Chapter 6

PROJECTILE MOTION

INTRODUCTION

It was inter-house sports day at Miono Secondary School, and Mr. Akilikubwa stood at the edge of the football field, clipboard in hand, recording student performances for the shot put competition. Kipanga stepped into the throwing circle, gripping a 4 kg shot put with determination. *"Sir,"* he called out confidently, *"I've been practicing! Watch me break the school record!"*

Mr. Akilikubwa smiled. *"Remember, Kipanga! It's not just about how hard you throw, but also the angle."*

"Angle?" **Kipanga** snorted. *"Sir, I'm strong! I'll just throw it as hard as I can straight forward!"* With a mighty grunt, he launched the shot put at nearly zero degrees above horizontal. The metal ball rocketed forward... and crashed into the ground barely three metres away, creating a small crater in the sandy field.

Kipute, waiting her turn, could not help but giggle. "*Kipanga, you threw it into the ground! That's not how shot put works!"*

"Fine!" **Kipanga** huffed, retrieving the ball. *"If throwing straight doesn't work, I'll throw it straight up!"* On his second attempt, he launched the shot put almost vertically. It soared magnificently into the sky, climbing higher and higher and higher... then came straight back down, landing half a metre from where he had thrown it. The students scattered as the heavy ball thudded back to earth.

Mr. Akilikubwa shook his head, trying not to laugh. *"Kipanga, you've just demonstrated the two extreme failures of projectile motion! Too flat, and gravity defeats you immediately. Too steep, and you waste all your energy fighting gravity instead of moving forward." He turned to Kipute. "Your turn. Show him how physics works."*

Kipute stepped up, closed her eyes briefly to estimate, and released the shot put at approximately 45 degrees. The ball traced a beautiful arc through the air, landing smoothly 8 metres away, a respectable throw!

"But how did you know the right angle?" **Kipanga** demanded, genuinely puzzled.

Mr. Akilikubwa grinned. *"Because unlike you, Kipute paid attention in Chapter 2 when we studied vertical motion under gravity. Now she's applying it to motion in two dimensions. Welcome to Chapter 6, everyone, where we learn that every thrown ball, launched javelin, or kicked football becomes a slave to the same beautiful mathematics. Master projectile motion, and you'll understand why goalkeepers kick at certain angles, why water fountains arc gracefully, and why Kipanga just lost five metres of his throwing distance by defying physics!"*

"Sir," **Kipanga** protested weakly, *"I was just... testing different approaches..."*

"Testing?" **Kipute** teased. *"Is that what we're calling it now?"*

By the end of this chapter, even Kipanga will understand exactly how to calculate the perfect angle for maximum distance. The mathematics that governs his embarrassing shot put attempts is the same mathematics that describes bullets, water spraying from hoses, and rockets **once their engines are no longer firing.** Let us explore how motion in two dimensions emerges from the simpler concepts we mastered in Chapter 2.

FUNDAMENTAL EQUATIONS OF PROJECTILE MOTION

A projectile *is any object that, once launched with an initial velocity, moves under the influence of gravity **alone***. Whether it is a shot put released by Kipanga, a stone thrown by hand, a javelin launched by an athlete, or water spraying from a hose, the motion follows the same fundamental principles. Once the object leaves contact with whatever launched it, the only force acting upon it (when air resistance is negligible) is the downward pull of gravity. *The path traced by a projectile through space* is called its **trajectory**.

Connecting to chapter 2: *Do you remember when we studied vertical motion under gravity in Chapter 2?* We learned that any object either thrown upward or dropped from rest experiences constant downward acceleration, $g = 9.8 m/s^2$. We also learned three equations of motion ($v = u + at$, $s = ut + \frac{1}{2}at^2$, and $v^2 = u^2 + 2as$) that perfectly describe this one-dimensional vertical motion. The beautiful insight of projectile motion is that these same principles still apply; we simply extend them to two dimensions.

The key to understanding projectile motion lies in recognising that it represents motion in two dimensions simultaneously, yet these two components of motion are completely independent of one another. A projectile exhibits uniform motion in the horizontal direction while simultaneously experiencing uniformly accelerated motion in the vertical direction due to gravity, exactly the same accelerated motion we studied in Chapter 2. This independence means that *what happens horizontally has no effect on what happens vertically, and vice versa*. This fundamental principle, known as **the independence of perpendicular components of motion**, allows us to analyse each direction separately using the equations we developed in Chapter 2 for one-dimensional motion.

Consider Kipanga's disastrous shot put throws from our introduction. From the moment the shot left his hand, it possessed both horizontal and vertical components of velocity. Horizontally, it moved forward at constant velocity as there is no horizontal forces to slow it or speed it up (ignoring air resistance). Vertically, however, gravity continuously acted downward, first slowing any upward motion, bringing it momentarily to rest at its highest point, then accelerating it downward (exactly as we learned in Chapter 2 for vertical motion). These two motions occurred simultaneously and independently, creating the parabolic arc.

To analyse projectile motion systematically, we establish a coordinate system with the x-axis along the horizontal direction and the y-axis pointing vertically upward. The origin is typically placed at the **point of projection** which *is the location from which the projectile is launched*. By **convention**, we take **upward** as the **positive y-direction**, which means that the acceleration due to gravity, acting downward, has a negative value: $\mathbf{a_y = -g = -9.8\ m/s^2}$. In the horizontal direction, where no forces act (again neglecting air resistance), the acceleration is zero: $a_x = 0$.

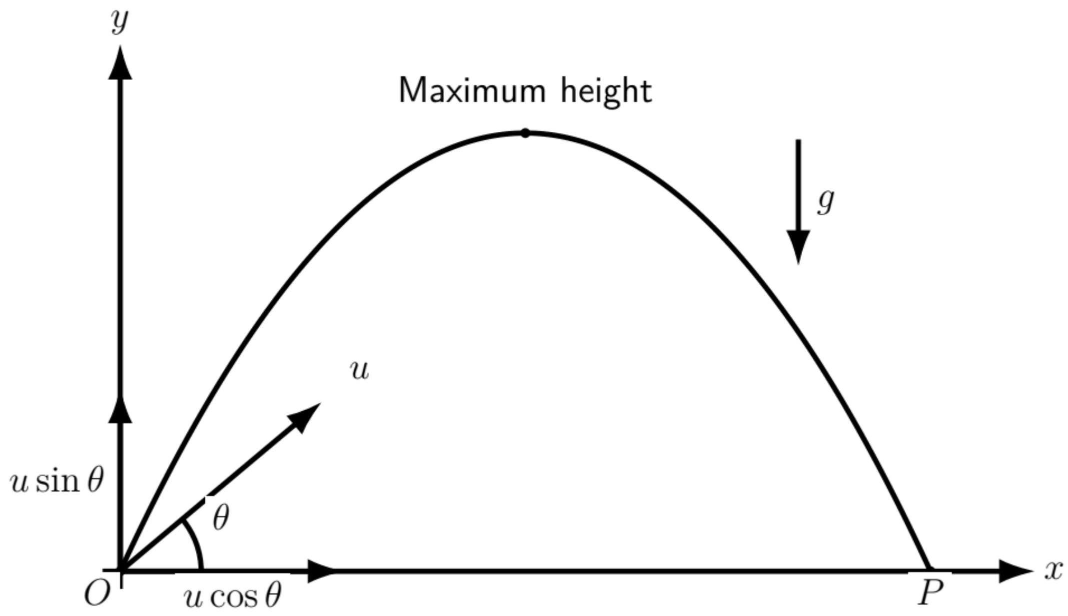

Figure: *Projectile launched from the origin O with initial velocity, u at an angle θ above the horizontal. The trajectory is parabolic. The initial velocity is resolved into horizontal component ucosθ and vertical component usinθ. The highest point represents the maximum height, and the projectile lands at point P on the x-axis while the acceleration due to gravity g acts vertically downward.*

Resolving initial velocity into components

When a projectile is launched from point O with initial velocity u at an angle θ above the horizontal, this velocity vector can be resolved into two perpendicular components using basic trigonometry. The horizontal component, u_x, represents the portion of the initial velocity directed along the x-axis, while the vertical component, u_y, represents the portion directed along the y-axis. From the geometry of the velocity vector triangle, these components are given by:

$$\mathbf{u_x = u\cos\theta}$$

$$\mathbf{u_y = u\sin\theta}$$

The initial velocity u forms the hypotenuse of a right triangle, with u_x and u_y as the two perpendicular sides. The horizontal component uses the cosine function because it lies adjacent to the angle θ, while the vertical component uses the sine function because it lies opposite to the angle. These trigonometric relationships ensure that the vector sum of the components equals the original velocity vector in both magnitude and direction.

Velocity as a function of time

Since the horizontal acceleration is zero, the horizontal component of velocity remains constant throughout the motion. Thus, at **any time t** after launch, the horizontal velocity is simply:

$$\mathbf{v_x = u\cos\theta = u_x}$$

This constancy reflects the fact that, in the absence of horizontal forces, there is nothing to change the projectile's horizontal velocity. *The projectile continues to move forward at the same horizontal rate from launch until landing.*

The vertical component of velocity, however, changes continuously under the influence of gravity. Applying the first equation of motion ($v = u + at$) to the vertical direction, with initial velocity $u_y = u\sin\theta$ and acceleration $a_y = -g$, we obtain:

$$\mathbf{v_y = u\sin\theta - gt}$$

The negative sign before g reflects gravity's downward action. Initially, when $t = 0$, the vertical velocity equals $u\sin\theta$, directed upward. As time progresses, gravity steadily reduces this upward velocity. At some point, v_y becomes zero; this marks the instant when the projectile reaches its maximum height and momentarily stops rising. Beyond this point, v_y becomes negative, indicating downward motion as the projectile begins to fall.

Displacement as a function of time

The **horizontal displacement** ($\mathbf{s_x}$) at any time t follows from the second equation of motion, which is: $s = ut + ½at^2$. With zero horizontal acceleration, the $½at^2$ term vanishes, leaving:

$$\mathbf{s_x = (u\cos\theta)t}$$

This linear relationship between horizontal displacement and time shows that the projectile covers equal horizontal distances in equal time intervals, which is a characteristic of uniform motion. Thus, the horizontal distance increases steadily for as long as the projectile remains in the air.

For **vertical displacement**, we again apply $s = ut + ½at^2$ to the vertical direction, substituting $u_y = u\sin\theta$ and $a_y = -g$:

$$\mathbf{s_y = (u\sin\theta)t - ½gt^2}$$

This equation contains two competing terms. *The first term, (usinθ)t, represents how high the projectile would rise if gravity did not act* (it increases linearly with time). *The second term, $-½gt^2$, represents how far gravity pulls the projectile downward* (it increases quadratically with time). The net vertical displacement at any instant results from these two opposing effects. Initially, the linear term dominates and the projectile rises. Eventually, the quadratic term overtakes the linear term, and the projectile begins to descend.

The trajectory equation

The equations for horizontal and vertical displacement both contain time as a parameter. By eliminating time between these two equations, we can obtain a direct relationship between s_x and s_y that describes the shape of the trajectory itself.

From $s_x = (u\cos\theta)t$, we can express time as:

$$t = \frac{s_x}{u\cos\theta}$$

Substituting this expression for t into the equation for vertical displacement:

$$s_y = (u\sin\theta)\left(\frac{s_x}{u\cos\theta}\right) - ½g\left(\frac{s_x}{u\cos\theta}\right)^2$$

Simplifying the first term:

$$\mathbf{s_y = s_x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)s_x^2}$$

This is the trajectory equation.

We may take s_y as *y*, and s_x as *x*; then the trajectory equation becomes:

$$\boldsymbol{y = x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2}$$

With this form, we can clearly see that the equation is in the standard form of a parabola equation:

$$\boldsymbol{y = bx + ax^2}$$

Where:

b (coefficient of s_x) = $\tan\theta$

a (coefficient of s_x^2) = $-\left(\frac{g}{2u^2\cos^2\theta}\right)$ (negative)

So the trajectory equation has the mathematical form of a parabola that opens downward.

Also it is important to note that both coefficients are constants for a given projectile launched with particular initial conditions. This reveals that *regardless of the initial velocity or launch angle, the path of a projectile is always parabolic* when air resistance is negligible.

Magnitude and direction of velocity at any instant

Although we analyse the horizontal and vertical components of velocity separately, the actual velocity of the projectile at any instant is a single vector pointing along the instantaneous direction of motion. The magnitude of this velocity vector can be found by combining its perpendicular components using the Pythagorean theorem:

$$\mathbf{v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta\ g - t)^2}}$$

The direction of the velocity vector at time t, measured as an angle α **from the horizontal**, is given by:

$$\tan\alpha = \frac{v_y}{v_x}$$

$$\boldsymbol{\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)}$$

As the projectile moves along its trajectory, both the magnitude and direction of its velocity change continuously. The horizontal component remains constant, but the changing vertical component causes both the velocity and direction to vary from moment to moment.

The Principle of Independence

Students often find it puzzling that we can treat horizontal and vertical motions as if they were completely separate, when clearly the projectile moves as a single object through space. This separation is justified by a fundamental principle of mechanics: ***perpendicular components of motion are independent.*** *What happens in one direction has no influence on what happens in a perpendicular direction, provided no forces act to couple the two directions.*

To understand this more clearly, imagine an observer on a train moving at constant velocity who drops a ball. Relative to the observer, the ball falls straight downward, taking time t to reach the floor. The fact that the train is moving horizontally does not alter the time required for the ball to fall (gravity acts only vertically and is unrelated to the train's horizontal motion). An observer standing beside the track sees the ball follow a parabolic path, but even from this external perspective, the time for the ball to fall remains t. Thus, the horizontal motion is superimposed on the vertical motion without affecting it.

This independence extends to all projectile motion. The time a football remains in the air depends only on its initial vertical velocity and the action of gravity; it does not matter whether the ball was kicked gently or powerfully in the horizontal direction. Similarly, the horizontal distance traveled depends on how long the projectile stays in the air (determined by vertical motion) and how fast it moves horizontally (determined by the horizontal component of initial velocity). The two components of motion proceed simultaneously and independently, yet together they produce the observed parabolic trajectory.

Addressing Common Misconceptions

Several persistent misconceptions about projectile motion deserve explicit attention. Let us address them directly:

Misconception 1: "Heavier objects follow different trajectories than lighter objects."

The truth: *Mass does not appear anywhere in our projectile equations!* A heavy stone and a light pebble, if thrown with the same initial velocity and angle, will follow identical paths. Examination of all the equations we have derived reveals no dependence on mass whatsoever.

Why students believe this: In everyday experience, air resistance affects lighter objects more severely than heavier ones, making them appear to follow different paths. But the fundamental projectile motion equations (which assume negligible air resistance) are completely independent of mass.

Misconception 2: "At the highest point, the velocity is zero."

The truth: At the maximum height, **only** the vertical component of velocity (v_y) becomes zero. The horizontal component remains unchanged at $v_x = u\cos\theta$. *The projectile never stops moving completely, even at the peak of its trajectory, it continues to travel forward.*

Why this matters: If the velocity were truly zero at the highest point, the projectile would simply drop straight downward from there. Instead, it continues its parabolic arc because horizontal motion persists throughout the flight.

Misconception 3: "Gravity acts differently during ascent than during descent."

The truth: Throughout the entire motion; rising, at the peak, and falling, *the acceleration due to gravity remains constant and always acts downward ($a = -g = -9.8m/s^2$ always),* exactly as we learned in Chapter 2. This never changes.

The apparent difference: On the way up, this downward acceleration opposes and reduces the upward velocity. On the way down, the same downward acceleration increases the downward velocity. The acceleration itself remains identical; only its effect on the velocity appears different depending on the direction of motion.

Misconception 4: "Throwing harder changes the shape of the trajectory."

The truth: *The trajectory is always parabolic, regardless of how hard you throw*. Throwing harder increases the horizontal distance and maximum height, creating a larger parabola, but the shape remains fundamentally the same (it is still a parabola). Throwing gently produces a smaller parabola, but still a parabola.

Relating to Kipanga's two attempts: In Kipanga's throwing attempts: one nearly horizontal, another nearly vertical, both failed. In the first case, he created a very flat parabola that hit the ground almost immediately. In the second, he created a very tall, narrow parabola that barely traveled forward. Kipute's 45-degree throw created a parabola with optimal proportions for maximum horizontal distance.

Application to Kipanga's shot put disaster

Returning to the opening scenario, we can now analyse what went wrong with Kipanga's two attempts and why Kipute succeeded.

Kipanga's first throw (nearly horizontal, say $\theta \approx 5°$):

If he threw with velocity $u = 10m/s$ at angle $\theta = 5°$:

- $u_x = 10\cos5° \approx 9.96m/s$ (almost all horizontal)
- $u_y = 10\sin5° \approx 0.87m/s$ (very little vertical)

*The small vertical component meant the shot barely rose above his release height before gravity pulled it down. With such **minimal time in the air**, even the large horizontal component could not carry it far.*

Kipanga's second throw (nearly vertical, say $\theta \approx 86°$):

If he threw with the same velocity at $\theta = 86°$:

- $u_x = 10\cos 86° \approx 0.70m/s$ (barely any horizontal)
- $u_y = 10\sin 86° \approx 9.98m/s$ (almost all vertical)

The large vertical component sent the shot high into the air, giving it plenty of time before landing. But with such minimal horizontal velocity, it barely moved forward during all that airtime!

Kipute's optimal throw (θ = 45°):

With $u = 10 m/s$ at $\theta = 45°$:

- $u_x = 10\cos 45° \approx 7.07 m/s$ (balanced horizontal)
- $u_y = 10\sin 45° \approx 7.07 m/s$ (balanced vertical)

This *balance between horizontal and vertical components gives maximum horizontal distance*. It allowed the shot to stay in the air long enough while also traveling forward effectively. The result: 8 metres compared to Kipanga's embarrassing 3 metres and 0.5 metres!

The mathematics of projectile motion transforms shot put from pure strength into applied science. Understanding these principles applies equally to javelin throws, long jumps, basketball shots, and yes, even Mr. Akilikubwa's favorite example, the perfect arc of water from a garden hose!

With these fundamental concepts now simmering nicely, let us serve them properly through a few worked examples and enjoy the flavour of physics in action!

BINDER Example 1

A stone is thrown with an initial velocity of $20 m/s$ at an angle of $30°$ above the horizontal. Take $g = 9.8\ m/s^2$. Calculate:

(a) the horizontal and vertical components of the initial velocity,
(b) the horizontal and vertical components of velocity after 1.5s.

Solution

(a) Horizontal component:

$$u_x = u\cos\theta = 20 m/s \times \cos 30° = \mathbf{17.3 m/s}$$

Vertical component:

$$u_y = u\sin\theta = 20 m/s \times \sin 30° = \mathbf{10 m/s}$$

(b) Horizontal velocity (remains constant):

$$v_x = u_x = \mathbf{17.3 m/s}$$

Vertical velocity (changes due to gravity):

$$v_y = u_y - gt = 10 m/s - (9.8 m/s^2)(1.5 s) = -4.7\ m/s$$

The negative sign indicates the stone is now moving downward.

So the vertical velocity is $-4.7\ m/s$ (or $4.7\ m/s$ downward)

Making Sense of the Answer: *After 1.5 seconds, the horizontal velocity remains unchanged at 17.3m/s because there's no horizontal force. The vertical velocity changed from +10m/s (upward) to −4.7 m/s (downward), meaning the stone has passed its maximum height and is now falling. The stone initially rose with vertical velocity 10m/s, slowed to zero at the peak, then began falling, reaching 4.7m/s downward after total time of 1.5s.*

Think Like a Physicist: *Notice that we could immediately tell the stone had passed its highest point because v_y became negative. At the exact moment of maximum height, $v_y = 0$. We can calculate when this occurred: $0 = 10 - 9.8t$, giving $t = 1.02s$. So the stone reached maximum height at $t = 1.02s$, and by $t = 1.5s$ it was already descending. This demonstrates how the vertical component alone determines when the projectile reaches its peak.*

BINDER Example 2

A ball is kicked horizontally from the top of a cliff with initial velocity $15 m/s$. Calculate the position and velocity of the ball after 2 seconds. Take $g = 9.8\ m/s^2$.

Solution

In this case, $u_x = 15 m/s, u_y = 0$ (horizontal launch), Horizontal displacement:

$$s_x = u_x t = 15\text{m}/\times 2\text{s} = 30\text{m}$$

Vertical displacement:

$$s_y = u_y t - \tfrac{1}{2}gt^2 = 0 - \tfrac{1}{2}(9.8\text{m/s}^2)(2\text{s})^2 = -19.6\text{m}$$

The negative sign indicates displacement downward from the launch point.

Hence**, the position of the ball is 30m horizontally from the cliff and 19.6m below the launch point.**

Horizontal velocity:

$$v_x = u_x = 15\text{m/s}$$

Vertical velocity:

$$v_y = u_y - gt = 0 - (9.8\text{m/s}^2)(2\text{s}) = -19.6\text{m/s}$$

Magnitude of velocity:

$$v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(15\text{m/s})^2 + (-19.6\text{m/s})^2} = 24.7\text{m/s}$$

Direction (below horizontal since the motion is downward):

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-19.6\text{m/s}}{15\text{m/s}}\right) = -52.6°$$

The negative sign means the angle was measured clockwise from the horizontal, and hence the direction of motion is below the horizontal.

Hence, **the velocity of the ball is 24.7m/s directed at $52.6°$ below horizontal.**

Making Sense of the Answer: *Even though the ball was kicked horizontally (no initial vertical velocity), after 2 seconds it has fallen 19.6m and gained significant downward velocity (19.6m/s). The horizontal velocity remained constant at 15m/s throughout. The final velocity (24.7m/s) is greater than the initial velocity (15m/s) because gravity has added a vertical component. The steep angle (52.6° below horizontal) shows the ball is now falling quite rapidly.*

Think Like a Physicist: *This example illustrates a special case of projectile motion: horizontal launch (θ = 0°). Even though there was no initial upward motion, the ball still follows a parabolic path as it falls. This is exactly what happens when you roll a ball off a table; it continues forward at constant horizontal velocity while simultaneously falling under gravity. Both motions combine to create the characteristic parabolic trajectory, even without an initial upward component.*

REAL Example 3

At Kariakoo market, vendors selling oranges often demonstrate the quality of their fruit by tossing an orange to potential customers standing several metres away. An experienced vendor can consistently land the orange gently in a customer's hands, while inexperienced vendors either throw too hard (orange sails over the customer) or too soft (orange falls short).

(a) Explain why the vendor must adjust both the velocity and angle of throw depending on how far away the customer stands.
(b) Explain what happens if the vendor uses the correct velocity but wrong angle.

Solution

(a) The horizontal distance travelled by the orange (the range) depends on both the initial velocity and the angle of projection. For a given distance to the customer, there are actually multiple combinations of velocity and angle that could work: a fast throw at a low angle, or a slower throw at a higher angle, can both reach the same spot. However, experienced vendors prefer a moderate velocity with optimal angle because:
- **High velocity at low angle:** The orange arrives with high horizontal velocity, making it difficult to catch and potentially bruising the fruit on impact
- **Low velocity at high angle:** The orange spends too long in the air, during which wind could deflect it.

The vendor instinctively calculates (through experience) the combination of velocity and angle that makes the orange arrive gently with low velocity, making it easy to catch.

(b) If the vendor uses correct velocity but wrong angle, two problems arise:

Case 1 - Angle too large (too steep):

Too much initial velocity goes into the vertical component, less into horizontal. The orange rises high, spending long time in the air, but does not travel far enough horizontally. It falls short of the customer, landing on the ground in front of them. (This is similar to Kipanga's second shot put attempt.)

Case 2 - Angle too small (too flat):

Too much initial velocity goes into horizontal component, too little into vertical. The orange does not rise high enough or stay in the air long enough. Despite moving forward quickly, it drops to the ground before reaching the customer. (This is similar to Kipanga's first attempt.)

Making Sense of the Answer: *This example demonstrates that projectile motion is not just abstract physics; it is embedded in everyday activities. Market vendors, though they have never studied our equations, understand projectile motion intuitively through experience. They know that "how hard" (initial velocity) and "which direction" (angle) are both crucial and must work together. This intuitive understanding is precisely what our mathematical framework makes explicit and quantifiable.*

Think Like a Physicist: *Notice how real-world projectile motion involves optimization: not just reaching a target, but reaching it in a way that minimizes certain undesirable effects (impact force, time in air, trajectory height, etc.). In sports, warfare, firefighting, and countless other applications, understanding the mathematics allows us to optimize these factors systematically rather than relying purely on trial and error. The vendor's experience represents thousands of unconscious calculations of which our equations let us perform those calculations consciously and precisely.*

HOT Example 4

A projectile is launched with initial velocity 25m/s at an angle of $53°$ above the horizontal from ground level. Taking $g = 9.8\ \text{m/s}^2$ and $\sin 53° = 0.8, \cos 53° = 0.6$:

(a) Derive the equation of the trajectory for this projectile.
(b) Calculate the height of the projectile when it is at horizontal distance 15m from the launch point.
(c) Calculate the horizontal distance at which the projectile returns to ground level.

Solution

(a) The general trajectory equation is:

$$s_y = s_x \tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)s_x^2$$

Substituting given values:

$$\tan 53° = \frac{\sin 53°}{\cos 53°} = \frac{0.8}{0.6} = 1.333$$

$$\frac{g}{2u^2\cos^2\theta} = \frac{9.8}{(2(25)^2(0.6)^2)} = 0.0218$$

Therefore:

$$s_y = 1.333s_x - 0.0218s_x^2$$

Thus the trajectory equation for this particular projectile is:

$$s_y = 1.333s_x - 0.0218s_x^2$$

(b) At $s_x = 15\text{m}$:

$$s_y = 1.333(15) - 0.0218(15)^2 = 15.1\ \text{m}$$

The projectile is at height of $\mathbf{15.1m}$ when it is 15m horizontally from launch.

(c) The projectile returns to ground level when $s_y = 0$:

$$0 = 1.333s_x - 0.0218s_x^2$$

$$0 = s_x(1.333 - 0.0218s_x)$$

This gives two solutions: $s_x = 0$ (the launch point); or

$$1.333 - 0.0218s_x = 0;\ s_x = \frac{1.333}{0.0218} = 61.1\text{m}$$

Hence, the projectile lands at horizontal distance 61.1m.

Making Sense of the Answer: *The trajectory equation, $s_y = 1.333s_x - 0.0218s_x^2$ is indeed a parabola (it has the form $y = ax + bx^2$ where b is negative). At $x = 15m$, the height is $15.1m$ which is still quite high, indicating the projectile is probably near its peak. The horizontal distance of $61.1m$ means the projectile travels this far before returning to launch height. Notice that the coefficient $1.333\ (= tan\,53°)$ determines the initial slope of the trajectory, while 0.0218 determines how quickly gravity curves it downward.*

Think Like a Physicist: *The trajectory equation is powerful because it relates position to position directly without time as an intermediate variable. This is useful when we care about "where" rather than "when." For instance, if there's an obstacle at a certain location, we can immediately check if the projectile will clear it by substituting that horizontal position and checking if s_y exceeds the obstacle height. The fact that we found two solutions when $s_y = 0$ (namely $s_x = 0$ and $s_x = 61.1$) makes physical sense: the projectile is at ground level at both launch ($x = 0$) and landing ($x = 61.1m$). The parabola crosses the x-axis at exactly two points, as expected for a symmetric trajectory launched from and returning to the same level.*

Summary of Fundamental Equations

For convenient reference, we collect here the essential equations governing projectile motion, valid when air resistance is negligible and the acceleration due to gravity is constant:

Initial velocity components:

$$u_x = u\cos\theta,\ u_y = u\sin\theta$$

Velocity at time any time t:

$$v_x = u\cos\theta,\ v_y = u\sin\theta - gt$$

Displacement at any time t:

$$s_x = (u\cos\theta)t,\ s_y = (u\sin\theta)t - ½gt^2$$

Trajectory equation:

$$s_y = s_x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)s_x^2$$

Magnitude of velocity at any instant:

$$v = \sqrt{(v_x)^2 + (v_y)^2}$$

Direction of velocity at any instant:

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

These equations form the mathematical foundation for analysing any projectile motion problem. In the following subtopic, we shall use them to derive expressions for important physical quantities such as time of flight, maximum height, and horizontal range.

KEY PARAMETERS OF THE TRAJECTORY

Having developed the fundamental equations governing projectile motion in the previous subtopic, we now turn our attention to deriving expressions for specific quantities that characterize the trajectory. When Kipanga asks "How far will my shot put travel?" or Kipute wonders "How high will the orange I toss reach at its peak?", they are asking about what we call the parameters of projectile motion. These parameters: time of flight, maximum height, and horizontal range, provide complete practical information about any projectile's trajectory. In this subtopic, we shall derive mathematical expressions for each parameter and discover some surprising relationships between them.

Time of flight, T

The **time of flight** *is the total time interval from the moment a projectile is launched until the moment it returns to the same vertical level from which it was projected.* This parameter is crucial because it determines how long the projectile remains in the air, which in turn affects how far it can travel horizontally.

Consider a projectile launched from ground level with initial velocity u at angle θ above the horizontal. We established in the previous subtopic that the vertical displacement at time t is given by:

$$s_y = (u\sin\theta)t - \tfrac{1}{2}gt^2$$

The projectile returns to ground level when $s_y = 0$. This occurs at two instants: $t = 0$ (the moment of launch) and at some later time T (the time of flight). Setting $s_y = 0$:

$$0 = (u\sin\theta)t - \tfrac{1}{2}gt^2$$

Factorising the expression:

$$0 = t(u\sin\theta - \tfrac{1}{2}gt)$$

This equation has two solutions:

$$t = 0 \text{ and } u\sin\theta - \tfrac{1}{2}gt = 0.$$

For the second solution, $t = T$, thus:

$$u\sin\theta = \tfrac{1}{2}gT$$

Solving for T:

$$\mathbf{T = \frac{2u\sin\theta}{g}}$$

This is the time of flight for a projectile launched and landing at the same level.

Notice the physical insight embedded in this formula: time of flight depends only on the vertical component of initial velocity (usinθ) and gravity. The horizontal component plays no role whatsoever in determining how long the projectile stays in the air. This makes perfect sense from our principle of independence: *the time spent in the air is determined entirely by vertical motion, exactly as if we had thrown the object straight upward with initial velocity usinθ, as we studied in* ***Chapter 2****.*

Connecting to Chapter 2: *Do you remember when we derived the time for an object thrown vertically upward to return to its starting point?* We found $t = \frac{2u}{g}$, where u was the upward initial velocity. Here, the vertical component usinθ plays exactly that role, giving us $T = \frac{2(u\sin\theta)}{g}$ The physics is identical; we have simply replaced u with usinθ to account for the angled launch.

Alternative derivation:

We can verify this result using another approach. The projectile reaches maximum height when $v_y = 0$. From the previous subtopic, we know that $v_y = u\sin\theta - gt$. Setting this to zero:

$$0 = u\sin\theta - gt_{max}$$

$$t_{max} = \frac{u\sin\theta}{g}$$

This is the time to reach maximum height (the upward journey). By symmetry, the downward journey takes equal time. Therefore, total time of flight is:

$$T = 2t_{max} = \frac{2(u\sin\theta)}{g}$$

This confirms our result and demonstrates the symmetric nature of projectile motion when launch and landing occur at the same level.

Maximum height, H

The **maximum height** *is the greatest vertical displacement achieved by the projectile above its launch point.* This occurs at the instant when the vertical component of velocity becomes zero, when the projectile stops rising and is about to begin falling.

We can derive the maximum height using the third equation of motion from Chapter 2:

$$v^2 = u^2 + 2as.$$

Applying this to vertical motion:

$$v_y^2 = u_y^2 + 2a_y s_y$$

At maximum height:

$$v_y = 0, u_y = u\sin\theta, a_y = -g, \text{and } s_y = H.$$

Substituting:

$$0 = (u\sin\theta)^2 - 2gH$$

Solving for H:

$$\mathbf{H = \frac{u^2\sin^2\theta}{2g}}$$

This elegant formula reveals that maximum height depends on the square of the vertical component of initial velocity. Doubling the vertical component quadruples the maximum height.

Alternative derivation:

We can also derive this using the time to maximum height and the displacement equation.

We found $t_{max} = \frac{u\sin\theta}{g}$.

Substituting into $s_y = (u\sin\theta)t - ½gt^2$:

$$H = u\sin\theta\left(\frac{u\sin\theta}{g}\right) - ½g\left(\frac{u\sin\theta}{g}\right)^2$$

$$H = \frac{u^2\sin^2\theta}{g} - ½\frac{g(u^2\sin^2\theta)}{g^2}$$

$$H = \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g}$$

$$\mathbf{H = \frac{u^2\sin^2\theta}{2g}}$$

Both methods yield the same result, providing mathematical confirmation of our formula.

An important observation: If we launch the projectile vertically upward ($\theta = 90°$), then $\sin^2 90° = 1$ (*at this angle, the maximum height becomes greatest for a given initial velocity*), and we get $H_{max} = \frac{u^2}{2g}$, which is exactly the formula we derived in Chapter 2 for maximum height of vertical projection. This serves as another verification that our projectile motion analysis correctly reduces to simpler cases.

Horizontal range, R

The **horizontal range** *is the total horizontal distance travelled by the projectile from launch point to landing point (when both are at the same level).* This is perhaps the most practically important parameter, as it determines how far the projectile will travel.

From the previous subtopic, horizontal displacement is given by:

$$s_x = (u\cos\theta)t$$

At landing ($t = T$), the horizontal displacement is equal to the range R:

$$R = (u\cos\theta)T$$

We already derived $T = \frac{2u\sin\theta}{g}$.

Substituting:

$$R = u\cos\theta\left(\frac{2u\sin\theta}{g}\right)$$

From which:

$$R = \frac{2u^2\cos\theta\sin\theta}{g}$$

This can be simplified by using the trigonometric identity: $2\sin\theta\cos\theta = \sin2\theta$:

$$\mathbf{R = \frac{u^2\sin2\theta}{g}}$$

This remarkably simple formula captures everything about horizontal range. Notice that range depends on the initial velocity squared, inversely on gravity, and on the sine of twice the angle of projection.

The factor $\sin2\theta$ deserves special attention. Since the maximum value of sine function is 1 (occurring when the angle is 90°), the range is maximized when:

$$2\theta = 90°$$

Which gives:

$$\theta = 45°$$

And the range formula becomes:

$$R_{max} = \frac{u^2}{g}$$

This is the famous result: *for a given initial velocity, maximum range occurs at 45° launch angle*. This explains why Kipute's 45° throw in our introduction travelled farther than both of Kipanga's attempts; she intuitively chose the optimal angle!

Physical interpretation: *Why does 45° give maximum range?* At angles below 45°, the projectile has greater horizontal velocity but insufficient vertical velocity, so its time of flight is small. At angles above 45°, the projectile has larger time of flight but moves too slow horizontally. The 45° angle perfectly balances these competing requirements: enough vertical component to provide adequate flight time, and enough horizontal velocity to cover large horizontal distance during that time.

Moreover, the factor $\sin2\theta$ has an important property. The sine function satisfies the identity:

$$\sin x = \sin(180 - x)$$

If we let $x = 2\theta$. Then:

$$\sin 2\theta = \sin(180 - 2\theta)$$

Since the horizontal range depends on $\sin2\theta$, the same range will be obtained when:

$$2\theta' = 180 - 2\theta$$

Solving for θ':

$$\boldsymbol{\theta' = 90 - \theta}$$

Hence, *a projectile launched at angle* $\boldsymbol{\theta}$ *will have the same horizontal range as one launched at an angle* $(\boldsymbol{90 - \theta})$**.** In other words: *two projectiles launched at complementary angles (angles that sum to 90°), have the same horizontal range*. For example, launches at 30° and 60° produce identical horizontal range, and the same is true for 20° and 70°.

Relationship between parameters

The three parameters we have derived: time of flight, maximum height, and range are not independent. They are intimately connected through the initial conditions u (velocity of projection) and θ (angle of projection). Let us explore some of these relationships.

From our formulas:

- $T = \frac{2u \sin \theta}{g}$
- $H = \frac{u^2 \sin^2\theta}{2g}$
- $R = \frac{u^2 \sin 2\theta}{g}$

We can express **maximum height in terms of time of flight:**

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{(u\sin\theta)^2}{2g}$$

But from $T = \frac{2u\sin \theta}{g}$:

$$u\sin\theta = \frac{gT}{2}$$

So:

$$H = \frac{\left(\frac{gT}{2}\right)^2}{2g} = \frac{g^2T^2}{4 \times 2g} = \frac{gT^2}{8}$$

Therefore:

$$\mathbf{H = \frac{gT^2}{8}}$$

This shows that for any projectile, maximum height is proportional to the square of time of flight. Thus, a projectile that stays in the air four times as long reaches sixteen times the height!

We can also relate **range to maximum height.**

Starting from:

$$R = \frac{u^2\sin2\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g}$$

And:

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin\theta\sin\theta}{2g}$$

Taking $\frac{R}{H}$:

$$\frac{R}{H} = \frac{2u^2\sin\theta\cos\theta}{g} \times \frac{2g}{u^2\sin\theta\sin\theta} = \frac{4\cos\theta}{\sin\theta} = 4\cot\theta$$

Hence:

$$\mathbf{R = 4H\cot\theta}$$

This relationship tells us that for a given maximum height, the range depends on the launch angle. Shallow angles (small θ) give large cotθ and thus large range for the same height. Steep angles give smaller range for the same height. It also shows that for a given angle of projection, if the maximum height increases, the range increases proportionally.

For the special case of 45° launch:

$$\cot 45^\circ = 1$$

So:

$$\mathbf{R = 4H}$$

Thus, *at 45° launch angle, the range is exactly four times the maximum height*. This is a useful rule of thumb: ***if a projectile launched at 45° reaches height H, it will land at distance 4H from its starting point.***

After all these derivations, the mathematics has revealed the structure of projectile motion. But equations alone can feel abstract until we see them at work. Before they start arguing among themselves, let us invite a few worked examples to settle the matter.

BINDER Example 5

A cricket ball is struck at 30m/s at an angle of 60° above horizontal. Calculate:

(a) the time of flight, (b) the maximum height reached, (c) the horizontal range. Take g = 9.8 m/s².

Solution

(a) Time of flight is given by:

$$T = \frac{2u\sin\theta}{g}$$

Substituting values:

$$T = \frac{2 \times 30\text{m/s} \times \sin 60^\circ}{9.8\text{m/s}^2} = 5.3\text{s}$$

The time of flight is 5.3s.

(b) Maximum height is given by:

$$H = \frac{u^2\sin^2\theta}{2g}$$

Substituting values:

$$H = \frac{(30\text{m/s})^2 \times \sin^2 60^\circ}{2 \times 9.8\text{m/s}^2} = 34.44\text{m}$$

The maximum height is 34.44m.

(c) Horizontal range is given by:

$$R = \frac{u^2\sin 2\theta}{g}$$

Substituting values:

$$R = \frac{(30\text{m/s})^2 \times \sin 120^\circ}{9.8\text{m/s}^2} = 79.55\text{m}$$

The horizontal range is 79.55m.

Making Sense of the Answer: *The ball stays in the air for over 5 seconds, reaches a height of 34 metres, and travels nearly 80 metres horizontally. These are realistic values for a powerfully struck cricket ball. Notice that 60° is quite steep, so it gives impressive height but not maximum range. If the same ball were struck at 45° instead, it would travel farther ($R_{max} = \frac{900}{9.8} = 91.8m$) but not as high ($H = \frac{900}{39.2} = 23m$).*

Think Like a Physicist: *Observe the complementary angle relationship: if we launched at 30° (the complement of 60°), we would get the same range (79.5m) but different flight time and height. We can verify: sin 2(30°) = sin 60° = 0.866, same as sin 2(60°) = sin 120° = 0.866. This symmetry is not coincidental; it is built into the trigonometry of projectile motion. In cricket, fielders must position themselves based on the likely range, which depends on both the batsman's power (determining u) and shot selection (determining θ).*

BINDER Example 6

A stone is thrown from ground level and reaches a maximum height of 20m. If the stone was thrown at 45°, calculate:

(a) the initial velocity of the stone, (b) the range of the stone, (c) the time of flight.

Take $g = 9.8\ m/s^2$.

Solution

(a) From the maximum height formula:

$$H = \frac{u^2 \sin^2\theta}{2g}$$

Making u the subject:

$$u = \sqrt{\frac{2gH}{\sin^2\theta}}$$

Substituting values:

$$u = \sqrt{\frac{2 \times 9.8\ m/s^2 \times 20m}{\sin^2 45}} = 28m/s$$

The initial velocity is $28m/s$.

(b) If the angle of projection is 45°:

$$R = 4H = 4 \times 20m = 80\ m$$

The range is 80m.

(c) Using:

$$T = \frac{2u\sin\theta}{g} = \frac{2 \times 28m/s \times \sin 45°}{9.8\ m/s^2} = 4.04s$$

The time of flight is 4.04s.

Making Sense of the Answer: *The special angle of 45° creates the elegant relationship R = 4H, making part (b) trivial once we know the height. The initial velocity of 28m/s is enough fast to reach 20 metres height. The 4-second flight time is substantial, giving plenty of time for the stone to travel the 80-metre range. This example demonstrates how knowing one parameter (height) and the launch angle allows us to determine all other parameters.*

Think Like a Physicist: *This example reverses the usual problem: instead of starting with initial conditions (u and θ) and finding the trajectory parameters, we start with a parameter (H) and work backward to find initial velocity. This "inverse problem" approach is common in real applications. For instance, if you observe a projectile reaching a certain height and range, you can deduce its initial velocity and launch angle. This is how military analysts estimate the capabilities of observed missile launches; they measure the trajectory and work backward to the launch conditions.*

BINDER Example 7

A javelin thrower achieves a throw of 60m when the javelin leaves her hand at 25m/s. Calculate:

(a) the two possible angles at which the javelin could have been launched,
(b) explain which angle is more realistic for javelin throwing. Take $g = 9.8\ m/s^2$.

Solution

(a) Javelin is thrown for maximum horizontal distance, so we are going to apply the range formula:

$$R = \frac{u^2 \sin 2\theta}{g}$$

Substituting values:

$$60\text{m} = \frac{(25\text{m/s})^2 \times \sin 2\theta}{9.8\text{m/s}^2}$$

From which:

$$\sin 2\theta = 0.941 \text{ or } 2\theta = \sin^{-1}(0.941)$$

Thus:

$$2\theta = 70.2° \text{ or } 2\theta = 180° - 70.2° = 109.8°$$

And:

$$\theta = \frac{70.2°}{2} = 35.1° \text{ or } \frac{109.8°}{2} = 54.9°$$

two possible angles are 35.1° and 54.9°.

(b) For javelin throwing, the smaller angle (35.1°) is more realistic.

Explanation:

In real situations, air resistance reduces the horizontal range, and this effect becomes more significant for steeper launch angles because the javelin spends more time in the air. Consequently, the optimum angle for maximum range becomes less than 45° (typically around 35°–40°).

Making Sense of the Answer: *The two angles (35.1° and 54.9°) are complementary (they sum to 90°), which is exactly what we expect from the theory: complementary angles give the same range. Both are physically possible, but aerodynamics favour the shallower trajectory. Notice that neither angle is exactly 45° because the range (60m) is less than the maximum possible range:* $R_{max} = \frac{u^2}{g} = \frac{625}{9.8} = 63.8m$. *The maximum range would require 45°.*

Think Like a Physicist: *This inverse problem requires finding angle from range has two solutions because of the symmetry in the sine function:* $\sin 2\theta_1 + \sin 2\theta_2$ *when* $2\theta_1 + 2\theta_2 = 180°$. *In sports, military applications, and firefighting, this symmetry creates interesting choices. In practice, other factors (obstacles, wind, equipment limitations) often make one choice clearly superior.*

BINDER Example 8

Two identical stones are thrown from ground level with the same initial velocity of 20m/s, but at different angles: stone A at 30° and stone B at 60°. Taking g = 9.8 m/s². Calculate:

(a) the horizontal range for each stone and verify they are equal,
(b) the maximum height reached by each stone,
(c) the time of flight for each stone.

Solution

(a) Range for stone A:

$$R_A = \frac{u^2 \sin 2\theta_A}{g} = \frac{(20\text{m/s})^2 \times \sin 60°}{9.8\text{m/s}^2} = 35.3\text{m}$$

Range for stone B:

$$R_B = \frac{u^2 \sin 2\theta_B}{g} = \frac{(20\text{m/s})^2 \times \sin 120°}{9.8\text{m/s}^2} = 35.3\text{m}$$

The horizontal range for each stone is 35.3m.

(b) Maximum height for stone A:

$$H_A = \frac{u^2\sin^2\theta_A}{2g} = \frac{(20\text{m/s})^2 \times \sin^2 30°}{2 \times 9.8\text{m/s}^2} = 5.1\text{m}$$

Maximum height for stone B:

$$H_B = \frac{u^2\sin^2\theta_B}{2g} = \frac{(20\text{m/s})^2 \times \sin^2 60°}{2 \times 9.8\text{m/s}^2} = 15.3\text{m}$$

The maximum height of stone A is 5.1m.

The maximum height of stone B is 15.3m. (Stone B goes 3 times higher!)

(c) Time of flight for stone A:

$$T_A = \frac{2u\sin\theta_A}{g} = \frac{2 \times 20\text{m/s} \times \sin 30°}{9.8\text{m/s}^2} = 2.04\text{s}$$

Time of flight for stone B:

$$T_B = \frac{2u\sin\theta_B}{g} = \frac{2 \times 20\text{m/s} \times \sin 60°}{9.8} = 3.54\text{s}$$

The time of flight for stone A is 2.04s.

The time of flight for stone B is 3.54s.

Making Sense of the Answer: *Although both stones travel the same horizontal distance, stone B accomplishes this by staying in the air longer while moving more slowly horizontally, while stone A covers the same distance by moving faster horizontally in less time. So the complementary angle relationship creates fascinating trade-offs: same range, but completely different flight characteristics. Stone A (30°): low, fast trajectory which is good for minimizing flight time. Stone B (60°): high, slow trajectory which is good for clearing obstacles. If there were a 10-metre wall in the path, only stone B would clear it (15.3m > 10m), even though both eventually land at the same spot. This explains why artillery can hit the same target with two different firing solutions: one "flat" trajectory for quick strikes, one "high" trajectory for clearing obstacles.*

Think Like a Physicist: *This example reveals a profound principle: there are often multiple ways to achieve the same outcome in physics, each with different trade-offs. Both angles reach 35.3m, but the "how" differs dramatically. In engineering and warfare, these choices matter: flat trajectories are faster but cannot clear obstacles; high trajectories clear obstacles but take longer and are more affected by wind. Similarly, in life, different paths can lead to the same destination: some fast and risky, some slow and safe! Physics teaches us to quantify these trade-offs rather than just acknowledge them qualitatively.*

REAL Example 9

During the annual school sports day, the long jump competition takes place on a sand pit. After the competition ends, Kipanga sits dejectedly on the bench; he managed only 4.8 metres while his classmate Musa won with 6.5 metres.

Kipanga: (frustrated) *"It's not fair, sir! I ran down that track just as fast as Musa, maybe even faster! But he jumped way farther. He must have longer legs or something."*

Mr. Akilikubwa: (sitting down beside him) *"Kipanga, I watched both your jumps carefully. Your running speed looked similar to Musa's, that's true. But tell me, what happened at the moment you left the takeoff board?"*

Kipanga: *"I jumped as hard as I could, sir! I remember going quite high. I could see the whole sand pit below me for a moment. It felt like flying!"*

Kipute: (joining them) *"I noticed that too! Kipanga jumped very high. Musa's jump looked different; he barely went upward. His jump was much flatter."*

Mr. Akilikubwa: (smiling) *"Interesting observations. But neither of you used the best possible angle for your running speed. Long jump actually hides an important secret of* ***projectile motion****."*

Kipanga: (puzzled) *"Projectile motion? Like the shot put we studied?"*

Mr. Akilikubwa: *"Exactly. The moment you leave the board; your body behaves like a projectile moving through the air. Two things determine how far you travel horizontally: how long you stay in the air, and how fast you move forward while airborne."*

Kipute: *"But sir, those seem to contradict each other! To stay in the air longer, you need to jump upward more. But to move fast horizontally, you need to jump forward more. You can't do both!"*

Mr. Akilikubwa: (enthusiastically) *"Exactly, Kipute! That's the heart of it! Your running speed gives you a certain amount of initial velocity. At takeoff, you must decide how to share this velocity between going up and going forward. Jump too steep like Kipanga did, and yes, you'll be airborne for a long time, but you're crawling forward so slowly that you waste all that airtime. Jump too flat like Musa, and yes, you're rocketing forward, but you hit the sand so quickly that you don't have time to travel far."*

Kipanga: *"So... there's a perfect angle in the middle somewhere?"*

Mr. Akilikubwa: *"Now you're thinking like a physicist! In ideal projectile motion, when the launch point and landing point are at the same height and air resistance is neglected, the mathematics shows that the maximum horizontal range occurs when the launch angle is 45°. At that angle, the initial velocity is shared equally between the horizontal and vertical directions. The vertical component keeps the projectile in the air long enough, while the horizontal component carries it forward efficiently. This balance produces the greatest possible range"*

Kipute: (excited) *"It's like that time we learned about the see-saw! The balance point is in the middle!"*

Mr. Akilikubwa: *"Beautiful analogy, Kipute! And here's the magical part: if you could somehow jump at exactly 45 degrees, the distance you travel horizontally will always be exactly four times the maximum height you reach. Always! Whether you're a child jumping 2 metres, or an Olympic athlete jumping 8 metres; 45 degrees makes the range exactly four times the height."*

Kipanga: (thoughtful) *"But sir, I've watched Olympic long jumpers on TV. They don't seem to go that high..."*

Mr. Akilikubwa: *"Ah! Sharp observation, Kipanga. You're right! Do you know why?"*

Kipanga: *"Um... they're making a mistake?"*

Mr. Akilikubwa: (laughing) *" "Not at all! They are actually being clever. Real long-jump motion is more complicated than the simple projectile model. The athlete does not land at exactly the same height as the take-off point, and air resistance also affects the motion. Because of these factors, the best take-off angle in real long jumping is usually slightly less than 45°."*

Kipute: *"So Kipanga wasn't entirely wrong to jump upward, and Musa wasn't entirely wrong to jump flat; they were both just too extreme?"*

Mr. Akilikubwa: *"Precisely! And here's your homework: both of you, practice finding that middle ground. Kipanga, try to drive your leading knee forward more, not just up. Musa…. well, he needs the opposite advice: lift that knee higher at takeoff. You're both capable of jumping over 6 metres if you find your 45-degree sweet spot."*

Kipanga: (grinning) *"Next year, sir, I'm going to beat Musa! Now that I know the secret physics formula!"*

Mr. Akilikubwa: *"That's the spirit! But remember, knowing the physics is one thing, training your body to execute it is another. Even knowing the optimal angle, it takes years of practice to actually achieve it consistently. Physics tells you the target; dedication gets you there!"*

Based on this dialogue, explain:

(a) Explain clearly why both Kipanga's high jump (steep angle) and Musa's flat jump (shallow angle) failed to achieve maximum distance, even though they had similar running speeds.
(b) Explain why the optimal take-off angle of 45° gives the maximum range, and explain the meaning of Mr. Akilikubwa's statement that "range equals four times the height" for 45-degree launches.
(c) Explain why Olympic long jumpers actually use angles slightly below 45° rather than exactly 45°, as Mr. Akilikubwa explained.

Solution

(a) Both jumpers failed because they did not balance the vertical and horizontal components of their initial velocity optimally.

Kipanga (too steep, $\theta > 45°$): By jumping at a steep angle, too much of his initial velocity went into the vertical component ($u\sin\theta$) and too little into the horizontal component ($u\cos\theta$).

Result: He rose very high and stayed in the air for a long time (large $T = 2u\sin\theta/g$), but his horizontal velocity was so small that he barely traveled forward during all that airtime. High time of flight but low horizontal velocity leads to mediocre range.

Musa (too flat, $\theta < 45°$): By jumping at a shallow angle, too much velocity went into horizontal component and too little into vertical.

Result: He moved forward very quickly (large $u\cos\theta$), but he did not stay in the air for long enough (small T because small $u\sin\theta$) to take full advantage of that velocity. High horizontal velocity but low time of flight leads to mediocre range again.

(b) Explanation for maximum range at 45°:

Mathematically, $R = \frac{u^2\sin 2\theta}{g}$. Since the maximum value of sine function is 1, range is maximized when $\sin 2\theta = 1$, which means $2\theta = 90°$, giving $\theta = 45°$.

At this angle, the initial velocity is distributed equally between the horizontal and vertical directions, giving the best balance between: sufficient **time of flight**, and sufficient **horizontal velocity** and the range becomes maximum.

Explanation on the range-height relationship:

For 45° launch:

- Maximum height: $H = \frac{u^2\sin^2 45°}{2g} = \frac{u^2 \times 0.5}{2g} = \mathbf{\frac{u^2}{4g}}$
- Range: $R = \frac{u^2\sin 90°}{g} = \frac{u^2}{g}$ (since $\sin 2 \times 45° = \sin 90° = 1$)

Therefore: $\frac{R}{H} = \frac{\frac{u^2}{g}}{\frac{u^2}{4g}} = 4$

So: $R = 4H$

This means that for a projectile launched at **45°**, the horizontal range is always **four times the maximum height reached**. For example, if the projectile rises to a maximum height of **2m**, then its range is **8m**.

(c) Real long-jump athletes like Olympic long jumpers usually use take-off angles slightly less than 45° because actual long-jump motion is not exactly the same as ideal projectile motion.

In the ideal model, the projectile is launched and lands at the same height and air resistance is neglected. Under those conditions, 45° gives maximum range. In real long jump, however:

- the athlete's landing position is effectively **lower than the take-off position**, and
- **air resistance** also affects the motion.

When the landing point is lower than the launch point, the projectile can stay in the air long enough even with a slightly smaller launch angle. Therefore, the angle for practical maximum range becomes slightly less than 45°.

Making Sense of the Answer: *The dialogue beautifully captures the central insight of projectile optimization: extreme strategies fail, balanced approaches succeed. Kipanga and Musa represent the two failure modes: too much vertical emphasis versus too much horizontal emphasis. Mr. Akilikubwa guides them to the middle ground where physics achieves the optimal result. The conversation also reveals that real-world applications (Olympic long jump) must account for factors beyond the idealized physics (like landing height differences), showing how theory guides practice even when perfect application is not possible.*

Think Like a Physicist: *This example demonstrates how physics provides both explanation and optimization strategy. Without physics, an athlete might spend years trying random angles, hoping to improve. With physics, we know immediately that 45° (or slightly below for long jump) is the target, and training can focus on achieving that specific goal. The dialogue format makes the abstract mathematics tangible: students see Kipanga's frustration, understand his confusion, and follow his journey from "it's not* *fair" to "now I know the secret formula." This is how physics transforms from formulas on paper into actionable knowledge that changes outcomes.*

HOT Example 10

A projectile is fired from ground level with initial velocity u at angle θ. At the highest point of its trajectory, the projectile explodes into two equal fragments. One fragment falls straight down from the explosion point and lands directly below.

(a) Show that the second fragment lands at a distance $\frac{3R}{2}$ from the launch point, where R is the range the projectile would have achieved without exploding.

(b) If the projectile was fired at 40m/s at 45°, calculate:
- (i) the range the projectile would have travelled if no explosion occurred,
- (ii) the position where each fragment lands,
- (iii) the horizontal velocity of the second fragment immediately after explosion.

Take g = 9.8 m/s².

Solution

(a) Before explosion:

At the highest point:

- Horizontal position $= \frac{R}{2}$ (halfway through the trajectory)
- Vertical velocity $= 0$
- Horizontal velocity $= u\cos\theta$ (unchanged throughout flight)

Total horizontal momentum before explosion:

$p_{before} = m(u\cos\theta)$ (where m is the projectile mass)

After explosion:

Fragment 1 (mass $m/2$) falls straight down; thus, horizontal velocity $= 0$.

Fragment 2 (mass $m/2$) has horizontal velocity v_2.

Conservation of horizontal momentum:

$$m(u\cos\theta) = \left(\frac{m}{2}\right)(0) + \left(\frac{m}{2}\right)v_2$$

Solving for v_2:

$\mathbf{v_2 = 2u\cos\theta}$ *(Fragment 2 has twice the original horizontal velocity!)*

Landing position:

Fragment 2 travels from horizontal position $R/2$ with velocity $2u\cos\theta$ for time t (time to fall from maximum height H).

But the time to reach the maximum height is equal to the falling time. Thus:

$$t = t_{max} = \frac{u\sin\theta}{g}$$

Additional horizontal distance traveled by fragment 2:

$$\Delta x = \mathbf{v_2} \times t = (\mathbf{2u\cos\theta}) \times \left(\frac{u\sin\theta}{g}\right) = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{u^2\sin 2\theta}{g} = \text{original range R}$$

Therefore: $\mathbf{\Delta x = R}$

Fragment 2 starts at horizontal position, $\mathbf{R/2}$ and travels additional distance $\mathbf{R}$:

Landing position of fragment 2 $= \frac{R}{2} + R = \frac{\mathbf{3R}}{\mathbf{2}}$

Hence, the second fragment lands at a distance $\frac{\mathbf{3R}}{\mathbf{2}}$ from the launch point.

(b) The solution of each part is as follows:

(i) Using:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(40\text{m/s})^2 \times \sin 90°}{9.8\text{m/s}^2} = 163.3\text{m}$$

The range the projectile would be 163.3m.

(ii) Fragment 1 falls vertically from height H at horizontal position $R/2$. So it lands at:

$$\frac{R}{2} = \frac{163.3\text{m}}{2} = 81.7\text{m}$$

Fragment 2 lands at:

$$\frac{3R}{2} = 3 \times \frac{163.3\text{m}}{2} = 245\text{m}$$

Fragment 1 lands at 81.7m, while fragment 2 lands at 245m.

(iii) Horizontal velocity of fragment 2:

$$v_2 = 2u\cos\theta = 2 \times 40\text{m/s} \times \cos 45° = 56.6\text{m/s}$$

Making Sense of the Answer: *Fragment 2 lands at 245m which is equivalent to a 50% farther than the original projectile would have traveled (163m)! This seems counterintuitive until we realize that the explosion doubles its horizontal velocity from 28.3 m/s to 56.6 m/s. Even though it only has half the time left to travel, the doubled horizontal velocity allows the fragment to travel a further horizontal distance of 163.3m (R) before reaching the ground. Fragment 1 lands exactly halfway along the original trajectory, which makes sense since it falls straight down from the peak.*

Think Like a Physicist: *This problem beautifully demonstrates momentum conservation in two dimensions. The explosion provides internal forces between fragments, but no external horizontal force acts on the system. Therefore, total horizontal momentum must be conserved. Since one fragment stops horizontally (falls straight down), the other must carry the entire horizontal momentum, and having half the mass, it must travel at twice the original horizontal velocity.*

HOT Example 11

A ball is thrown from ground level at 20m/s at an angle of 40° toward a wall. The wall is 25m away and 3.5m high. Taking $g = 9.8\ \text{m/s}^2$ and using sin 40° = 0.643, cos 40° = 0.766, sin 80° = 0.985:

(a) Show that the ball clears the wall.
(b) Calculate by how much the ball clears the top of the wall.
(c) Calculate where the ball lands beyond the wall.
(d) If the wall height were increased, determine the maximum wall height that the ball could just clear.

Solution

(a) We need to find the height of the ball when it is at horizontal position of $s_x = 25\text{m}$.

Using the trajectory equation:

$$s_y = s_x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)s_x^2$$

Where: $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$s_y = s_x\left(\frac{\sin 40°}{\cos 40°}\right) - \left(\frac{g}{2u^2\cos^2 40°}\right)s_x^2$$

Substituting values:

$$s_y = 25\text{m}\left(\frac{0.643}{0.766}\right) - \left(\frac{9.8\ \text{m/s}^2}{2(20\text{m/s})^2(0.766)^2}\right)(25\text{m})^2 = 7.91\text{m}$$

Since $s_y = 7.91\text{m} > 3.5\text{m}$ (wall height), **the ball clears the wall**.

(b) $\text{Clearance} = \text{height of ball} - \text{height of wall}$

$$\text{Clearance} = (7.91 - 3.5)\text{m} = 4.41\text{m}$$

The ball clears the wall by 4.41m.

(c) Total horizontal distance travelled by ball:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin\theta\cos\theta}{g} = \frac{2(20\,m/s)^2(0.643)(0.766)}{9.8\,m/s^2} = 40.2m$$

Distance beyond wall $= 40.2m - 25m = 15.2m$

The ball lands 15.2m beyond the wall.

(d) For the ball to just clear the wall: $s_y = h$ at $s_x = 25m$.

But from (a):

When $s_x = 25m$, $s_y = 7.91m = h$

The maximum wall height is 7.91m (as long as the wall stays at 25m distance).

Making Sense of the Answer: *The ball reaches 7.91m height at the wall location, clearing the 3.5m wall by a comfortable margin of 4.41m which is more than the wall height itself! This is not surprising since the ball reaches maximum height of 8.43m (by applying maximum height formula) at its peak (which occurs before the wall because R/2 is smaller than 25m). If the wall were higher than 7.91m, the ball would hit it. The ball's trajectory is already descending when it reaches the wall position, which is why it lands only 15.1 m beyond the wall rather than much farther.*

Think Like a Physicist: *In real applications (sports, military, construction), clearance calculations are critical: will the basketball clear the defender's outstretched hand? Will the artillery shell clear the hill? Will the water from the hose reach over the fence? The mathematics gives definitive answers. Notice how the maximum clearable wall height depends on where the wall is positioned; obstacles at different distances pose different challenges, even for the same projectile. A wall placed at the peak position (R/2 which is about 20m for our example) is the easiest to clear because the projectile is at its maximum height there.*

Summary of key formulas

For a projectile launched at angle θ with initial velocity u from and returning to the same level:

Time of flight: $T = \frac{2u\sin\theta}{g}$

Maximum height: $H = \frac{u^2\sin^2\theta}{2g}$

Horizontal range: $R = \frac{u^2\sin 2\theta}{g}$

Special relationships:

Maximum range occurs at $\theta = 45°$: $R_{max} = \frac{u^2}{g}$

At 45°, the range is four times the height: $R = 4H$

Complementary angles (θ and $90° - \theta$) give the same range

Maximum height relates to time of flight: $H = \frac{gT^2}{8}$

These formulas, combined with those from previous subtopic, provide complete tools for analysing any projectile motion problem where launch and landing occur at the same level. In the next subtopic, we shall extend our analysis to more complex cases where these conditions do not hold.

PROJECTILES FROM HEIGHT AND ON INCLINED PLANES

In the previous two subtopics, we developed a complete mathematical framework for projectile motion under the assumption that the projectile is launched and lands at the same horizontal level. While this idealized case is fundamental for understanding the basic principles, it represents only a limited range of real-world situations. In practice, projectiles are often launched and land at different heights. For example, a basketball player releases the ball about 2m above the ground toward a hoop 3m high. A stone may be thrown from the edge of a cliff toward the ocean below. A skier may launch off a slope and land farther down the same incline. In each of these situations, the projectile does not return to its original launch height, and therefore the formulas developed earlier must be modified to account for the difference in vertical levels.

This subtopic extends our analysis to two important special cases that frequently arise in practice: **projectiles launched from a height above the landing level**, and **projectiles moving on inclined planes**. In both situations, the projectile does not return to the same vertical level from which it was launched.

We shall see that the fundamental principles remain unchanged. The horizontal and vertical components of velocity still act independently, and the trajectory of the projectile remains parabolic. However, the specific parameters such as the time of flight, horizontal range, and landing velocity must be recalculated to account for the asymmetry between the launch and landing conditions.

Horizontal Projection from a Height

Consider the simplest extension of our previous analysis: a projectile launched horizontally from height **h** above the ground. This scenario occurs when you roll a ball off a table, kick a football from a cliff edge, or drop supplies from a moving aircraft, and other many familiar circumstances. Since the initial velocity is horizontal, there is no upward component. The projectile therefore begins falling immediately while simultaneously moving forward.

To analyse the motion clearly, let us first define a suitable coordinate system and specify the initial conditions.

Let the origin be at the point of projection, with the x-axis horizontal (direction of initial velocity) and the y-axis vertical (upward positive). The initial conditions are:

- $\mathbf{u_x = u}$ (horizontal initial velocity).
- $\mathbf{u_y = 0}$ (no vertical component).
- **Initial position:** $(0, 0)$.
- **Landing position:** $(R, -h)$ where R is the range and h is the launch height.

Time of flight (time to hit the ground)

The vertical displacement equation gives:

$$s_y = u_y t - \tfrac{1}{2}gt^2$$

At landing: $s_y = -h$ (negative because downward from origin) and $u_y = 0$:

$$-h = 0 - \tfrac{1}{2}gt^2$$

Making **t** the subject:

$$\mathbf{t = \sqrt{\frac{2h}{g}}}$$

This is exactly the same result we obtained in **Chapter 2**: *fall time depends only on height, not on horizontal velocity*. Whether the ball rolls off the table slowly or is strongly kicked horizontally, **the time to hit the ground** remains identical, determined solely by **h** and **g**.

Horizontal range

During time t, the projectile travels horizontally at constant velocity $u_x = u$:

$$\mathbf{R} = ut = \mathbf{u} \times \sqrt{\frac{\mathbf{2h}}{\mathbf{g}}}$$

This elegant formula reveals that range is proportional to both initial velocity and the square root of height. Doubling the launch height increases range by factor $\sqrt{2} \approx 1.41$. Doubling the initial velocity doubles the range directly.

Landing velocity

At time $t = \sqrt{\frac{2h}{g}}$, the velocity components are:

$v_x = u$ (unchanged horizontal velocity)

$v_y = u_y - gt$

$v_y = 0 - g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$ (The negative sign means the velocity is **downward**).

The magnitude of landing velocity:

$$\mathbf{v} = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{\mathbf{u^2 + 2gh}}$$

The landing angle α satisfies:

$$\tan\alpha = \frac{v_y}{v_x}$$

However, since v_y is negative while v_x is positive, the value of $\tan\alpha$ becomes negative. This indicates that the angle α lies **below the horizontal direction**. To determine the magnitude of this angle directly, we take the absolute value of v_y. Tus:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right)$$

Where α represents the **angle below the horizontal**.

Notice that landing velocity exceeds initial velocity (v > u) because gravity has added a vertical component of velocity during the fall.

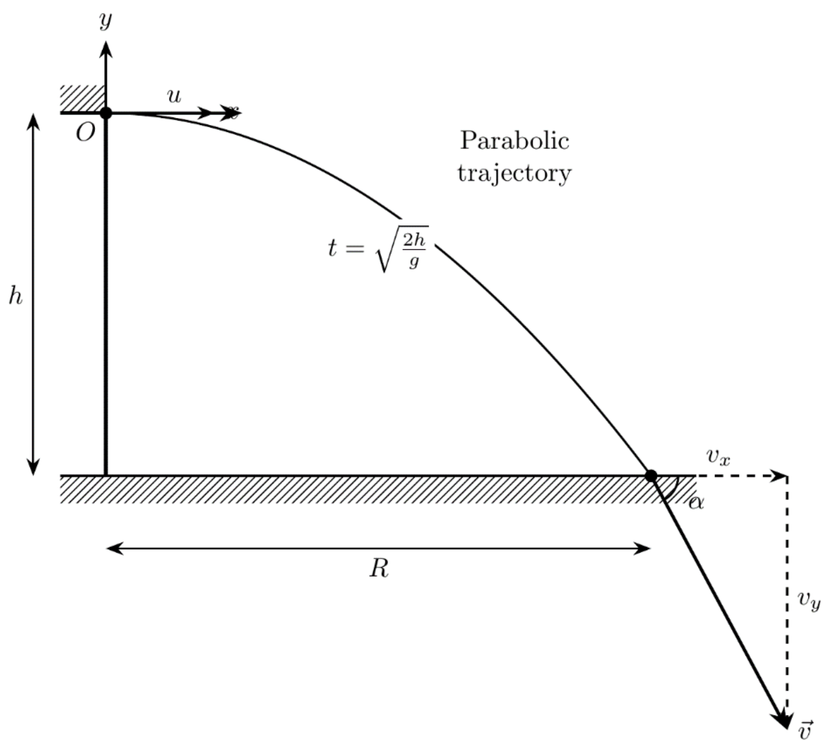

Figure: *Horizontal projection from a height h. The projectile is launched with horizontal velocity u and follows a parabolic trajectory, reaching the ground after time* $t = \sqrt{\frac{2h}{g}}$ *at horizontal range R. The landing velocity has horizontal and vertical components* v_x *and* v_y*.*

Projection at an Angle from a Height

The more general case occurs when a projectile is launched at angle **θ** above horizontal from height **h**. This describes situations like a basketball shot, a water fountain on elevated ground, or an archer shooting from a castle wall. The analysis combines our previous work on angled projection with the modifications introduced by the height difference.

Now, let us specify the initial conditions for this case:

- $\mathbf{u_x = u\cos\theta}$ (horizontal component)
- $\mathbf{u_y = u\sin\theta}$ (vertical component, upward)
- **Initial position:** $(0, 0)$
- **Landing position:** $(R, -h)$

Time of flight

Again, this requires solving the vertical displacement equation:

$$s_y = (u\sin\theta)t - ½gt^2$$

At landing, $s_y = -h$:

$$-h = (u\sin\theta)t - ½gt^2$$

Rearranging to standard quadratic equation in t:

$$½gt^2 - (u\sin\theta)t - h = 0$$

Using the quadratic formula with: $a = ½g, b = -(u\sin\theta), c = -h$:

$$t = \frac{\left(u\sin\theta \pm \sqrt{(u\sin\theta)^2 + 2gh}\right)}{g}$$

Since $\sqrt{(u\sin\theta)^2 + 2gh} > u\sin\theta$, the negative root make the numerator negative and therefore give a **negative value of t**. Because negative time has no physical meaning, that root must be rejected. Hence, only the **positive root** is physically meaningful, and therefore:

$$\mathbf{t = \frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g}}$$

Comparing with level-ground formula: When $h = 0$, this reduces to:

$$t = \frac{\left(u\sin\theta + \sqrt{(u\sin\theta)^2}\right)}{g} = \frac{(u\sin\theta + u\sin\theta)}{g} = \frac{2u\sin\theta}{g}$$

This matches our result we derived earlier, confirming that our new formula generalizes the previous one.

Notice the following important physical insight: The term $\sqrt{(u\sin\theta)^2 + 2gh}$ is always greater than $u\sin\theta$ when $h > 0$. So time of flight from a height is always longer than for the same launch on level ground. The projectile gets "bonus time" in the air from the additional height.

Range

Substituting the time of flight into $R = (u\cos\theta)t$:

$$\mathbf{R = u\cos\theta\left(\frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g}\right)}$$

This formula can be simplified but is often left in this form for calculation purposes.

Maximum height above launch point

The projectile reaches maximum height when $v_y = 0$:

$$v_y = u\sin\theta - gt_{max} = 0$$

$$t_{max} = \frac{u\sin\theta}{g}$$

The height above launch point is:

$$\mathbf{H} = (u\sin\theta)t_{max} - ½gt_{max}^2 = \mathbf{\frac{u^2\sin^2\theta}{2g}}$$

This is the same as the formula we derived earlier. *The maximum height above the launch point depends only on the vertical component of initial velocity, regardless of the absolute elevation of the launch point.*

Keep in mind that the maximum height above the ground is $H + h$ (the height above launch plus the launch elevation).

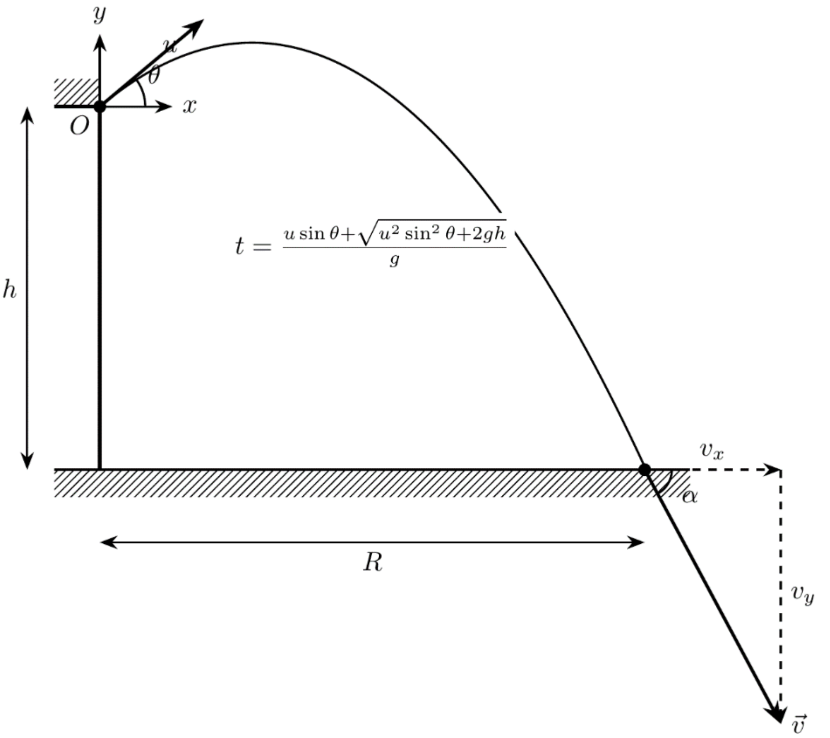

Figure: *Projectile launched from a height h above the ground with initial velocity u at angle θ. The projectile follows a parabolic trajectory and lands at horizontal range R. The landing velocity has horizontal and vertical components v_x and v_y, making an angle α below the horizontal.*

For now, give theory a short holiday and let us sharpen our understanding with some useful worked examples.

BINDER Example 12

A ball is rolled off a table 1.2m high with horizontal velocity 3m/s. Taking g = 9.8 m/s²; Calculate:

(a) the time taken for the ball to reach the floor,
(b) the horizontal distance from the table edge where the ball lands
(c) the velocity of the ball just before it hits the floor.

Solution

(a) Time for the ball to reach the floor is given by:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1.2\text{m}}{9.8\text{m/s}^2}} = 0.495\text{s}$$

The time is 0.495s.

(b) Horizontal distance is given by:

$$R = ut = 3\text{m/s} \times 0.495\text{s} = 1.49\text{m}$$

The horizontal distance is 1.49m.

(c) Velocity components at landing:

$v_x = u = 3\text{m/s}$ (unchanged)

$v_y = \sqrt{2gh} = \sqrt{2 \times 9.8\text{m/s}^2 \times 1.2\text{m}} = 4.85\text{m/s}$ (downward)

Magnitude: $v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(3m/s)^2 + (4.85m/s)^2} = 5.7m/s$

Angle below horizontal:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{4.85}{3}\right) = 58.3°$$

The velocity is $5.7m/s$ at 58.3° below horizontal.

Making Sense of the Answer: *The ball takes half a second to fall 1.2 m, reasonable for free fall. The landing velocity (5.7m/s) exceeds initial velocity (3m/s) because gravity added vertical component of velocity.*

Think Like a Physicist: *In projectile motion, the horizontal and vertical components of motion are independent. Gravity affects only the vertical motion, which is why the fall time depends solely on the height (and of course the vertical component of velocity).*

BINDER Example 13

A stone is thrown horizontally from the top of a cliff 45 m high with velocity 15m/s. Taking the value of g = 9.8 m/s²; calculate:

(a) how long the stone is in the air,
(b) how far from the base of the cliff the stone lands,
(c) the angle at which the stone strikes the ground.

Solution

(a) Time of flight:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 45m}{9.8m/s^2}} = 3.03s$$

The stone stays in the air for 3.03s.

(b) Range:

$$R = ut = 15m/s \times 3.03s = 45.45m$$

The stone lands 45.5m from the base of the cliff.

(c) Landing velocity components:

$$v_x = 15m/s$$

$$v_y = \sqrt{2gh} = \sqrt{2 \times 9.8m/s^2 \times 45.45} = \sqrt{882} = 29.85m/s$$

Landing angle:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{29.85}{15}\right) = 63.4°$$

The stone strikes at 63.4° below horizontal.

Making Sense of the Answer: *The 3s fall time and 45.45m range are both substantial, reflecting the considerable cliff height. The steep landing angle (63°) shows the stone is falling quite rapidly by the time it reaches the ground (the vertical velocity (29.7m/s) is nearly double the horizontal velocity (15m/s).*

Think Like a Physicist: *Compare this to dropping the stone straight down (u = 0). The fall time would be identical (3.03s), but the landing would occur at the cliff base (R = 0) instead of 45.45m away. The horizontal motion is "free"; it adds range without costing any flight time. This principle explains why supply drops from aircraft are so effective: the aircraft's forward velocity gives range without reducing the time available for parachutes to deploy.*

BINDER Example 14

A projectile is launched at 25m/s at 37° above horizontal from a platform 5m above ground level. Taking g = 9.8 m/s², calculate:

(a) the time of flight,
(b) the range,
(c) the maximum height above the ground.

Solution

(a) Time of flight from height is given by:

$$t = \frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g} = \frac{25\sin37° + \sqrt{(25\sin37°)^2 + 2 \times 9.8 \times 5}}{9.8} = 3.37s$$

The time of flight is 3.37s.

(b) Range is given by:

$$R = (u\cos\theta)t = 25\cos37° \times 3.37 = 67.3m$$

The range is 67.3m.

(c) Maximum height above the ground

$$= H + h = \frac{u^2\sin^2\theta}{2g} + h = \frac{(25\sin37°)^2}{2 \times 9.8} + 5 = 16.55m$$

The height is 16.55m.

Making Sense of the Answer: *The time of flight (3.37s) is longer than it would be for level ground launch (T = 3.06s) because the 5m platform height provides extra airtime. The range (67.3m) reflects this longer flight time combined with the horizontal velocity of about 20 m/s.*

Think Like a Physicist: *The maximum height above the launch point (11.55m) depends only on the vertical velocity component, exactly as in level-ground projection. The 5m platform height simply adds to this, giving 16.55m total above ground. This clear separation of effects shows that the vertical motion is independent of the initial elevation.*

REAL Example 15

A construction worker accidentally kicks a hammer off the roof of a building. At the same moment, another worker standing on the roof simply drops a similar hammer without giving it any horizontal push. Explain why both hammers reach the ground at the same time, even though one travels forward while the other falls straight down.

Solution

Both hammers reach the ground at the same time because the time taken to fall depends only on the vertical motion. This is due to the fact that in both cases, the vertical motion begins with zero vertical velocity and the only force acting vertically is gravity. Therefore, each hammer falls with the same acceleration and from the same height leading to equal time of fall. The horizontal velocity only changes the landing position, not the landing time.

Making Sense of the Answer: *The kicked hammer moves forward while falling, but gravity acts only vertically. Therefore, both hammers fall for the same duration, even though one lands farther from the building.*

Think Like a Physicist: *Projectile motion can be separated into independent horizontal and vertical motions. Horizontal motion determines* ***where*** *the object lands, while vertical motion determines* ***when*** *it reaches the ground.*

REAL Example 16

During a rescue operation, supplies are dropped from a moving helicopter flying horizontally over a disaster area. Explain why the supplies land some distance ahead of the point directly below the helicopter, even though they are simply released and not thrown forward.

Solution

When the supplies are released from the helicopter, they already possess the same horizontal velocity as the helicopter at that instant. After release, there is no horizontal force acting on the supplies (neglecting air resistance), so they continue to move forward with this horizontal velocity while falling under gravity.

Consequently, they land some distance ahead of the point directly below the helicopter at the moment of release.

Making Sense of the Answer: *Although the supplies are simply dropped, they already have the same forward velocity as the helicopter. During the fall they keep moving forward while gravity pulls them downward.*

Think Like a Physicist: *An object released from a moving body retains the horizontal velocity it had at the moment of release. Without a horizontal force to change it, that velocity remains constant while gravity controls the vertical motion.*

HOT Example 17

A basketball player shoots from 2m above the ground, releasing the ball at 7.5m/s at 50° above horizontal. The basket is 3.05 m above the ground and 5m away horizontally. Taking g = 9.8 m/s², sin 50° = 0.766, cos 50° = 0.643:

(a) Calculate the time for the ball to reach the basket's horizontal position.
(b) Determine whether the ball passes through the basket or misses (too high or too low).
(c) Calculate the velocity of the ball when it reaches the basket location.
(d) If the shot misses the basket, determine by how much it misses and, without performing further calculations, suggest possible corrections.

Solution

(a) The horizontal displacement after any time, t is given by: $x = (x\cos\theta)t$

Thus, the time to reach horizontal distance of 5m:

$$t = \frac{x}{u\cos\theta} = \frac{5\text{m}}{7.5\text{m/s} \times 0.643} = 1.04\text{s}$$

The time is 1.04s.

(b) Height of ball at $t = 1.04\text{s}$ (measured from release point at 2m):

$$s_y = (u\sin\theta)t - ½gt^2$$

$$s_y = (7.5\text{m/s} \times 0.766)(1.04\text{s}) - ½(9.8\text{m/s}^2)(1.04\text{s})^2 = 0.67\text{m}$$

The vertical displacement is 0.67m above release point.

Height above ground $= s_y + h_o = 0.67\text{m} + 2\text{m} = 2.67\text{m}$

The ball vertical position above the ground $= 2.67\text{m} < 3.05\text{m}$ (basket position above the ground)

Hence, the ball **misses too low** (passes under the basket).

(c) Velocity at $t = 1.04\text{s}$:

$$v_x = u\cos\theta = 7.5\text{m/s} \times 0.643 = 4.82\text{m/s}$$

$$v_y = u\sin\theta - gt = 7.5\text{m/s} \times 0.766 - 9.8\text{m/s}^2 \times 1.04 = -4.45\text{ m/s (downward)}$$

Magnitude: $v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(4.82\text{m/s})^2 + (-4.45\text{m/s})^2} = 6.56\text{m/s}$

Direction: $\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{4.45}{4.82}\right) = 42.7°$ below horizontal.

The velocity is 6.56m/s at $42.7°$ below horizontal.

(d) But basket height is 3.05m.

Difference $= 3.05\text{m} - 2.67\text{m} = 0.38\text{m}$

The shot misses by $\mathbf{0.38m\ low}$.

Possible corrections:

1. Increasing angle of projection.
2. Increasing velocity of projection (initial velocity).

3. Launching the ball from higher position (increasing arm extension or jumping higher)

Making Sense of the Answer: The ball reaches correct horizontal position but is 38cm too low; a clear miss! By the time it arrives (1.04 s), it is already descending (v_y negative), having peaked earlier.

Think Like a Physicist: *Basketball shooting is applied projectile motion. The 2-metre release height helps by reducing vertical distance to the 3.05m basket (only 1.05m to climb), making the trajectory less extreme. Professional players develop intuition through thousands of shots, but physics is unforgiving: wrong velocity or angle means a miss, regardless of skill.*

HOT Example 18

A stone is thrown from the top of a cliff at 15m/s at 30° below the horizontal. The cliff is 40m high. Taking $g = 9.8\ m/s^2$, calculate:

(a) Calculate the time taken for the stone to reach the ground.
(b) Calculate the horizontal distance from the cliff base where the stone lands.
(c) Calculate the magnitude and direction of stone's velocity when it hits the ground.

Solution

Since the angle of projection is **below the horizontal**, the ball is thrown **downward**. Therefore, in the calculations the angle is taken as **negative**, according to the usual sign convention.

Components of initial velocity:

$u_x = u\cos(-30°) = 15\text{m/s} \times \cos(-30°) = 13\text{m/s}$

$u_y = u\sin(-30°) = 15\text{m/s} \times \sin(-30°) = -7.5\text{m/s}$ (downward)

(a) Using vertical displacement equation:

$s_y = u_y t - ½gt^2$

At $s_y = -40\text{m}$ (downward displacement is taken as negative)

$-40 = -7.5t - ½(9.8)t^2$

$4.9t^2 + 7.5t - 40 = 0$

Solving the quadratic equation by using calculator gives the practical (positive) value of $t = 2.19\text{s}$

The time taken is 2.19s.

Alternative solution

By using formula (where $u\sin\theta = -7.5\text{m/s}$):

$$t = \frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g} = \frac{-7.5 + \sqrt{(-7.5)^2 + 2 \times 9.8 \times 40}}{9.8} = 2.19\text{s}$$

The time is 2.19s.

(b) The horizontal distance from the cliff base:

$R = u_x t = 13\text{m/s} \times 2.19\text{s} = 28.47\text{m}$

The horizontal distance is 28.47m.

(c) Final velocity components:

$v_x = u_x = 13\text{m/s}$

$v_y = u_y - gt = -7.5\text{m/s} - 9.8\text{m/s}^2(2.19\text{s}) = -29\ \text{m/s}$

Magnitude: $v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(13\text{m/s})^2 + (-29\ \text{m/s})^2} = \mathbf{31.8m/s}$

Direction: $\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{29}{13}\right) = \mathbf{65.9°}$ **below horizontal.**

Making Sense of the Answer: *Starting with downward velocity (−7.5m/s) and falling for 2.19s adds significant vertical velocity, reaching −29m/s vertically. The steep landing angle (66°) shows the stone is falling rapidly, with vertical velocity more than double the horizontal velocity.*

Think Like a Physicist: *Throwing downward at 30° gives the stone an initial downward "head start" of 7.5m/s, reducing fall time compared to throwing horizontally (which would be about 2.86s). This trade-off between faster impact versus more horizontal distance, appears in many applications, from cliff diving to aerial delivery of supplies.*

Projectiles on Inclined Planes

An entirely different special case arises when a projectile is launched from and lands on an inclined plane. This situation occurs in many practical contexts, for example: a skier launching from a jump on a slope, a ball thrown uphill along a mountain path, or supplies released onto an inclined conveyor belt. In such cases, the landing point does not lie on a horizontal surface. Instead, the projectile returns to the **sloping surface**, which introduces a natural asymmetry. Depending on the launch direction, the projectile may land "below" its launch point (**downhill projection**) or "above" it (**uphill projection**), measured along the incline.

To analyse this situation, consider an inclined plane that makes an angle **β** with the horizontal. A projectile is launched from a point on the plane with initial velocity **u** at an angle **θ** measured from the horizontal (not from the incline).

Two coordinate systems may be used for this analysis:

1) The horizontal and vertical axes (the standard x–y system).
2) Axes parallel and perpendicular to the incline.

Although both approaches are valid, we use standard x-y, which connects directly to our previous work and makes the mathematics more straightforward.

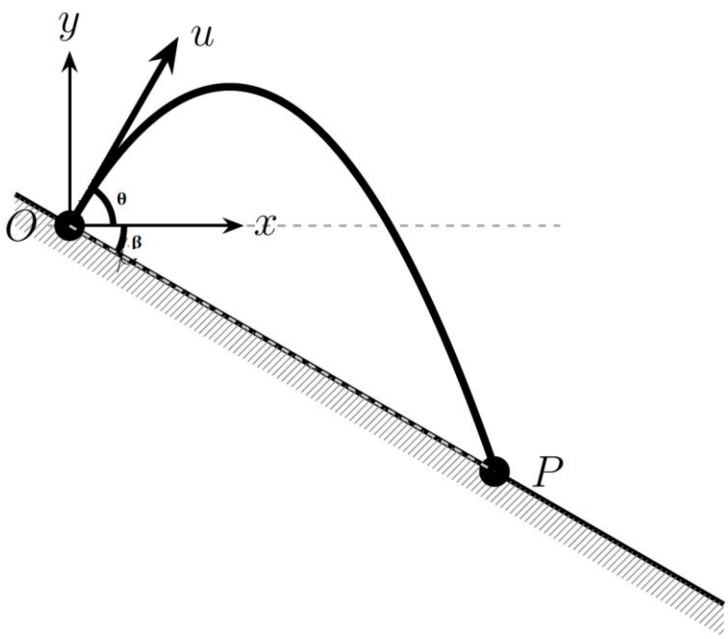

Figure: ***Downhill projection on an inclined plane.*** *A projectile is launched with initial velocity u at an angle θ above the horizontal from a point on an incline that makes an angle β below the horizontal. The projectile follows a parabolic path and lands at point P on the slope.*

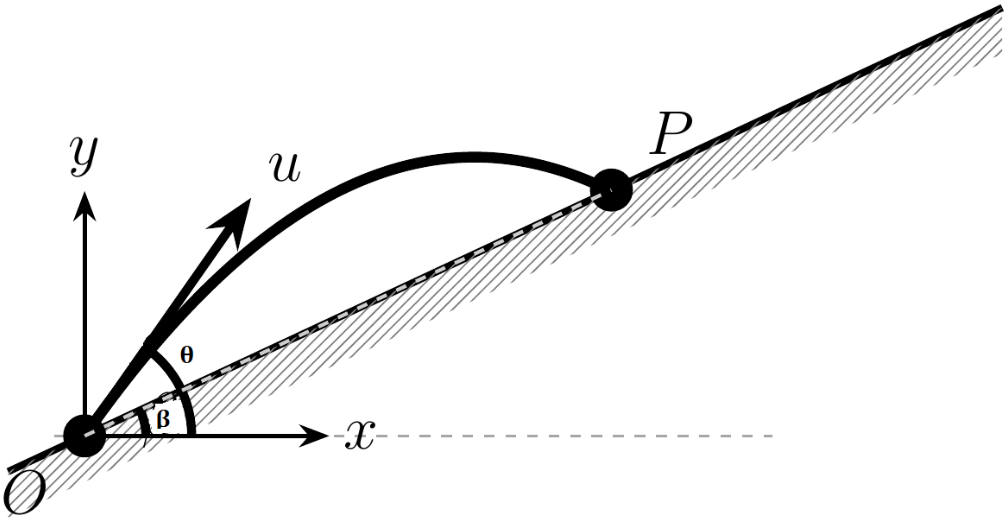

Figure: ***Uphill projection on an inclined plane.*** *A projectile is launched with initial velocity u at an angle θ above the horizontal from a point on an incline that makes an angle β above the horizontal. The projectile follows a parabolic path and lands at point P on the slope.*

From diagrams:

- θ is the **angle of projection** (between u and the horizontal axis).
- β is the **angle of the incline** relative to the horizontal.
- OP is the **range along the incline** (the distance measured along the sloping surface from the point of projection O to the landing point P). This distance will be denoted by **L**.

Important: Even though the projectile is launched from an inclined plane, the angle of projection is still defined relative to the horizontal, not relative to the slope. If the angle θ were measured from the incline itself, the corresponding angle of projection with respect to the horizontal would be $\theta - \beta$ for downhill projection and $\theta + \beta$ for uphill projection.

Range along the incline, L

The derivation and resulting formula are the same for both **uphill** and **downhill** projections. The only difference arises when applying the formula: for **uphill projection**, the angle β is taken as **positive** (above the horizontal), whereas for **downhill projection**, β is taken as **negative** (below the horizontal). Therefore, for the purpose of this derivation, we shall consider the **uphill case.**

Writing the incline equation

From the diagram:

$$\tan\beta = \frac{y}{x}$$

From which, the equation of the inclined plane is:

$$y = x\tan\beta \ldots \text{(i)}$$

(It is equivalent to general equation of a straight line through the origin, $y = mx$, where m is the slope. Here the slope of the incline is $\tan\beta$).

Comparing with trajectory equation:

$$y = x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2 \ldots \text{(ii)}$$

The projectile lands where it intersects the incline. At this point, the coordinates (x, y) must satisfy **both** equations simultaneously.

Thus, by equating (i) and (ii):

$x\tan\beta = x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$

Dividing by x throughout (if $x \neq 0$)

$\tan\beta = \tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)x$

Making x the subject:

$\mathbf{x} = \left(\frac{2u^2\cos^2\theta}{g}\right)(\tan\theta - \tan\beta)$

Simplifying (tanθ − tanβ)

$\tan\theta - \tan\beta = \frac{\sin\theta}{\cos\theta} - \frac{\sin\beta}{\cos\beta} = \frac{\sin\theta\cos\beta - \cos\theta\sin\beta}{\cos\theta\cos\beta}$

Using the sine difference formula: $\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\sin\theta\cos\beta - \sin\beta\cos\theta = \sin(\theta - \beta)$

Therefore:

$$\mathbf{\tan\theta - \tan\beta = \frac{\sin(\theta - \beta)}{\cos\theta\cos\beta}}$$

Substituting back into expression for x

$\mathbf{x} = \left(\frac{2u^2\cos^2\theta}{g}\right)\left(\frac{\sin(\theta - \beta)}{\cos\theta\cos\beta}\right) = \mathbf{\frac{2u^2\cos\theta\sin(\theta - \beta)}{g\cos\beta}}$

This is the **horizontal distance to the landing point**.

Finding the range L along the incline

Also from the diagram:

$\cos\beta = \frac{x}{OP} = \frac{x}{L}$ or $\mathbf{x} = L\cos\beta$

Equating:

$L\cos\beta = \frac{2u^2\cos\theta\sin(\theta - \beta)}{g\cos\beta}$

$\mathbf{L = \frac{2u^2\cos\theta\sin(\theta - \beta)}{g\cos^2\beta}}$

This is the **range along the inclined plane.**

Simplifying the formula

$2\cos\theta\sin(\theta - \beta)$ can be simplified by using the product-to-sum formula:

$2\cos A\sin B = \sin(A + B) - \sin(A - B)$

With $A = \theta$ and $B = \theta - \beta$:

$$\mathbf{2\cos\theta\sin(\theta - \beta)} = \sin(\theta + (\theta - \beta)) - \sin(\theta - (\theta - \beta)) = \mathbf{\sin(2\theta - \beta) - \sin\beta}$$

Substituting into our expression for L:

$L = \frac{u^2(\sin(2\theta - \beta) - \sin\beta)}{g\cos^2\beta}$

Hence, the simplified formula for the **range along the inclined plane** is:

$$\mathbf{L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)}$$

This compact result is remarkably general. It applies equally well to both **uphill** and **downhill** projections; the only difference lies in the sign of β. For an uphill incline β is positive, while for a downhill incline β is taken as negative as mentioned earlier.

A useful check on the correctness of this formula is obtained by considering the special case where the plane becomes horizontal. If $\beta = 0$, the incline disappears and the situation reduces to ordinary ground-level projection. Substituting $\beta = 0$ gives:

$$\mathbf{L} = \frac{u^2}{g\cos^2 0}(\sin(2\theta - 0) - \sin 0) = \frac{\mathbf{u^2 \sin 2\theta}}{\mathbf{g}},$$

which is exactly the familiar **range formula for level ground**. This confirmation shows that the inclined-plane result is simply a natural extension of the standard projectile theory developed earlier.

Having obtained the expression for the range, we can now ask an important question: ***for what angle of projection will the range along the incline be greatest?*** Since the θ containing term sin(2θ−β) determines the variable part of the expression (β is constant for given incline, so other factors with β are constant, they do not affect the maximum), the range is maximized when:

$$\sin(2\theta - \beta) = 1$$

This occurs when:

$$2\theta - \beta = 90°$$

Solving for θ gives:

$$\boldsymbol{\theta} = \frac{90° + \beta}{2} = \mathbf{45°} + \frac{\boldsymbol{\beta}}{\mathbf{2}}$$

Hence, the **optimal angle** for maximum range along the incline is given by:

$$\boldsymbol{\theta}_{\mathbf{optimal}} = \mathbf{45°} + \frac{\boldsymbol{\beta}}{\mathbf{2}}$$

This result also connects beautifully with the familiar case of projection on level ground. When the incline disappears ($\beta = 0$), the expression reduces immediately to $\theta = 45°$, the well-known angle that produces **maximum horizontal range** on a flat surface.

Time of flight

We can derive this by using the vertical displacement equation.

The vertical position at any time t is given by: $y = (u\sin\theta)t - \frac{1}{2}gt^2$

At landing (time $t = T$), the projectile is on the incline at horizontal position x: $y = x\tan\beta$

And also: $x = (u\cos\theta)T$

So: $y = (u\cos\theta)T \times \tan\beta$

The vertical displacement equation after time $t = T$ (at landing) is:

$$y = (u\sin\theta)T - \tfrac{1}{2}gT^2$$

Where: $y = (u\cos\theta)T \times \tan\beta$

Substituting:

$$(u\cos\theta)T \times \tan\beta = (u\sin\theta)T - \tfrac{1}{2}gT^2$$

Divide by T (if $T \neq 0$):

$$(u\cos\theta)\tan\beta = u\sin\theta - \tfrac{1}{2}gT$$

$$\tfrac{1}{2}gT = u\sin\theta - (u\cos\theta)\tan\beta = u\sin\theta - (u\cos\theta)\frac{\sin\beta}{\cos\beta} = u\sin\theta - \frac{u\cos\theta\sin\beta}{\cos\beta}$$

$$\tfrac{1}{2}gT = \frac{u(\sin\theta\cos\beta - \cos\theta\sin\beta)}{\cos\beta} = \frac{u(\sin(\theta - \beta))}{\cos\beta}$$

Hence, by making T the subject:

$$\mathbf{T} = \frac{\mathbf{2u(sin(\theta - \beta))}}{\mathbf{gcos\beta}}$$

After a rather lengthy chain of algebra and trigonometric manipulation, it is easy for the main physical ideas to become lost in the mathematics. Before those ideas begin to drift away, let us pause for a moment and anchor them with a few carefully chosen worked examples. These examples will translate the formulas we have just derived into concrete situations, showing how the theory predicts the motion of projectiles on inclined surfaces and revealing the practical meaning behind the equations.

BINDER Example 19

A skier launches off a jump on a slope inclined at 30° below horizontal. The skier leaves the jump with velocity 20m/s at 25° above horizontal. Use $g = 9.8\ m/s^2$ to calculate:

(a) how far down the slope (measured along the slope) the skier lands,
(b) the time the skier spends in the air before landing.

Solution

Interpreting the data:

$\theta = 25°$ (above horizontal).

$\beta = -30°$ (negative for downhill).

(a) Range along the incline is given by:

$$L = \frac{u^2}{gcos^2\beta}(sin(2\theta - \beta) - sin\beta)$$

Substituting values:

$$L = \frac{(20m/)^2}{9.8\ m/s^2 \times cos^2(-30)}(sin(2 \times 25° + 30°) - sin(-30°)) = 80.8m$$

The skier lands at distance of 80.8m down the slope.

(b) From $x = (ucos\theta)t$

Just before landing:

- $t = T$
- $x = Lcos\beta$

Thus:$(ucos\theta)T = Lcos\beta$

From which:

$$T = \frac{Lcos\beta}{ucos\theta} = \frac{80.8m \times cos(-30°)}{20m/s \times cos\,25°} = 3.86s$$

The time the skier spends in the air is 3.86s.

Alternative solution

By using the time of flight formula for an incline projection:

$$T = \frac{2u(sin(\theta - \beta))}{gcos\beta}$$

Substituting values:

$$T = \frac{2 \times 20m/s \times sin(25° + 30°)}{9.8\ m/s^2 \times cos(-30°)} = \mathbf{3.86s}$$

Making Sense of the Answer: *The skier travels 80.8m along the slope in just 3.86s. The 25° launch angle is close to the optimal 30° for maximum range in 30° downslope, explaining the large range achieved.*

Think Like a Physicist: *The optimal angle for maximum range on a* $-30°$ *inclined plane is:* $45° + \frac{-30°}{2} = 30°$. *The skier's 25° is close, that is why good range was achieved. Interestingly, experienced ski jumpers naturally learn to use angles below 45° when jumping downhill. The mathematics simply confirms what skilled athletes discover through practice.*

BINDER Example 20

A ball is thrown with velocity 18m/s up a hill inclined at 25° to the horizontal. Take $g = 9.8$ m/s²:

(a) Calculate the optimal angle of projection for maximum range up the slope,
(b) Calculate the maximum range achieved at this optimal angle,
(c) Compare this with the maximum range the ball would achieve on horizontal ground.

Solution

(a) The optimal angle is given by:

$$\theta_{optimal} = 45° + \frac{\beta}{2}$$

Where, $\beta = 25°$ (uphill, positive)

$$\theta_{optimal} = 45° + \frac{25°}{2} = 57.5°$$

The optimal angle is 57.5° above the horizontal.

(b) Range along the incline is given by:

$$L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)$$

For optimal angle, ($\theta = 57.5°$), L becomes maximum.

Substituting values:

$$L_{max} = \frac{(18\text{m/s})^2}{9.8\text{ m/s}^2 \times \cos^2(25°)}(\sin(2 \times 57.5° - 25°) - \sin 25°) = 23.2\text{m}$$

The maximum range is 23.2m.

(c) For horizontal ground, range is given by:

$$R = \frac{u^2 \sin 2\theta}{g}$$

The optimal angle of projection for maximum range is 45°.

$$R_{max} = \frac{(18\text{m/s})^2 \sin 90°}{9.8\text{ m/s}^2} = 33.1\text{m} > L_{max}(23.2\text{m})$$

The maximum range on horizontal ground is greater than that on an uphill inclined plane because, when moving uphill, the ball must overcome both **gravity** and the **rising slope**.

Making Sense of the Answer: *The optimal 57.5° angle is steeper than the familiar 45° because the ball must climb the 25° slope. The uphill range (23.2m) is therefore smaller than the range on horizontal ground (33.1m), as expected; with the same initial velocity, a projectile cannot travel farther uphill than it can on level ground.*

Think Like a Physicist: *This illustrates how changing the reference frame (level ground versus inclined plane) changes the optimal strategy. On level ground, 45° maximizes horizontal distance. On a 25° upslope, the optimal angle shifts to 57.5° to maximize distance along that slope but the actual range achieved is still less than what is possible on flat ground. Physics respects the fundamental constraint: going uphill is harder than going on level ground.*

REAL Example 21

At a ski training ground, Kipute notices that ski jumpers launching from a downward slope rarely jump at the familiar 45° angle taught in physics for maximum range on level ground. Instead, their launch angles appear noticeably smaller.

(a) Explain why the optimal launch angle on a downhill slope is less than 45°.
(b) Explain why ski jumpers can travel much farther (measured along the slope) than they could on flat ground with the same launch velocity.

Solution

(a) On a downhill slope, the ground falls away beneath the jumper, giving extra flight time. A flatter angle (less than 45°) maximizes horizontal distance while the falling ground provides the vertical component naturally. The optimal angle is $45° + \frac{\beta}{2}$; for downhill β is negative, so optimal angle < 45°.
(b) There two reasons that combine to give much greater range:

1) **Falling time:** The falling ground gives the jumper extra time in the air compared to flat ground, allowing more horizontal distance to accumulate.
2) **Measurement along slope:** Distance is measured along the slope, not horizontally. Since jumper lands at a much lower elevation, the sloped path from launch to landing is geometrically longer than just the horizontal distance.

Making Sense of the Answer: *The launch angle for downslope may appear quite shallow, but this is reasonable because the ground drops away beneath the projectile as it moves forward. This increases the time before it meets the surface again, allowing the projectile to travel a much greater distance along the slope.*

Think Like a Physicist: *This illustrates optimization in physics: what is "optimal" depends on constraints. On flat ground, 45° balances horizontal and vertical motion equally. On a downslope, the ground falling away changes the optimization; you do not need as much vertical velocity because the slope provides vertical descent. Nature finds the angle that maximizes the goal (range along slope) given the constraints (gravity + sloped landing). Engineers designing ski jumps use this physics to set ramp angles for maximum safety and distance.*

REAL Example 22

A student throws a ball with the same velocity both uphill and downhill on identical slopes (same angle magnitude, opposite directions). The ball travels much farther down the slope than up the slope.

(a) Explain why the downhill throw travels farther even though the slope angles have the same steepness.
(b) Explain why the optimal throwing angle is different for uphill versus downhill throws.

Solution

(a) The difference comes from flight time:

For downhill: The ground falls away, so the ball has more flight time before landing. The projectile gets extra time from the descending slope, and gravity helps the descent.

For uphill: The ground rises to meet the ball, cutting flight time short. The ball must fight both gravity and the rising slope.

Consequently, the same initial velocity and slope steepness, but downhill range is significantly larger than that of uphill because of dramatically different flight times.

(b) The optimal angle must balance vertical velocity against the slope direction:

For downhill: Ground falls away, so the aim is flatter angle which is **less than 45°** to maximize horizontal velocity. This is because, gravity and the falling slope together provide sufficient vertical descent time. This is justified by optimal angle formula, $\theta_{optimal} = 45° + \frac{\beta}{2}$; for downhill β is negative, so optimal angle < 45°.

For uphill: Ground rises, so the aim is steeper angle which is **greater than 45°** to give more vertical velocity to overcome both gravity and the rising slope. Again, this is justified by optimal angle formula, $\theta_{optimal} = 45° + \frac{\beta}{2}$; for uphill β is positive, so optimal angle $>$ 45°.

Making Sense of the Answer: *To make sense the difference, think in this way: downhill, you are working* ***with*** *gravity and the falling ground. Uphill, you are working* ***against*** *both. It is like the difference between running downhill versus uphill at the same speed; going down is much easier and you cover more ground. The physics quantifies this intuition precisely.*

Think Like a Physicist: *This problem demonstrates broken symmetry. The setup looks symmetric (same slope angle, just opposite directions), but gravity breaks the symmetry. In physics, apparent symmetries do not always produce symmetric results; you must identify what breaks the symmetry. Here, gravity's downward direction means "downhill" and "uphill" are fundamentally different, even if the slope angles match.*

HOT Example 23

An agricultural irrigation system uses a water pump to spray water onto a sloped field. The nozzle is positioned at the bottom of the field. The field slopes upward at 15° to the horizontal. The water leaves the nozzle at 12m/s at an adjustable angle θ above horizontal. Taking $g = 9.8\ m/s^2$:

(a) Calculate the nozzle angle that maximizes the distance up the slope where water lands.
(b) Calculate the maximum distance up the slope that can be irrigated.
(c) If the field extends 25m up the slope, determine whether this irrigation system can adequately water the entire field.
(d) Without performing further calculations, suggest possible modifications to reach the full 25m if needed.

Solution

(a) The optimal angle is given by:

$$\theta_{\text{optimal}} = 45° + \frac{\beta}{2}$$

Where, $\beta = 15°$ (uphill, positive)

$$\theta_{\text{optimal}} = 45° + \frac{15°}{2} = 52.5°$$

The optimal nozzle angle is 52.5° above the horizontal.

(b) Maximum distance up the slope is found by range formula along the incline:

$$L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)$$

For optimal angle, $(\theta = 52.5°)$, L becomes maximum.

Substituting values:

$$L_{\max} = \frac{(12\text{m/s})^2}{9.8\ \text{m/s}^2 \times \cos^2(15°)}(\sin(2 \times 52.5° - 15°) - \sin 15°) = 11.7\text{m}$$

The maximum distance up slope is 11.7m.

(c) Field extends 25m, but system reaches only 11.7m:

Shortfall = $25\text{m} - 11.7\text{m} = 13.3\text{m}$

Hence, the irrigation system **cannot** adequately water the entire field. It falls short by 13.3m.

(d) Possible modifications are:
1. **Increase the water velocity** by using higher-power pressure pump.
2. **Reposition the nozzle** by moving it partway up the slope.
3. **Use multiple nozzles** placed at different positions along the slope.

Making Sense of the Answer: The 52.5° angle is steeper than 45° because of the upward slope. The 11.7m range is limited because we are throwing uphill, so the water must overcome both gravity and the 15° rising ground.

Think Like a Physicist: *This demonstrates practical engineering constraints. The physics clearly shows the system is inadequate for a 25m field because no angle adjustment can extend 12m/s water beyond 11.7m on a 15° upslope. Physics sets fundamental limits; engineering works within or around them.*

APPLICATIONS OF PROJECTILE MOTION

The parabolic paths we have studied mathematically appear throughout the natural and engineered world. From ancient battlefields to modern sports arenas, from water fountains to spacecraft trajectories, projectile motion shapes our physical reality. Here are some of the countless ways these principles manifest in real life:

1. Basketball and free throws

Professional players intuitively understand the 45° rule and how height affects range. Shot tracking technology now analyzes launch angle, velocity, and arc height in real-time, using projectile equations to predict whether a shot will score before the ball reaches the hoop. The optimal arc for a free throw is typically 52° due to the elevated hoop.

2. Artillery and military ballistics

For centuries, armies have relied on projectile motion to aim cannons and mortars. The maximum range at 45° on level ground was discovered empirically by gunners long before Newton formalized the mathematics. Modern artillery computers instantly solve the projectile equations, accounting for wind, air resistance, and terrain elevation to hit targets kilometers away.

3. Irrigation and Water Distribution

Agricultural sprinklers use projectile motion principles to distribute water evenly across fields. Engineers design nozzle angles and pressures to achieve specific ranges, often placing systems on slopes where the inclined plane formulas determine optimal configurations. The parabolic water arcs you see from fountains and garden sprinklers are pure projectile motion.

4. Long jump and athletic performance

Olympic long jumpers launch at angles between $20-25°$ (not 45°!) because they carry substantial horizontal velocity from their run-up. Sports scientists use projectile analysis to optimize takeoff angles for maximum distance. The same principles apply to triple jump, javelin throw, shot put, and discus, each requiring different optimal angles based on initial conditions.

5. Ski jumping and winter sports

Ski jump designers use downhill projectile formulas to create safe landing zones. Jumpers launch at angles around 30° (much less than 45°) because the downward-sloping landing hill provides the vertical component naturally. The world's longest ski jumps exceed 250 metres, with flight times over 7 seconds; all predictable using our inclined plane equations.

6. Firefighting and emergency response

Fire hoses project water in parabolic arcs to reach elevated windows and distant flames. Firefighters adjust nozzle angles to maximize horizontal range or achieve specific heights, applying projectile principles under pressure. Aerial firefighting aircraft drop water or retardant, calculating release points to account for the aircraft's velocity and the target's location below.

7. Video games and animation

Every video game with jumping, throwing, or shooting implements projectile motion equations in its physics engine. Game developers use the same formulas we have studied to create realistic ball trajectories, bullet paths, and character jumps. Angry Birds, basketball simulations, and first-person shooters all run projectile calculations thousands of times per second.

8. Goalkeeper strategies in football

When a goalkeeper kicks a football from their hands, they are solving a projectile problem: what angle gives maximum distance downfield? Professional keepers typically launch at $30-40°$, balancing range against "hang time" that allows teammates to reach the landing zone. Goalkeepers intuitively account for wind and other factors, applying projectile principles.

9. Cliff diving and extreme sports

Professional cliff divers leap from heights exceeding 25 metres, with horizontal launch velocities that determine how far from the cliff they land. Divers must account for projection from height to ensure they clear dangerous rocks below and land safely in deep water. A miscalculation of just one degree can mean the difference between safety and catastrophe.

10. Fireworks displays

Pyrotechnic shells launched from mortars trace parabolic paths skyward before exploding at their maximum height. Display designers calculate launch angles and velocities to create patterns at specific altitudes and locations. Large displays coordinate hundreds of projectiles, each following calculated trajectories to burst at precisely timed moments.

11. Volcanic eruptions and natural phenomena

Nature's most spectacular demonstrations of projectile motion occur during volcanic eruptions, where molten rock follows parabolic trajectories reaching hundreds of metres. Geologists use these paths to estimate eruption velocities and predict hazard zones. Even simpler phenomena like water spraying from a broken pipe, obeys the same mathematical laws we have mastered.

We have explored the theory, derived the formulas, and seen how projectile motion shapes the world around us. Now it is time to put these principles to work through a diverse collection of problems that will test your understanding and sharpen your problem-solving skills. Ready? Let us tackle some projectiles in miscellaneous worked examples!

MISCELLANEOUS WORKED EXAMPLES ON PROJECTILE MOTION

Example 24

(a) Two projectiles are launched simultaneously from the same point with the same velocity but at complementary angles. Explain why they land at the same horizontal distance.
(b) A goalkeeper kicks a ball at 20m/s at 35° above horizontal. Taking $g = 9.8\ m/s^2$:
 (i) Calculate the maximum height reached by the ball.
 (ii) Calculate the time the ball stays in the air.
 (iii) Calculate the horizontal distance the ball travels.
 (iv) Explain why goalkeepers often prefer angles around 30−35° rather than the theoretical optimum of 45°.

Solution

(a) A projectile launched at the **smaller angle, has greater horizontal velocity and thus** moves forward faster but stays in the air for a **shorter time.** A projectile launched at the **larger complementary angle** moves forward more slowly but remains in the air for a **longer time.** These effects balance, so both land at the **same horizontal distance.**

This is supported by range formula $R = (u^2 \sin 2\theta)/g$ which depends on $\sin 2\theta$. For complementary angles θ_1 and θ_2 where $\theta_1 + \theta_2 = 90°$, we have $2\theta_1 + 2\theta_2 = 180°$. This means $2\theta_2 = 180° - 2\theta_1$, so $\sin 2\theta_2 = \sin(180° - 2\theta_1) = \sin 2\theta_1$. Since both angles produce the same $\sin 2\theta$ value, they give identical ranges.

(b) The solution of each part is as follows:
(i) Maximum height:

$$H = \frac{u^2 \sin^2\theta}{2g} = \frac{(20\text{m/s})^2 \times \sin^2 35°}{2 \times 9.8\text{m/s}^2} = 6.71\text{ m}$$

The maximum height is 6.71m.

(ii) Time of flight:

$$T = \frac{2u\sin\theta}{g} = \frac{2 \times 20\text{m/s} \times \sin 35°}{9.8\text{m/s}^2} = 2.34\text{ s}$$

The ball stays in the air for 2.34s.

(iii) Horizontal range:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(20\text{m/s})^2 \times \sin 70°}{9.8\text{m/s}^2} = 38.4\text{m}$$

The ball travels 38.4m horizontally.

(iv) Although 45° gives maximum range mathematically, goalkeepers prefer 30° −35° for practical reasons. Such lower trajectories have smaller maximum height and shorter time of flight, allowing the ball to reach teammates faster. As a result, opponents have less time to intercept, and teammates can more easily judge and control the descending ball with moderate downward velocity (higher trajectories lead to very high downward velocity of the ball). Therefore, the slightly reduced range as result of choosing slight smaller angle is acceptable for the tactical advantages gained.

Example 25

(a) A ball is thrown from the edge of a cliff. Explain why throwing it at an angle above horizontal can produce greater horizontal range than throwing it horizontally with the same velocity.

(b) From the top of a 20m building, stone A is thrown horizontally at 15m/s while stone B is thrown at 15m/s at 30° above horizontal. Taking $g = 9.8\ m/s^2$:

(i) Calculate the time taken for stone A to hit the ground.

(ii) Calculate the time taken for stone B to hit the ground.

(iii) Determine which stone lands farther and by how much.

Solution

(a) When thrown at a **moderate** angle above horizontal from a height, the projectile rises first before falling, which increases total flight time compared to horizontal throw. Although the horizontal velocity component is reduced ($u\cos\theta < u$), the increased flight time outweighs the decreased horizontal velocity. So the projectile stays in the air longer, allowing it to travel farther horizontally despite moving horizontally at a slower rate.

(b) The solution of each part is as follows:

(i) Time of flight for horizontal projection at height is given by:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 20m}{9.8m/s^2}} = 2.02s$$

Stone A takes 2.02s to hit the ground.

(ii) Time of flight from height at an angle is given by:

$$t = \frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g} = \frac{15\sin30° + \sqrt{(15\sin30°)^2 + 2 \times 9.8 \times 20}}{9.8} = 2.93s$$

Stone B takes 2.93s to hit the ground.

(iii) Range of stone A:

$R_A = ut = 15m/s \times 2.02s = 30.3m$

Range of stone B:

$$R_B = (u\cos\theta)t = (15m/s \times \cos 30°) \times 2.93s = 38.1m$$

$$\text{Difference} = R_B - R_A = (38.1 - 30.3)m = 7.8m$$

Stone B lands **7.8m** farther than stone A.

Example 26

(a) A projectile is launched at angle θ from ground level. Explain why the projectile's speed at any height h during ascent equals its speed at the same height h during descent.

(b) A tennis ball is served from height 2.4m with velocity 25m/s horizontally. The net is 12m away and stands 0.9m high. Taking $g = 9.8\ m/s^2$:

(i) Determine whether the ball clears the net, and if so, by what margin.

(ii) Calculate the total horizontal distance the ball travels before hitting the ground.

Solution

(a) By energy conservation, at any height h, the gravitational potential energy is the same whether the projectile is ascending or descending. Since total mechanical energy remains constant (no air resistance), the kinetic energy at height h must be identical in both cases. Equal kinetic energy means equal speed (since $KE = ½mv^2$).

(b) The solution for each part is as follows:

Calculating time to reach net's horizontal position:

$$t = \frac{x}{u} = \frac{12m}{25m/s} = 0.48s$$

Time to reach net is 0.48s.

Calculating the ball's height when it reaches the net:

Decrease in height: $s_y = ½gt^2 = ½ \times 9.8m/s^2 \times (0.48s)^2 = 1.13m$

Height at net = $h_o - s_y = 2.4m - 1.13m = 1.27m$ above ground.

The ball is at height of 1.27m when it reaches the net.

Comparing the ball's height versus the net's height:

$\text{Net height} = 0.9m < \text{Ball height } (1.27m)$ (the ball will clear the net)

Clearance = $1.27m - 0.9m = 0.37m$

(i) Hence, the ball clears the net by margin of 0.37m.

The horizontal range is given by:

$$R = ut = u \times \sqrt{\frac{2h}{g}} = 25m/s \times \sqrt{\frac{2 \times 2.4m}{\frac{9.8m}{s^2}}} = 17.5m$$

(ii) The total horizontal distance the ball travels before hitting the ground is 17.5m.

Example 27

(a) Explain the difference between speed and velocity for a projectile, using the example of a ball at its maximum height.

(b) A cricket ball is thrown at 22m/s at 40° above horizontal from ground level. Taking g=9.8m/s²:

(i) Calculate the ball's speed when it is 8m above ground on its way up.

(ii) Calculate the ball's speed when it is 8m above ground on its way down.

(iii) Calculate the ball's velocity (magnitude and direction) at maximum height.

(iv) Explain why the velocities at 8m (ascending vs descending) are different even though speeds are equal.

Solution

(a) Speed is a scalar quantity (magnitude only) measuring how fast an object moves. Velocity is a vector quantity (magnitude and direction) describing both speed and direction. At maximum height, a projectile has zero vertical velocity component but non-zero horizontal velocity component, so it still has speed (equals horizontal component ucosθ) even though vertical velocity is zero. The velocity vector is purely horizontal at this point.

(b) Initial components:

$u_y = u\sin\theta = 22m/s \times \sin 40° = 14.15\ m/s$

$u_x = u\cos\theta = 22m/s \times \cos 40° = 16.85\ m/s$ (constant)

(i) At h = 8 m (ascending), using $v^2 = u^2 - 2gh$:

$v_y^2 = (14.15m/s)^2 - 2 \times 9.89.8m/s^2 \times 8m;\ v_y = 6.59m/s$ (upward)

Total speed: $v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(16.85m/s)^2 + (6.59m/s)^2} = 18.1m/s$

Speed at 8m going up is 18.1m/s.

(ii) At h = 8 m (descending):
$v_y = -6.59m/s$ (downward, same magnitude)
$v_x = 16.85m/s$ (unchanged)

Speed: $v = \sqrt{(16.85m/s)^2 + (-6.59m/s)^2} = 18.1m/s$

Speed at 8m coming down is $18.1m/s$.

(iii) At maximum height: $v_y = 0, v_x = 16.85m/s$

Velocity magnitude $= 16.85\ m/s$, direction = horizontal

At maximum height: velocity is $16.85m/s$ horizontally.

(iv) Although speeds are equal (18.1m/s in both cases), velocities differ because velocity includes direction. Ascending: vertical component points upward; descending: vertical component points downward. The horizontal component remains the same, but the different vertical directions make the velocity vectors different even though their magnitudes (speeds) are identical.

Example 28

(a) A package dropped from a moving aircraft follows a curved path as seen from the ground. Explain why the pilot sees it fall straight down.
(b) A ball is thrown at 30m/s at 50° from ground level. Taking $g = 9.8\ m/s^2$: Calculate the time interval during which the ball is below a height of 20m.

Solution

(a) This is because the motion is observed from different reference frames. From the ground observer's viewpoint, the package has horizontal velocity (inherited from the aircraft) and vertical velocity (from gravity), creating a curved path. From the pilot's viewpoint, both aircraft and package share the same horizontal velocity, so relative to the pilot, the package has zero horizontal motion and only falls vertically.
(b) Calculating the time of flight

$$T = \frac{2u\sin\theta}{g} = \frac{2 \times 30m/s \times \sin 50°}{9.8} = 4.69s$$

Total time of flight is 4.69s.

Calculating the times at which the ball is at height 20m

$s_y = (u\sin\theta)t - ½gt^2$

$20 = (30\sin 50)t - ½(9.8)t^2$

$20 = 23.0t - 4.9t^2$

$4.9t^2 - 23t + 20 = 0$

$t_1 = 1.15s$ (smaller time, ascending)

$t_2 = 3.54s$ (larger time, descending)

Calculating the time interval during which the ball is above 20m

Time interval: $\Delta t = t_2 - t_1 = 3.54s - 1.15s = 2.39s$

The ball remains above 20m for 2.39s.

Calculating the time of flight

$$T = \frac{2u\sin\theta}{g} = \frac{2 \times 30m/s \times \sin 50°}{9.8} = 4.69s$$

Total time of flight is 4.69s.

Finding difference between the time of flight and the time interval above 20m

Time interval below 20m = Total flight time − Time interval above 20m

$$= T - \Delta t = 4.69s - 2.39s = 2.3s$$

The time interval during which the ball is below a height of 20m is 2.3s.

Example 29

(a) A stone is thrown downward at an angle below horizontal from a cliff. Explain how this affects the range compared to throwing it horizontally.

(b) During a match, a goalkeeper attempts to pass the ball directly to a teammate positioned 50m away on level ground. However, an opposing defender stands directly between the goalkeeper and the teammate, positioned 1.5m in front of the teammate. The goalkeeper can kick the ball with a maximum velocity of 24m/s. Take 9.8m/s^2:

(i) Determine whether the goalkeeper can reach the teammate with a direct kick on level ground.

(ii) Calculate the two possible launch angles that would make the ball land exactly at the teammate's position.

(iii) For each angle found in (ii), calculate the maximum height reached by the ball.

(iv) If the defender is 1.8m tall and remains standing upright without jumping, determine which of the angles found in (ii) would allow the ball to pass over the defender's head before reaching the teammate.

Solution

(a) Throwing the stone downward below the horizontal gives it an initial downward velocity, so it reaches the ground in a shorter time (reduced time of flight) than if it were thrown horizontally. At the same time, its horizontal component of velocity is reduced to ucosθ, which is less than u. Therefore, the stone moves forward more slowly and for a shorter time, so its horizontal range is smaller than for a horizontal throw.

(b) The solution for each part is as follows:

(i) Finding maximum possible range on the level ground (at 45°):

$$R_{max} = \frac{u^2}{g} = \frac{(24m/s)^2}{9.8m/s^2} = 58.8m > 50m$$

Since the maximum horizontal distance travelled by the ball (58.8m) exceeds the horizontal separation between the goalkeeper and the teammate (50m), the goalkeeper can reach the teammate with a direct kick.

(ii) For level ground projection:

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\sin 2\theta = \frac{Rg}{u^2} = \frac{50m \times 9.8m/s^2}{(24m/s)^2} = 0.8507$$

$$2\theta = \sin^{-1}(0.8507) = 58.25° \text{ or } 180° - 58.3° = 121.75°$$

Hence, $\theta = 29.1°$ or $60.9°$

The two possible angles are 29.1° and 60.9°.

(iii) For $\theta = 29.1°$:

$$H_1 = \frac{u^2 \sin^2\theta}{2g} = \frac{(24m/s)^2 \times \sin^2 29.1°}{2 \times 9.8m/s^2} = 6.93m$$

At 29.1°, maximum height is 6.93m.

For $\theta = 60.1°$:

$$H_2 = \frac{u^2 \sin^2\theta}{2g} = \frac{(24m/s)^2 \times \sin^2 60.9°}{2 \times 9.8m/s^2} = 22.4m$$

At 60.9°, maximum height is 22.4m.

(iv) Since the opposing defender is 1m in front of the teammate and between the goalkeeper and the teammate, the distance from goalkeeper to defender is $50m - 1.5m = 48.5m$.

Using $t = \frac{x}{u\cos\theta}$ (From $x = (u\cos\theta)t$)

Time taken by the ball to reach the defender when $\theta = 29.1°$:

$$t = \frac{48.5\text{m}}{24\text{m/s} \times \cos 29.1°} = 2.31\text{s}$$

Vertical displacement for $\theta = 29.1°$:

$$s_y = h = u_y - \frac{1}{2}gt^2 = (24\text{m/s} \times \sin 29.1°)2.31\text{s} - \frac{1}{2} \times 9.8\text{m/s}^2 \times (2.31\text{s})^2 = 0.81\text{m} < 1.8\text{m}$$

Since the height of the ball is less than the height of the defender, the ball will not pass over the defender's head.

Time taken by the ball to reach the defender when $\theta = 60.9°$:

$$t = \frac{48.5\text{m}}{24\text{m/s} \times \cos 60.9°} = 4.16\text{s}$$

Vertical displacement for $\theta = 60.9°$:

$$s_y = h = u_y - \frac{1}{2}gt^2 = (24\text{m/s} \times \sin 60.9°)4.16\text{s} - \frac{1}{2} \times 9.8\text{m/s}^2 \times (4.16\text{s})^2 = 2.52\text{m} > 1.8\text{m}$$

Since the height of the ball is greater than the height of the defender, the ball will pass over the defender's head.

Hence, the suitable angle is $60.9°$.

Alternative solution for (b)(iv)

By using trajectory equation:

$$y = x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$$

For $\theta = 29.1°$:

$$y = 48.5\tan 29.1° - \left(\frac{9.8}{2 \times 24^2\cos^2 29.1}\right)48.5^2 = 0.82\text{m} \approx 0.81\text{m as before.}$$

For $\theta = 60.9°$:

$$y = 48.5\tan 60.9° - \left(\frac{9.8}{2 \times 24^2\cos^2 60.9}\right)48.5^2 = 2.57\text{m} \approx 2.52\text{m as before.}$$

Example 30

(a) A projectile is launched with a certain velocity from the ground. Explain why increasing the launch angle beyond 45° reduces the horizontal range on level ground.

(b) Ball A is thrown from ground level with velocity of 20 m/s at angle 60° above horizontal. One second later, ball B is thrown from the same point at angle 45° above horizontal. Find the magnitude of initial velocity u_B so that both balls land at the same time. Take $g = 9.8$ m/s².

Solution

(a) Increasing the angle beyond 45° directs too much velocity upward (vertical component) and not enough forward (horizontal component). Although a larger launch angle results in a greater maximum height and hence a longer time of flight, the reduced horizontal velocity means it covers less horizontal distance during that time. The optimal 45° balances vertical and horizontal components to maximize range.

(b) Time of light of Ball A (launched at t = 0):

$$T_A = \frac{2u_A\sin\theta_A}{g} = \frac{2 \times 20\text{m/s} \times \sin 60°}{9.8\text{m/s}^2} = 3.54\text{s}$$

Ball A lands at $t = 3.54\text{s}$.

For ball B to land at the same time as ball A, it must land at absolute time t = 3.54s.

Since ball B is launched at t = 1s, (1s delay) its flight time is:

$$T_B = 3.54\text{s} - 1\text{s} = 2.54\text{s}$$

Again, using time of flight formula:

$$T_B = \frac{2u_B \sin\theta_B}{g}$$

From which:

$$u_B = \frac{gT_B}{2\sin\theta_B} = \frac{9.8m/s^2 \times 2.54s}{2 \times \sin 45°} = 17.6m/s$$

For both balls to land simultaneously, the magnitude of initial velocity u_B must be $17.6m/s$.

Example 31

(a) On an inclined plane, explain why the optimal launch angle for maximum range is **not** 45° but shifts depending on the plane's slope.

(b) A ball is thrown at 18m/s from the bottom of a hill sloping upward at 25°. Take g = 9.8 m/s²:

- (i) Calculate the optimal launch angle (from horizontal) for maximum range up the slope.
- (ii) Calculate this maximum range along the slope.
- (iii) Calculate how high above the launch point the ball lands.

Solution

(a) On level ground, 45° balances horizontal and vertical components optimally. On an upward slope at angle β, the ground rises to meet the projectile, shortening effective flight time. To compensate, the launch must be steeper to gain more vertical velocity and extend time aloft despite the rising ground. Thus the optimal angle increases by half the slope angle:

$$\theta_{optimum} = 45° + \frac{\beta}{2}$$

Conversely, on a downward slope (β negative), the ground falls away, allowing flatter trajectories, and the optimal angle decreases by half the slope angle:

$$\theta_{optimum} = 45° - \frac{|\beta|}{2}$$

Hence, the slope breaks the symmetry that makes 45° optimal on level ground.

(b) For upward slope β = 25°:

(i) $\theta_{optimal} = 45° + \frac{\beta}{2} = 45° + 12.5° = 57.5°$

Optimal angle is 57.5° above horizontal.

(ii) Maximum range is found from range formula along the incline:

$$L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)$$

For optimal angle, ($\theta = 57.5°$), L becomes maximum.

Substituting values:

$$L_{max} = \frac{(18m/s)^2}{9.8m/s^2 \times \cos^2(25°)}(\sin(2 \times 57.5° - 25°) - \sin 25°) = 23.24m$$

The maximum range along the slope is 23.24m.

(iii) **Vertical height gain *h*** is found from:

$$\mathbf{h = L\sin\beta} \qquad \left(\text{since } \sin\beta = \frac{h}{L}\right)$$

Substituting values:

$$h = 23.24m \times \sin 25° = 9.82m$$

The ball lands 9.82m above launch point.

Example 32

(a) A projectile's trajectory is parabolic and symmetric. Explain why the projectile spends equal time ascending and descending when launched from and landing on level ground.

(b) Two projectiles are launched simultaneously from the same point at the same velocity 20m/s: projectile A on level ground at 45°, and projectile B down a 30° slope at optimal angle for that slope. Taking g = 9.8 m/s²:

(i) Calculate the range and flight time for projectile A on level ground.

(ii) Calculate the optimal angle and range for projectile B on the downslope.

(iii) Calculate the flight time for projectile B.

(iv) When projectile A lands, determine how far projectile B has travelled along the slope from the launch point. Hence calculate the additional distance along the slope that projectile B will travel before it lands.

Solution

(a) By symmetry, the upward path mirrors the downward path. Starting from ground with vertical velocity usinθ (upward), the ball decelerates uniformly at rate *g* until $v_y = 0$ at maximum height. During descent from maximum height back to ground level, it accelerates uniformly at rate *g* from $v_y = 0$ back to usinθ (downward). Since both phases involve the same change in velocity (usinθ) and the same acceleration (g), they take equal time.

(b) The solution for each part is as follows:

(i) At 45°:

$$R_A = \frac{u^2}{g} = \frac{(20\text{m/s})^2}{9.8\text{m/s}^2} = 40.8\text{ m}$$

$$T_A = \frac{2u\sin\theta}{g} = \frac{2 \times 20\text{m/s} \times \sin 45°}{9.8\text{m/s}^2} = 2.89\text{s}$$

For projectile A:

Horizontal range is 40.8m,

Flight time is 2.89s.

(ii) Projectile B (30° downslope, $\beta = -30°$):

Optimal angle:

$$\theta_{optimal} = 45° + \frac{\beta}{2} = 45° - \frac{30°}{2} = 30°$$

Using:

$$L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)$$

Substituting values:

$$L = \frac{(20\text{m/s})^2}{9.8\text{m/s}^2 \times \cos^2(-30°)}(\sin(2 \times 30° + 30°) - \sin(-30°)) = 81.63\text{m}$$

For projectile B:

Optimal angle is 30°,

Horizontal range is 81.63m along slope.

(iii) Flight time of projectile B:

$$T_B = \frac{2u\sin(\theta - \beta)}{g\cos\beta} = \frac{2 \times 20\text{m/s} \times \sin(30° + 30°)}{9.8\text{m/s}^2 \times \cos(-30°)} = 4.08s$$

Flight time of projectile B is 4.08s.

(iv) Projectile A lands at 2.89s. At this time, projectile B is still in flight since its time of flight (4.08s) is greater than 2.89s.

Horizontal position of B at t = 2.89s:

$$x = (u\cos\theta)t = (20\text{m/s} \times \cos 30°) \times 2.89\text{s} = 50.05\text{m}$$

Displacement along the slope, *l*:

From:

$$\cos\theta = \frac{x}{l};\ l = \frac{x}{\cos\theta} = \frac{50.05\text{m}}{\cos 30°} = 57.8\text{m}$$

When A lands, B has travelled 57.8m along slope.

Remaining distance B will travel $= 81.63\text{m} - 57.8\text{m} = 23.8\text{m}$

B will travel another 23.8m along the slope before landing.

Example 33

(a) In long jump, athletes try to jump forward as far as possible. Does jumping very high help increase the horizontal distance? Explain.

(b) A military aircraft is flying horizontally at a constant velocity of 60m/s at a height of 200m above a defended region. At the instant the aircraft passes directly above a defence base, it releases a bomb. At the same instant, the air-defence system launches an interceptor projectile from the ground with a velocity of 90m/s at an angle **θ** above the horizontal in order to destroy the bomb before it reaches the ground. Neglect air resistance and take $g = 9.8\ \text{m/s}^2$; determine:

(i) The angle of projection **θ** required for the interceptor to collide with the bomb in mid-air.

(ii) The time after release when the interception occurs.

(iii) The position where the collision occurs.

Solution

(a) The effect of jumping very high on the achieved horizontal distance depends on the reason behind that high jump. If it is caused by a greater take-off (initial) velocity at the same angle, it helps to achieve greater horizontal distance. This can be seen from the relation: $R = 4H\cot\theta$, from which if the **angle θ is fixed**, then **cotθ is constant**. Therefore, $\mathbf{R \propto H}$. So in this case, a higher jump means a longer jump.
However, if the athlete jumps higher by using a larger take-off angle, the horizontal component of velocity is reduced, and the horizontal distance (range) may decrease.
Therefore, jumping higher helps only when it comes from greater take-off velocity without losing too much horizontal velocity.

(b) The solution for each part is as follows:

(i) Horizontal positions must be equal at collision.

For bomb, b (released from aircraft): Bomb velocity = Aircraft velocity $= 60\text{m/s}$ (horizontally):

$$x_b = 60t$$

For interceptor, i:

$$x_i = (90\cos\theta)t$$

At collision: $\mathbf{x_b = x_i}$

$$60t = (90\cos\theta)t$$

$$\cos\theta = \frac{60}{90} = 0.6667$$

$$\theta = \cos^{-1}(0.6667) = 48.19°$$

The required angle of projection is 48.19°.

(ii) The vertical positions must also be equal. Therefore, if the bomb has descended through a vertical distance **b**, the interceptor must have risen through a vertical distance **200 − b**.

For bomb: $u_y = 0\text{m/s}$ (aircraft was flying horizontally), $s_y = -\mathbf{b}$ (downward, negative)

Using $s_y = -\tfrac{1}{2}gt^2$ (for $u_y = 0\text{m/s}$)

Substituting values:

$-b = -\tfrac{1}{2} \times 9.8 \times t^2$

$b = 4.9t^2$ ... (i)

For interceptor:

Using $s_y = u_y t - ½gt^2$

Substituting values:

$200 - b = (90\sin 48.19°)t - ½ \times 9.8 \times t^2$

$b = 200 + 4.9t^2 - (90\sin 48.19°)t$ ... (ii)

At collision:

Equating (i) and (ii):

$4.9t^2 = 200 + 4.9t^2 - (90\sin 48.19°)t$

$$t = \frac{200}{90\sin 48.19°} = 2.99\text{s}$$

The interception occurs after 2.99s.

Alternative solution for (b)(ii)

The quicker method of solving (b)(ii) is by using relative motion concept:

Relative vertical acceleration, $a_R = a_i - a_b = -9.8\text{m/s}^2 - (-9.8\text{m/s}^2) = 0$

And;

Relative initial vertical velocity, $u_R = u_i - u_b = 90\sin 48.19° - 0\text{m/s} = 90\sin 48.19°$

Then, the equation $s_y = u_y t + ½a_y t^2$ becomes:

$$s_y = u_R t + \frac{1}{2}a_R t^2$$

Substituting values:

$200 = (90\sin 48.19°)t + 0$

$$\mathbf{t} = \frac{200}{90\sin 48.19°} = \mathbf{2.99s}$$

(iii) At $t = 2.99$s:

Horizontal position:

$x = 60\text{m/s} \times 2.99\text{s} = 179.4\text{m}$

Vertical position:

$y = 200 - b = 200 - (½)(9.8)(2.99)^2 = 156.2\text{m}$

They meet at position (179.4m, 156.2m); that is, **179.4m horizontally from the defence base and 156.2m above the ground**.

Having enjoyed the full combination of ideas, it is time to sharpen our thinking; the Digging Deeper Exercise is ready in the next page.

DIGGING DEEPER EXERCISE 6

EXERCISE 6A: BINDER QUESTIONS

Question 1

A stone is thrown horizontally from a cliff. Explain why the horizontal velocity remains constant throughout the flight while the vertical velocity continuously increases.

Question 2

Explain why does a projectile trajectory form a parabola rather than any other curve?

Question 3

Explain why a projectile launched from a height travels farther horizontally than the same projectile launched at the same angle and velocity from ground level.

Question 4

At the highest point of a projectile's trajectory, explain why the velocity is not zero even though the vertical velocity component is zero.

Question 5

Explain why throwing a ball downward at an angle from a cliff results in shorter horizontal range than throwing it horizontally at the same velocity.

Question 6

Two identical balls are thrown simultaneously from the same cliff: one horizontally and one at 30° above horizontal, both with the same velocity. Explain which ball hits the ground first and why.

Question 7

A ball is thrown at certain angle from ground level. Explain why increasing the initial velocity increases both the maximum height and the horizontal range.

Question 8

Explain why a projectile's horizontal range on level ground cannot exceed $\frac{u^2}{g}$ regardless of launch angle.

EXERCISE 6B: REAL QUESTIONS

Question 9

Why does water from a fountain follow a curved path?

Question 10

Explain why a fountain designed to spray water to a specific distance on level ground requires choosing between two different nozzle angles.

Question 11

A firefighter directs a water jet from a hose to extinguish flames on the upper floor of a building. The flames are at the same horizontal distance whether the water is aimed at 30° or 60° above horizontal. However, the firefighter chooses the 30° angle. Explain the practical advantages of using the lower angle despite both angles reaching the same distance.

Question 12

During a basketball practice session, Kipanga attempts several three-point shots. After many misses, he becomes frustrated.

"Sir, I'm shooting with good power, but the ball keeps hitting the front rim!" **Kipanga** complains to Mr. Akilikubwa.

Mr. Akilikubwa observes, *"Your shots are too flat; you're using too much horizontal velocity and not enough arc."*

Kipute adds, *"I notice that when you shoot flatter, the ball arrives at the basket while still rising or level. But when I shoot with more arc, the ball is descending when it goes through the hoop."*

"Exactly right, Kipute," says **Mr. Akilikubwa.** *"That's the key difference between a good shot and a miss."*

Explain why a ball arriving at the basket while still rising or traveling horizontally is more likely to miss than a ball that is descending.

Question 13

A gardener uses a hose to water plants at various distances. He notices that tilting the nozzle slightly downward from horizontal reduces the distance the water travels. Explain why aiming downward decreases the range even though it seems like it should help the water "fall" toward the target faster.

Question 14

Mountain rescue teams sometimes need to drop supplies to stranded climbers. When dropping from a hovering helicopter, the package lands some distance away from the helicopter rather than directly below. Explain why this happens and how the pilot can ensure the package lands at the desired location.

Question 15

A stone falls from a tall bridge into a river below. A observer standing on the bridge sees the stone fall straight down. However, a person on a moving boat passing under the bridge at the moment of release sees the stone follow a curved path. Explain how both observers can be correct.

Question 16

In the sport of shot put, athletes must release the shot from within a throwing circle and the shot must land in a marked sector. Why do shot putters release at angles around 35−40° rather than 45°, which gives maximum range in theory?

EXERCISE 6C: HOT QUESTIONS

Take $g = 9.8ms^{-2}$

Question 17

A ball is projected from the ground with velocity $25ms^{-1}$. A wall 15m high stands 40m away.

(a) Derive the expression for the height of the projectile as a function of horizontal distance.
(b) Determine the minimum angle of projection required for the ball to just clear the wall.

Question 18

A projectile is fired with velocity $40ms^{-1}$ toward a target located 50m horizontally away and 30m above the launch point. Determine the possible angles of projection that allow the projectile to hit the target.

Question 19

A projectile is launched with velocity $20ms^{-1}$ at an angle 45° above the horizontal toward a plane inclined at 30° above the horizontal.

(a) Determine the horizontal distance from the point of projection where the projectile strikes the plane.
(b) Hence determine the distance measured along the slope.

Question 20

A ball is dropped from rest from a height of 40m. At the same instant another ball is projected horizontally with velocity $15ms^{-1}$ from a point 30m away.

(a) Determine the time taken for the horizontally projected ball to reach the vertical line beneath the falling ball.
(b) Determine the vertical positions of both balls at that instant.
(c) State whether the two balls collide.

Question 21

Projectile A is fired vertically upward with velocity $25ms^{-1}$. At the same instant projectile B is fired from ground 40m away toward A with velocity $30ms^{-1}$ at 45°. Determine whether the two projectiles collide.

Question 22

A cart moves along a straight horizontal track away from the point of projection with a constant velocity of $5ms^{-1}$. At the instant the cart is at a distance x from the launch point, a projectile is fired from the ground with velocity $25ms^{-1}$ at an angle of 45° to the horizontal. Find the distance x from the launch point at which the cart must be located at the instant of projection so that the projectile lands in the cart.

Question 23

A projectile launched from level ground with speed u can achieve the same range R with two different launch angles θ_1 and θ_2 (where $\theta_1 > \theta_2$). Let H_1 and H_2 be the maximum heights and T_1 and T_2 the times of flight for the two trajectories.

Prove the following four results:

(a) $\theta_1 + \theta_2 = 90°$

(b) $\frac{T_1}{T_2} = \tan\theta_1$

(c) $\frac{H_1}{H_2} = \tan^2\theta_1$

(d) $H_1H_2 = R^2/16$

Question 24

A projectile is launched from level ground with speed u at angle θ above the horizontal.

(a) Show that the condition for the maximum height H to equal the horizontal range R is $\tan\theta = 4$.

(b) Calculate the launch angle.

(c) If $u = 30ms^{-1}$, calculate the common value of H and R.

Question 25

Two footballers stand 60m apart on level ground. At the same instant, each kicks a ball toward the other. Player A kicks at 30° above the horizontal with speed $25ms^{-1}$. Player B kicks at 60° above the horizontal.

(a) Find the speed at which Player B must kick for the two balls to collide in mid-air.

(b) Find the time and height of the collision.

(c) Find the horizontal distance from Player A to the collision point.

ANSWERS

EXERCISE 6A

1. In the horizontal direction, there is no force acting on the stone (ignoring air resistance), so by Newton's first law, the horizontal velocity remains constant. In the vertical direction, gravity acts continuously downward with constant acceleration $g = 9.8\ m/s^2$, causing the vertical velocity to increase steadily.

2. The trajectory is parabolic because horizontal displacement increases linearly with time ($x = ut$) while vertical displacement varies with the square of time ($y = ut - ½gt^2$). When we eliminate time t from these equations, we get $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$, which is a quadratic equation in x. Any quadratic relationship between y and x represents a parabola. Thus, the parabolic shape results from the combination of uniform horizontal motion and uniformly accelerated vertical motion.

3. Launching from height h increases the total flight time. The projectile first completes its normal parabolic arc (as if on level ground), then continues falling through the additional height h before landing. This extra descent time allows more horizontal distance to accumulate since horizontal velocity remains constant. This is supported by formula $t = \frac{u\sin\theta + \sqrt{(u\sin\theta)^2 + 2gh}}{g}$, which shows that flight time increases with h, and since $R = (u\cos\theta)t$, greater time means greater range (horizontal distance).

4. At maximum height, only the vertical velocity component becomes zero ($v_y = 0$), but the horizontal velocity component remains unchanged throughout flight ($v_x = u\cos\theta$). So the total velocity vector at maximum height is purely horizontal with magnitude $v = u\cos\theta$. Consequently, the projectile continues moving forward even at its peak because no horizontal force acts to stop it; it simply has no upward or downward motion at that instant.

5. Throwing downward adds initial downward velocity, causing faster descent and thus reduced time of flight. Additionally, the horizontal velocity component is reduced to $u\cos\theta < u$ for any downward angle. Both factors work against range: shorter time of flight and slower horizontal velocity combine to produce less horizontal distance. Horizontal throw maximizes horizontal velocity (all velocity directed forward) and maximizes flight time among, giving maximum horizontal range.

6. The horizontal throw hits first. Both start at the same height, but the angled throw has an upward vertical component that makes it rise first before falling, while the horizontal throw has zero initial vertical velocity and begins falling immediately. Since both experience the same downward acceleration g, the one that starts descending immediately (horizontal throw) reaches ground sooner. The angled throw must first decelerate upward, stop at maximum height, then accelerate downward, taking longer overall.

7. Both maximum height $H = \frac{u^2\sin^2\theta}{2g}$ and range $R = \frac{u^2\sin 2\theta}{g}$ are proportional to u^2, so doubling velocity quadruples their amounts. Physically, greater initial velocity provides more kinetic energy. In the vertical direction, this converts to greater potential energy (peak of greater height). In the horizontal direction, increased initial velocity means greater velocity maintained throughout flight ($v_x = u\cos\theta$ unchanged) combined with longer flight time (from greater height) produces much greater range.

8. Range formula $R = \frac{u^2\sin 2\theta}{g}$ shows R is proportional to $\sin 2\theta$. The maximum value of sine function is 1, which is achieved when $2\theta = 90°$, giving $\theta = 45°$. At this angle, $R = \frac{u^2}{g}$, which is the maximum possible range for given velocity u. Any other angle gives $\sin 2\theta < 1$, resulting in $R < \frac{u^2}{g}$. This represents a fundamental limit imposed by the combination of initial kinetic energy and gravitational acceleration; no choice of angle can overcome this limit.

EXERCISE 6B

9. When the water leaves the nozzle, it has an initial horizontal velocity. After leaving the nozzle, gravity acts downward while the horizontal motion continues unchanged (neglecting air resistance). The combination of constant horizontal motion and downward acceleration produces a parabolic path.

10. For any target distance R less than maximum range ($R < \frac{u^2}{g}$), the range equation $R = \frac{u^2\sin 2\theta}{g}$ has two solutions for 2θ: one acute angle, and its supplement. These correspond to two launch angles that are complementary (summing to 90°). One angle is shallow (below 45°), giving a low, fast arc; the other is steep (above 45°), giving a high, slow arc. Both reach the same distance but follow different paths. Fountain designers typically **choose** the steeper angle for aesthetic appeal as the higher arc is more visually dramatic and impressive.

11. Although both angles reach the same horizontal distance (complementary angles), the 30° angle has several practical advantages.

1. The water arrives faster because it travels a lower, shorter arc, allowing quicker response to the fire.

2. The lower trajectory means water arrives with greater horizontal velocity component, providing more impact force to penetrate smoke and reach the base of flames.
3. The flatter arc is easier to aim and control as the firefighter can see the water path more clearly and make quick adjustments.
4. The 30° jet wastes less water on excessive height (water spends less time in the air and thus experiences less loss due to air resistance and dispersion), directing more energy toward forward reach rather than vertical climb.

12. When the ball approaches the rim while rising or moving nearly horizontally, it has little or upward vertical velocity. If it strikes the rim, this vertical motion tends to deflect the ball upward and away from the basket. When the ball approaches while descending, it has a downward vertical velocity. If it hits the rim, this downward momentum helps guide the ball into the basket rather than causing it to bounce away. Additionally, a rising ball that misses the hoop continues upward and away, whereas a descending ball that is close to the rim may still drop through the basket.

13. Aiming downward reduces range because of two compounding effects. First, the downward angle reduces the horizontal component of velocity. Second, adding downward initial velocity makes the water hit the ground faster, reducing flight time. Slower horizontal velocity and shorter time of flight combine to produce much less horizontal distance.

14. When a package is released from a helicopter, it retains the horizontal velocity it had at the moment of release. Even if the helicopter appears to hover, it usually has small horizontal motions due to wind or position adjustments. The package therefore continues with this horizontal motion while gravity pulls it downward, producing a parabolic path that may land away from the release point.

15. Both observers are correct because they describe motion from different reference frames. The bridge observer is stationary relative to the stone's release point. Since the stone is simply dropped (zero horizontal velocity relative to the bridge), it falls straight down under gravity; pure vertical motion.

The boat observer moves horizontally relative to the bridge. From the boat's reference frame, the stone has horizontal velocity equal and opposite to the boat's motion at the moment of release. The boat observer sees the stone combine this horizontal component with vertical acceleration due to gravity, creating a parabolic trajectory.

16. Shot putters release below 45° because they do not launch from ground level; they release from shoulder height. When launching from an elevated position, the optimal angle for maximum range is less than 45°. The additional height provides extra flight time even without much upward velocity, so directing more velocity horizontally (flatter angle) produces greater range than 45° would. Additionally, in real conditions air resistance cannot be neglected and it (air resistance) reduces the horizontal range more strongly for higher angles due to longer time of flight at larger angle. As a result, the optimal angle becomes smaller than 45°.

EXERCISE 6C

17. (a) $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$ (b) 45.2°

18. $\theta_1 = 41°$, $\theta_2 = 80°$

19. (a) 28.3m (b) 32.7m

20. (a) 2s (b) $y_1 = 20.4$m, $y_2 = 20.4$m (c) Since both balls have the same horizontal and vertical position, the **balls collide**.

21. Since heights differ, **collision does not occur** (At t =1.89s; y_A (29.8m) $\neq y_B$(22.6m))

22. x=45.75m

23. (a) From $R = \frac{u^2\sin2\theta}{g}$, equal ranges require $\sin2\theta_1 = \sin2\theta_2$. Since $\theta_1 \neq \theta_2$:

$$2\theta_1 = 180° - 2\theta_2 \quad \Rightarrow 2\theta_1 + 2\theta_2 = 180° \Rightarrow \theta_1 + \theta_2 = 90°$$

(b) $T_1 = \frac{2u\sin\theta_1}{g}$ and $T_2 = \frac{2u\cos\theta_1}{g}$:

$$\frac{T_1}{T_2} = \frac{\sin\theta_1}{\cos\theta_1} = \tan\theta_1$$

(c) $H_1 = \frac{u^2\sin^2\theta_1}{2g}$ and $H_2 = \frac{u^2\cos^2\theta_1}{2g}$:

$$\frac{H_1}{H_2} = \frac{\sin^2\theta_1}{\cos^2\theta_1} = \tan^2\theta_1$$

(d) The product:

$$H_1 \times H_2 = \frac{u^2\sin^2\theta_1}{2g} \times \frac{u^2\cos^2\theta_1}{2g} = \frac{u^4\sin^2\theta_1\cos^2\theta_1}{4g^2} = \frac{u^4\sin^2 2\theta_1}{16g^2}$$

Since $R = \frac{u^2\sin2\theta_1}{g}$:

$$R^2 = \frac{u^4 \sin^2 2\theta_1}{g^2}$$

Therefore:

$$H_1 H_2 = \frac{R^2}{16}$$

24. (a) Maximum height: $H = \frac{u^2 \sin^2\theta}{2g}$

Range: $R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin\theta\cos\theta}{g}$

Setting $H = R$:

$$\frac{u^2 \sin^2\theta}{2g} = \frac{2u^2 \sin\theta\cos\theta}{g}$$

Dividing both sides by $u^2 \sin\theta/g$:

$$\frac{\sin\theta}{2} = 2\cos\theta$$

$$\tan\theta = 4$$

(b) $76°$ (c) 43.2m

25. (a) $14.43 ms^{-1}$ (b) 2.08s, 4.8m (c) 45m from Player A.