not a real force. In the reference frame of the ground (an inertial frame), only the inward centripetal force exists.*

REAL Example 13

Kipanga, a Form Six student preparing for his physics practical session, decided to demonstrate circular motion using a simple setup. He tied a 0.5kg stone to a string and began whirling it in a horizontal circle of radius 0.8m in the school playground. However, the string he used could withstand a maximum tension of only 50N before breaking. Curious about the limits of the experiment, Kipanga wondered how fast he could safely whirl the stone without snapping the string.

(a) Explain what provides the centripetal force for the stone's circular motion.
(b) Calculate the maximum speed at which the stone can be whirled before the string breaks.
(c) Explain what happens to the stone if this maximum speed is exceeded.

Solution

(a) In this situation, tension in the string provides the centripetal force. The string pulls the stone inward toward the centre of the circle. This inward pull continuously deflects the stone's velocity, bending its path into a circle.
(b) At maximum speed, tension reaches its maximum value before breaking:

$$F_c = T_{max} = \frac{mv_{max}^2}{r}$$

Making v_{max} the subject:

$$v_{max} = \sqrt{\frac{rT_{max}}{m}}$$

Substituting values:

$$v_{max} = \sqrt{\frac{0.8m \times 50N}{0.5kg}} = 8.94m/s$$

The maximum safe speed is approximately 8.9m/s.

(c) If the speed exceeds 8.9 m/s, the required centripetal force mv^2/r exceeds the maximum tension the string can provide (50N). The string cannot supply the needed inward force, so it breaks. At the instant of breaking, the centripetal force vanishes. With no force to deflect its path, the stone immediately obeys Newton's first law and continues moving in a straight line along the tangent to the circular path at the breaking point.

Making Sense of the Answer: *The faster you whirl the stone, the greater the required centripetal force (because $F \propto v^2$). Eventually, you demand more force than the string cannot provide. The string does not break because of "centrifugal force throwing the stone outward." It breaks because you are asking for more inward force than it can deliver.*

Think Like a Physicist: *When analyzing circular motion, always identify what provides the centripetal force (tension, friction, gravity, normal reaction, etc.), then set it equal to the required mv^2/r or $m\omega^2r$. This equation can then be solved for any unknown: speed, radius, force, or mass.*

HOT Example 14

A car of mass 1500 kg travels around a level (unbanked) circular curve of radius 80m. The coefficient of static friction between the tyres and the dry road surface is 0.65. Take g = 9.8 m/s².

(a) Calculate the maximum speed at which the car can negotiate the curve without sliding.

(b) After rainfall, the coefficient of static friction drops to 0.30. Calculate the new maximum safe speed.

(c) Determine the percentage reduction in maximum safe speed caused by the wet road.

Solution

(a) For a car on a level curve, friction provides the centripetal force. The car slides when the required centripetal force exceeds the maximum static friction.

At maximum speed:

$$\frac{mv_{max}^2}{r} = \mu_s mg \text{ or } \frac{v_{max}^2}{r} = \mu_s g$$

Making v_{max} the subject:

$$v_{max} = \sqrt{\mu_s gr}$$

Where: $\mu_s = 0.65\ g = 9.8\ m/s^2, r = 80m$

Substituting:

$$v_{max} = \sqrt{0.65 \times 9.8 m/s^2 \times 80m} = \mathbf{22.6 m/s}$$

The maximum safe speed on the dry road is 22.6 m/s (about 81.3 km/h).

(b) With $\mu_s = 0.30$:

$$v_{max} = \sqrt{\mu_s gr}$$

Substituting:

$$v_{max} = \sqrt{0.30 \times 9.8 m/s^2 \times 80m} = \mathbf{15.3 m/s}$$

The maximum safe speed on the wet road is 15.3 m/s (about 55.2 km/h).

(c) Percentage reduction:

$$\% \text{ reduction} = \frac{v_{dry} - v_{wet}}{v_{dry}} \times 100\%$$

$$\% \text{ reduction} = \frac{(22.6 - 15.3) m/s}{22.6 m/s} \times 100\% = 32.3\%$$

The maximum safe speed is reduced by approximately 32%.

Making Sense of the Answer: *Notice that the maximum safe speed depends on the square root of μ_s. So even though friction dropped by more than half (from 0.65 to 0.30), the speed dropped by only about a third. However, the required force depends on v^2, so even a moderate speed increase beyond the safe limit causes a dramatic increase in the force demanded from the tyres.*

Think Like a Physicist: *The mass cancels in the equation for maximum safe speed:* $v_{max} = \sqrt{\mu_s gr}$

This means the maximum safe speed on a level curve does not depend on the mass of the vehicle. A heavy lorry and a light car have the same maximum safe speed on the same curve, assuming identical tyres and road conditions.

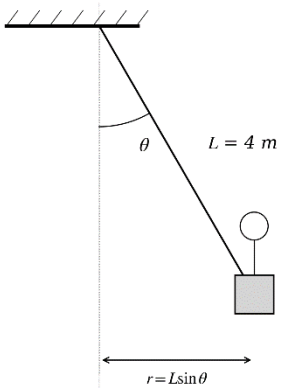
HOT Example 15

A fairground ride consists of chairs hanging from chains 4m long attached to a rotating circular platform. When rotating at constant speed, the chains make an angle of 30° with the vertical.

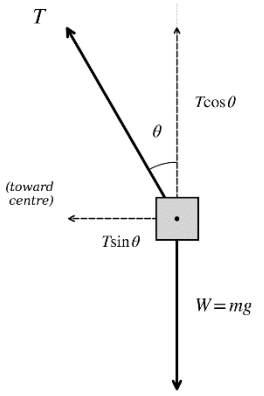
(a) Draw a free-body diagram for a chair with a person sitting in it during rotation.
(b) Explain what force provides the centripetal force for the circular motion.
(c) Calculate the radius of the circular path followed by the chair.
(d) Calculate the speed of rotation.

Solution

(a) Free-body diagram for chair + person (mass m):



(i) Physical setup



(ii) Free-body diagram

(b) The tension T has two components:

Vertical component: Tcos 30° (upward)

Horizontal component: Tsin 30° (toward centre of rotation)

Hence, **the centripetal force is provided by horizontal component of tension, Tsin 30°.**

(c) From the geometry (length L = 4.0 m, angle θ = 30°):

The radius r is the horizontal distance from the rotation axis:

$$r = L\sin\theta = 4\text{m} \times \sin 30° = 2\text{m}$$

The radius of the circular path is 2m.

(d) Vertical equilibrium (no vertical acceleration):

$$T\cos\theta = mg$$

$$T = \frac{mg}{\cos\theta}$$

Horizontally (centripetal force equation):

$$T\sin\theta = \frac{mv^2}{r}$$

Substituting $T = \frac{mg}{\cos\theta}$:

$$\left(\frac{mg}{\cos\theta}\right) \times \sin\theta = \frac{mv^2}{r}$$

Cancelling m and simplifying $\sin\theta / \cos\theta$:

$$g\tan\theta = \frac{v^2}{r}$$

From which:

$$v = \sqrt{rg\tan\theta} = \sqrt{2\text{m} \times 9.8\text{ms}^{-2} \times \tan 30°} = 3.36\text{m/s}$$

The speed of rotation is 3.36m/s.

Making Sense of the Answer: *The tilted chain shows that the tension must both **support the combined weight of rider and chair**, and **pull the chair toward the centre of the circle**. The vertical component balances the weight mg, while the horizontal component provides the centripetal force. Because the chain is*

inclined at 30°, the chair moves in a circular path of radius 2m. To maintain this angle during rotation, the ride must move at a speed of 3.36 m/s.

Think Like a Physicist: *This problem shows how circular motion problems often involve both centripetal force equations and equilibrium in other directions. The vertical forces balance (no vertical acceleration), giving one equation. The horizontal force provides centripetal acceleration, giving another equation. Solving these simultaneously yields the motion parameters. This pattern appears in many circular motion scenarios: banked curves, conical pendulums, and vertical circles.*

These examples demonstrate the power and versatility of the centripetal force equation. By identifying what physical force provides the centripetal force and setting it equal to the required mv^2/r, we can solve a vast array of circular motion problems. In the next subtopic, we will apply these principles specifically to horizontal circular motion, including detailed analysis of conical pendulums and banking.

HORIZONTAL CIRCULAR MOTION

In the previous subtopic, we developed the formulas for centripetal acceleration and centripetal force, and we applied them to several situations: a car on a track, a stone on a string, the moon orbiting the Earth, and even a fairground ride. Along the way, we derived the maximum safe speed on a level curve, $v_{max} = \sqrt{\mu_s gr}$, and saw clearly why our bus at Kilimani Hill could not hold the road.

But those examples only introduced the ideas. In this subtopic, we examine horizontal circular motion in full detail. We will use free-body diagrams and Newton's second law applied carefully in two directions, just as we learned in Chapter 4. Every situation we meet here involves one central question: *what real force provides the centripetal force?* The answer changes from one situation to the next, but the method remains the same. Identify the forces, resolve them, and set the net inward component equal to $\frac{mv^2}{r}$.

We begin with the simplest case: a car negotiating a level curve.

Car on a Level (Unbanked) Curve

In the previous subtopic, we showed that the maximum safe speed on a level curve is $v_{max} = \sqrt{\mu_s gr}$. We arrived at this result quickly by equating friction to centripetal force. Now we approach the same situation more carefully, using a proper free-body diagram and Newton's second law in both directions. This systematic method will serve us well when we meet more complex situations later.

Setting up the problem

Consider a car of mass **m** moving at constant speed **v** around a horizontal circular curve of radius **r** on a flat road. The road is not banked; it is perfectly level.

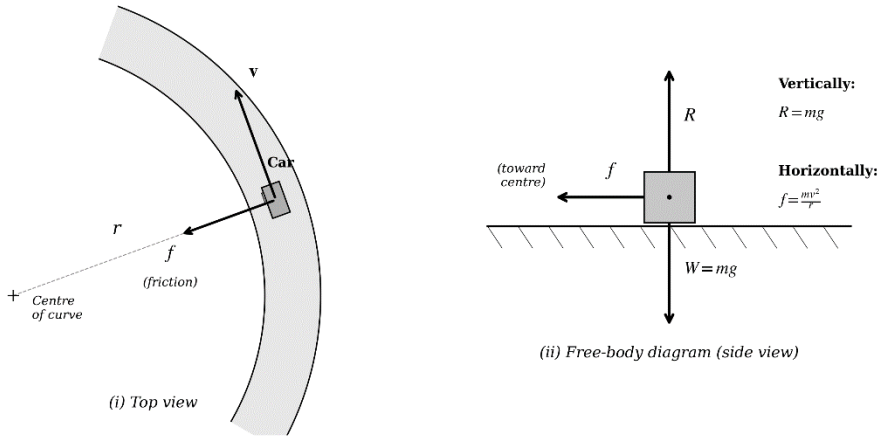


Figure: ***A car negotiating a level (unbanked) circular curve.*** *(i) Top view showing velocity tangent to the path and friction directed toward the centre of the curve. (ii) Free-body diagram: the normal reaction R balances the weight mg vertically, while static friction f provides the centripetal force horizontally.*

Identifying the forces

Three forces act on the car:

(i) **Weight** (W = mg), acting vertically downward through the centre of mass.

(ii) **Normal reaction** (R), acting vertically upward from the road surface, perpendicular to the road.

(iii) **Static friction** (f), acting horizontally toward the centre of the circular path. This is the force that prevents the tyres from sliding sideways across the road surface.

Notice that *friction here acts sideways (toward the centre of the curve), not forward or backward along the direction of motion. Many students picture friction as always opposing motion along the road. In circular motion on a level surface, it is the sideways friction between tyre and road that bends the car's path into a curve.*

Applying Newton's second law

We apply Newton's second law in two perpendicular directions:

Vertically (no acceleration):

The car does not accelerate up or down, so the net vertical force is zero:

$$R - mg = 0$$

From which:

$$\mathbf{R = mg}$$

Horizontally (toward the centre):

The only horizontal force is friction, and it acts toward the centre, providing the centripetal acceleration:

$$f = \frac{mv^2}{r}$$

This is the force that the road must supply to keep the car on the circular path. If the road cannot supply this force, the car slides.

The maximum safe speed

The maximum friction the road can provide is limited by:

$$f_{max} = \mu_s R = \mu_s mg$$

The car remains on the curve as long as the required centripetal force does not exceed this maximum friction:

$$\frac{mv^2}{r} \leq \mu_s mg$$

At the maximum safe speed, these are equal:

$$\frac{mv_{max}^2}{r} = \mu_s mg$$

Cancelling m and making v_{max} the subject:

$$\mathbf{v_{max} = \sqrt{\mu_s gr}}$$

It is important for you to understand that:

The mass *m* cancels completely from the equation. *This means the maximum safe speed on a level curve does not depend on the mass of the vehicle.* A fully loaded truck and an empty saloon car have exactly the same maximum safe speed, provided their tyres have the same coefficient of friction with the road. This surprises many students, but the physics is clear: *a heavier vehicle needs more centripetal force, but it also has more friction available (because $R = mg$ is larger), and these two effects cancel exactly.*

The result depends on three factors only:

(iv) The coefficient of static friction μ_s between the tyres and the road (tyre condition, road surface, wetness).

(v) The acceleration due to gravity g.

(vi) The radius r of the curve (sharper curves are more dangerous).

With the ideas now set in place, let us anchor them through worked examples.

BINDER Example 16

A car moves at 15m/s around a horizontal circular curve of radius 60m on a level road. The coefficient of static friction between the tyres and the road is 0.5. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the centripetal acceleration of the car.

(b) Determine whether the car can safely negotiate the curve at this speed.

Solution

(a) Using:

$$a = \frac{v^2}{r}$$

Where: $v = 15\text{m/s}$, $r = 60\text{m}$

Substituting:

$$a = \frac{(15\text{m/s})^2}{60\text{m}} = 3.75\text{m/s}^2$$

The centripetal acceleration is 3.75m/s^2.

(b) The maximum centripetal acceleration that friction can provide is given by:

$$a_{max} = \frac{v_{max}^2}{r}$$

But from:

$$\frac{mv_{max}^2}{r} = \mu_s mg;$$

$$\frac{v_{max}^2}{r} = \mu_s g$$

Hence:

$$a_{max} = \mu_s g = 0.5 \times 9.8\text{m/s}^2 = 4.9\text{m/s}^2 > 3.75\text{m/s}^2(\text{required})$$

Since the required centripetal acceleration (3.75m/s^2) is less than the maximum available (4.9m/s^2), the friction is sufficient. The car can safely negotiate the curve.

Alternative solution

By comparing velocities:

$$v_{max} = \sqrt{\mu_s gr} = \sqrt{0.5 \times 9.8\text{m/s}^2 \times 60\text{m}} = 17.15\text{m/s} > 15\text{m/s (given)}$$

Since the maximum allowable velocity is greater than the given velocity, the car can safely negotiate the curve.

Making Sense of the Answer: *The car needs 3.75 m/s² of centripetal acceleration, but friction can provide up to 4.9 m/s². There is a safety margin. However, if the speed increases or the road becomes wet (reducing μ_s), that margin shrinks. At some critical speed, the required acceleration equals the maximum available, and beyond that, the car slides.*

Think Like a Physicist: *Comparing accelerations or velocities is a quick way to check whether circular motion is possible without calculating forces. If $\frac{v^2}{r} > \mu_s g$ or $v_{max} > v$, the car will slide.*

REAL Example 17

During a road safety lesson, **Kipanga** argues: *"A heavy truck is more dangerous on a curve than a small car, because the truck needs much more centripetal force and the road cannot provide it."*

Kipute disagrees: *"The truck is heavier, but it also presses harder on the road, so friction is stronger too."*

Who is correct? Explain clearly.

Solution

Kipute is correct.

Explanation

A heavier vehicle does require more centripetal force to maintain circular motion, since $F_c = \frac{mv^2}{r}$ increases with mass. However, the maximum friction the road can provide also increases with mass, since $f_{max} = \mu_s mg$.

At maximum speed:

$$\frac{mv_{max}^2}{r} = \mu_s mg$$

The mass m appears on both sides and cancels, giving $v_{max} = \sqrt{\mu_s gr}$, which is independent of mass.

Therefore, the maximum safe speed is the same for both vehicles, provided they have the same tyre-road friction coefficient and travel the same curve. The truck's greater weight demands more force, but the road supplies more friction in exact proportion. The two effects cancel.

Making Sense of the Answer: *This is why speed limit signs on curves do not show different limits for trucks and cars. The safe speed depends on the curve's radius and the road condition, not on the vehicle's mass.*

Think Like a Physicist: *Whenever mass cancels from a physics result, it signals something deep: the phenomenon does not depend on the amount of matter involved. This is the same reason all objects fall with the same acceleration under gravity (mass cancels in $mg = ma$). Here, mass cancels for a similar reason: both the required force and the available force are proportional to mass.*

HOT Example 18

A driver travelling at 25m/s approaches a curve on a level road. The coefficient of static friction between the tyres and the road is 0.4. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the minimum radius of curve the driver can safely negotiate at this speed.

(b) If the road curves with a radius of only 120m, determine whether the driver must slow down, and if so, find the maximum safe speed for that curve.

(c) The driver enters the 120m curve at 25m/s without slowing down. Calculate the centripetal force required and the maximum friction force available for a car of mass 1400kg. Use these values to explain what happens to the car.

Solution

(a) At the maximum safe speed:

$$\frac{mv_{max}^2}{r_{min}} = \mu_s mg$$

Making r_{min} the subject:

$$r_{min} = \frac{v^2}{\mu_s g}$$

Where: $v = 25\text{m/s}$, $\mu_s = 0.40$, $g = 9.8\text{m/s}^2$

Substituting:

$$r_{min} = \frac{(25\text{m/s})^2}{0.40 \times 9.8\text{m/s}^2} = 159.4\text{m}$$

The minimum safe radius is 159.4m.

(b) Since the actual radius (120m) is less than the minimum safe radius (159.4m), the driver **must slow down**.

The maximum safe speed for the 120m curve:

$$v_{max} = \sqrt{\mu_s gr}$$

Substituting:

$$v_{max} = \sqrt{0.40 \times 9.8\text{m/s}^2 \times 120\text{m}} = 21.7\text{m/s}$$

The maximum safe speed is 21.7m/s (about 78km/h).

So the driver must slow from 25m/s to at most 21.7m/s.

(c) Required centripetal force at 25m/s:

$$F_{required} = \frac{mv^2}{r} = \frac{1400\text{kg} \times (25\text{m/s})^2}{120\text{m}} = 7291.7\text{N}$$

Maximum friction available:

$$f_{max} = \mu_s mg = 0.40 \times 1400\text{kg} \times 9.8\text{m/s}^2 = 5488\text{N} < 7291.7\text{N(required)}$$

The required centripetal force (7291.7N) exceeds the maximum friction (5488N) by 1803.7N. Since friction cannot supply the force needed to maintain circular motion at this speed, the tyres lose grip and the car slides outward from the curve, following a tangential path in accordance with Newton's first law.

Making Sense of the Answer: *The numbers tell the story clearly: the car demands 7292N of inward force, but the road can only offer 5488 N. The deficit of about 1804 N means the car cannot follow the curve. It is not a matter of the driver turning the steering wheel harder; the physics simply does not allow it. This is why speed limits on sharp curves are not suggestions; they are calculated from the laws of physics.*

Think Like a Physicist: *Rearranging $v_{max} = \sqrt{\mu_s gr}$ to find $r_{min} = \frac{v^2}{\mu_s g}$ is a useful trick. It tells you the minimum curve radius you can safely negotiate at a given speed. Highway engineers use exactly this formula when designing roads.*

With the car on the level curve now thoroughly understood, let us see what happens when a cyclist, rather than a car, must negotiate a curved path. The physics is the same, but the human body reveals it in a beautifully visible way.

Cyclist on a Curved Rough Level Road

When a car turns a corner, the physics is hidden inside the vehicle. The tyres grip, friction acts sideways, and the car follows the curve. But when a cyclist takes the same corner, something remarkable happens that the car cannot show: **the cyclist leans inward.**

This leaning is not a style choice. It is a physical necessity. A cyclist who tries to turn a corner while sitting perfectly upright will topple outward. The lean is the body's way of ensuring that the forces acting on the bicycle and rider can produce the required centripetal force while maintaining balance.

Why must the cyclist lean?

A bicycle is a narrow vehicle. Unlike a car, which has four widely spaced wheels, a bicycle makes contact with the ground along a very thin line. If the cyclist sits upright and turns, friction acts horizontally at the contact point (just as for a car), but the cyclist's weight acts vertically through the centre of mass, which is high above the ground. These two forces would then create a net *turning effect* (**torque**) that tips the cyclist outward.

To prevent this, the cyclist leans the entire bicycle inward. This shifts the line of action of the forces so that they balance not only in terms of net force, but also in terms of torque. The result is stable circular motion without toppling.

The forces acting on the cyclist

Consider the following diagram:

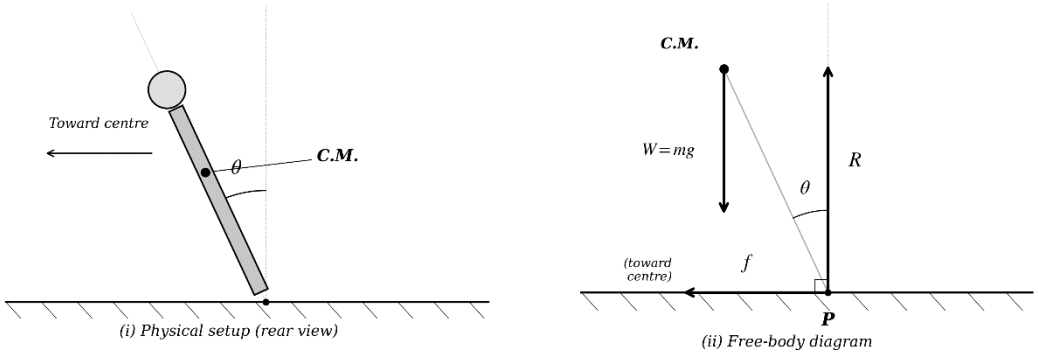


Figure: ***A cyclist negotiating a level circular curve, shown from behind (rear view).*** *(i) The cyclist and bicycle lean inward at angle θ from the vertical. The centre of mass (C.M.) is displaced toward the centre of the curve. (ii) Free-body diagram: weight W = mg acts downward through the centre of mass, while the normal reaction R and friction f act at the contact point P.*

Two forces act on the cyclist-bicycle system:

1. **Weight** (W = mg), acting vertically downward through the centre of mass.
2. **Contact force from the road**, which has two components at the point of contact:

- **Normal reaction** (R), acting vertically upward, perpendicular to the road surface.
- **Static friction** (f), acting horizontally toward the centre of the curve.

These are the same forces as for the car on a level curve. The difference is that for the cyclist, the angle of lean θ connects them geometrically.

Applying Newton's second law

Vertically (no acceleration):

$$R = mg$$

Horizontally (toward the centre):

$$f = \frac{mv^2}{r}$$

These are identical to the equations for a car on a level curve.

Finding the angle of lean

For the cyclist to remain balanced (no toppling), the net force must pass through the contact point. This requires the resultant of weight and friction to act along the line of the bicycle frame. Geometrically, this gives:

$$\tan\theta = \frac{f}{R} = \frac{mv^2/r}{mg}$$

From which:

$$\boldsymbol{\tan\theta = \frac{v^2}{rg}}$$

Where θ is the angle of lean measured from the vertical.

This equation reveals several important insights:

1. The angle of lean increases with speed. Faster turns require greater lean.
2. The angle of lean increases as the radius decreases. Sharper turns require greater lean.

3. The angle of lean does not depend on mass. A heavy cyclist and a light cyclist lean at the same angle for the same speed and radius.

Notice that this is the same expression, $\tan\theta = \frac{v^2}{rg}$, that will appear again in the conical pendulum and in banking of roads. The physics is universal; only the physical situation changes.

Maximum speed before skidding

The cyclist skids when friction reaches its maximum:

$$f_{max} = \mu_s mg$$

At maximum speed:

$$\frac{mv_{max}^2}{r} = \mu_s mg$$

$$\mathbf{v_{max} = \sqrt{\mu_s gr}}$$

This is exactly the same result as for the car.

Maximum lean angle before skidding

Since $\tan\theta = \frac{v^2}{rg}$, a larger speed requires a larger lean angle. Therefore, the maximum speed and the maximum lean angle, θ_{max} occur at the same instant: just before skidding. The cyclist cannot lean beyond θ_{max} because doing so would require a higher speed that demands more friction than the road can provide.

We showed earlier that the lean angle satisfies:

$$\tan\theta = \frac{f}{R} = \frac{f}{mg}$$

At maximum speed, friction reaches its limit:

$$f_{max} = \mu_s mg$$

Substituting:

$$\tan\theta_{max} = \frac{\mu_s mg}{mg}$$

From which:

$$\tan\theta_{max} = \mu_s$$

Hence, at this maximum speed, the lean angle satisfies:

$$\mathbf{\tan\theta_{max} = \mu_s}$$

This is a remarkable result. The maximum lean angle depends only on the coefficient of static friction between the tyres and the road. It does not depend on the radius of the curve or the mass of the cyclist. A heavier cyclist, a lighter cyclist, a wide curve, a tight curve; none of these matter. Only the grip of the tyres on the road determines how far the cyclist can lean before skidding.

With the ideas laid out clearly, let us practise them through worked examples.

BINDER Example 19

A cyclist rounds a level circular curve of radius 25m at a constant speed of 10m/s. Take $g = 9.8\ m/s^2$.

(a) Calculate the angle at which the cyclist must lean from the vertical.

(b) If the cyclist doubles the speed to 20 m/s on the same curve, find the new lean angle.

Solution

(a) Using:

$$\tan\theta = \frac{v^2}{rg}$$

Where: $v = 10$ m/s, $r = 25$ m, $g = 9.8$ m/s^2

Substituting:

$$\tan\theta = \frac{(10 \text{ m/s})^2}{25 \text{ m} \times 9.8 \text{ m/s}^2} = 0.408$$

$$\theta = \tan^{-1}(0.408) = 22.2°$$

The cyclist must lean at $22.2°$ from the vertical.

(b) At $v = 20$ m/s:

$$\tan\theta = \frac{(20 \text{ m/s})^2}{25 \text{ m} \times 9.8 \text{ m/s}^2} = 1.633$$

$$\theta = \tan^{-1}(1.633) = 58.5°$$

The new lean angle is $58.5°$.

Making Sense of the Answer: *Doubling the speed more than doubled the lean angle, from 22.2° to 58.5°. This is because the lean angle depends on* v^2*, not* v*. A 58.5° lean is extreme and very difficult to maintain in practice, which explains why cyclists slow down significantly before entering sharp curves.*

Think Like a Physicist: *If the calculated lean angle becomes very large (say above 45°), the required friction is enormous and the cyclist is almost certainly going to skid. In practice, lean angles beyond about 40° are dangerous on ordinary road surfaces.*

REAL Example 20

Kipanga rides his bicycle fast around a sharp corner near the school. **Kipute** notices that he leans his body and the bicycle strongly toward the inside of the curve. She asks: *"Why do you lean so much when turning? You don't lean when riding straight."*

Explain why a cyclist must lean inward when negotiating a curve.

Solution

When riding in a straight line, only two forces act on the cyclist: weight downward and the normal reaction upward. These are balanced, and no lean is needed.

When turning, however, a horizontal centripetal force is required toward the centre of the curve. This force is provided by static friction at the contact point between the tyres and the road. Friction acts horizontally at ground level, while the weight acts vertically through the centre of mass, which is high above the ground.

If the cyclist remains upright, friction and weight act along different lines, creating a turning effect that would topple the cyclist outward. By leaning inward, the cyclist shifts the centre of mass so that the combined effect of weight and friction passes through the contact point. This eliminates the toppling tendency and allows stable circular motion. The faster the turn or the sharper the curve, the greater the required friction and the more the cyclist must lean to maintain balance.

Making Sense of the Answer: *Leaning is not about courage or style. It is the cyclist's only way to keep the forces balanced during a turn. A car does not need to lean because it has four widely spaced wheels that prevent toppling. A bicycle, with only two narrow contact points, must lean to achieve the same balance.*

Think Like a Physicist: *The lean angle satisfies* $\tan\theta = \frac{v^2}{rg}$*. This same expression appears in the conical pendulum and in road banking, because the underlying physics is the same: balancing a horizontal centripetal requirement against a vertical gravitational force.*

HOT Example 21

A cyclist of mass 70 kg (including the bicycle) rounds a level curve of radius 30m. The coefficient of static friction between the tyres and the rough road surface is 0.55. Take $g = 9.8$ m/s^2.

(a) Calculate the maximum speed at which the cyclist can negotiate the curve without skidding.

(b) Find the lean angle at maximum speed.

(c) At maximum speed, calculate the friction force and the normal reaction.

(d) If the road becomes wet and the coefficient of friction drops to 0.25, determine the new maximum speed and the new lean angle.

Solution

(a) Using:

$$v_{max} = \sqrt{\mu_s gr}$$

Where: $\mu_s = 0.55$, $g = 9.8\ m/s^2$, $r = 30\ m$

Substituting:

$$v_{max} = \sqrt{0.55 \times 9.8\ m/s^2 \times 30\ m} = 12.7\ m/s$$

The maximum speed is 12.7 m/s (about 45.7 km/h).

(b) At maximum speed:

$$\tan\theta_{max} = \mu_s = 0.55$$

$$\theta_{max} = \tan^{-1}(0.55) = 28.8°$$

The lean angle at maximum speed is 28.8°.

(c) Normal reaction:

$$R = mg = 70\ kg \times 9.8\ m/s^2 = 686N$$

Friction force at maximum speed:

$$f = \mu_s mg = 0.55 \times 686\ N = 377.3N$$

The normal reaction is 686N and the friction force is 377.3N.

(d) With $\mu_s = 0.25$:

$$v_{max} = \sqrt{0.25 \times 9.8\ m/s^2 \times 30\ m} = 8.6\ m/s$$

$$\tan\theta_{max} = 0.25; \quad \theta_{max} = \tan^{-1}(0.25) = 14°$$

The new maximum speed on the wet road is 8.6 m/s (about 30.9 km/h) and the lean angle drops to 14°.

Making Sense of the Answer: *On a wet road, the cyclist must reduce speed from 12.7 m/s to 8.6 m/s, a reduction of about 32%. The lean angle also drops sharply, from 28.8° to 14°. This makes sense: less friction means less centripetal force available, so the cyclist must go slower and lean less. Attempting the original speed on a wet road would cause the tyres to skid sideways.*

Think Like a Physicist: *At maximum speed, $\tan\theta_{max} = \mu_s$. This is elegant: the maximum lean angle depends only on friction, regardless of speed, radius, or mass. It provides a quick way to estimate the danger of a curve: if the required lean angle exceeds* $\tan^{-1}(\mu_s)$*, skidding is inevitable.*

Having explored the cyclist, we notice something interesting: the expression $\tan\theta = \frac{v^2}{rg}$ keeps appearing. It appeared for the cyclist's lean angle, and it will appear again in our next subtopic, the conical pendulum, where a mass on a string traces a horizontal circle. The angle is different, the physical setup is different, but the mathematics is remarkably the same. Let us see why.

Conical Pendulum

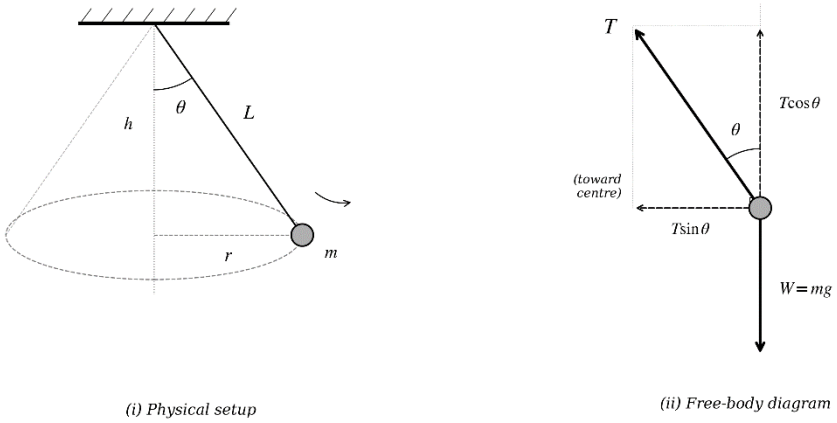
Imagine tying a small stone to the end of a string, holding the other end fixed above your head, and then setting the stone in motion so that it traces a horizontal circle. The string sweeps out the surface of a cone as the stone revolves, and the arrangement is called a **conical pendulum**.

Unlike the cyclist or the car, there is no friction here. The only forces acting on the stone are its weight and the tension in the string. Yet the stone follows a perfectly circular horizontal path. *How does it manage this without friction?* The answer lies in the geometry: the string is not vertical but inclined at an angle θ to the vertical, and it is the horizontal component of the tension that provides the centripetal force.

This is a beautiful example of how a single force (tension) can serve two roles simultaneously: its vertical component supports the weight, while its horizontal component maintains the circular motion.

Setting up the problem

Consider a small object of mass m attached to a string of length **L**, with the other end fixed to a point directly above the centre of the circular path. The object moves at constant speed in a horizontal circle of radius **r**. The string makes a constant angle **θ** with the vertical.



(i) Physical setup *(ii) Free-body diagram*

Figure: A conical pendulum. *(i) A mass m attached to a string of length L revolves in a horizontal circle of radius r. The string makes angle θ with the vertical and sweeps out a cone. (ii) Free-body diagram: only tension T and weight W = mg act on the mass. The vertical component Tcos θ balances the weight, while the horizontal component Tsin θ provides the centripetal force.*

Identifying the forces

Only two forces act on the object:

1) **Weight** (W = mg), acting vertically downward.
2) **Tension** (T), acting along the string, directed from the object toward the fixed point.

There is no friction, no normal reaction, and no additional horizontal force. The centripetal force must come entirely from the tension.

Applying Newton's second law

We resolve the tension into vertical and horizontal components:

Vertically (no acceleration):

The object does not move up or down, so the net vertical force is zero:

$$T\cos\theta - mg = 0$$

From which:

$$T\cos\theta = mg \quad \ldots (i)$$

Horizontally (toward the centre):

The horizontal component of tension provides the centripetal force:

$$T\sin\theta = \frac{mv^2}{r} \quad \ldots (ii)$$

Finding the relationship between angle and speed

Dividing equation (ii) by equation (i):

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2/r}{mg}$$

The tension T and mass m cancel:

$$\mathbf{\tan\theta = \frac{v^2}{rg}}$$

This is the same expression we met for the cyclist's lean angle. The physics is different, the forces are different, but the geometry of balancing a horizontal centripetal requirement against a vertical gravitational force produces the same result.

It is important for you to understand that:

The angle θ does not depend on the mass of the object. A heavy stone and a light stone, whirled on the same string at the same speed, will both make the same angle with the vertical. This is because mass appears in both the centripetal force requirement (mv^2/r) and the weight (mg), and cancels when we divide.

What the angle does depend on is the speed v and the radius r. A faster speed or a smaller radius requires a larger angle.

Finding the tension

From equation (i):

$$\mathbf{T = \frac{mg}{\cos\theta}}$$

Since $\cos\theta < 1$ for any angle $\theta > 0°$, the tension is always greater than the weight. This is because the string has two duties: it must support the weight vertically and simultaneously pull the object inward. This double duty means the tension must exceed mg.

Relating radius to string length

From the geometry of the cone:

$$\mathbf{r = L\sin\theta}$$

This connects the radius of the circular path to the string length and the angle. It is the same geometric relationship we used for the fairground chair ride in Example 15.

Deriving the period of revolution

The period T_p (we use T_p instead of the usual T because in this subtopic the symbol T is used for tension; using the same letter for both would cause confusion) can be found by combining our results. Starting from:

$$\tan\theta = \frac{v^2}{rg}$$

We substitute $v = \frac{2\pi r}{T_p}$:

$$\tan\theta = \frac{4\pi^2 r}{T_p^2 g}$$

Making T_p^2 the subject:

$$T_p^2 = \frac{4\pi^2 r}{g\tan\theta}$$

Now substituting $r = L\sin\theta$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$T_p^2 = \frac{4\pi^2 L\sin\theta}{g \times \frac{\sin\theta}{\cos\theta}} = \frac{4\pi^2 L\cos\theta}{g}$$

From which:

$$\mathbf{T_p = 2\pi\sqrt{\frac{L\cos\theta}{g}}}$$

Understanding what this equation reveals

This is a striking result. The period depends on the string length L, the angle θ, and gravity g. It does not depend on the mass of the object.

As the angle θ increases (faster rotation), cosθ decreases, and the period becomes shorter. The object revolves faster. In the limiting case where θ approaches 90° (string nearly horizontal), the period approaches zero and the speed becomes very large, but the tension becomes extremely large because $T = \frac{mg}{\cos\theta}$ increases without bound.

In the opposite limit, as θ approaches 0° (string nearly vertical), the period approaches $2\pi\sqrt{\frac{L}{g}}$, which is the period of a simple pendulum of the same length. This makes physical sense: at very small angles, the conical pendulum barely traces a circle and its motion resembles a simple pendulum.

With the theory complete, let us work through some examples.

BINDER Example 22

A small ball of mass 0.3 kg is attached to a string of length 0.8 m and whirled as a conical pendulum. The string makes an angle of 40° with the vertical. Take $g = 9.8\ m/s^2$.

(a) Calculate the tension in the string.

(b) Calculate the radius of the circular path.

(c) Calculate the speed of the ball.

Solution

(a) Using:

$$T\cos\theta = mg$$

Making T the subject:

$$T = \frac{mg}{\cos\theta}$$

Where: $m = 0.3\ kg$, $g = 9.8\ m/s^2$, $\theta = 40°$

Substituting:

$$T = \frac{0.3\ kg \times 9.8\ m/s^2}{\cos 40°} = 3.84N$$

The tension is 3.84 N.

(b) Using:

$$r = L\sin\theta$$

Where: $L = 0.8m$, $\theta = 40°$

Substituting:

$$r = 0.8\ m \times \sin 40° = 0.514\ m$$

The radius is 0.514 m.

(c) Using:

$$T\sin\theta = \frac{mv^2}{r}$$

Making v the subject:

$$v = \sqrt{\frac{T\sin\theta \cdot r}{m}}$$

Substituting:

$$v = \sqrt{\frac{3.84\ \text{N} \times 0.643 \times 0.514\ \text{m}}{0.3\ \text{kg}}} = \sqrt{4.23} = 2.06\text{m/s}$$

The speed is 2.06 m/s.

Making Sense of the Answer: *The tension (3.84N) is greater than the weight (2.94N), as expected. The string must support the weight vertically and pull the ball inward simultaneously. At a 40° angle, the radius is about 0.51m, which is reasonable for a 0.8m string. The speed of 2.06m/s is modest, consistent with a gentle swing.*

Think Like a Physicist: *Always find the tension first using $T\cos\theta = mg$, because tension appears in both the vertical and horizontal equations. Once you have T, the rest follows easily.*

REAL Example 23

Kipanga ties a small stone to a string and whirls it in a horizontal circle above his head. He notices that when he spins the stone faster, the string rises and the angle with the vertical increases. He asks **Kipute**: *"Why does the string go higher when I spin faster? Shouldn't gravity keep pulling it down the same way?"*

Explain why the angle increases when the speed increases.

Solution

Gravity does pull the stone downward with the same force (mg) regardless of speed. However, as the speed increases, the stone requires a larger centripetal force to maintain its circular path, since the centripetal force is proportional to v^2.

The only horizontal force available is the horizontal component of the tension: $T\sin\theta$. To increase this horizontal component, either the tension must increase, or the angle must increase, or both. In practice, as the speed increases, the string rises to a larger angle θ. A larger angle means a greater proportion of the tension is directed horizontally (larger $\sin\theta$), providing the extra centripetal force needed.

The relationship $\tan\theta = \frac{v^2}{rg}$ confirms this: increasing v increases $\tan\theta$, which means θ increases.

So gravity has not changed, but the balance between vertical support and horizontal pull has shifted. The string rises because the stone's circular motion demands more horizontal force, and the only way to get it is by tilting the string further from the vertical.

Making Sense of the Answer: *Think of it this way: at low speed, the stone barely needs any centripetal force, so the string hangs almost vertically and nearly all the tension goes toward supporting the weight. At high speed, the centripetal demand is large, so the string must tilt outward to redirect more tension horizontally.*

Think Like a Physicist: *The angle is a visible indicator of how much centripetal force the system demands. A large angle means the circular motion is demanding a large fraction of the available tension. This is why conical pendulums are used in some engineering speed governors: the angle directly indicates the rotation speed.*

HOT Example 24

A conical pendulum has a string of length 1.2m. The period of revolution is 1.8s. Take $g = 9.8\ \text{m/s}^2$ and $\pi = 3.14$.

(a) Calculate the angle the string makes with the vertical.

(b) Calculate the radius of the circular path.

(c) Calculate the tension in the string if the mass is 0.5 kg.

(d) Determine the speed of the object.

Solution

(a) Using:

$$T_p = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

Making $\cos\theta$ the subject:

$$T_p^2 = 4\pi^2 \times \frac{L\cos\theta}{g}$$

$$\cos\theta = \frac{T_p^2 g}{4\pi^2 L}$$

Where: $T_p = 1.8$ s, $g = 9.8$ m/s^2, $L = 1.2$ m

Substituting:

$$\cos\theta = \frac{(1.8\text{ s})^2 \times 9.8\text{ m/s}^2}{4 \times (3.14)^2 \times 1.2\text{ m}} = 0.671$$

$$\theta = \cos^{-1}(0.671) = 47.9°$$

The angle is 47.9°.

(b) Using:

$$r = L\sin\theta = 1.2\text{ m} \times \sin 47.9° = 1.2 \times 0.742 = 0.890\text{ m}$$

The radius is 0.890 m.

(c) Using:

$$T = \frac{mg}{\cos\theta} = \frac{0.5\text{ kg} \times 9.8\text{ m/s}^2}{0.671} = 7.30\text{ N}$$

The tension is 7.30 N.

(d) Using:

$$v = \frac{2\pi r}{T_p} = \frac{2 \times 3.14 \times 0.890\text{ m}}{1.8\text{ s}} = \frac{5.59\text{ m}}{1.8\text{ s}} = 3.1\text{m/s}$$

The speed is 3.1 m/s.

Making Sense of the Answer: *An angle of nearly 48° is quite large; the string is closer to horizontal than vertical. This tells us the rotation is fairly fast. The tension (7.3N) is about 1.5 times the weight (4.9N), confirming that the string works hard to provide both vertical support and horizontal centripetal force. The speed of 3.1m/s over a radius of 0.89m gives a reasonable revolution time of 1.8 s.*

Think Like a Physicist: *When the period is given instead of the speed, use the period formula directly to find the angle. Do not try to find the speed first; it creates unnecessary steps. The formula* $\cos\theta = \frac{T_p^2 g}{4\pi^2 L}$ *gives the angle in one calculation.*

With the conical pendulum now understood, we recognise a pattern that runs through all of horizontal circular motion. The leaning cyclist and the conical pendulum both produce the same central relationship: $\tan\theta = \frac{v^2}{rg}$. In each case, the angle θ connects the vertical force (gravity) to the horizontal force (centripetal). This same relationship will appear once more in our next subtopic, where engineers deliberately tilt the road surface to help vehicles navigate curves safely. Welcome to banking of roads.

Banking of Roads

On a level curve, the centripetal force comes entirely from friction. But friction is unreliable. Rain reduces it. Worn tyres reduce it. Oil spills destroy it. And when friction fails, vehicles slide off the road, as we learned at Kilimani Hill.

Engineers asked a simple but powerful question: *What if the road itself could help provide the centripetal force, even without any friction at all?*

The answer is **banking:** *tilting the road surface so that the outer edge is higher than the inner edge.* On a banked curve, the road surface is no longer horizontal. The normal reaction, which is always perpendicular to the surface, now tilts inward. This tilted normal reaction has a horizontal component that points toward the centre of the curve, providing some or all of the centripetal force.

Banking does not eliminate friction; it reduces the dependence on friction. A well-banked curve at the correct speed requires no friction at all. At other speeds, friction assists or opposes the motion, but the banking carries the main load.

Case 1: Ideal banking (no friction)

Consider a car of mass **m** moving at constant speed **v** around a banked curve of radius **r**. The road surface makes an angle θ with the horizontal. We analyse the ideal case where friction is zero (perfectly smooth surface).

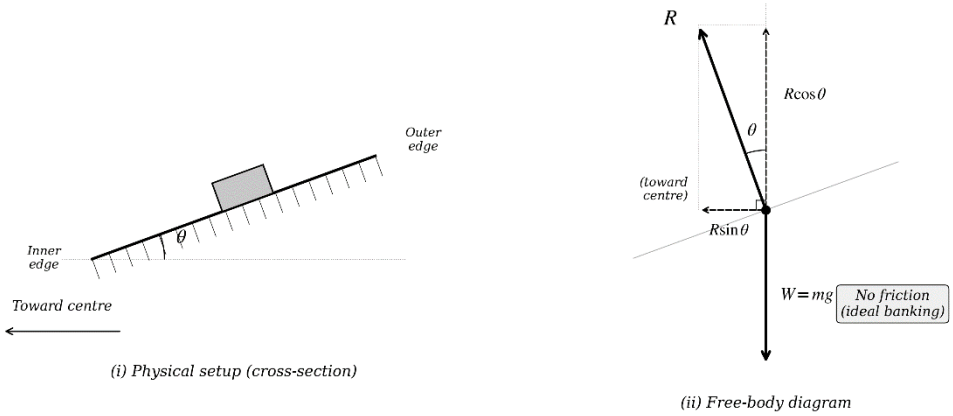


Figure: ***A car on a banked (tilted) curve, shown in cross-section.*** *(i) The road surface makes angle θ with the horizontal, with the outer edge higher than the inner edge. (ii) Free-body diagram for ideal banking (no friction): the normal reaction R is perpendicular to the tilted surface. Its vertical component Rcosθ balances the weight, while its horizontal component Rsinθ provides the centripetal force.*

Identifying the forces

With no friction, only two forces act on the car:

1) **Weight** (W = mg), acting vertically downward.
2) **Normal reaction** (R), acting perpendicular to the banked surface. Because the surface is tilted, R is no longer vertical; it tilts inward.

Resolving the normal reaction

We resolve R into two components:

- **Vertical component:** Rcosθ (upward)
- **Horizontal component:** Rsinθ (toward the centre of the curve)

Applying Newton's second law

Vertically (no acceleration):

$$R\cos\theta = mg \quad \ldots (i)$$

Horizontally (toward the centre):

$$R\sin\theta = \frac{mv^2}{r} \quad \ldots (ii)$$

Finding the banking angle

Dividing equation (ii) by equation (i):

$$\frac{R\sin\theta}{R\cos\theta} = \frac{mv^2/r}{mg}$$

R and m cancel:

$$\mathbf{\tan\theta = \frac{v^2}{rg}}$$

This is the same expression we derived for the cyclist's lean angle and the conical pendulum. The physics is universal: *whenever gravity provides vertical equilibrium and a horizontal component of some force provides centripetal acceleration, this relationship emerges.*

The design speed

The equation $\tan\theta = \frac{v^2}{rg}$ can be rearranged to give the speed for which the banking angle is perfectly matched:

$$\mathbf{v_{design} = \sqrt{rg\tan\theta}}$$

At this speed, the car negotiates the curve without any friction at all. The entire centripetal force comes from the horizontal component of the normal reaction. This is called the **design speed** of the banked curve.

It is important for you to understand that:

The design speed does not depend on the mass of the vehicle. A motorcycle, a car, and a fully loaded bus all have the same design speed on the same banked curve, because mass cancels from the equation. This is the same mass-independence we observed for the maximum safe speed on a level curve, the cyclist's lean angle, and the conical pendulum angle.

What happens at speeds other than the design speed?

If the car travels at exactly the design speed, no friction is needed.

If the car travels **faster** than the design speed, the required centripetal force exceeds what the banking alone provides. Friction must act *inward* (down the slope) to supply the extra centripetal force. Without sufficient friction, the car slides outward (up the slope).

If the car travels **slower** than the design speed, the banking provides more inward force than needed. Friction must act *outward* (up the slope) to prevent the car from sliding inward (down the slope).

At the design speed, the forces are perfectly balanced and friction is zero. This is why highway engineers choose the banking angle based on the expected speed of traffic on that section of road.

Case 2: Banked curve with friction (maximum speed)

In real life, roads are not frictionless. When a car travels faster than the design speed on a banked curve, the horizontal component of the normal reaction alone cannot supply the centripetal force required. The car tends to slide outward (up the slope). Friction therefore acts to prevent this sliding, which means friction acts **down the slope** (toward the inner edge), helping to push the car inward.

Why does friction act down the slope?

This is the point that confuses many students. On a level road, friction acts horizontally toward the centre of the curve. On a banked road, however, if the speed is too high, the vehicle tends to slide up the slope toward the outside of the curve. Friction therefore acts **down the slope** toward the centre to oppose this motion. In general, friction always acts in the direction that opposes the motion that would occur if friction were absent.

Identifying the forces

Three forces now act on the car:

3) **Weight** (W = mg), acting vertically downward.
4) **Normal reaction** (R), acting perpendicular to the banked surface (tilted inward from the vertical).
5) **Friction** (f), acting along the banked surface, directed down the slope (toward the inner edge).

Resolving the forces

Each force must be resolved into vertical and horizontal components. This is where careful thought is needed, because both R and f contribute to both directions.

Normal reaction R (perpendicular to road, tilted at angle **θ from the vertical**):

- Vertical component: $R\cos\theta$ (upward)
- Horizontal component: $R\sin\theta$ (toward centre)

Friction f (along the road surface, directed down the slope, at angle **θ from the horizontal**):

- Vertical component: $f\sin\theta$ (downward, because friction points down the slope)
- Horizontal component: $f\cos\theta$ (toward centre, because the down-slope direction has a horizontal inward component)

This is the key insight: friction has a **vertical component that acts downward**, reducing the vertical support, and a **horizontal component that acts toward the centre**, adding to the centripetal force. **Both R and f contribute to the centripetal force**, but friction slightly undermines the vertical balance.

Applying Newton's second law

Vertically (no acceleration):

The upward forces must balance the downward forces:

$$R\cos\theta = mg + f\sin\theta$$

Rearranging:

$$R\cos\theta - f\sin\theta = mg \quad \ldots (i)$$

Notice that friction's vertical component ($f\sin\theta$) appears on the same side as weight. Friction acting down the slope pulls the car slightly downward, so the normal reaction must support both the weight and this downward component of friction.

Horizontally (toward the centre):

Both the horizontal component of R and the horizontal component of f point toward the centre:

$$R\sin\theta + f\cos\theta = \frac{mv^2}{r} \quad \ldots (ii)$$

Banking and friction work together to provide the centripetal force.

Finding the maximum speed

The maximum speed occurs when friction reaches its limit:

$$f = \mu_s R$$

Substituting $f = \mu_s R$ into equation (i):

$$R\cos\theta - \mu_s R\sin\theta = mg$$

$$R(\cos\theta - \mu_s\sin\theta) = mg \quad \ldots (iii)$$

Substituting $f = \mu_s R$ into equation (ii):

$$R\sin\theta + \mu_s R\cos\theta = \frac{mv_{max}^2}{r}$$

$$R(\sin\theta + \mu_s\cos\theta) = \frac{mv_{max}^2}{r} \quad \ldots (iv)$$

Dividing equation (iv) by equation (iii) to eliminate both R and m:

$$\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta} = \frac{v_{max}^2}{rg}$$

Dividing numerator and denominator of the left side by $\cos\theta$:

$$\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta} = \frac{v_{max}^2}{rg}$$

Making v_{max} the subject:

$$\mathbf{v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)}}$$

Checking the formula against known results

A good formula should reduce to simpler cases when we set certain quantities to zero. Let us verify:

- If $\theta = 0$ (level road, no banking): $\tan\theta = 0$, so the formula gives $v_{max} = \sqrt{rg \times \mu_s} = \sqrt{\mu_s gr}$, which is our result we established earlier.
- If $\mu_s = 0$ (no friction): the formula gives $v_{max} = \sqrt{rg\tan\theta}$, which is the design speed from Case 1.

Both limiting cases check out, giving us confidence that the derivation is correct.

Case 3: Banked curve with friction (minimum speed)

So far we have found the maximum speed. But on a banked curve, there is also a **minimum** safe speed, and this surprises many students.

When the car travels slower than the design speed, the horizontal component of the normal reaction provides more inward force than the circular motion requires. The car tends to slide **down the slope** (toward the inner edge). Friction must now act **up the slope** (toward the outer edge) to prevent this inward sliding.

Notice the reversal: *at high speed, friction acts down the slope to help the car turn; at low speed, friction acts up the slope to stop the car from sliding inward. At exactly the design speed, friction is zero.*

Identifying the forces

The same three forces act on the car: weight $W = mg$ (downward), normal reaction R (perpendicular to road, tilted inward), and friction f (along the road surface). But now friction acts **up the slope** instead of down.

Resolving the forces

Because friction is reversed, its components change sign compared to Case 2:

Friction f (up the slope):

- Vertical component: $f\sin\theta$ (**upward**, helping to support the weight)
- Horizontal component: $f\cos\theta$ (**away from centre**, opposing the centripetal force)

Applying Newton's second law

Vertically (no acceleration):

$$R\cos\theta + f\sin\theta = mg \quad \ldots (v)$$

Compare this with equation (i) from Case 2: the friction term has changed from $-f\sin\theta$ to $+f\sin\theta$ because friction now has an upward vertical component.

Horizontally (toward the centre):

$$R\sin\theta - f\cos\theta = \frac{mv^2}{r} \quad \ldots (vi)$$

Compare with equation (ii) from Case 2: the friction term has changed from $+f\cos\theta$ to $-f\cos\theta$ because friction now opposes the centripetal direction.

Finding the minimum speed

At the minimum speed, friction reaches its limit: $f = \mu_s R$.

Substituting into equations (v) and (vi):

$$R(\cos\theta + \mu_s\sin\theta) = mg \quad \ldots (vii)$$

$$R(\sin\theta - \mu_s\cos\theta) = \frac{mv_{min}^2}{r} \quad \text{... (viii)}$$

Dividing (viii) by (vii):

$$\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta} = \frac{v_{min}^2}{rg}$$

Dividing numerator and denominator by $\cos\theta$:

$$\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta} = \frac{v_{min}^2}{rg}$$

Making v_{min} the subject:

$$\mathbf{v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)}}$$

Understanding what this equation reveals

Compare the minimum and maximum speed formulas side by side:

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)} \qquad v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)}$$

The pattern is elegant: wherever the maximum speed formula has a $+$, the minimum speed formula has a $-$, and vice versa. This reflects the reversal of friction direction.

An important special case arises when $\tan\theta \leq \mu_s$. In this case, the numerator $(\tan\theta - \mu_s)$ becomes zero or negative, which means $v_{min} = 0$. Physically, this means friction is strong enough to hold the car stationary on the banked surface without sliding. The car can safely travel at any speed from zero up to v_{max}.

However, if the banking is steep (large θ) and friction is weak (small μ_s), then $\tan\theta > \mu_s$, and there is a real minimum speed. A car travelling too slowly on such a curve will slide down the slope toward the inner edge.

The complete picture

Every banked curve with friction has a safe speed range:

$$v_{min} \leq v \leq v_{max}$$

Below v_{min}, the car slides inward (down the slope). Above v_{max}, the car slides outward (up the slope). At the design speed $v_{design} = \sqrt{rg\tan\theta}$, friction is zero and the ride is smoothest.

The worked examples that follow will sharpen all of these ideas.

BINDER Example 25

A highway curve has a radius of 200m. The road is banked at an angle of 12° to the horizontal. Take $g = 9.8\ m/s^2$.

(a) Calculate the design speed of the curve.

(b) A car of mass 1200 kg travels at the design speed. Calculate the normal reaction acting on the car.

Solution

(a) Using:

$$v_{design} = \sqrt{rg\tan\theta}$$

Where: $r = 200$ m, $g = 9.8\ m/s^2$, $\theta = 12°$

Substituting:

$$v_{design} = \sqrt{200\ m \times 9.8\ m/s^2 \times \tan 12°} = 20.4\ m/s$$

The design speed is 20.4 m/s (about 73.4 km/h).

(b) Using:

$$R\cos\theta = mg$$

Making R the subject:

$$R = \frac{mg}{\cos\theta}$$

Where: $m = 1200$ kg, $g = 9.8$ m/s^2, $\theta = 12°$
Substituting:

$$R = \frac{1200\text{ kg} \times 9.8\text{ m/s}^2}{\cos 12°} = \frac{11760\text{N}}{0.978} = 12024\text{N}$$

The normal reaction is 12024N.

Making Sense of the Answer: *The design speed of about 73 km/h is typical for a gently banked highway curve. The normal reaction (12024N) is slightly larger than the weight (11760N) because the normal force must support the weight vertically while also providing centripetal force horizontally. This is the same effect we saw in the conical pendulum, where tension exceeded the weight.*

Think Like a Physicist: *At the design speed, friction is zero, so the only contact force is the normal reaction. The car feels exactly as if it were driving on a perfectly smooth surface. Passengers experience no sideways sensation because the banking angle matches the speed perfectly.*

REAL Example 26

On a school trip to Iringa, the bus passes through a mountain road with several sharp curves. At one particular curve, **Kipanga** looks out the window and sees that the road is visibly tilted, with the outer edge much higher than the inner edge.

Kipanga (alarmed): *"Mr. Akilikubwa! Look at this road! It is slanting badly. I think the road has been damaged by heavy rains. Should we report it to TANROADS?"*

Mr. Akilikubwa (smiling): *"Kipanga, if you report that road, the engineers at TANROADS will laugh. They are the ones who built it that way on purpose."*

Kipanga (confused): *"On purpose? Why would anyone build a slanting road?"*

Kipute (grinning): *"Because physics said so, Kipanga."*

Explain why the road is deliberately tilted (banked) at curves, and what advantage this provides over a flat road.

Solution

On a level (flat) curve, the centripetal force required to keep the bus on its circular path must come entirely from friction between the tyres and the road. If friction is insufficient (due to rain, worn tyres, or excessive speed), the bus slides off the curve.

A banked road is deliberately tilted so that the outer edge is higher than the inner edge. This causes the normal reaction from the road to tilt inward. The horizontal component of this tilted normal reaction points toward the centre of the curve and provides centripetal force.

At the design speed, the banking alone supplies all the centripetal force needed, and friction is not required at all. This means the bus can safely negotiate the curve even on a wet or icy surface, provided it travels near the design speed. Thus banking greatly reduces dependence on friction and makes the curve safer under all road conditions.

The engineers at TANROADS calculated the banking angle using $\tan\theta = \frac{v^2}{rg}$, based on the expected traffic speed and the radius of the curve.

Making Sense of the Answer: *Kipanga's instinct was understandable. A tilted road looks wrong if you do not know why it is tilted. But the tilt is deliberate engineering, not damage. Every banked curve on a highway represents a careful calculation: the angle is chosen so that normal vehicles at normal speeds require little or no friction to stay on the road.*

Think Like a Physicist: *Banking is nature's way of saying: "If you cannot trust friction, tilt the surface." The same principle appears in athletics (track bicycle racing), railway curves, and aircraft turns. Whenever something must follow a curved path reliably, banking is the engineer's first tool.*

HOT Example 27

(a) A curved section of road has a radius of 150m and is banked at an angle of 15°. The coefficient of static friction between the tyres and the road surface is 0.4. Take $g = 9.8\ m/s^2$.

(b) Compare this maximum speed with: the design speed of the curve (no friction), and the maximum speed if the same curve were level (unbanked) with the same friction. What does this comparison reveal about the advantage of combining banking with friction?

Solution

(a) Using the result derived for the maximum speed on a banked curve with friction:

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)}$$

Where: $r = 150\ m$, $g = 9.8\ m/s^2$, $\theta = 15°$, $\mu_s = 0.4$

Substituting:

$$v_{max} = \sqrt{150 m/s \times 9.8\ m/s^2 \times \left(\frac{\tan 15° + 0.4}{1 - 0.4\tan 15°}\right)} = 33.2 m/s$$

The maximum speed on the banked curve is 33.2 m/s (about 119.4 km/h).

(b) Comparisons:

On a level road (no banking, $\theta = 0$), the maximum speed depends on friction alone:

$$v_{max,\ level} = \sqrt{\mu_s gr} = \sqrt{0.40 \times 9.8 \times 150} = \sqrt{588} = 24.2\ m/s \text{ (about 87.3 km/h)}$$

The design speed of the banked curve (no friction):

$$v_{design} = \sqrt{rg\tan\theta} = \sqrt{150 \times 9.8 \times 0.2679} = \sqrt{393.8} = 19.8\ m/s \text{ (about 71.4 km/h)}$$

So the three scenarios give:

- Banking alone (no friction): 19.8 m/s
- Friction alone (no banking): 24.2 m/s
- Banking and friction together: 33.2 m/s

The combination of banking and friction supports a maximum speed that is significantly higher than either banking or friction could achieve alone.

Making Sense of the Answer: *Banking alone gives 71 km/h. Friction alone gives 87 km/h. But together they give 119 km/h, which is far more than either alone. This is not simply addition; the two effects reinforce each other through the geometry of the tilted surface. This is why highway engineers never rely on friction alone for high-speed curves; they always bank the road as well.*

Think Like a Physicist: *Notice that the formula has* $1 - \mu_s\tan\theta$ *in the denominator. If* $\mu_s\tan\theta$ *approaches 1, the denominator approaches zero and the maximum speed becomes very large. Physically, this means a steeply banked road with high friction can support extremely high speeds.*

With banking of roads now understood, we have seen how engineers use geometry to assist circular motion. The final application in horizontal circular motion is simpler but equally instructive: an object placed on a rotating platform. Let us see what keeps it from sliding off, and what happens when the rotation becomes too fast.

Object on a Rotating Platform

Place a coin on a spinning turntable. At low speed, the coin stays in place. Increase the speed gradually, and at some critical point the coin suddenly slides outward and flies off the edge. This simple observation contains a complete physics lesson.

The coin is not glued to the turntable. The only horizontal force available to keep it moving in a circle is **static friction** between the coin and the surface. As the turntable spins faster, the required centripetal force

increases. The moment this required force exceeds the maximum static friction, the coin can no longer maintain circular motion and it slides outward.

Setting up the problem

Consider an object of mass m placed at a distance **r** from the centre of a horizontal turntable that rotates at angular velocity $\boldsymbol{\omega}$.

Identifying the forces

Three forces act on the object:

(i) **Weight** (W = mg), acting vertically downward.

(ii) **Normal reaction** (R), acting vertically upward. Since the surface is horizontal, R is vertical (unlike the banked road).

(iii) **Static friction** (f), acting horizontally toward the centre of the turntable. This is the only force available to provide centripetal acceleration.

Applying Newton's second law

Vertically (no acceleration):

$$R = mg$$

Horizontally (toward the centre):

$$f = m\omega^2 r$$

Be careful: We use $m\omega^2 r$ rather than $\frac{mv^2}{r}$ here because angular velocity ω is the natural quantity for a rotating platform: every point on the turntable shares the same ω, but points at different radii have different linear speeds.

The critical condition for sliding

The object remains in place as long as friction can supply the required centripetal force:

$$m\omega^2 r \leq \mu_s mg$$

Sliding occurs when the required centripetal force exceeds the maximum friction. At the critical point:

$$m\omega_{max}^2 r = \mu_s mg$$

Making ω_{max} the subject:

$$\boldsymbol{\omega_{max} = \sqrt{\frac{\mu_s g}{r}}}$$

This can also be expressed as a maximum linear speed at radius r (from $v = \omega r$):

$$\boldsymbol{v_{max} = \sqrt{\mu_s g r}}$$

Or as a maximum frequency (from $\omega = 2\pi f$):

$$\boldsymbol{f_{max} = \frac{1}{2\pi}\sqrt{\frac{\mu_s g}{r}}}$$

It is important for you to understand that:

The formula $\omega_{max} = \sqrt{\frac{\mu_s g}{r}}$ reveals something that many students find surprising: *objects farther from the centre slide off first. Why?*

At a larger radius r, the maximum angular velocity ω_{max} is smaller. So when the turntable gradually speeds up, objects near the edge reach their sliding limit before objects near the centre. The outer objects fly off while the inner objects remain safely in place.

This makes physical sense: at the same angular velocity, an object at larger radius has greater linear speed ($v = \omega r$) and therefore needs greater centripetal force ($m\omega^2 r$). The required force is proportional to r, but the available friction ($\mu_s mg$) is the same regardless of position. So objects farther out are always the first to lose the battle against inertia.

With the theory in hand, let us explore through worked examples.

BINDER Example 28

A small coin of mass 20 g is placed 15cm from the centre of a horizontal turntable. The coefficient of static friction between the coin and the surface is 0.35. Take $g = 9.8\ m/s^2$.

Calculate the maximum angular velocity at which the turntable can rotate before the coin slides off.

Solution
Using:

$$\omega_{max} = \sqrt{\frac{\mu_s g}{r}}$$

Where: $\mu_s = 0.35$, $g = 9.8\ m/s^2$, $r = 0.15\ m$
Substituting:

$$\omega_{max} = \sqrt{\frac{0.35 \times 9.8\ m/s^2}{0.15\ m}} = 4.78\ rad/s$$

The maximum angular velocity is 4.78 rad/s.

Making Sense of the Answer: *At 4.78 rad/s, the turntable completes about* $\frac{4.78}{2\pi} = 0.76$ *revolutions per second. This is a gentle spin. The coin slides at a modest speed because friction is relatively weak (*$\mu_s = 0.35$*) and the coin is fairly far from the centre (15cm). Moving the coin closer to the centre would allow faster rotation before sliding.*

Think Like a Physicist: *Notice that mass does not appear in the formula. A heavy coin and a light coin at the same radius, with the same friction coefficient, slide off at exactly the same angular velocity. Mass cancels for the same reason as in every other horizontal circular motion problem: both the required force and the available friction are proportional to mass.*

REAL Example 29

During a physics demonstration, **Mr. Akilikubwa** places several coins at different distances from the centre of a spinning turntable. He slowly increases the speed. **Kipute** observes that the coin nearest the edge slides off first, then the next one, and the coin closest to the centre is the last to slide.

Kipanga is puzzled: *"All the coins are on the same turntable spinning at the same speed. Why don't they all slide off at the same time?"*

Explain why the outermost coin slides first.

Solution

All the coins share the same angular velocity ω because they are on the same rigid turntable. However, the centripetal force required to keep each coin moving in a circle is $f = m\omega^2 r$, which is proportional to the radius r.

A coin at a larger radius needs a greater centripetal force. The maximum friction available is $\mu_s mg$, which is the same for all coins (assuming equal mass and the same surface). So the coin at the largest radius demands the most force while having the same friction limit as the others. So it reaches the sliding threshold first.

As the turntable speeds up further, the sliding threshold is reached at progressively smaller radii, so the coins slide off one by one, starting from the outermost.

Making Sense of the Answer: *Think of it as a competition between demand and supply. Every coin has the same friction supply (*$\mu_s mg$*). But the outer coins have greater centripetal demand (*$m\omega^2 r$*). The coin with the greatest demand loses the competition first.*

Think Like a Physicist: *The critical angular velocity is* $\omega_{max} = \sqrt{\frac{\mu_s g}{r}}$. *Since* ω_{max} *decreases as* r *increases, outer objects always have a lower threshold. This principle applies to any rotating system: centrifuges, spin dryers, and even planetary rings all obey the same rule.*

HOT Example 30

Two coins are placed on a horizontal turntable. Coin A (mass 10g) is placed 10cm from the centre, and coin B (mass 25g) is placed 20cm from the centre. The coefficient of static friction is 0.4 for both coins. The turntable starts from rest and its angular velocity increases gradually. Take $g = 9.8\ m/s^2$.

Which coin slides first, and at what angular velocity does it happen? At that instant, what fraction of the maximum friction is being used by the other coin?

Solution

The maximum angular velocity before sliding is:

$$\omega_{max} = \sqrt{\frac{\mu_s g}{r}}$$

For coin A ($r = 0.1$m):

$$\omega_{max,\,A} = \sqrt{\frac{0.4 \times 9.8}{0.10}} = 6.26 \text{ rad/s}$$

For Coin B ($r = 0.20$ m):

$$\omega_{max,\,B} = \sqrt{\frac{0.4 \times 9.8}{0.20}} = 4.43 \text{ rad/s}$$

Since $\omega_{max,\,B} < \omega_{max,\,A}$, **coin B slides first** at $\omega = 4.43$ rad/s.

At $\omega = 4.43$ rad/s, the friction force on coin A is:

$$f_A = m_A \omega^2 r_A = 0.010 \text{ kg} \times (4.43 \text{rad/s})^2 \times 0.10 \text{ m} = 0.0196 \text{ N}$$

The maximum friction on coin A is:

$$f_{max,\,A} = \mu_s m_A g = 0.4 \times 0.01\text{m} \times 9.8 \text{ m/s}^2 = 0.0392\text{N}$$

Fraction used:

$$\frac{f_A}{f_{max,\,A}} = \frac{0.0196}{0.0392} = 0.5$$

At the instant coin B slides off, coin A is using exactly **half** (50%) of its available friction. It has a comfortable safety margin.

Making Sense of the Answer: *Coin B slides first despite being heavier. Mass is irrelevant; only radius matters. Coin B is twice as far from the centre, so it needs twice the centripetal force per unit mass. At the moment coin B reaches its limit, coin A is only halfway to its own limit because it sits at half the radius. The factor of exactly one-half is not coincidence: since* $r_A = \frac{1}{2} r_B$*, and friction used is proportional to* $\omega^2 r$*, the ratio is* $\frac{r_A}{r_B} = \frac{1}{2}$.

Think Like a Physicist: *This problem tests whether you truly understand that radius, not mass, determines which object slides first. Many students instinctively expect the heavier coin to slide first because it "has more inertia." But more mass also means more friction, and the two cancel. Only radius remains as the deciding factor.*

That brings horizontal circular motion to a close. From the simplest case of a car on a flat road, through the leaning cyclist, the conical pendulum, the banked curve, and finally the rotating platform, one principle has guided every analysis: identify the real forces, resolve them, and set the net inward component equal to $\frac{mv^2}{r}$ or $m\omega^2 r$.

In the next subtopic, we leave the horizontal plane and enter a world where gravity no longer sits quietly in the background. In vertical circular motion, gravity actively participates: it helps the centripetal force at the top and opposes it at the bottom. The speed changes, the tension changes, and new critical conditions emerge. The physics becomes richer, but the method remains the same.

VERTICAL CIRCULAR MOTION

Understanding Vertical Circular Motion

In horizontal circular motion, gravity stayed in the background. It held objects down, balanced normal reactions, and determined the weight. But it never directly helped or opposed the centripetal force. The centripetal force came from friction, tension, or the horizontal component of a normal reaction, while gravity quietly maintained vertical equilibrium.

In vertical circular motion, everything changes. The circular path is now oriented vertically, and gravity acts along the same line as the centripetal direction at certain points. At the top of the circle, gravity pulls the object **toward the centre** and helps maintain the circular path. At the bottom, gravity pulls the object **away**

from the centre and must be overcome. At the sides, gravity acts perpendicular to the radius and contributes nothing to the centripetal force.

This means the forces on the object change continuously as it moves around the loop. The tension in a string, or the normal reaction from a track, is different at every position. The speed also changes: the object slows down as it rises and speeds up as it falls, because gravity does work on it throughout the motion. Vertical circular motion is therefore **not uniform**: the speed is not constant.

Despite this complexity, the method remains exactly the same as in horizontal circular motion. At any instant, we apply Newton's second law in the radial direction (toward the centre) and set the net inward force equal to $\frac{mv^2}{r}$, where v is the speed at that particular point. The only difference is that the forces contributing to the net inward force change with position.

Forces at different positions in the vertical circle

To understand vertical circular motion, we must examine what happens at three key positions: the **bottom**, the **top**, and the **side** (at the level of the centre). Consider an object of mass m attached to a string of length r and swung in a vertical circle. At each position, we identify the forces and apply Newton's second law toward the centre.

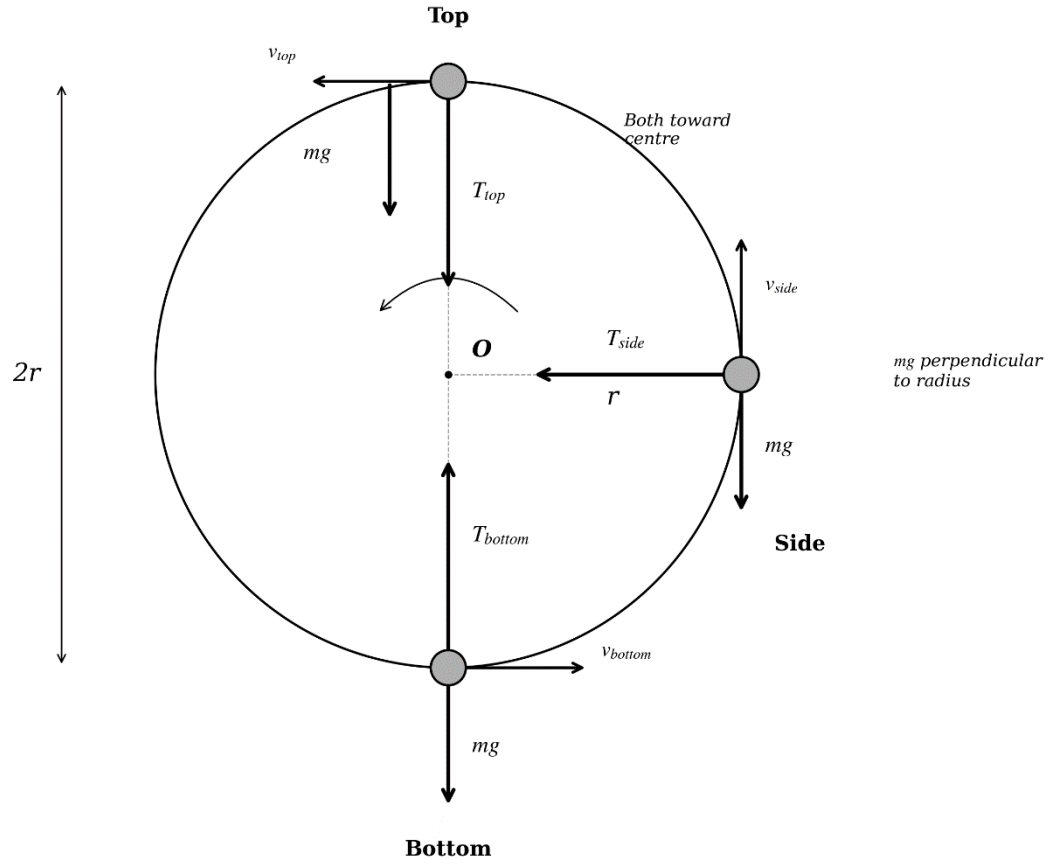


Figure: ***Forces on an object at three key positions in a vertical circle.*** *At the bottom, tension acts toward the centre (upward) and weight acts away from it (downward). At the top, both tension and weight act toward the centre (downward). At the side, tension acts toward the centre (horizontally) while weight acts perpendicular to the radius (downward). The velocity is tangent to the circle at each position.*

At the bottom of the circle

At the lowest point, the centre of the circle is directly above the object. The forces are:

- Tension T_{bottom} acts upward (toward the centre).
- Weight mg acts downward (away from the centre).

Applying Newton's second law toward the centre (upward at this point):

$$T_{bottom} - mg = \frac{mv_{bottom}^2}{r}$$

From which:

$$\mathbf{T_{bottom} = mg + \frac{mv_{bottom}^2}{r}}$$

The tension at the bottom is **greater** than the weight. The string must support the weight against gravity and simultaneously provide the centripetal force. *This is the position where the tension is maximum and the string is most likely to break.*

At the top of the circle

At the highest point, the centre is directly below the object. The forces are:

- Tension T_{top} acts downward (toward the centre).
- Weight mg acts downward (also toward the centre).

Both forces point in the same direction: toward the centre. Applying Newton's second law toward the centre (downward at this point):

$$T_{top} + mg = \frac{mv_{top}^2}{r}$$

From which:

$$\mathbf{T_{top} = \frac{mv_{top}^2}{r} - mg}$$

The tension at the top is **less** than at the bottom. Gravity now assists the centripetal force, so the string does not need to work as hard. *This is the position where the tension is minimum. If the speed is too low, the tension becomes zero or the formula gives a negative value, meaning the string goes slack and the object falls out of the circular path.*

At the side of the circle (level with the centre)

At the position level with the centre, the forces are:

- Tension T_{side} acts horizontally toward the centre.
- Weight mg acts vertically downward (perpendicular to the radius at this point).

Since weight is perpendicular to the radial direction, it does not contribute to the centripetal force. Applying Newton's second law toward the centre:

$$T_{side} = \frac{mv_{side}^2}{r}$$

At this position, the tension provides the entire centripetal force by itself, and gravity has no radial effect. However, gravity does cause the speed to change as the object moves through this position.

Summary of tension at three key positions

$$T_{bottom} = mg + \frac{mv_{bottom}^2}{r} \qquad \text{(maximum)}$$

$$T_{side} = \frac{mv_{side}^2}{r}$$

$$T_{top} = \frac{mv_{top}^2}{r} - mg \qquad \text{(minimum)}$$

Tension at a general angle

The bottom, top, and side are three special positions. *But what if you need the tension when the string makes an angle θ measured from the lowest point (the bottom)?*

At a general angle θ from the bottom, the object is at a height $h = r - r\cos\theta = r(1 - \cos\theta)$ above the lowest point. The string points from the object toward the centre, and the weight mg acts vertically downward.

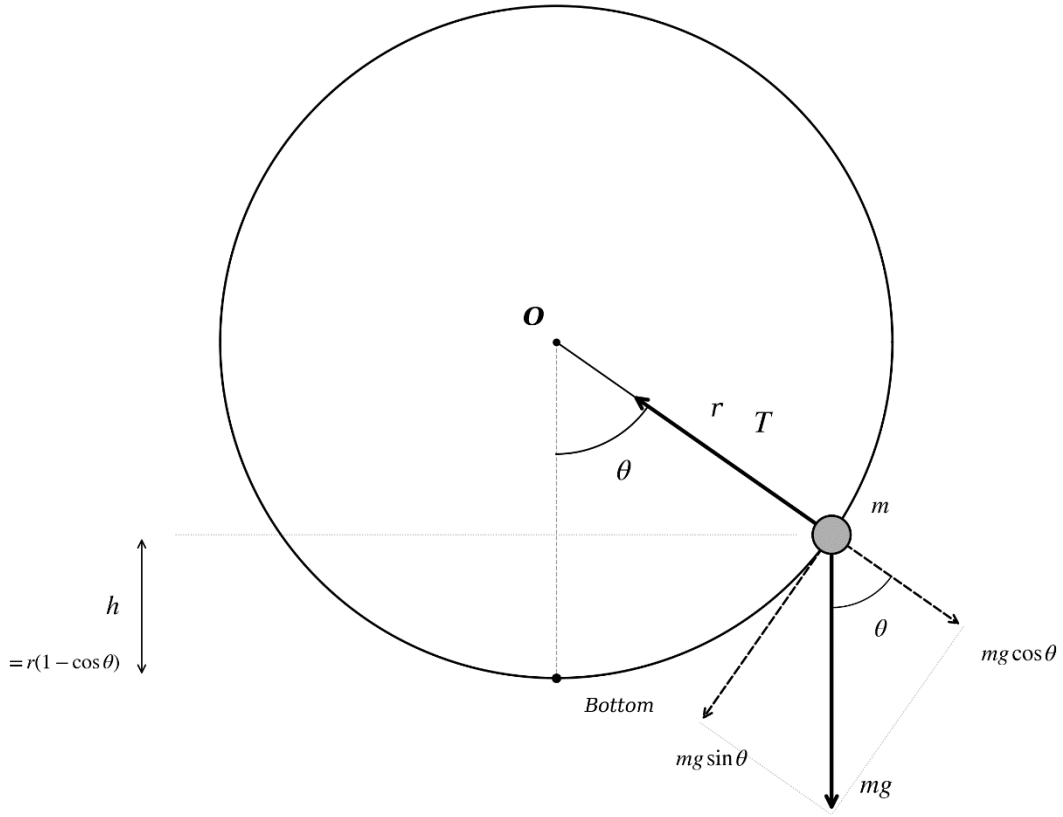


Figure: ***An object at a general angle θ from the bottom of a vertical circle.*** *Tension T acts along the string toward the centre O. Weight mg acts vertically downward and is resolved into two components: mgcosθ along the radius (away from the centre) and mgsinθ along the tangent. The angle between mg and the outward radial direction equals θ. The height above the bottom is* $h = r(1 - cos\theta)$.

The component of the weight along the radius (toward or away from the centre) is $mg\cos\theta$. At angle θ from the bottom, this component acts away from the centre (opposing the centripetal direction).

Applying Newton's second law toward the centre:

$$T - mg\cos\theta = \frac{mv^2}{r}$$

From which:

$$\mathbf{T = mg\cos\theta + \frac{mv^2}{r}}$$

This is the **general expression for tension at any point** in the vertical circle. Let us verify that it gives our earlier results at the three special positions:

At the **bottom** ($\theta = 0°$): $\cos 0° = 1$, so $T = mg + \frac{mv^2}{r}$. Correct.

At the **side** ($\theta = 90°$): $\cos 90° = 0$, so $T = \frac{mv^2}{r}$. Correct.

At the **top** ($\theta = 180°$): $\cos 180° = -1$, so $T = -mg + \frac{mv^2}{r} = \frac{mv^2}{r} - mg$. Correct.

All three special cases emerge naturally from the general formula. This confirms that the general expression is consistent.

Finding speed at any angle using energy conservation

To use the general tension formula, we need the speed v at angle θ. Using conservation of energy between the bottom and the position at angle θ:

$$\frac{1}{2}mv_{bottom}^2 = \frac{1}{2}mv^2 + mgh$$

$$\frac{1}{2}v_{bottom}^2 = \frac{1}{2}v^2 + gh$$

$$v^2 = v_{bottom}^2 - gh$$

Where $h = r(1 - \cos\theta)$ (you can easily deduce it from diagram by using geometry):

$$v^2 = v_{bottom}^2 - 2gr(1 - \cos\theta)$$

Substituting into the general tension formula:

$$T = mg\cos\theta + \frac{m}{r}\left(v_{bottom}^2 - 2gr(1 - \cos\theta)\right)$$

$$T = mg\cos\theta + \frac{mv_{bottom}^2}{r} - 2mg(1 - \cos\theta)$$

$$\mathbf{T = \frac{mv_{bottom}^2}{r} - 2mg + 3mg\cos\theta}$$

This powerful expression gives the tension at any angle θ in terms of the speed at the bottom alone. It shows clearly that:

- Tension decreases as θ increases from 0° to 180° (because cosθ decreases), confirming that the bottom has maximum tension and the top has minimum tension.
- The tension depends on cosθ, so it changes smoothly and continuously around the circle.

It is important for you to understand that:

The speed v is different at each position. As the object rises from bottom to top, it loses kinetic energy and gains gravitational potential energy, so it slows down. As it descends from top to bottom, it speeds up. This means $v_{bottom} > v_{side} > v_{top}$ for an object that completes the full circle.

To find the relationship between speeds at different positions, we use **conservation of energy**, which we will apply in the worked examples and in the next subtopic.

The difference between the tension at the bottom and at the top

A useful result can be obtained by subtracting the tension at the top from the tension at the bottom:

$$T_{bottom} - T_{top} = \left(mg + \frac{mv_{bottom}^2}{r}\right) - \left(\frac{mv_{top}^2}{r} - mg\right)$$

$$T_{bottom} - T_{top} = 2mg + \frac{m}{r}\left(v_{bottom}^2 - v_{top}^2\right)$$

Using conservation of energy between the bottom and top of the circle:

$$\frac{1}{2}mv_{bottom}^2 = \frac{1}{2}mv_{top}^2 + mg(2r)$$

From which:

$$v_{bottom}^2 - v_{top}^2 = 4gr$$

Substituting:

$$T_{bottom} - T_{top} = 2mg + \frac{m}{r} \times 4gr = 2mg + 4mg$$

$$\mathbf{T_{bottom} - T_{top} = 6mg}$$

This is an elegant result: regardless of the speed, the difference in tension between the bottom and the top is always exactly 6mg. This result holds for any object completing a full vertical circle on a string, provided no energy is lost to friction or air resistance.

With the framework established, let us practise through worked examples.

BINDER Example 31

A ball of mass 0.4 kg is attached to a string of length 0.5 m and swung in a vertical circle. At the bottom of the circle, the speed is 6 m/s. Take $g = 9.8\ m/s^2$.

(a) Calculate the tension in the string at the bottom of the circle.

(b) State whether this is the maximum or minimum tension in the string during the motion. Explain briefly.

Solution

(a) Using:

$$T_{bottom} = mg + \frac{mv_{bottom}^2}{r}$$

Where: $m = 0.4\ kg$, $g = 9.8\ m/s^2$, $v_{bottom} = 6\ m/s$, $r = 0.5\ m$

Substituting:

$$T_{bottom} = 0.4kg \times 9.8\ m/s^2 + \frac{0.4kg \times (6\ m/s^2)^2}{0.5m} = 32.7N$$

The tension at the bottom is 32.7N.

(b) This is the **maximum** tension. At the bottom, the string must support the weight and provide centripetal force simultaneously, making the tension greatest at this position.

Making Sense of the Answer: *The weight is only 3.92N, but the tension is 32.7N, more than eight times the weight. The centripetal force demand (*28.8N*) dominates because the speed is high and the radius is small. This is why strings and ropes can snap at the bottom of a vertical swing even when the object is light.*

Think Like a Physicist: *At the bottom, tension and weight pull in opposite directions, so the net inward force is* $T - mg$. *At the top, they pull in the same direction, so the net inward force is* $T + mg$. *Always draw the forces first and identify which direction is "toward the centre" at each position.*

BINDER Example 32

A ball of mass 0.5kg is attached to a string of length 0.8m and swung in a vertical circle. At the top of the circle, the tension in the string is 5N. Take $g = 9.8\ m/s^2$.

Determine the tension in the string at the bottom of the circle.

Solution

This problem can be solved directly using the result $T_{bottom} - T_{top} = 6mg$:

$$T_{bottom} = T_{top} + 6mg$$

Substituting:

$$T_{bottom} = 5N + 6 \times 0.5kg \times 9.8\ m/s^2 = 34.4N$$

The tension at the bottom is 34.4N.

Making Sense of the Answer: *The tension increases by exactly* $6mg = 29.4\ N$ *from top to bottom. This is a fixed difference that depends only on the mass and gravity, not on the speed. Even if the ball were moving faster or slower (as long as it completes the circle), this difference would be the same. The result is powerful because it allows us to find one tension from the other without knowing the speed at either point.*

Think Like a Physicist: *The result* $T_{bottom} - T_{top} = 6mg$ *is worth memorising. It provides a shortcut in many vertical circular motion problems. However, remember that it applies only when energy is conserved (no friction or air resistance) and only for motion on a string where tension provides the centripetal force.*

REAL Example 33

Kipanga swings a stone tied to a string in a vertical circle. He notices that the string feels much tighter when the stone passes through the bottom of the circle than when it passes through the top. He says to **Kipute**: *"The stone feels heavier at the bottom. Does its weight change during the swing?"*

Help Kipute answer Kipanga's question.

Solution

The weight of the stone (mg) does not change; gravity pulls it downward with the same force at every point in the circle. What changes is the **tension** in the string.

At the bottom, the centre of the circle is above the stone. The string must pull upward with enough force to both support the weight and provide the centripetal force toward the centre. So the tension at the bottom is $T_{bottom} = mg + \frac{mv^2}{r}$, which is greater than the weight.

At the top, the centre is below the stone. Both gravity and tension pull toward the centre (downward). Gravity already provides part of the centripetal force, so the string needs to supply less. The tension at the top is $T_{top} = \frac{mv^2}{r} - mg$, which is less than at the bottom.

What Kipanga feels as "heavier" is not a change in weight but an increase in tension. The string must work hardest at the bottom and easiest at the top. The difference is always exactly 6mg.

Making Sense of the Answer: *It is the same illusion as feeling heavier when a lift accelerates upward. Your weight has not changed, but the support force (tension or normal reaction) has increased. At the bottom of a vertical circle, the string provides an upward acceleration (centripetal), making it feel like the object is heavier.*

Think Like a Physicist: *The feeling of heaviness or lightness is determined by the contact force (tension or normal reaction), not by the actual weight.*

HOT Example 34

A ball of mass 0.25kg is attached to a string of length 0.6m and set in motion in a vertical circle. The speed at the bottom of the circle is 4.5m/s. Take $g = 9.8\ m/s^2$.

Determine the angle from the bottom at which the string goes slack.

Solution

The string goes slack when the tension becomes zero. Using the general expression for tension at angle θ from the bottom:

$$T = \frac{mv_{bottom}^2}{r} - 2mg + 3mg\cos\theta$$

Setting $T = 0$:

$$0 = \frac{mv_{bottom}^2}{r} - 2mg + 3mg\cos\theta$$

Making $\cos\theta$ the subject:

$$3mg\cos\theta = 2mg - \frac{mv_{bottom}^2}{r}$$

$$\cos\theta = \frac{2}{3} - \frac{v_{bottom}^2}{3gr}$$

Where: $v_{bottom} = 4.5\ m/s$, $g = 9.8\ m/s^2$, $r = 0.6\ m$

Substituting:

$$\cos\theta = \frac{2}{3} - \frac{(4.5\ m/s)^2}{3 \times 9.8\ m/s^2 \times 0.6\ m} = -0.481$$

$$\theta = \cos^{-1}(-0.481) = 118.8°$$

The string goes slack at 118.8° from the bottom.

Making Sense of the Answer: *The angle is between 90° (the side) and 180° (the top). This means the ball passes through the side position successfully but loses tension before reaching the top. At 118.8° from the bottom, the ball is well above the centre of the circle. After the string goes slack, the ball follows a projectile path (parabola) rather than continuing along the circle. If the speed at the bottom were greater, the string would go slack at a larger angle (closer to the top), or not at all if the speed is sufficient to complete the full circle.*

Think Like a Physicist: *The formula* $cos\theta = \frac{2}{3} - \frac{v_{bottom}^2}{3gr}$ *tells us directly whether the ball completes the circle. If the right side is less than* -1 *(meaning* $cos\theta < -1$*, which is impossible), the tension never reaches zero and the ball completes the full circle. This happens when* $v_{bottom}^2 > 5gr$*, a result we will derive in the next subtopic.*

The ideas are now in place. In the next subtopic, we examine the most important question in vertical circular motion: *what is the minimum speed needed to maintain a complete vertical circle?* This is where strings go slack, water falls from buckets, and roller coasters must be carefully engineered.

Minimum Speed and Critical Condition of Object on a String

In the previous subtopic, we derived the tension at the top of the circle as:

$$T_{top} = \frac{mv_{top}^2}{r} - mg$$

An important question arises from this equation: *what happens if the speed at the top is very small?*

As v_{top} decreases, the term $\frac{mv_{top}^2}{r}$ becomes smaller and the tension decreases. At some critical speed, the tension reaches zero. If the speed drops below this critical value, the formula gives a negative tension, but a string cannot push; it can only pull. So a negative tension is physically meaningless. It means the string has gone slack and the object has left the circular path.

Finding the minimum speed at the top

At the critical condition, the string is on the verge of going slack:

$$T_{top} = 0$$

Substituting into the equation for tension at the top:

$$0 = \frac{mv_{min,top}^2}{r} - mg$$

$$\frac{mv_{min,top}^2}{r} = mg$$

Making $v_{min,top}$ the subject:

$$\mathbf{v_{min,top} = \sqrt{gr}}$$

At this critical speed, gravity alone provides exactly the centripetal force needed at the top of the circle. If the speed becomes any smaller, gravity would be more than enough for the circular motion at that speed. However, this would require a circular path of a smaller radius while the object cannot actually move in that smaller circle because the string fixes the radius. As a result, the object leaves the circular path and begins to fall under gravity.

Again, **you have to understand this:** The minimum speed at the top does not depend on mass. A heavy ball and a light ball on the same string both need the same minimum speed $\sqrt{gr}$ at the top to complete the circle. This is because both the centripetal force requirement ($\frac{mv^2}{r}$) and the gravitational force (mg) are proportional to mass, and mass cancels.

Finding the minimum speed at the bottom for a complete circle

Knowing the minimum speed at the top is useful, but in practice we control the speed at the bottom (where we launch or swing the object). The question becomes: *what minimum speed must the object have at the bottom so that it still has at least* $\sqrt{gr}$ *at the top?*

We use conservation of energy between the bottom and the top. The height difference is 2r (the full diameter of the circle).

$$\frac{1}{2}mv_{bottom}^2 = \frac{1}{2}mv_{top}^2 + mg(2r)$$

At the critical condition, $v_{top} = \sqrt{gr}$, so $v_{top}^2 = gr$:

$$\frac{1}{2}mv_{min,bottom}^2 = \frac{1}{2}m(gr) + mg(2r)$$

$$\frac{1}{2}v_{min,bottom}^2 = \frac{1}{2}gr + 2gr = \frac{5}{2}gr$$

$$\mathbf{v_{min,bottom} = \sqrt{5gr}}$$

This is a key result in vertical circular motion. To complete a full vertical circle on a string, the speed at the bottom must be at least $\sqrt{5gr}$. Below this speed, the string goes slack somewhere before reaching the top.

Connecting to the general angle result

In the previous subtopic, the Think Like a Physicist section for Example 34 noted that the ball completes the circle when $v_{bottom}^2 > 5gr$. We have now derived this condition formally. The two results are consistent: $v_{min,bottom} = \sqrt{5gr}$ is the boundary between completing the circle and falling out of it.

Summary of critical speeds for a vertical circle on a string:

$$v_{min,top} = \sqrt{gr}$$

$$v_{min,bottom} = \sqrt{5gr}$$

Tension at the bottom when the ball just completes the circle

When $v_{bottom} = \sqrt{5gr}$:

$$T_{bottom} = mg + \frac{mv_{bottom}^2}{r} = mg + \frac{m(5gr)}{r} = mg + 5mg = 6mg$$

At the critical condition, the tension at the bottom is exactly 6mg. This is a useful benchmark: if the string can withstand a tension of at least 6mg, the object can just complete the vertical circle.

With the critical conditions established, let us put them to work.

BINDER Example 35

A ball of mass 0.3kg is attached to a string of length 0.5m and swung in a vertical circle. Take $g = 9.8m/s^2$.

(a) Calculate the minimum speed at the top for the ball to complete the circle.

(b) Calculate the minimum speed at the bottom for the ball to complete the circle.

(c) Find the tension in the string at the bottom when the ball has exactly the minimum speed to complete the circle.

Solution

(a) Using:

$$v_{min,top} = \sqrt{gr}$$

Where: $g = 9.8m/s^2$, $r = 0.5m$

Substituting:

$$v_{min,top} = \sqrt{9.8m/s^2 \times 0.5m} = \sqrt{4.9m^2/s^2} = 2.21m/s$$

The minimum speed at the top is 2.21m/s.

(b) Using:

$$v_{min,bottom} = \sqrt{5gr}$$

Substituting:

$$v_{min,bottom} = \sqrt{5 \times 9.8m/s^2 \times 0.5m} = \sqrt{24.5m^2/s^2} = 4.95m/s$$

The minimum speed at the bottom is 4.95m/s.

(c) Using:

$$T_{bottom} = 6mg$$

Substituting:

$$T_{bottom} = 6 \times 0.3kg \times 9.8m/s^2 = 17.64N$$

The tension at the bottom is 17.64N.

Making Sense of the Answer: *The minimum speed at the bottom (4.95m/s) is more than double the minimum speed at the top (2.21m/s). This is because the ball must have enough kinetic energy at the bottom to climb a height of $2r = 1m$ and still arrive at the top with speed 2.21m/s. Most of the kinetic energy at the bottom is consumed by the climb against gravity. The tension at the bottom ($17.64N$) is six times the weight ($2.94N$), confirming that the string is under considerable stress even at the minimum speed.*

Think Like a Physicist: *The ratio $\frac{v_{min,bottom}}{v_{min,top}} = \frac{\sqrt{5gr}}{\sqrt{gr}} = \sqrt{5} \approx 2.24$. This ratio is universal: the minimum speed at the bottom is always $\sqrt{5}$ times the minimum speed at the top, regardless of the radius or the mass.*

REAL Example 36

During a school science fair, **Kipanga** fills a bucket with water and swings it in a vertical circle over his head. To everyone's amazement, the water stays inside the bucket even when the bucket is upside down at the top of the circle. **Kipute** watches carefully and then says:

"I bet if you swing it slowly, the water will fall on your head."

Kipanga grins and slows down. Sure enough, as the swing becomes slower, water splashes out of the bucket and drenches him from above. The class erupts in laughter.

Mr. Akilikubwa, wiping tears of laughter from his eyes, says: *"Kipanga has just demonstrated the minimum speed condition. Now, Kipute, explain to the class: why did the water stay in the bucket at high speed, and why did it fall out when Kipanga slowed down?"*

Help Kipute explain to the class.

Solution

When the bucket is at the top of the vertical circle, it is upside down. At this point, the centre of the circle is directly below. Both the weight of the water and any contact force from the bucket act downward (toward the centre).

At high speed, the water needs a large centripetal force ($\frac{mv^2}{r}$) to follow the circular path. Gravity alone provides mg, but this is not enough. The bucket base must push the water downward to supply the extra centripetal force. Since the bucket pushes down on the water, the water pushes up on the bucket (Newton's third law). The water stays in contact with the bucket and remains inside.

When Kipanga slows down, the required centripetal force decreases. At the critical speed $v = \sqrt{gr}$, gravity alone provides exactly the centripetal force needed, and the bucket base exerts zero force on the water. Below this speed, gravity provides more downward force than needed for the circular path. The water wants to fall faster than the bucket, so it separates from the bucket base and falls out, landing on Kipanga's head.

In short: the water stays in when the required centripetal acceleration ($\frac{v^2}{r}$) exceeds g, and falls out when $\frac{v^2}{r} <$ g.

Making Sense of the Answer: *The bucket trick works because at high speed, the water's inertia is so large that gravity cannot pull it away from the circular path. The water "wants" to fly off tangentially but the bucket forces it to curve. At low speed, gravity dominates over inertia, and the water simply falls. The critical speed* $\sqrt{gr}$ *is the balance point between these two regimes.*

Think Like a Physicist: *This is exactly the same physics as the string going slack. Replace "bucket base pushing water" with "string pulling ball" and the analysis is identical. In both cases, the contact force reaches zero at* $v = \sqrt{gr}$ *at the top. The bucket trick is simply a dramatic demonstration of the minimum speed condition.*

HOT Example 37

A ball on a string of length 1.2m is set in motion in a vertical circle. The speed at the bottom is 6.5m/s. Take $g = 9.8\text{m/s}^2$.

Determine whether the ball completes the full circle, and if not, find the angle from the bottom at which the string goes slack.

Solution

First, check whether the speed at the bottom is sufficient to complete the circle.

The minimum speed required at the bottom is:

$$v_{min,bottom} = \sqrt{5gr} = \sqrt{5 \times 9.8\text{m/s}^2 \times 1.2\text{m}} = \sqrt{58.8\text{m}^2/\text{s}^2} = 7.67\text{m/s}$$

Since the actual speed at the bottom (6.5m/s) is less than the minimum required (7.67m/s), the ball does **not** complete the full circle. The string goes slack before the ball reaches the top.

To find the angle where the string goes slack, set $T = 0$ in the general formula:

$$\cos\theta = \frac{2}{3} - \frac{v_{bottom}^2}{3gr}$$

Substituting:

$$\cos\theta = \frac{2}{3} - \frac{(6.5\text{m/s})^2}{3 \times 9.8\text{m/s}^2 \times 1.2\text{m}} = -0.531$$

$$\theta = \cos^{-1}(-0.531) = 122.1°$$

The string goes slack at $122.1°$ from the bottom.

Making Sense of the Answer: *The ball fails to complete the circle because 6.5m/s is well below the required 7.67m/s. It goes slack at 122.1° from the bottom, which is about 32° past the horizontal (side) position and roughly 58° short of the top. The ball makes it more than two-thirds of the way around but runs out of speed before reaching the top. After the string goes slack, the ball follows a parabolic projectile path.*

Think Like a Physicist: *This problem requires two distinct skills: first, recognising that the speed is insufficient (comparing* v_{bottom} *with* $\sqrt{5gr}$*), and second, finding exactly where the string fails (using the general tension formula with* $T = 0$*). A common mistake is to assume that if the ball cannot reach the top, the string must go slack at the top. In reality, it goes slack at whatever angle the tension first becomes zero, which is generally below the top.*

That brings the critical speed conditions to a close. We now know when a string goes slack and what speed is needed to prevent it. In the next subtopic, we replace the string with a track: an object moves inside a circular loop (like a roller coaster). The physics is similar, but with one important difference: the track pushes rather than pulls, and the critical condition involves the normal reaction rather than tension.

Loop-the-Loop

A roller coaster climbs slowly to the top of a tall slope, pauses for a breathless moment, then plunges downward, gaining terrifying speed. Moments later, it enters a vertical loop and the passengers find themselves upside down, pressed into their seats, screaming with a mixture of terror and delight. The car races around the inside of the loop and emerges safely at the bottom.

How does this work? Why don't the passengers fall when they are upside down? And how do engineers decide how tall the initial slope must be?

The answers come from the same physics we developed for the ball on a string, with one crucial difference. A string pulls. A track pushes. The string exerts tension directed inward along its length. The track exerts a **normal reaction** directed perpendicular to its surface. For a circular loop, the inner surface pushes the object toward the centre at every point. So in the loop-the-loop, normal reaction replaces tension as the force that works alongside gravity to maintain circular motion.

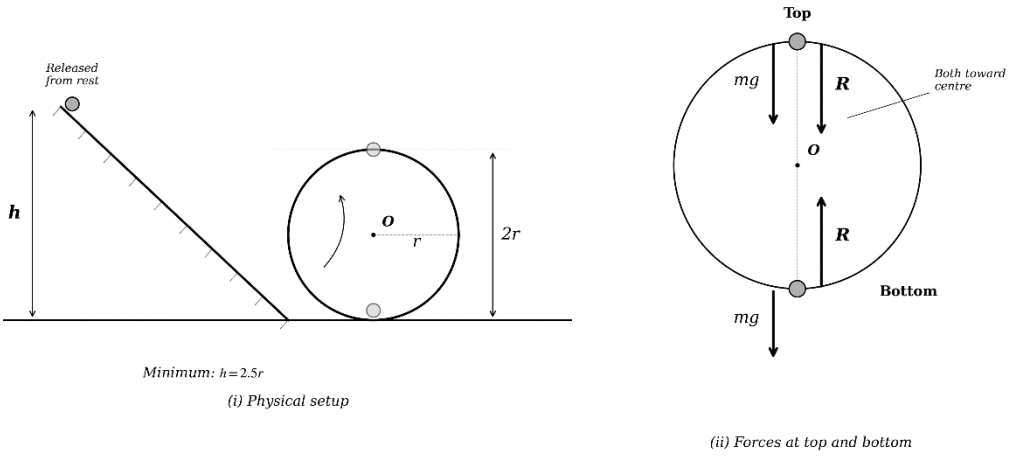


Figure: ***Loop-the-loop.*** *(i) A ball is released from rest at height h on a ramp and enters a vertical circular loop of radius r. The minimum release height for a complete loop is h = 2.5r. (ii) Forces on the ball at the top and bottom of the inside loop. At the top, both the normal reaction R and weight mg act downward toward the centre. At the bottom, R acts upward toward the centre while mg acts downward away from it.*

Forces at the top of an inside loop

At the top of a loop, the object is on the **inside** of the track. The centre of the circle is directly below. Both forces act downward (toward the centre):

- Weight mg acts downward.
- Normal reaction R from the track acts downward (the track is above the object and pushes it toward the centre).

Applying Newton's second law toward the centre (downward):

$$mg + R = \frac{mv_{top}^2}{r}$$

From which:

$$\mathbf{R = \frac{mv_{top}^2}{r} - mg}$$

Compare this carefully with the string result: $T_{top} = \frac{mv_{top}^2}{r} - mg$. The formula is identical in structure. The only change is the symbol: R instead of T.

Forces at the bottom of the loop

At the bottom, the object is on the inside of the track. The centre is directly above. The forces oppose each other:

- Weight mg acts downward (away from centre).
- Normal reaction R acts upward (toward the centre).

Applying Newton's second law toward the centre (upward):

$$R - mg = \frac{mv_{bottom}^2}{r}$$

$$\mathbf{R = mg + \frac{mv_{bottom}^2}{r}}$$

Again, identical in structure to $T_{bottom} = mg + \frac{mv_{bottom}^2}{r}$.

The critical condition: losing contact

For the string, the critical condition was $T = 0$ (string goes slack). For the track, the critical condition is $R = 0$ (object loses contact with the track surface).

Setting $R = 0$ at the top:

$$0 = \frac{mv_{min,top}^2}{r} - mg$$

$$\mathbf{v_{min,top} = \sqrt{gr}}$$

The same result as for the string. At this critical speed, gravity alone provides the centripetal force at the top, and the track exerts no force on the object. Below this speed, the object separates from the track.

There is, however, a subtle but important difference. When a string goes slack, the object falls inward (toward the centre) because nothing holds it outward. When an object loses contact with the **inside** of a loop, it also falls away from the track; but now that means falling inward as well (the track is on the outside of the object's position). In both cases, the object leaves the circular path and follows a projectile trajectory.

Minimum release height for a complete loop

Roller coasters and toy loop-the-loop tracks launch the object from a height, converting gravitational potential energy into kinetic energy. The practical question is: *from what minimum height must the object be released to complete the loop?*

Let the loop have radius r, and let the object be released from rest at height h above the bottom of the loop.

Using conservation of energy between the release point and the top of the loop:

$$mgh = \frac{1}{2}mv_{top}^2 + mg(2r)$$

At the critical condition, $v_{top}^2 = gr$:

$$mgh_{min} = \frac{1}{2}m(gr) + mg(2r)$$

$$gh_{min} = \frac{1}{2}gr + 2gr = \frac{5}{2}gr$$

$$\mathbf{h_{min} = \frac{5}{2}r = 2.5r}$$

The object must be released from a height of at least 2.5r above the bottom of the loop. For a loop of radius 2m, the minimum release height is 5m. Engineers always design the initial slope to be taller than this to provide a safety margin.

The worked examples that follow will bring these concepts to life and make them concrete.

BINDER Example 38

A small ball rolls without friction along a track that includes a vertical circular loop of radius 0.4m. The ball is released from rest at a height h above the bottom of the loop. Take $g = 9.8 m/s^2$.

(a) Find the minimum height from which the ball must be released to complete the loop.

(b) At this minimum release height, find the speed of the ball at the top of the loop.

(c) Find the normal reaction on the ball at the top of the loop at the minimum speed.

Solution

(a) Using:

$$h_{min} = \frac{5}{2}r = \frac{5}{2} \times 0.4\text{m} = 1\text{m}$$

The minimum release height is 1m above the bottom of the loop.

(b) At the minimum condition, the speed at the top is:

$$v_{min,top} = \sqrt{gr} = \sqrt{9.8\text{m/s}^2 \times 0.4\text{m}} = \sqrt{3.92\text{m}^2/\text{s}^2} = 1.98\text{m/s}$$

The speed at the top is 1.98m/s.

(c) At the minimum speed, the normal reaction at the top is:

$$R = 0\text{N}$$

At the critical condition (minimum speed), gravity provides all the centripetal force and the track exerts no force on the ball.

Making Sense of the Answer: *The release height (1m) is 2.5 times the loop radius (0.4m). The ball arrives at the top of the loop with just enough speed (1.98m/s) for gravity to maintain the circular path alone. The track barely touches the ball at this point. Any lower release height and the ball falls away from the track before reaching the top.*

Think Like a Physicist: *Part (c) is deceptively simple but tests deep understanding. Many students calculate R using the full formula and get zero, then doubt their answer. Trust the physics: the minimum speed condition is defined as the speed at which R = 0. If you set up the problem correctly and get R = 0 at the minimum speed, that confirms your working, not a mistake.*

REAL Example 39

A toy car track includes a vertical loop. A child releases the toy car from the top of a ramp. On the first attempt, the car enters the loop, climbs partway up the inside, then falls away from the track before reaching the top. On the second attempt, the child places the car higher on the ramp, and this time it goes all the way around.

Explain why increasing the release height allows the car to complete the loop.

Solution

When the car is released from rest on the ramp, its gravitational potential energy converts into kinetic energy as it descends. A higher release point means more potential energy is available, so the car arrives at the bottom of the loop with greater speed.

As the car climbs inside the loop, it loses kinetic energy and gains potential energy. At the top of the loop, the car must still have enough speed for gravity to provide the centripetal force needed to maintain contact with the track (at minimum, $v_{top} = \sqrt{gr}$).

On the first attempt, the release height was too low. The car arrived at the bottom with insufficient speed, and its kinetic energy was exhausted before reaching the top. The car slowed to the point where gravity could no longer maintain the circular path, and it separated from the track.

On the second attempt, the higher release point provided more kinetic energy. The car reached the top of the loop with speed at or above $\sqrt{gr}$, so the track maintained contact and the car completed the loop.

The minimum release height is 2.5r above the bottom of the loop.

Making Sense of the Answer: *Every loop-the-loop toy teaches this lesson: if the ramp is too short, the car falls. The child learns by trial and error what physics calculates precisely. The factor of 2.5 accounts for the height of the loop (2r) plus the extra kinetic energy needed at the top ($\frac{1}{2}r$ worth of height).*

Think Like a Physicist: *The minimum height of* $2.5r$ *is measured from the bottom of the loop, not from the ground. If the ramp connects smoothly to the loop at ground level, then* $2.5r$ *is the height above the ground. But if the loop is elevated, the total height above the ground would be* $2.5r$ *plus the height of the platform.*

HOT Example 40

A ball of mass 0.2kg is released from rest at the top of a frictionless track. The track descends and enters a vertical circular loop of radius 0.5m. The release point is 2m above the bottom of the loop. Take $g = 9.8\text{m/s}^2$.

(a) Show that the ball completes the loop.

(b) Determine the speed and the normal reaction on the ball at the top of the loop.

Solution

(a) The minimum release height for a complete loop is:

$$h_{min} = 2.5r = 2.5 \times 0.5\text{m} = 1.25\text{m}$$

Since the actual release height (2m) exceeds the minimum (1.25m), the ball completes the loop.

(b) Using conservation of energy between the release point (height $h = 2\text{m}$ above the bottom) and the top of the loop (height $2r = 1\text{m}$ above the bottom):

$$mgh = \frac{1}{2}mv_{top}^2 + mg(2r)$$

Making v_{top} the subject:

$$v_{top}^2 = 2g(h - 2r)$$

$$v_{top} = \sqrt{2g(h - 2r)}$$

Substituting:

$$v_{top} = \sqrt{2 \times 9.8\text{m/s}^2 \times (2\text{m} - 1\text{m})} = 4.43\text{m/s}$$

The speed at the top is 4.43m/s.

Normal reaction at the top:

$$R = \frac{mv_{top}^2}{r} - mg = \frac{0.2\text{kg} \times 19.6\text{m}^2/\text{s}^2}{0.5\text{m}} - 0.2\text{kg} \times 9.8\text{m/s}^2 = 5.88\text{N}$$

The normal reaction at the top is 5.88N.

Making Sense of the Answer: *The release height is* $2m$*, well above the minimum of* $1.25m$*. The ball arrives at the top with* $4.43m/s$*, which is more than double the minimum of* $\sqrt{gr} = 2.21m/s$*. As a result, the normal reaction is* $5.88N$*, which is about three times the weight (*$1.96N$*). The ball presses firmly against the track at the top. The extra height translates directly into extra speed and extra contact force.*

Think Like a Physicist: *Part (a) requires a quick comparison (*h *versus* $2.5r$*), not a lengthy calculation. Recognising when a simple check answers the question saves time and shows confidence. Part (b) combines energy conservation (to find* v_{top}*) with Newton's second law (to find R). These two tools, used in sequence, solve almost every loop-the-loop problem.*

With the loop-the-loop understood, one final scenario remains. *What happens when a vehicle travels over the top of a hill?* The circular path is now on the **outside** of the curve rather than the inside, and the physics flips in a surprising way. Instead of needing a minimum speed to stay on the track, there is a **maximum** speed beyond which the vehicle leaves the road. Let us see why.

Vehicle over a Hill or Hump

Every situation we have studied so far, namely the ball on a string and the loop-the-loop, involves motion on the **inside** of a circular path. The object presses against the inner surface, and the contact force points toward the centre.

But what happens when an object moves over the ***outside*** *of a curved surface?* A car cresting a hill, a motorcycle riding over a hump, or a roller coaster passing over the top of a rise; in all these cases, the object is on the outside of the curve, and the physics reverses in an important way.

On the inside of a loop, the critical question was: *what is the minimum speed to maintain contact?* Too slow, and the object falls away inward.

On the outside of a curve, the critical question flips: *what is the maximum speed before the object leaves the surface?* Too fast, and the object lifts off.

Forces at the top of a hill

Consider a vehicle of mass m travelling at speed v over the top of a circular hill of radius r. At the highest point, the centre of the circular path is directly below the vehicle.

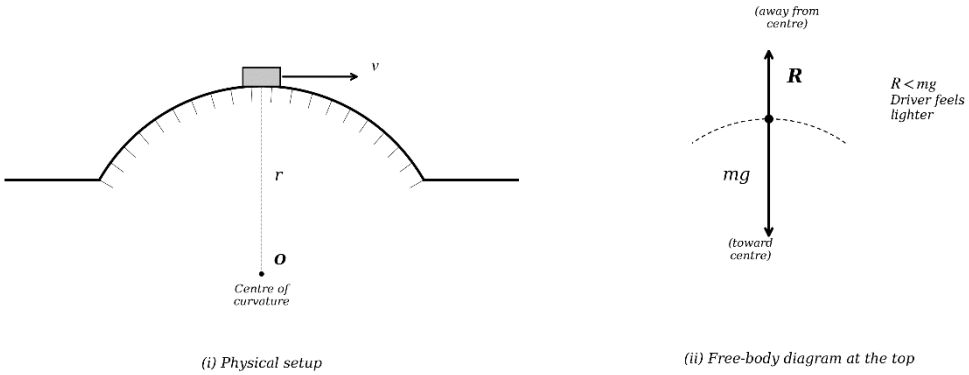


Figure: ***A vehicle at the top of a circular hill.*** *(i) The centre of curvature O is below the road surface. The car travels at speed v over the crest. (ii) Free-body diagram at the top: weight mg acts downward toward the centre, while the normal reaction R acts upward away from the centre. Since R < mg, the driver feels lighter than normal.*

The forces are:

- Weight mg acts downward (toward the centre).
- Normal reaction R acts upward (away from the centre).

This is the key difference from the inside of a loop. Here, weight acts toward the centre and the normal reaction acts away from it. They oppose each other.

Applying Newton's second law toward the centre (downward at the top of the hill):

$$mg - R = \frac{mv^2}{r}$$

From which:

$$\mathbf{R = mg - \frac{mv^2}{r}}$$

Compare this with the top of an inside loop: $R = \frac{mv^2}{r} - mg$. The terms are reversed. On a hill, increasing speed **decreases** the normal reaction; inside a loop, increasing speed **increases** it.

Apparent weight and the sensation of lightness

The normal reaction R is what the driver feels as their "weight." At the top of a hill:

$$R = mg - \frac{mv^2}{r} = m\left(g - \frac{v^2}{r}\right)$$

Since $\frac{v^2}{r}$ is subtracted from g, the apparent weight is less than the true weight. The driver feels lighter at the top of the hill. This is the familiar stomach-dropping sensation you feel when a car goes over a bump at high speed.

The faster the car moves, the lighter the driver feels, because $\frac{v^2}{r}$ grows larger and R becomes smaller.

The critical speed: leaving the road

As speed increases, R decreases. At some critical speed, R reaches zero. The road no longer pushes the car, and the car is on the verge of lifting off the surface.

Setting $R = 0$:

$$0 = mg - \frac{mv_{max}^2}{r}$$

$$\frac{mv_{max}^2}{r} = mg$$

$$\mathbf{v_{max} = \sqrt{gr}}$$

At this speed, gravity alone provides exactly the centripetal force needed. The car follows the curved hilltop without any help from the road. Beyond this speed, the car would need more centripetal force than gravity can provide, so it lifts off and becomes briefly airborne.

Notice that $v_{max} = \sqrt{gr}$ is the same expression as $v_{min,top}$ for the inside loop. The critical speed has the same magnitude, but the meaning is opposite: for the inside loop it is a **minimum** (below which contact is lost), and for the hilltop it is a **maximum** (above which contact is lost).

Weightlessness at the critical speed

When $R = 0$, the driver has zero apparent weight. This is the condition of **weightlessness**. The driver is not truly weightless (gravity still acts), but there is no contact force, so they feel as though they are floating. This is the same sensation astronauts experience in orbit, where $R = 0$ and gravity provides all the centripetal force.

The worked examples bring all of this to life.

BINDER Example 41

A car of mass 1200kg travels over a hill whose crest has a radius of curvature of 50m. The speed at the top is 15m/s. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the normal reaction on the car at the top of the hill.

(b) Express this as a percentage of the car's actual weight.

Solution

(a) Using:

$$R = mg - \frac{mv^2}{r}$$

Where: $m = 1200\text{kg}$, $g = 9.8\text{m/s}^2$, $v = 15\text{m/s}$, $r = 50\text{m}$

Substituting:

$$R = 1200\text{kg} \times 9.8\text{m/s}^2 - \frac{1200\text{kg} \times (15\text{m/s})^2}{50\text{m}} = 11760\text{N} - 5400\text{N} = 6360\text{N}$$

The normal reaction at the top is 6360N.

(b) The actual weight is $W = mg = 11760\text{N}$.

$$\frac{R}{W} \times 100\% = \frac{6360\text{N}}{11760\text{N}} \times 100\% = 54\%$$

The apparent weight is 54% of the actual weight.

Making Sense of the Answer: *At 15m/s over a 50m-radius hill, the car loses nearly half its apparent weight. Passengers would definitely feel the "stomach drop." If the speed were higher, the effect would be more dramatic. At about 22m/s, the apparent weight would reach zero.*

Think Like a Physicist: *The fraction* $\frac{R}{W} = 1 - \frac{v^2}{gr}$ *tells you immediately how "light" the driver feels. When* $\frac{v^2}{gr} = 0.5$, *the driver feels half their weight. When* $\frac{v^2}{gr} = 1$, *the driver feels weightless. This ratio is a quick diagnostic tool.*

REAL Example 42

On a family road trip, **Kipanga** notices that every time the car goes over a small hill at high speed, he feels his body lift slightly off the seat. He grabs the door handle in alarm. **Kipute**, sitting next to him, laughs: *"Relax, Kipanga. You are not flying. The road just stopped pushing you as hard."*

Explain what Kipute means, and why Kipanga feels lighter at the top of the hill.

Solution

When the car is on level ground, the road pushes upward on the car with a normal reaction equal to the weight: $R = mg$. Kipanga feels his full weight pressing him into the seat.

At the top of a hill, the car follows a curved path, and the centre of curvature is below the road. Part of gravity is "used" to provide the centripetal force for this curved motion. The remaining gravity is balanced by the normal reaction, which is now less than the weight: $R = mg - \frac{mv^2}{r}$.

Since the seat pushes Kipanga upward with less force than his weight, he feels lighter. His body does not actually leave the seat (unless the speed is extremely high), but the reduced contact force creates the sensation of lifting. This is what Kipute means: the road has not disappeared; it is simply pushing with less force because some of gravity's pull is being redirected into centripetal acceleration.

Making Sense of the Answer: *The same physics explains why roller coasters are thrilling. The hills are designed so that passengers feel light (or even weightless) at the top. The sensation is entirely due to the reduced normal reaction, not any change in gravity itself.*

Think Like a Physicist: *What you feel as "weight" is never gravity itself; it is the contact force (normal reaction, tension, or seat force) acting on you. Gravity cannot be felt directly. This is why astronauts in orbit feel weightless even though gravity is still pulling them: there is no contact force because both the astronaut and the spacecraft are in free fall together.*

HOT Example 43

A car travels along a road that passes over a circular hump of radius 40m. Take $g = 9.8\text{m/s}^2$.

At what speed does the car lose contact with the road at the top of the hump? Explain what happens to the car immediately after it loses contact.

Solution

The car loses contact when $R = 0$:

$$v_{max} = \sqrt{gr} = \sqrt{9.8\text{m/s}^2 \times 40\text{m}} = \sqrt{392\text{m}^2/\text{s}^2} = 19.8\text{m/s}$$

The car loses contact at 19.8m/s (about 71.3km/h).

At this speed, gravity provides exactly the centripetal force needed to follow the curved road surface. Beyond this speed, gravity cannot supply enough centripetal force for the road's curvature. The car cannot follow the tight curve of the hilltop and instead follows a gentler parabolic path (projectile motion). The car becomes briefly airborne, rising above the road surface before gravity brings it back down.

The car does not fly upward dramatically. It simply follows a trajectory with less curvature than the road, so a gap opens between the car and the road surface. When gravity eventually curves the car's path enough to meet the road again (on the downward slope), the car lands (often with a bump).

Making Sense of the Answer: *The critical speed of about 71km/h is surprisingly achievable on ordinary roads. A sharp hump with 40m radius is not unusual in hilly terrain. This is why speed bumps are effective: even a gentle hump at moderate speed creates the light sensation, and a sharper hump at higher speed can genuinely cause the car to leave the ground. Road engineers design hill crests with large radii specifically to prevent this at legal speeds.*

Think Like a Physicist: *After losing contact, the car is a projectile. Its horizontal velocity continues (ignoring air resistance) and gravity acts downward. The parabolic path it follows has less curvature than the circular road, so it rises above the road at the top and rejoins the road further down the slope. The landing point depends on the car's speed and the road's geometry; a problem that combines circular motion with projectile motion.*

With this final subtopic, vertical circular motion is complete. From the ball on a string to the loop-the-loop and now the hilltop, we have seen how gravity and contact forces interact differently depending on whether the object is on the inside or outside of the circular path. The critical speed $\sqrt{gr}$ appears in every case, but its meaning depends on the geometry: minimum speed for the inside, maximum speed for the outside.

Throughout this chapter, we have built circular motion from first principles and applied it to idealised situations: smooth tracks, uniform strings, frictionless loops. But circular motion is not confined to textbook problems. It governs the orbits of satellites, the design of centrifuges, the banking of aircraft in flight, the spin cycle of a washing machine, and the motion of charged particles in magnetic fields. In the next subtopic, we step outside the classroom and see how the physics we have mastered finds its way into the real world.

APPLICATIONS OF CIRCULAR MOTION

The equations are powerful, but they were never meant to live only on paper. Every formula we derived in this chapter was born from nature and returns to nature. In this subtopic, we leave the idealised world of smooth strings and frictionless tracks, and see circular motion at work in the machines, vehicles, and systems that shape everyday life and advanced technology alike.

1. Satellites in Orbit

A satellite orbiting the Earth is in continuous free fall. It falls toward the Earth every moment, but because it moves sideways fast enough, the Earth's surface curves away beneath it at the same rate. The result is a circular (or nearly circular) orbit.

For a satellite at height h above the Earth's surface, the gravitational force provides the centripetal force:

$$\frac{GMm}{(R_E + h)^2} = \frac{mv^2}{R_E + h}$$

From which the orbital speed is:

$$v = \sqrt{\frac{GM}{R_E + h}}$$

Where G is the gravitational constant, M is the mass of the Earth, and R_E is the Earth's radius. The mass of the satellite cancels, confirming that all satellites at the same altitude orbit at the same speed, regardless of their mass.

For a low-Earth orbit (h much smaller than R_E), the orbital speed is approximately 7.9km/s and the period is about 90 minutes. The International Space Station orbits at this speed, completing roughly 16 orbits every day.

A special case is the **geostationary orbit**, where the satellite's period matches the Earth's rotation period (24 hours). At this altitude (about 35,786km above the equator), the satellite appears stationary relative to the ground. Communication satellites and weather satellites are placed in geostationary orbits for this reason.

2. Centrifuges

A centrifuge is a machine that spins samples at high angular velocity to separate substances of different densities. In a hospital laboratory, blood samples are spun in a centrifuge to separate red blood cells (heavier) from plasma (lighter). In industrial settings, centrifuges separate cream from milk, uranium isotopes from one another, and sediment from wastewater.

The principle relies on density difference. When the centrifuge spins, the surrounding fluid creates a pressure gradient that supplies centripetal force proportional to the **fluid's** density. A particle denser than the fluid requires more centripetal force than the fluid can supply at that radius, so it drifts outward. A particle less dense than the fluid drifts inward. This is how separation occurs.

The **relative centrifugal force (RCF)** compares the centripetal acceleration to gravitational acceleration:

$$\text{RCF} = \frac{\omega^2 r}{g}$$

A laboratory centrifuge spinning at 3000rpm with a radius of 0.15m produces an RCF of about 1500. This means particles experience an effective "gravitational pull" 1500 times stronger than normal gravity, causing separation to occur in minutes rather than the hours or days it would take under gravity alone.

3. Banking of Aircraft

When an aircraft turns in flight, there is no road to provide friction. The centripetal force must come entirely from the aircraft itself. The pilot banks (tilts) the aircraft so that the lift force, which is always perpendicular to the wings, has a horizontal component pointing toward the centre of the turn.

The analysis is identical to the banked road with no friction:

$$\tan\theta = \frac{v^2}{rg}$$

Where θ is the bank angle, v is the aircraft's speed, and r is the radius of the turn. A faster turn or a tighter radius requires a steeper bank angle. Commercial aircraft typically bank at angles up to about $25°$ during normal turns, while fighter jets may bank at $60°$ or more during sharp manoeuvres.

At a bank angle of $60°$, the passengers experience an apparent weight of $\frac{mg}{\cos 60°} = 2mg$, meaning they feel twice as heavy. This is described as pulling "$2g$." Fighter pilots undergo special training to withstand accelerations of up to $9g$ during extreme turns.

4. Washing Machine Spin Cycle

During the spin cycle of a washing machine, the drum rotates at high speed. Clothes are pressed against the inner wall of the drum, and the drum wall provides the normal reaction that acts as the centripetal force:

$$R = m\omega^2 r$$

But water in the clothes is not held by the fabric strongly enough to maintain this circular motion. When the required centripetal force exceeds the weak adhesive forces holding the water to the fabric, the water separates from the clothes and moves outward through the holes in the drum.

This is not "centrifugal force throwing water outward." What actually happens is that the drum wall pushes the clothes inward (centripetal force), but the water, lacking sufficient inward force, cannot follow the circular path. It continues tangentially through the drum holes, following Newton's first law. The effect is the same as the Kilimani Hill bus: when the available inward force is insufficient, the object (water, in this case) leaves the circular path.

5. Circular Motion of Charged Particles in Magnetic Fields

When a charged particle moves through a magnetic field, it experiences a force perpendicular to its velocity. This force does not change the particle's speed (because it acts perpendicular to the motion), but it continuously changes the direction of motion. The result is circular motion.

For a particle of charge q and mass m moving at speed v perpendicular to a uniform magnetic field of strength B, the magnetic force provides the centripetal force:

$$qvB = \frac{mv^2}{r}$$

From which the radius of the circular path is:

$$r = \frac{mv}{qB}$$

This principle is used in **cyclotrons** (particle accelerators that accelerate charged particles along spiral paths), **mass spectrometers** (which separate ions by mass based on their circular path radii), and the large particle accelerators at research centres such as CERN, where protons travel in circular paths of several kilometres radius at speeds close to the speed of light.

6. Artificial Gravity in Space Stations

In the weightless environment of space, astronauts' bones weaken and muscles deteriorate over time. One proposed solution is to build a rotating space station. As the station spins, the floor (the outer wall) pushes astronauts inward, providing centripetal force. This inward push feels exactly like gravity to the astronauts standing on the inside of the outer wall.

For the artificial gravity to match Earth's gravity:

$$\omega^2 r = g$$

$$\omega = \sqrt{\frac{g}{r}}$$

A station of radius 100m would need to rotate at $\omega = \sqrt{\frac{9.8}{100}} = 0.313$rad/s, completing about one revolution every 20 seconds. This is slow enough to be comfortable for the inhabitants.

This concept has appeared in many science fiction films, but the physics is real and straightforward: it is nothing more than the centripetal acceleration $\omega^2 r$ mimicking the acceleration due to gravity g.

7. Speed Governors

The conical pendulum, which we have studied, has a direct engineering application: the **centrifugal governor**, for controlling steam engine speed. Two heavy balls are attached to a rotating shaft by hinged arms. As the shaft spins faster, the balls rise outward (increasing θ), just as in the conical pendulum. This outward movement is linked mechanically to a valve that reduces the steam supply, slowing the engine. When the engine slows, the balls drop, opening the valve again.

The system is self-regulating: the angle of the balls automatically adjusts to maintain a constant engine speed. The relationship $\tan\theta = \frac{v^2}{rg}$ ensures that each angle corresponds to a specific rotation speed, providing precise feedback control.

This was one of the earliest examples of **automatic feedback control** in engineering, and it is a beautiful demonstration of the conical pendulum at work outside the physics classroom.

From satellites silently circling the Earth to the washing machine noisily spinning in the kitchen, from the blood centrifuge in Muhimbili hospital to the International Space Station floating above our heads, circular motion is everywhere. The equations we derived in this chapter are not abstract mathematics; they are the operating instructions of the universe.

And speaking of operating instructions, it is time to put everything together. In the next section, the concepts of this entire chapter: centripetal acceleration, horizontal motion, vertical motion, energy conservation, and critical conditions will stop being polite and start working together in miscellaneous worked examples. If the individual subtopics were the ingredients, the miscellaneous examples are the full meal. Sharpen your appetite, and don't forget a calculator!

MISCELLANEOUS WORKED EXAMPLES ON UNIFORM CIRCULAR MOTION

Example 44

(a) Explain why a satellite in a circular orbit around the Earth is accelerating even though its speed is constant.

(b) A satellite orbits the Earth at a height of 200km above the surface. At this altitude, the acceleration due to gravity is 9.6m/s^2. The radius of the Earth is 6400km. Calculate the orbital speed and the period of revolution of the satellite.

Solution

(a) Acceleration is defined as the rate of change of velocity. Velocity is a vector quantity that has both magnitude (speed) and direction. Although the satellite's speed is constant, its direction of motion changes continuously as it follows the circular path. This continuous change in the direction of the velocity vector constitutes a centripetal acceleration directed toward the centre of the Earth. Therefore, the satellite is accelerating even at constant speed.

(b) The orbital radius is:

$$r = R_E + h = 6400\text{km} + 200\text{km} = 6600\text{km} = 6.6 \times 10^6\text{m}$$

At this altitude, gravity provides the centripetal force. For the orbital speed:

$$\frac{mv^2}{r} = mg'$$

Where $g' = 9.6\text{m/s}^2$ is the gravitational acceleration at that height.

Making v the subject:

$$v = \sqrt{g'r} = \sqrt{9.6\text{m/s}^2 \times 6.6 \times 10^6\text{m}} = \sqrt{6.336 \times 10^7\text{m}^2/\text{s}^2} = 7960\text{m/s}$$

The orbital speed is 7960m/s.

For the period:

$$T = \frac{2\pi r}{v} = \frac{2 \times 3.14 \times 6.6 \times 10^6\text{m}}{7960\text{m/s}} = \frac{4.145 \times 10^7\text{m}}{7960\text{m/s}} = 5207\text{s}$$

$$T = \frac{5207}{60}\text{min} = 86.8\text{min}$$

The period of revolution is 5207s (about 86.8 minutes).

Example 45

(a) A student claims: *"If the net force on a body is always perpendicular to its velocity, the body cannot accelerate because the force does no work."* Explain why the student's reasoning is only partially correct.

(b) A particle of mass 0.05kg moves in a horizontal circle of radius 0.4m at constant speed. The centripetal force acting on the particle is 2N. Calculate the speed and the number of revolutions the particle completes per second.

Solution

(a) The student is correct that a force perpendicular to the velocity does no work and therefore does not change the kinetic energy or the speed of the body. However, the student is wrong to conclude that the body cannot accelerate. A body can accelerate without any change in speed, provided its direction changes. This is because, acceleration is the rate of change of velocity, not the rate of change of speed. A perpendicular force changes the *direction* of the velocity without changing its magnitude.

(b) Using:

$$F = \frac{mv^2}{r}$$

Making v the subject:

$$v = \sqrt{\frac{Fr}{m}} = \sqrt{\frac{2\text{N} \times 0.4\text{m}}{0.05\text{kg}}} = \sqrt{16\text{m}^2/\text{s}^2} = 4\text{m/s}$$

The speed is 4m/s.

The number of revolutions per second (frequency):

$$f = \frac{v}{2\pi r} = \frac{4\text{m/s}}{2 \times 3.14 \times 0.4\text{m}} = \frac{4}{2.513}\text{s}^{-1} = 1.59\text{rev/s}$$

The particle completes 1.59 revolutions per second.

Example 46

(a) Explain why road surfaces at sharp bends are made rough rather than smooth.

(b) A curved section of road has a radius of 120m and is banked at 18° to the horizontal. The coefficient of static friction between the tyres and the road is 0.35. Take $g = 9.8\text{m/s}^2$.

Determine the range of speeds at which a car can safely negotiate this curve without sliding.

Solution

(a) When a vehicle negotiates a curve on a flat road, the centripetal force required to maintain the circular path is provided entirely by static friction between the tyres and the road surface. A rough surface provides a higher coefficient of static friction than a smooth surface, allowing a greater maximum centripetal force and therefore a higher safe speed on the curve. A smooth surface would provide very little friction, and vehicles would slide outward even at low speeds.

(b) **Maximum speed** (friction acts down the slope, supplementing centripetal force):

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)}$$

Where: $r = 120\text{m}$, $g = 9.8\text{m/s}^2$, $\theta = 18°$, $\mu_s = 0.35$, $\tan 18° = 0.3249$

$$v_{max} = \sqrt{120\text{m} \times 9.8\text{m/s}^2 \times \frac{0.3249 + 0.35}{1 - 0.35 \times 0.3249}} = 29.9\text{m/s}$$

The maximum safe speed is 29.9m/s.

Minimum speed (friction acts up the slope, preventing inward sliding):

$$v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)}$$

$$v_{min} = \sqrt{120\text{m} \times 9.8\text{m/s}^2 \times \frac{0.3249 - 0.35}{1 + 0.35 \times 0.3249}} = \sqrt{1176 \times \frac{-0.0251}{1.1137}}$$

The numerator $(\tan\theta - \mu_s) = 0.3249 - 0.35 = -0.0251$ is negative.

Since $\tan\theta < \mu_s$, the minimum speed is **zero**. Friction is strong enough to prevent the car from sliding inward down the banked surface at any speed, including rest. The car can safely travel at any speed from 0 to 29.9m/s on this curve.

The safe speed range is $0 \leq v \leq 29.9\text{m/s}$.

Example 47

(a) Kipanga claims that a heavier person is more likely to fall off a spinning ride than a lighter person because *"more mass means more centrifugal force pulling them outward."* Is Kipanga correct? Explain.

(b) A spinning ride has a radius of 4m and rotates at 1.5rev/s. A child of mass 35kg and an adult of mass 80kg sit at the same radius. Calculate the centripetal force required for each person, and show that both require the same coefficient of friction to remain seated.

Solution

(a) Kipanga is wrong in two accounts:

First, there is no outward "centrifugal force" acting on the rider. When the ride spins, friction between the rider and the seat pushes the rider inward, providing the centripetal force needed for circular motion. If friction is insufficient, the rider does not get "pulled outward" by any force. Instead, the rider's body, obeying Newton's first law, continues in a straight line (tangentially), which from the ride's perspective appears as sliding outward.

Second, a a heavier person is not more likely to slide off. A heavier person requires more centripetal force ($F = m\omega^2 r$), but a heavier person also has more weight, which to the same proportion, increases the normal reaction and therefore the maximum friction available ($f_{max} = \mu mg$), providing more centripetal acceleration. Mathematically, when we set the required centripetal force equal to the maximum friction, the mass cancels:

$$m\omega^2 r = \mu mg \Longrightarrow \mu = \frac{\omega^2 r}{g}$$

Hence, the coefficient of friction needed is independent of mass. Both the child and the adult slide off at the same rotation speed.

(b) Angular velocity:

$$\omega = 2\pi f = 2 \times 3.14 \times 1.5 s^{-1} = 9.42 rad/s$$

For the child ($m = 35kg$):

$$F_c = m\omega^2 r = 35kg \times (9.42rad/s)^2 \times 4m = \mathbf{12424N}$$

$$\mu = \frac{F_c}{mg} = \frac{12424N}{35kg \times 9.8m/s^2} = \mathbf{36.22}$$

For the adult ($m = 80kg$):

$$F_c = 80kg \times (9.42rad/s)^2 \times 4m = \mathbf{28397N}$$

$$\mu = \frac{F_c}{mg} = \frac{28397N}{80kg \times 9.8m/s^2} = \mathbf{36.22}$$

Both require $\mu = 36.22$, confirming that the required friction coefficient is independent of mass.

Example 48

(a) A stone is whirled in a vertical circle on a string. State the position in the circle where the string is most likely to break, and explain why.

(b) A stone of mass 0.2kg is attached to a string of length 0.8m and whirled in a vertical circle. The speed at the top of the circle is 5m/s. The string can withstand a maximum tension of 14N. The centre of the circle is 4m above the ground. Take $g = 9.8m/s^2$.

(i) Find the speed at the bottom of the circle.

(ii) Calculate the tension at the bottom and determine whether the string breaks.

(iii) If it breaks, find the time taken by stone to reach the ground and the horizontal distance covered.

Solution

(a) The string is most likely to break at the **bottom** of the circle. At this position, the tension must support the weight and provide the centripetal force simultaneously ($T_{bottom} = mg + \frac{mv^2}{r}$), making the tension maximum.

(b) The solution of each part is as follows:

(i) Using conservation of energy between the top (height 2r above the bottom) and the bottom:

$$\frac{1}{2}mv_{bottom}^2 = \frac{1}{2}mv_{top}^2 + mg(2r)$$

$$v_{bottom}^2 = v_{top}^2 + 4gr = (5m/s)^2 + 4 \times 9.8m/s^2 \times 0.8m = 25 + 31.36 = 56.36m^2/s^2$$

$$v_{bottom} = 7.51m/s$$

(ii) Tension at the bottom:

$$T_{bottom} = mg + \frac{mv_{bottom}^2}{r} = 0.2kg \times 9.8m/s^2 + \frac{0.2kg \times 56.36m^2/s^2}{0.8m} = 16.05N$$

Since $16.05N > 14N$, the string breaks at the bottom.

(iii) At the bottom, the velocity is horizontal (7.51m/s). The height of the bottom above the ground:

$$h = 4m - 0.8m = 3.2m$$

The stone becomes a projectile with $u_x = 7.51m/s$ and $u_y = 0$.

Time to reach the ground:

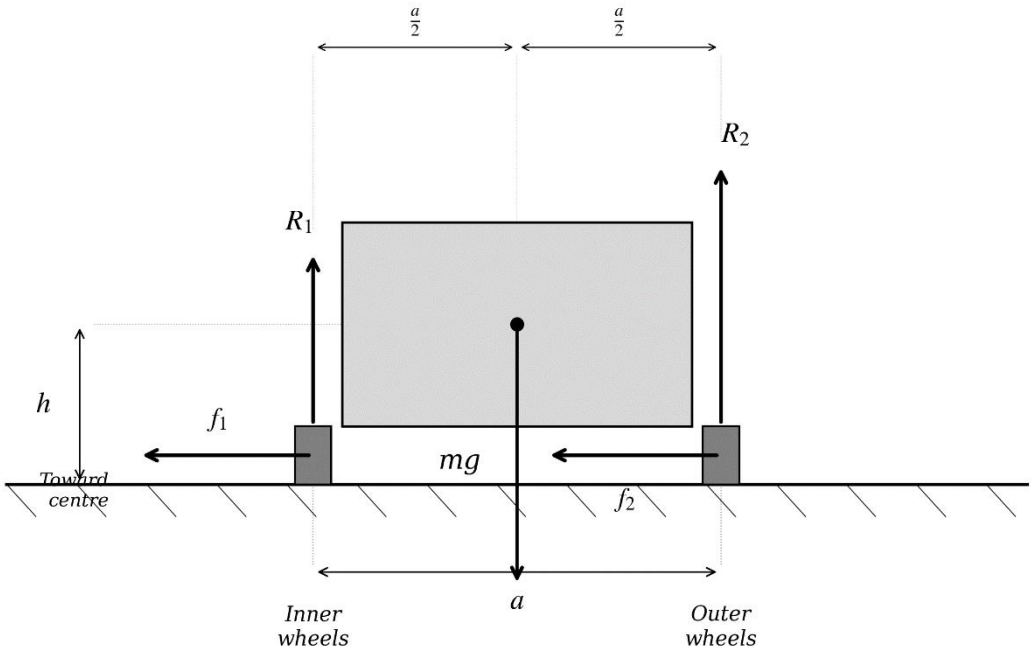
$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 3.2m}{9.8m/s^2}} = \sqrt{0.653s^2} = \mathbf{0.808s}$$

Horizontal distance:

$$x = u_x \times t = 7.51m/s \times 0.808s = \mathbf{6.07m}$$

Example 49

The figure below shows a rear view of a car of mass **m** rounding a level curved road of radius **r** at speed **v.** The lateral distance between the inner and outer wheels is **a**, and the centre of gravity **G** is at height **h** above the ground. $\mathbf{R_1}$ and $\mathbf{R_2}$ are the normal reactions on the inner and outer wheels, and $\mathbf{f_1}$ and $\mathbf{f_2}$ are the friction forces on the inner and outer wheels.



(a) By considering the forces and taking moments about G, derive expressions for:

(i) the normal reaction $\mathbf{R_1}$ on the inner wheels,
(ii) the normal reaction $\mathbf{R_2}$ on the outer wheels.

(b) Hence show that:

(i) the maximum speed before the car overturns is $v_{overturn} = \sqrt{\frac{arg}{2h}}$,
(ii) the maximum speed before the car skids is $v_{skid} = \sqrt{\mu_s gr}$,
(iii) the car skids before overturning if $\mu_s < \frac{a}{2h}$, and overturns before skidding if $\mu_s > \frac{a}{2h}$.

Solution

(a) Three equations govern the system:

Vertical equilibrium:

$$R_1 + R_2 = mg \quad \ldots(1)$$

Horizontal (centripetal force from friction):

$$f_1 + f_2 = \frac{mv^2}{r} \quad \ldots(2)$$

Taking moments about G:

Each wheel is at horizontal distance $\frac{a}{2}$ from G. The friction forces act at the ground, a vertical distance h below G. Taking anticlockwise as positive:

R_2 acts upward at distance $\frac{a}{2}$ to the right of G → anticlockwise (positive).

R_1 acts upward at distance $\frac{a}{2}$ to the left of G → clockwise (negative).

$(f_1 + f_2)$ acts horizontally at distance h below G → clockwise (negative).

$$R_2 \times \frac{a}{2} - R_1 \times \frac{a}{2} - (f_1 + f_2) \times h = 0$$

$$\frac{a}{2}(R_2 - R_1) = (f_1 + f_2) \times h$$

Substituting (2):

$$\frac{a}{2}(R_2 - R_1) = \frac{mv^2 h}{r} \quad \ldots (3)$$

(i) Finding R_1:

From (1): $R_2 = mg - R_1$. Substituting into (3):

$$\frac{a}{2}(mg - 2R_1) = \frac{mv^2 h}{r}$$

$$mg - 2R_1 = \frac{2mv^2 h}{ar}$$

$$\boxed{\mathbf{R_1 = \frac{m}{2}\left(g - \frac{2v^2 h}{ar}\right)}}$$

As v increases, R_1 decreases. The inner wheels progressively lose contact with the ground.

(ii) Finding R_2:

Substituting R_1 into (1):

$$R_2 = mg - \frac{m}{2}\left(g - \frac{2v^2 h}{ar}\right) = \frac{mg}{2} + \frac{mv^2 h}{ar}$$

$$\boxed{\mathbf{R_2 = \frac{m}{2}\left(g + \frac{2v^2 h}{ar}\right)}}$$

As v increases, R_2 increases. The outer wheels carry progressively more load.

(c) **(i) Maximum speed before overturning:**

The car overturns when $R_1 = 0$:

$$0 = g - \frac{2v_{overturn}^2 h}{ar}$$

$$\boxed{\mathbf{v_{overturn} = \sqrt{\frac{arg}{2h}}}}$$

(ii) Maximum speed before skidding:

The car skids when total friction reaches its limit: $f_1 + f_2 = \mu_s(R_1 + R_2) = \mu_s mg$

$$\mu_s mg = \frac{mv_{skid}^2}{r}$$

$$\boxed{\mathbf{v_{skid} = \sqrt{\mu_s gr}}}$$

(iii) Condition for skidding vs overturning:

Skidding occurs first if $v_{skid} < v_{overturn}$:

$$\mu_s gr < \frac{arg}{2h}$$

$$\boxed{\mu_s < \frac{a}{2h} \Longrightarrow \textit{car skids first (safer)}}$$

$$\boxed{\mu_s > \frac{a}{2h} \Longrightarrow \textit{car overturns first (dangerous)}}$$

Example 50

(a) State two design features that make a vehicle less likely to overturn on a curved road.

(b) A bus has its wheels 2m apart laterally and its centre of gravity is 1.2m above the ground. It rounds a level curve of radius 60m. The coefficient of static friction between the tyres and the road is 0.45. Take $g = 9.8\text{m/s}^2$.

(i) Calculate the maximum speed before the bus overturns.
(ii) Calculate the maximum speed before the bus skids.
(iii) Determine which happens first, and comment on the safety of this vehicle.
(iv) A sports car has wheels 1.8m apart and centre of gravity 0.4m above the ground. On the same road with the same friction, determine which happens first for the sports car.

Solution

(a) (1) A low centre of gravity. (2) A wide wheel base.

(b) The solution of each part is as follow:

(i) For the bus ($a = 2\text{m}$, $h = 1.2\text{m}$):

$$v_{\text{overturn}} = \sqrt{\frac{arg}{2h}} = \sqrt{\frac{2\text{m} \times 60\text{m} \times 9.8\text{m/s}^2}{2 \times 1.2\text{m}}} = \mathbf{22.1m/s}$$

(ii) Using:

$$v_{\text{skid}} = \sqrt{\mu_s gr} = \sqrt{0.45 \times 9.8\text{m/s}^2 \times 60\text{m}} = \mathbf{16.3m/s}$$

(iii) Using:

$$\frac{a}{2h} = \frac{2\text{m}}{2.4\text{m}} = 0.833$$

Since $\mu_s(0.45) < \frac{a}{2h}(0.833)$, the bus **skids first** at 16.3m/s. This is the safer outcome. However, if the road surface improves (higher μ_s approaching 0.833), the bus could overturn instead which is a serious danger for tall vehicles.

(iv) For the sports car ($a = 1.8\text{m}$, $h = 0.4\text{m}$):

$$\frac{a}{2h} = \frac{1.8\text{m}}{0.8\text{m}} = 2.25$$

Since $\mu_s(0.45) \ll \frac{a}{2h}(2.25)$, the sports car **skids first** with a very large safety margin against overturning. Its low centre of gravity and wide track make overturning virtually impossible under normal conditions.

Example 51

(a) Distinguish between the apparent weight of a driver at the top of a hill and at the bottom of a valley, in terms of the direction of centripetal acceleration.

(b) A car travels at 25m/s along a road that passes over a hill of radius 60m followed by a valley of radius 80m. Calculate the apparent weight of the driver (mass 70kg) at the top of the hill and at the bottom of the valley. Take $g = 9.8\text{m/s}^2$.

Solution

(a) At the top of a hill, the centripetal acceleration is directed **downward** (toward the centre of curvature below the road). The normal reaction is therefore less than the weight, so the driver feels lighter.

At the bottom of a valley, the centripetal acceleration is directed **upward** (toward the centre of curvature above the road). The normal reaction is therefore greater than the weight, so the driver feels heavier.

(b) **At the top of the hill ($r = 60\text{m}$):**

$$R_{hill} = mg - \frac{mv^2}{r} = 70kg \times 9.8m/s^2 - \frac{70kg \times (25m/s)^2}{60m} = 686N - 729.2N = -43.2N$$

The result is negative. This means the car has **lost contact with the road** at the top of the hill at this speed. The driver is momentarily airborne and experiences weightlessness ($R = 0$).

Hence, the apparent weight at the top of the hill is 0N.

Checking the answer

You may check the answer by using the fact that, *if $R = 0$, the speed exceeds the critical speed for this hill.*

$$v_{max} = \sqrt{gr} = \sqrt{9.8 \times 60} = 24.2m/s$$

Since $25m/s > 24.2m/s$, the car leaves the road.

At the bottom of the valley ($r = 80m$):

$$R_{valley} = mg + \frac{mv^2}{r} = 70kg \times 9.8m/s^2 + \frac{70kg \times (25m/s)^2}{80m} = 686N + 546.9N = 1232.9N$$

The apparent weight at the bottom of the valley is $1232.9N$, (which is 1.80 times the actual weight (686N). The driver feels almost twice as heavy).

Example 52

(a) Explain why a pendulum hanging from the ceiling of a car swings outward when the car rounds a curve at constant speed.

(b) A small ball of mass 0.1kg is suspended by a light string from the ceiling of a car. The car rounds a horizontal curve of radius 15m at constant speed 12m/s. Find the angle the string makes with the vertical and the tension in the string. Take $g = 9.8m/s^2$.

Solution

(a) When the car rounds a curve, the ball must accelerate toward the centre of the curve to follow the car's circular path. The only horizontal force available to provide this centripetal acceleration is the horizontal component of the tension in the string. For the tension to have a horizontal component, the string must swing outward from the vertical. The ball does not experience an outward force; rather, the string tilts outward to provide the inward (centripetal) force needed for circular motion.

(b) Let θ be the angle the string makes with the vertical.

Vertically: $T\cos\theta = mg$

Horizontally (toward centre): $T\sin\theta = \frac{mv^2}{r}$

Dividing:

$$\tan\theta = \frac{v^2}{rg} = \frac{(12m/s)^2}{15m \times 9.8m/s^2} = \frac{144}{147} = 0.9796$$

$$\theta = \tan^{-1}(0.9796) = 44.4°$$

Tension:

$$T = \frac{mg}{\cos\theta} = \frac{0.1kg \times 9.8m/s^2}{\cos 44.4°} = \frac{0.98N}{0.7147} = 1.37N$$

The string makes $44.4°$ with the vertical and the tension is 1.37N.

Example 53

(a) Explain why, for an object moving in a vertical circle on a string, the minimum speed at the top required to maintain a complete circle does not depend on the mass of the object.

(b) Two balls A (mass 0.3kg) and B (mass 0.5kg) are connected by a light rigid rod and move inside a smooth vertical circular track of radius 1m. At a certain instant, A is at the top and B is at the bottom.

The speed of each ball at this instant is 4m/s. Take $g = 9.8m/s^2$. Calculate the normal reaction on each ball from the track at this instant.

Solution

(a) At the top of the vertical circle, only gravity provides the centripetal force. A heavier object requires a larger centripetal force $\left(\frac{mv^2}{r}\right)$, but it also experiences a greater gravitational force (mg). Since both the required centripetal force and the gravitational force increase in direct proportion to mass, the effect of mass change cancels. Mathematically, setting the tension to zero at the top gives: $mg = \frac{mv_{min}^2}{r}$. The mass m appears on both sides and cancels, giving $v_{min} = \sqrt{gr}$, which depends only on g and r. Hence, the minimum speed at the top required to maintain the circular motion is the same for all objects, regardless of their mass.

(b) Since both balls are on a smooth track (not a string), the track pushes them. At each position, Newton's second law is applied toward the centre.

Ball A at the top (centre is below):

Both R_A and m_Ag act downward (toward centre):

$$R_A + m_Ag = \frac{m_Av^2}{r}$$

$$R_A = \frac{m_Av^2}{r} - m_Ag = \frac{0.3kg \times (4m/s)^2}{1.0m} - 0.3kg \times 9.8m/s^2 = 4.8N - 2.94N = 1.86N$$

Ball B at the bottom (centre is above):

R_B acts upward (toward centre), m_Bg acts downward (away from centre):

$$R_B - m_Bg = \frac{m_Bv^2}{r}$$

$$R_B = \frac{m_Bv^2}{r} + m_Bg = \frac{0.5kg \times (4m/s)^2}{1.0m} + 0.5kg \times 9.8m/s^2 = 8.0N + 4.9N = 12.9N$$

The normal reaction on A is 1.86N and on B is 12.9N.

Example 54

(a) Explain why a small block placed inside a smooth rotating hemispherical bowl rises up the inner surface as the angular speed of rotation increases.

(b) A small block of mass 0.1kg sits inside a smooth hemispherical bowl of internal radius 0.3m. The bowl rotates about its vertical axis of symmetry. The block is in equilibrium at a position where the line from the centre of the bowl to the block makes an angle of 40° with the vertical. Take $g = 9.8m/s^2$.

(i) Show that the angular speed of the bowl is given by $\omega = \sqrt{\frac{g}{R\cos\theta}}$, where R is the radius of the bowl and θ is the angle from the vertical.

(ii) Calculate the angular speed and the normal reaction on the block.

Solution

(a) As the bowl spins faster, the block requires a greater centripetal force to maintain circular motion at its current radius. The only horizontal force available is the horizontal component of the normal reaction from the bowl surface. To increase this horizontal component, the block must move to a position where the bowl surface is more steeply inclined; that is, higher up the bowl where θ is larger. This increases $R\sin\theta$ (the horizontal component), providing the greater centripetal force needed.

(b) The solution of each part is as follows:

(i) The block moves in a horizontal circle of radius $r = R\sin\theta$. The normal reaction N acts perpendicular to the bowl surface, which at this position is directed along the line from the block toward the centre of the bowl. Resolving N:

Vertically: $N\cos\theta = mg$

Horizontally: $N\sin\theta = m\omega^2 r = m\omega^2 R\sin\theta$

Dividing:

$$\frac{\sin\theta}{\cos\theta} = \frac{\omega^2 R\sin\theta}{g}$$

The $\sin\theta$ cancels:

$$\frac{1}{\cos\theta} = \frac{\omega^2 R}{g}$$

$$\boxed{\omega = \sqrt{\frac{g}{R\cos\theta}}}$$

(iii) Angular speed:

$$\omega = \sqrt{\frac{9.8\text{m/s}^2}{0.3\text{m} \times \cos 40°}} = 6.53\text{rad/s}$$

Normal reaction:

$$N = \frac{mg}{\cos\theta} = \frac{0.1\text{kg} \times 9.8\text{m/s}^2}{\cos 40°} = \frac{0.98\text{N}}{0.766} = 1.28\text{N}$$

The angular speed is 6.53rad/s and the normal reaction is 1.28N.

Example 55

(a) A student says: *"At the top of a vertical circle, the ball is weightless because the velocity is horizontal."* Identify and correct two errors in this statement.

(b) A ball of mass 0.25kg moves in a vertical circle of radius 0.5m on a string. The speed at the bottom is 5.5m/s. Take $g = 9.8\text{m/s}^2$. Find the angle from the bottom at which the tension in the string equals the weight of the ball.

Solution

(a) **Error 1:** The ball is not weightless at the top. Weight (mg) acts on the ball at every point in the circle, including the top. The ball may have zero *apparent* weight (zero tension) at the critical speed, but its actual weight never changes.

Error 2: The direction of velocity does not determine whether the ball is weightless. A ball moving horizontally at the bottom of the circle is certainly not weightless. Weightlessness (zero contact force) depends on the balance between gravity and centripetal requirements, not on the direction of velocity.

(b) Using the general expression:

$$T = \frac{mv_{bottom}^2}{r} - 2mg + 3mg\cos\theta$$

Setting $T = mg$:

$$mg = \frac{mv_{bottom}^2}{r} - 2mg + 3mg\cos\theta$$

$$3mg\cos\theta = 3mg - \frac{mv_{bottom}^2}{r}$$

$$\cos\theta = 1 - \frac{v_{bottom}^2}{3gr}$$

Substituting:

$$\cos\theta = 1 - \frac{(5.5\text{m/s})^2}{3 \times 9.8\text{m/s}^2 \times 0.5\text{m}} = 1 - \frac{30.25}{14.7} = 1 - 2.059 = -1.059$$

Since $\cos\theta = -1.059 < -1$, which is impossible, the tension **never equals the weight** at any point in the circle. The tension is always greater than mg throughout the motion.

Checking of the answer

This can be verified: at the bottom, $T_{bottom} = mg + \frac{mv^2}{r} = 0.25 \times 9.8 + \frac{0.25 \times 30.25}{0.5} = 17.58\text{N}$, which is far greater than $mg = 2.45\text{N}$.

At the top, using energy conservation: $v_{top}^2 = v_{bottom}^2 - 4gr = 30.25 - 19.6 = 10.65\text{m}^2/\text{s}^2$

$$T_{top} = \frac{mv_{top}^2}{r} - mg = \frac{0.25 \times 10.65}{0.5} - 2.45 = 2.88\text{N}$$

Even at the top, the tension (2.88N) exceeds the weight (2.45N). The speed is too high for the tension to ever drop to mg.

Example 56

(a) Explain why a ball released from the top of a smooth hemispherical dome will eventually leave the surface before reaching the bottom.

(b) A small ball of mass 0.15kg sits on top of a smooth hemispherical dome of radius 2m. It is given a tiny push and begins to slide down the surface. Take $g = 9.8\text{m/s}^2$.

(i) By considering forces along the radius and using energy conservation, show that the ball leaves the surface when $\cos\theta = \frac{2}{3}$, where θ is measured from the vertical.

(ii) Calculate the angle, the height above the ground, and the speed at the point where the ball leaves the dome.

Solution

(a) As the ball slides down the dome, it speeds up (gaining kinetic energy from lost potential energy). At any point on the dome, the component of weight along the radius toward the centre ($mg\cos\theta$) must provide both the centripetal force and the normal reaction. As the ball speeds up, the centripetal force requirement ($\frac{mv^2}{r}$) increases while $mg\cos\theta$ decreases (because θ increases as the ball descends). At some angle, the required centripetal force exceeds $mg\cos\theta$, the normal reaction drops to zero, and the ball leaves the surface.

(b) The solution of each part is as follows:

(i) At angle θ from the vertical, applying Newton's second law toward the centre (along the radius, inward):

$$mg\cos\theta - R = \frac{mv^2}{r}$$

The ball leaves the surface when $R = 0$:

$$mg\cos\theta = \frac{mv^2}{r} \quad \ldots(1)$$

Using energy conservation from the top ($\theta = 0$, $v = 0$) to angle θ:

The ball has descended a vertical height $h = r - r\cos\theta = r(1 - \cos\theta)$

$$\frac{1}{2}mv^2 = mgr(1 - \cos\theta)$$

$$v^2 = 2gr(1 - \cos\theta) \quad \ldots(2)$$

Substituting (2) into (1):

$$mg\cos\theta = \frac{m \times 2gr(1-\cos\theta)}{r}$$

$$g\cos\theta = 2g(1-\cos\theta)$$

$$\cos\theta = 2 - 2\cos\theta$$

$$3\cos\theta = 2$$

$$\boxed{\cos\theta = \frac{2}{3}}$$

(ii) Angle:

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2°$$

Height above the ground at the point of leaving:

The ball is on the *outside* of the dome. The centre of the hemisphere is at the base (ground level), and the top of the dome is at height r above the ground. At angle θ from the vertical, the ball is at height:

$$h = r\cos\theta = 2m \times \frac{2}{3} = 1.33m \text{ above the ground}$$

Speed at the point of leaving, from equation (2):

$$v^2 = 2gr(1-\cos\theta)$$

$$v = \sqrt{2gr(1-\cos\theta)} = \sqrt{2 \times 9.8 m/s^2 \times 2m \times \left(1-\frac{2}{3}\right)} = 3.61m/s$$

The ball leaves the dome at $\theta = 48.2°$ from the vertical, at a height of 1.33m above the ground, with a speed of 3.61m/s.

And with that, the miscellaneous worked examples take their final bow. Every concept in this chapter: from the gentle lean of a cyclist to the terrifying physics of overturning, from satellites silently orbiting overhead to a bucket of water that defied gravity (and soaked Kipanga), has been tested, twisted, combined, and calculated. If you survived all of that, congratulations! But do not relax just yet. The Digging Deeper Exercise is waiting on the next page, and it has no solutions to hold your hand. Just you, the physics, and a calculator. Good luck! You are ready!

DIGGING DEEPER EXERCISE 7

EXERCISE 7A: BINDER QUESTIONS

Question 1

A body moves in a circular path at constant speed. State whether each of the following is zero, constant, or changing:

(a) speed

(b) velocity

(c) centripetal acceleration

(d) centripetal force

Question 2

A body moves in a circular path at constant speed.

(a) Explain why the body is accelerating even though its speed does not change.

(b) State the direction of this acceleration.

Question 3

A stone tied to a string is whirled in a horizontal circle on a smooth table. The string suddenly breaks.

(a) Describe the path of the stone immediately after the string breaks.

(b) Explain your answer.

Question 4

Distinguish between angular velocity and linear velocity of an object in circular motion. State how they are related.

Question 5

A body moves in a circle of radius r with constant angular velocity ω.

(a) Write the centripetal acceleration in terms of v and r.

(b) Write the centripetal acceleration in terms of ω and r.

(c) Explain why both expressions give the same result.

Question 6

State the real force that provides the centripetal force in each of the following situations:

(a) a car rounding a level curve

(b) a satellite orbiting the Earth

(c) a ball on a string swung in a vertical circle

(d) a car on a banked curve with no friction

(e) a charged particle moving in a magnetic field

Question 7

Explain why the maximum safe speed of a car on a level circular curve does not depend on the mass of the car.

Question 8

In vertical circular motion on a string:

(a) State the position where the tension in the string is maximum. Explain why.
(b) State the position where the tension in the string is minimum. Explain why.

Question 9

Explain what is meant by the "design speed" of a banked curve, and describe what happens when a vehicle travels (a) faster than the design speed, and (b) slower than the design speed.

Question 10

A car moves over the top of a hill. Explain why the driver feels lighter at the top, and state the condition under which the car would lose contact with the road.

Question 11

Compare the critical speed condition at the top of a vertical loop for two cases:

(a) an object on the inside of a circular loop (e.g. a ball inside a loop-the-loop track)

(b) an object on the outside of a circular curve (e.g. a car at the top of a hill)

State how the critical speed differs in meaning between the two cases.

EXERCISE 7B: REAL QUESTIONS

Question 12

Passengers in a bus travelling along a straight road suddenly feel thrown to the left when the bus turns sharply to the right. Explain why this happens, and state whether any force actually pushes the passengers to the left.

Question 13

On the Dar es Salaam–Morogoro highway, sharp curves have road signs showing a recommended speed limit. Explain, using physics, how the recommended speed for a curve is determined and why different curves have different recommended speeds.

Question 14

During a motorcycle race, riders lean their bodies sharply toward the inside of a bend while turning. Explain why leaning is necessary and what would happen if a rider tried to turn without leaning.

Question 15

On a school trip, the bus crosses a bridge that has a gentle hump at the centre. As the bus goes over the hump at speed, **Kipanga** grabs his seat and shouts: *"My stomach just dropped! I think we are falling!"*

Kipute laughs and says: *"We are not falling, Kipanga. The road just stopped pushing you as hard."*

Explain what Kipute means. Why does Kipanga feel the sensation of his stomach dropping?

Question 16

Explain why a wet road is more dangerous at a curve than on a straight section of road.

Question 17

A washing machine removes water from clothes during the spin cycle by rotating the drum at high speed. Explain how the water is separated from the clothes.

Question 18

Tall vehicles such as buses and lorries are more likely to overturn on sharp curves than low vehicles such as saloon cars. Explain why.

Question 19

Kipanga watches a news report about a satellite orbiting the Earth. He turns to **Mr. Akilikubwa** and asks: *"If gravity pulls the satellite toward the Earth, why doesn't it fall down and crash?"*

Mr. Akilikubwa smiles and says: *"It is falling, Kipanga. It just keeps missing the ground."*

Explain what Mr. Akilikubwa means.

Question 20

Engineers build roads with banked curves rather than relying entirely on friction. Explain the advantage of banking, and state why banking alone is not sufficient for all speeds.

Question 21

A student observes that astronauts inside the International Space Station float as if there is no gravity. The student concludes that there is no gravity in space. Explain why this conclusion is incorrect.

EXERCISE 7C: HOT QUESTIONS

Question 22

A car of mass 1200kg rounds a level curve of radius 80m at 25m/s. The coefficient of static friction is 0.60. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the centripetal force required.

(b) Calculate the maximum friction force available.

(c) Determine whether the car can safely negotiate the curve at this speed.

Question 23

A conical pendulum has a string of length 1.2m and the bob has mass 0.4kg. The bob moves in a horizontal circle with a period of 2s. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the angle the string makes with the vertical.

(b) Calculate the tension in the string.

(c) Calculate the radius of the circular path and the speed of the bob.

Question 24

A student solves the following problem: *"A ball on a string moves in a vertical circle of radius 0.5m. The speed at the bottom is 4m/s. Find the tension at the top."*

The student writes: $T_{top} = \frac{mv_{bottom}^2}{r} - mg$, using the speed at the bottom directly in the tension formula at the top.

(a) Explain the error in the student's approach.

(b) For a ball of mass 0.3kg, carry out the correct calculation. If the ball does not reach the top, determine the angle from the bottom at which the string goes slack. Take $g = 9.8\text{m/s}^2$.

Question 25

A curved road of radius 100m is banked at 20°. The coefficient of static friction between the tyres and the road is 0.30. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the design speed of the curve.

(b) Determine the safe speed range.

Question 26

A ball of mass 0.2kg is attached to a string of length 0.7m and swung in a vertical circle. The speed at the bottom is 6m/s. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the tension at the bottom.

(b) Calculate the speed and the tension at the top.

(c) Find the angle from the bottom at which the tension equals 8N.

Question 27

A small ball rolls along a frictionless track and enters a vertical circular loop of radius 0.5m. The ball is released from rest at a height $h = 1.5m$ above the bottom of the loop. Take $g = 9.8m/s^2$.

(a) Show that the ball completes the loop.

(b) Calculate the speed and the normal reaction at the top of the loop for a ball of mass 0.1kg.

(c) Calculate the normal reaction at the bottom of the loop.

Question 28

An engineer designs a level circular curve of radius 70m for a road where the expected traffic speed is 60km/h. The road surface has coefficient of static friction 0.55. Take $g = 9.8m/s^2$.

(a) Calculate the maximum safe speed on this curve.

(b) Determine whether the curve is safe for the expected traffic speed.

(c) If the engineer decides to bank the road instead, calculate the banking angle required for the expected speed to be the design speed.

(d) With this banking angle and the same friction, calculate the new maximum safe speed.

Question 29

A car of mass 1500kg travels at 20m/s over a road that has a valley (dip) of radius 50m. Take $g = 9.8m/s^2$.

(a) Calculate the normal reaction on the car at the bottom of the valley.

(b) Express the apparent weight of the driver (mass 75kg) as a multiple of the driver's actual weight.

(c) Calculate the speed at which the apparent weight would be twice the actual weight.

Question 30

A stone of mass 0.3kg is whirled in a vertical circle of radius 0.9m. At the top of the circle, the speed is 4m/s. Take $g = 9.8m/s^2$.

(a) Calculate the tension at the top.

(b) Find the tension at the bottom.

(c) Calculate the speed at the bottom.

Question 31

A student claims that a heavier ball needs a greater minimum speed at the bottom to complete a vertical circle than a lighter ball on the same string.

(a) Calculate the minimum speed at the bottom for a 0.2kg ball and a 0.5kg ball, each on a string of length 0.8m. Take $g = 9.8m/s^2$.

(b) Comment on the student's claim.

Question 32

A small ball of mass 0.1kg is suspended by a light string from the roof of a car. The car travels at constant speed around a horizontal circular track of radius 20m. The string makes an angle of 30° with the vertical. Take $g = 9.8m/s^2$.

(a) Calculate the speed of the car.

(b) Calculate the tension in the string.

(c) If the speed doubles, calculate the new angle the string makes with the vertical.

Question 33

A car has wheels 1.5m apart and its centre of gravity is 0.9m above the ground. It rounds a level curve of radius 40m. The coefficient of static friction is 0.70. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the maximum speed before overturning.

(b) Calculate the maximum speed before skidding.

(c) Determine which happens first.

(d) Calculate the normal reactions on the inner and outer wheels when the car travels at 15m/s.

Question 34

A ball on a string of length 1m is set in motion in a vertical circle. The speed at the bottom is 5.5m/s. Take $g = 9.8\text{m/s}^2$.

(a) Determine whether the ball completes the full circle.

(b) If not, find the angle from the bottom at which the string goes slack.

(c) Find the speed at that angle.

Question 35

An engineer designs a roller coaster loop of radius 6m. The loaded car has mass 600kg. Safety regulations require that the normal reaction on the car at the top of the loop must be at least 0.5mg. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the minimum speed at the top to meet the safety requirement.

(b) Calculate the minimum height of the launch ramp above the bottom of the loop.

(c) Calculate the normal reaction at the bottom of the loop when launched from this minimum height. Assess whether the structure must withstand more than 6mg.

Question 36

A small ball sits on top of a smooth hemispherical dome of radius 3m. It is given a tiny push and slides down the surface. Take $g = 9.8\text{m/s}^2$.

(a) Find the angle from the vertical and the speed at the point where the ball leaves the dome.

(b) Determine the height above the ground at which the ball leaves the dome.

(c) After leaving the dome, the ball becomes a projectile. At the point of leaving, determine the horizontal and vertical components of the velocity.

ANSWERS

EXERCISE 7A

1. (a) Constant. (b) Changing (direction changes continuously). (c) Changing (magnitude is constant but direction changes continuously, always pointing toward the centre). (d) Changing (same reason as centripetal acceleration — constant magnitude but changing direction).

2. (a) Acceleration is the rate of change of velocity, which is a vector. Although the speed (magnitude) is constant, the direction of the velocity changes continuously as the body moves around the circle. A change in direction of velocity constitutes an acceleration. (b) The acceleration is directed toward the centre of the circular path (centripetal).

3. (a) The stone moves in a straight line along the tangent to the circle at the point where the string broke. (b) When the string breaks, the centripetal force vanishes. With no net force acting on it, the stone continues moving at constant velocity in the direction it was travelling at the instant of the break, in accordance with Newton's first law.

4. Angular velocity (ω) is the rate of change of angular displacement, measured in rad/s. Linear velocity (v) is the rate of change of linear displacement along the circular path, measured in m/s. They are related by $v = \omega r$, where r is the radius of the circular path.

5. (a) $a = \frac{v^2}{r}$ (b) $a = \omega^2 r$ (c) Since $v = \omega r$, substituting into $\frac{v^2}{r}$ gives $\frac{(\omega r)^2}{r} = \omega^2 r$. Hence, both are equivalent forms of the same physical quantity expressed using different variables.

6. (a) Static friction between the tyres and the road. (b)Gravitational force of the Earth on the satellite. (c) The component of tension directed toward the centre (assisted or opposed by gravity depending on position). (d) The horizontal component of the normal reaction from the banked road surface. (e) The magnetic force on the moving charged particle.

7. The required centripetal force is $\frac{mv^2}{r}$ and the maximum friction available is μmg. Both are proportional to mass m. When equated, the mass cancels, giving $v_{max} = \sqrt{\mu gr}$, which is independent of mass. A heavier car needs more centripetal force but also has proportionally more friction available.

8. (a) At the bottom. The tension must support the weight (acting away from centre) and provide the centripetal force (toward centre) simultaneously, so $T_{bottom} = mg + \frac{mv^2}{r}$, which is the largest value.

(b) At the top. Both tension and weight act toward the centre, so gravity assists the centripetal force and the string needs to supply less: $T_{top} = \frac{mv^2}{r} - mg$, which is the smallest value.

9. The design speed is the speed at which a vehicle can negotiate the banked curve without any friction. At this speed, the horizontal component of the normal reaction alone provides the required centripetal force.

(a) Faster than design speed: the banking alone cannot provide enough centripetal force. Friction must act down the slope (inward) to supplement the centripetal force. Without sufficient friction, the car slides outward (up the slope).

(b) Slower than design speed: the banking provides more centripetal force than needed. Friction must act up the slope (outward) to prevent the car from sliding inward (down the slope). Without friction, the car slides inward.

10. At the top of a hill, the centripetal acceleration is directed downward (toward the centre of curvature below the road). Part of the driver's weight is "used" to provide this centripetal acceleration, so the normal reaction (which the driver feels as weight) is reduced: $R = mg - \frac{mv^2}{r}$. The driver feels lighter because the support force is less than the actual weight.

The car loses contact with the road when $R = 0$, which occurs at speed $v = \sqrt{gr}$.

11. (a) Inside a loop: the critical speed at the top is a **minimum** speed. Below $v = \sqrt{gr}$, the object loses contact with the track and falls away from the circular path. The normal reaction is zero at the critical speed.

(b) Outside a curve (hill): the critical speed at the top is a **maximum** speed. Above $v = \sqrt{gr}$, the object loses contact with the road and becomes airborne. The normal reaction is zero at the critical speed.

In both cases, the critical speed has the same magnitude $\sqrt{gr}$, but for the inside loop it is a minimum to maintain contact, while for the outside curve it is a maximum before contact is lost.

EXERCISE 7B

12. When the bus turns right, friction between the seats and the passengers provides the centripetal force needed to follow the curved path. However, the passengers' bodies, due to inertia, tend to continue in a straight line (Newton's first law). This makes them feel as though they are being pushed to the left. No real force pushes them to the left; they simply resist the change in direction while the bus turns beneath them.

13. The recommended speed is based on the maximum speed at which a vehicle can safely negotiate the curve without sliding. This depends on the radius of the curve and the coefficient of friction between the tyres and the road: $v_{max} = \sqrt{\mu gr}$. A sharp curve (small radius) has a lower safe speed than a gentle curve (large radius). Different curves have different radii, so their recommended speeds differ.

14. When a motorcycle turns, friction at the contact point provides the centripetal force horizontally, while the weight acts vertically through the centre of mass high above the ground. If the rider does not lean, these two forces create a net torque (turning effect of a force) that would topple the motorcycle outward. Leaning inward shifts the centre of mass so that the resultant of weight and

friction passes through the contact point, eliminating the toppling torque. Without leaning, the motorcycle would tip outward and crash.

15. At the top of the hump, the bus follows a curved path whose centre of curvature is below the road. Part of the gravitational force is used to provide the centripetal acceleration downward, so the normal reaction from the seat on Kipanga is reduced: $R = mg - \frac{mv^2}{r}$. Kipanga feels lighter because the support force is less than his actual weight. The "stomach dropping" sensation is caused by this reduced contact force.

16. On a straight road, friction is mainly needed for braking and accelerating. On a curve, friction must provide the centripetal force to keep the vehicle on the circular path. When the road is wet, the coefficient of friction decreases. On a straight road, this mainly affects stopping distance. On a curve, it directly reduces the maximum safe speed ($v_{max} = \sqrt{\mu gr}$), and a vehicle travelling at a speed that was safe on a dry road may now exceed the reduced safe speed and slide off the curve.

17. During the spin cycle, the drum wall provides the centripetal force that keeps the clothes moving in a circle. The water trapped in the fabric requires the same centripetal force to follow the circular path. However, the adhesive forces holding the water to the fabric are too weak to supply the required centripetal force at high speed. The water, unable to maintain circular motion, moves tangentially outward through the holes in the drum, following Newton's first law. The clothes, pressed firmly against the drum wall, remain inside.

18. When a vehicle rounds a curve, friction at ground level provides the centripetal force, while the centre of gravity is above the ground. This creates a torque that tends to tip the vehicle outward about the outer wheels. A tall vehicle has a high centre of gravity, which increases the arm of the overturning torque. A low, wide vehicle has a low centre of gravity and large wheel separation, which increases the restoring torque resisting overturning. The maximum speed before overturning is $v = \sqrt{\frac{arg}{2h}}$, which is smaller for tall vehicles (large h, small a) and larger for low, wide vehicles (small h, large a).

19. The satellite is indeed falling toward the Earth due to gravity. However, it also has a large horizontal velocity. As it falls toward the Earth, it moves forward fast enough that the Earth's surface curves away beneath it at the same rate. The satellite continuously falls toward the Earth but never reaches it because the curvature of its path matches the curvature of the Earth. Gravity provides the centripetal force that bends the satellite's path into a circle. Mr. Akilikubwa's description is accurate: orbiting is continuous free fall where the falling object keeps "missing" the ground.

20. On a banked curve, the road surface is tilted so that the normal reaction has a horizontal component directed toward the centre of the curve. This component provides centripetal force without relying on friction. The advantage is that the vehicle can negotiate the curve safely even on a wet or icy surface, as long as it travels near the design speed.

However, banking alone is sufficient only at the design speed ($v = \sqrt{rg\tan\theta}$). At higher speeds, the banking cannot supply enough centripetal force, and friction is needed to make up the difference. At lower speeds, the banking provides too much inward force, and friction is needed to prevent the vehicle from sliding inward. Therefore, banking reduces dependence on friction but cannot eliminate it at all speeds.

21. Gravity does act on the astronauts and the space station. The astronauts appear weightless not because gravity is absent, but because both the astronauts and the station are in free fall together. Gravity provides the centripetal force for the orbital motion, and since everything inside the station accelerates toward the Earth at the same rate, there is no relative force between the astronauts and the station walls. With no contact force (normal reaction), the astronauts feel weightless. Weightlessness is the absence of a contact force, not the absence of gravity.

22. (a) 9375N (b) 7056N (c) Since the required force (9375N) exceeds the maximum friction (7056N), the car **cannot** safely negotiate the curve. It will skid outward.

23. (a) 34.1° (b) 4.73N (c) $r = 0.673$m, $v = 2.11$m/s

24. (a) The speed is different at different positions in a vertical circle. The student used v_{bottom} in the formula for the top, but the correct speed at the top (v_{top}) must first be found using energy conservation.

(b) Since v_{top}^2 is negative, the ball **never reaches the top**. (b) 114.9

25. (a) 18.9m/s (b) $7.52\text{m/s} \leq v \leq 27\text{m/s}$

26. (a) 12.25N (b) $v_{top} = 2.93$m/s, $T_{top} = 0.49$N (c) 73.9°

27. (a) Since $h(1.5\text{m}) > h_{min}(1.25\text{m})$, the ball completes the loop. (b) $v_{top} = 3.13$m/s, $R_{top} = 0.98$N (c) $R_{bottom} = 6.86$N

28. (a) 19.4m/s

(b) Since expected speed (16.7m/s) < maximum speed (19.4m/s), the curve is safe for the expected traffic speed on a dry road.

(c) 22.1° (d) 29.1m/s

29. (a) 26700N (b) 1.82 (c) 22.1m/s

30. (a) 2.39N (b) 20.03N (c) 7.16m/s

31. (a) For 0.2kg ball: 6.26m/s, For 0.5kg ball: 6.26m/s

(b) The student's claim is **wrong**. The minimum speed does not depend on mass. The formula $v_{min} = \sqrt{5gr}$ contains no mass term. This is because both the centripetal force requirement and the gravitational force increase in equal proportion with mass, so mass cancels.

32. (a) 10.6m/s (b) 1.13N (c) 66.4°

33. (a) 18.1m/s (b) 16.6m/s (c) the car **skids first** (Hint: compare speeds or coefficient of friction and $\frac{a}{2h}$)

(d) Inner wheel: 1.525m, outer wheel: 8.275m; where m is the mass of the car in kg.

34. (a) Since $v_{bottom}(5.5m/s) < v_{min,bottom}$ (7m/s), the ball does **not** complete the circle. (b) 111.3° (c) 1.88m/s

35. (a) 9.39m/s (b) 16.5m (c) 38220N (c) $\frac{R_{bottom}}{mg} = 6.5$. The structure must withstand 6.5mg at the bottom, which is greater than 6mg. Hence, the engineer must design for forces exceeding 6mg.

36. (a) $\theta = 48.2°$, $v = 4.43m/s$ (b) 2m (c) Horizontal component: 2.95m/s, Vertical component: 3.3m/s