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FOUNDATIONS OF PHYSICS

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Enjoy & Master Advanced Physics
Foundations of Physics: Measurement, Motion & Mechanics

Ngaiza Series of Advanced Physics

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PREFACE

This book was written because you deserve better.

Advanced Level physics demands deep understanding, yet many students arrive in Form Five having learned to memorise formulas without knowing where they come from and to copy worked examples without understanding the reasoning behind them. They memorise what they do not understand, pass examinations they cannot apply, and leave Form Six believing that physics is a collection of equations to be recalled under pressure. That is not physics. That is memorisation dressed as understanding.

Physics is the study of how the universe works. It is the reason the Nyerere Bridge stands, rockets leave the atmosphere, and a phone screen responds to your touch. It is powerful, deeply satisfying, and far more logical than its reputation suggests. But understanding it requires a different kind of textbook: one that explains *why* before *how*, that derives every formula instead of dropping it from the sky, and that tests your thinking, not just your memory. That is what this book sets out to be.

Why Mechanics?

Every branch of Advanced Level physics, from electricity to waves to thermal physics, uses the language and tools of mechanics. Force, energy, momentum, and motion are not just one topic among many; they are the foundation on which every other topic is built. A student who truly understands mechanics can learn any other branch of physics with confidence. A student who does not will struggle in every chapter that follows, no matter how hard they work. This is why this book is dedicated entirely to mechanics, and why we have given it the depth it deserves.

How this book is organised

The book is divided into three parts:

Part One: Foundations of Motion and Measurement lays the groundwork. It includes two chapters (Chapters 2 and 3) that revisit topics introduced at O-level: linear motion and Newton's laws. These chapters are not strictly in the Advanced Level syllabus, but do not be tempted to skip them. The A-level treatment demands a depth of understanding that O-level only introduced. Important concepts of motion and the Newton's laws are built here with the rigour that later chapters assume. Students who skip Part One invariably struggle in Parts Two and Three. Students who invest in it find that the rest of the book flows naturally.

Part Two: Particle Mechanics covers the core A-level syllabus: equilibrant forces, momentum and collisions, projectile motion, uniform circular motion, and gravitation.

Part Three: Extended Mechanics takes mechanics further into rotation of rigid bodies and fluid mechanics.

Within each chapter, the content is built in layers. Concepts are developed through narrative explanations, followed by in-text worked examples in three tiers:

BINDER examples anchor the basics: if you can do these, you own the concept.

REAL examples connect physics to everyday Tanzanian life, often through character dialogue, showing that physics is not confined to the laboratory.

HOT examples push you to examination standard and beyond, combining multiple concepts and demanding strategy.

Each in-text example ends with **Making Sense of the Answer** (which checks whether the result is physically reasonable) and **Think Like a Physicist** (which teaches a transferable problem-solving habit you can carry to any future problem).

At the end of each chapter, **Miscellaneous Worked Examples** combine ideas from across the chapter in the way real examinations do, followed by a **Digging Deeper Exercise** with questions in the same three tiers (BINDER, REAL, HOT).

At the end of each part, **Examination Questions** modelled on NECTA, Mock, Regional, and Commonwealth papers test everything from that part.

Near the end of the book, you will find complete answers to every Digging Deeper question, including full explanations for the explanatory questions, not just numerical answers. Full worked solutions to all

Examination Questions are also provided. Nothing is left unanswered. If you are stuck, the solution is there to learn from, but only after you have attempted the problem honestly with the book closed.

Three people will accompany you through every chapter.

Kipanga is the student who volunteers first and thinks afterwards. He is confident, energetic, and wrong more often than he would like to admit. But he learns from every mistake, and his mistakes will teach you too, because they are the same mistakes real students make in real classrooms.

Kipute is the student who sits in the second row with her notebook already open. She is curious, analytical, and asks the questions that separate good students from great ones. When Kipute challenges an idea, pay attention: the answer usually contains something important.

Mr. Akilikubwa is the teacher who never raises his voice but somehow makes every student listen. His explanations use vivid analogies drawn from everyday Tanzanian life, and his patience is legendary. When he speaks, the concept lands and stays.

These three characters are not decorations. They carry the teaching. Their dialogues address the exact misconceptions, confusions, and “almost-right” ideas that real students struggle with every day.

The team behind this book

Enjoy and Master Advanced Physics is the product of the Ngaiza Education Hub (NEH). It was built by a team of experienced physics educators who share a common conviction: that Tanzanian students deserve world-class teaching materials written for their context, their syllabus, and their aspirations.

Writers: Mr. Amir Juma, Mr. Kelvin Mashu, Mr. Endrew Mariki, and Mr. Morris Mulokozi.

Editors: Dr. Aloyce Julian and Dr. Alex Muganyizi.

Designer: Mr. Frank Mwita.

I am grateful to every member of this team for their dedication, their expertise, and their refusal to settle for “good enough.” I also thank the students whose hard questions and honest frustrations shaped every chapter. They reminded us, constantly, that a textbook which cannot be understood by the student it was written for has failed, no matter how correct its physics may be.

To every Form Five student opening this book for the first time: physics is not as hard as its reputation. It is as logical as a well-told story, as satisfying as a puzzle that clicks, and as useful as any skill you will ever learn. Give it your honest effort. This book will give you its honest best in return.

Ngaiza, Lusima

Series Creator and Director

This book is dedicated to every Tanzanian student who was told physics is too hard. It is not. And this book will prove it to you.

STUDENT SURVIVAL GUIDE

What Your Teacher Wants to Tell You But Runs Out of Time

Mr. Akilikubwa once told his Form Five class on the first day of the year: “*I can teach you every formula in the syllabus. But if you do not know how to study, how to fail, how to revise, and how to sit an examination, the formulas will not save you. So before we open a single chapter, let us talk about the things nobody puts in a physics textbook.*”

This is that talk.

How to Study Physics

Physics is not a reading subject. You cannot learn physics by reading notes the way you read history or literature. Physics enters through the pen, not the eyes. If your hand is not writing, your brain is resting.

Solve problems with the book closed. This is the single most important study habit in physics. Open the book, study the concept, study the worked example, then close the book and try a similar problem from memory. If you cannot do it, you have discovered exactly where your understanding breaks. That is not failure; that is the most useful information you can get. Go back, study the gap, close the book, and try again. Repeat until you can solve it without help.

Ask “why,” not “what.” A student who asks “what is the formula for kinetic energy?” will memorise $KE = \frac{1}{2}mv^2$ and forget it after the exam. A student who asks “*why does kinetic energy depend on v^2 and not v ?*” will understand it, remember it, and recognise it in problems they have never seen before. Every formula in physics has a reason for its form. Find that reason, and the formula remembers itself.

Draw before you calculate. Before writing a single equation, draw the physical situation. Label every force, every velocity, every distance. A clear diagram is worth more than a page of algebra, because it shows you what the problem actually looks like. Many students lose marks not because they cannot do the mathematics but because they set up the wrong equation, and they set up the wrong equation because they never drew the picture.

Carry units through every line of working. A number without a unit is a number without meaning. If you write “ $v = 25$ ” you have said nothing. Write “ $v = 25\text{ms}^{-1}$ ” and you have said something precise. But more importantly, carrying units through your calculation catches errors automatically. If you are calculating a force and your answer comes out in metres, something has gone wrong, and the units will tell you before the examiner does.

Convert to SI before substituting. Every formula in this book is written in SI units. If the question gives you a length in centimetres or a speed in kilometres per hour, convert first, substitute second. This single habit prevents more errors than any other piece of advice in this guide.

Study in short focused blocks. Two hours of concentrated problem-solving, fully committed with no distractions, teaches more than six hours of interrupted reading with conversations, daydreaming, and breaks every fifteen minutes. When you sit down to study physics, commit fully for a defined period, then stop and rest. Quality defeats quantity every time.

Do not skip the BINDER examples. They look simple. They are simple. That is the point. They build the foundation that the REAL and HOT examples stand on. A student who skips BINDER examples and jumps to HOT examples is a student who skips the ground floor and tries to enter a building through the third-floor window. It does not work, and the landing is painful.

How to Fail Productively

You will get problems wrong. Every student does. The question is not whether you fail, but what you do in the five minutes after you fail.

The unproductive response: erase your working, open the solution, copy it neatly into your notebook, and move on feeling satisfied. Your notebook looks complete. Your brain learned nothing. It was a spectator watching someone else solve the problem.

The productive response: stop. Do not erase. Look at your wrong answer and ask: where exactly did this go wrong? Was it the physics (wrong concept), the mathematics (algebraic mistake), or the setup (wrong

diagram, wrong equation chosen)? Find the exact point of failure. Fix that one thing. Close the solution. Try the problem again from the beginning.

Kipanga fails constantly in this book. He measures rooms with his feet, claims love is a physical quantity, and proves that trouble has dimensions of energy. But he is present for every correction, and by the end of each chapter he understands the concept because he engaged with his own mistakes instead of hiding from them. That is the model. Fail like Kipanga: openly, frequently, and productively.

How to Revise

Most revision looks productive but is not. Be honest with yourself about which category your revision falls into.

What does not work: re-reading notes (your eyes move, your brain rests), highlighting text (colouring is not understanding), recopying notes neatly (beautiful handwriting is art, not physics), and reading worked solutions without attempting the problem first (you are watching someone else exercise and expecting to get fit).

What works:

Solve past examination papers under timed conditions. This is the single most effective revision strategy for physics. It simulates the real pressure, exposes your weak topics, and trains you to manage time. Do not check the answers until you have finished the entire paper.

Explain a concept out loud as if teaching someone who knows nothing. If you cannot explain it clearly in your own words, you do not understand it yet. The places where you stumble are the places that need more work.

Teach a friend. Teaching forces you to organise your thinking and exposes gaps instantly. When your friend asks “but why?” and you cannot answer, you have found your revision target for the evening.

Work through the Digging Deeper exercises with the book closed. The textbook is your teacher. The examination is your test. You must practise performing without the teacher in the room.

Write your own summary after finishing each chapter, not the night before the exam. Summarising forces you to decide what matters. If you cannot compress a chapter into one page of key ideas and formulas, you have not understood it well enough. The act of writing the summary is the revision; the summary itself is just a souvenir.

Revise your weakest topic first, not your favourite. The topic you enjoy is the topic you already understand. The topic you avoid is the topic that will appear on the paper. Physics has a talent for examining exactly what you hoped it would not.

How to Discuss

Physics is not a solo sport. The best students do not study alone; they discuss, argue, challenge, and teach each other. A good discussion group is one of the most powerful learning tools available to you. A bad one is a waste of everyone’s time. Here is how to make yours good.

Come prepared. A discussion group is not a place to see the material for the first time. Read the topic, attempt the problems, and identify your specific questions before the meeting. A student who arrives saying “I do not understand Chapter 6” wastes the group’s time. A student who arrives saying “I do not understand why the normal force on a banked road has a horizontal component” starts a productive conversation in thirty seconds.

Explain, do not dictate. When you understand something that others do not, explain the reasoning, not just the answer. Saying “the answer is 25 newtons” teaches nothing. Saying “the weight has two components on an inclined plane because...” teaches everyone, including yourself. The moment you explain a concept, you discover whether you truly understand it.

Challenge each other respectfully. If a group member gives an explanation that sounds wrong, do not accept it silently and do not dismiss it rudely. Say: “I am not sure about that, can you explain why?” A wrong explanation that is challenged and corrected teaches more than a correct explanation that is passively accepted.

Solve problems individually first, then compare. The group should not solve a problem together from the start. Each member should attempt it alone, then the group compares methods and answers. If two members get different answers, finding the disagreement is the most valuable activity the group can do.

Rotate the teacher role. Each meeting, a different member explains the key concept from the topic. If you can teach it, you own it. If you cannot, the group helps you, and you try again the following week.

How to Face NECTA

The examination does not test whether you are intelligent. It tests whether you can communicate your understanding clearly, under pressure, in a fixed amount of time. These are skills that can be practised and improved.

Read the question twice. The most common reason students lose marks is answering a question that was not asked. Read every question twice before writing a single word. The second reading almost always reveals something the first reading missed.

Know what the command words mean. Examination questions use specific words that demand specific responses. Using the wrong approach for a given command word costs marks even when your physics is correct.

State — Write the fact in one or two sentences. No explanation, no derivation, no justification. Just the answer, briefly and directly.

Define — Give the formal, precise, textbook definition of the term. Every word must be exact. “Force is mass times acceleration” is not a definition; “Force is the rate of change of momentum” is.

What is meant by — Similar to “define” but allows a broader explanation in your own words. You may include an example or a physical description alongside the formal meaning.

Explain — Give the reason why, not just the fact. “The ball accelerates” is a statement. “The ball accelerates because a net downward force (gravity) acts on it” is an explanation. Every “explain” answer needs a “because.”

What and Explain what — These are **not** the same. “What happens when the string breaks?” asks for the fact only: “The ball flies off tangentially.” But “Explain what happens when the string breaks” asks for the fact **and** the reasoning: “The ball flies off tangentially because the centripetal force is removed and, by Newton’s first law, the ball continues in a straight line in the direction of its velocity at the instant of release.” “What” asks for identification. “Explain what” asks for identification plus reasoning.

Describe — Give a detailed account of what happens, step by step or feature by feature. No derivation or calculation is expected. Focus on the process, the sequence of events, or the key features of a phenomenon.

Discuss — Consider the topic from multiple angles. Present arguments, evidence, advantages, disadvantages, or different viewpoints. A “discuss” answer is longer and more reflective than an “explain” answer.

Account for — Explain why something happens the way it does. Similar to “explain” but usually applied to an observation or experimental result that may seem surprising. The examiner wants the underlying reason.

Justify — Give evidence or reasoning to support a statement or decision. You are defending a claim, not merely explaining it. “Why is this the correct answer?” is the question behind every “justify.”

Outline — Give the main points or steps without detailed explanation. An outline of a derivation shows the key stages without full algebra. An outline of an experiment lists the main steps without minor procedural details.

List — Write the items one by one. No explanation, no sentences. Just the items, clearly separated.

Show that — Start from known principles and arrive at the answer that the question has already given you. You must show every step. The examiner is checking your method, not your answer.

Derive — Build the formula from scratch, starting from fundamental definitions or laws. Every step must be shown and justified.

Calculate — Substitute values into a formula and compute the numerical result. Show your working and include units.

Distinguish — State the differences between two things directly, side by side. Do not define each one separately and hope the examiner spots the difference. Compare them explicitly.

Hence — You must use the result you have just obtained. No alternative method is accepted.

Hence or otherwise — The previous result is the intended method, but an alternative approach is accepted. If you cannot see how to use the previous result, try your own method.

Sketch — Draw a diagram showing the correct shape and key features (intercepts, turning points, asymptotes, labels) but without a precise numerical scale. Neatness matters but exact values do not.

Draw — Produce an accurate diagram with a proper scale, plotted points, and labelled axes. Precision and correct values are expected.

Show all working. A correct final answer with no working may receive zero marks. A wrong final answer with correct working and a small arithmetic error may receive almost full marks. Your working is your insurance. Write it clearly, line by line, with units at every step.

If your answer looks unreasonable, say so. If you calculate that a car accelerates at 500ms^{-2} or a pendulum has a period of 300 seconds, something has gone wrong. Examiners respect a student who writes: “This answer seems unreasonably large, which suggests an error in my calculation.” It shows physical awareness, even when the arithmetic has failed.

The “show that” trap. When the answer is given in the question, the temptation is to work backwards from it. Examiners know this trick. They can distinguish between a derivation that arrives at the answer naturally (full marks) and one that has been reverse-engineered to force the given result (few marks or zero). Always work forward from first principles. Let the answer appear at the end as a consequence of your reasoning, not as a target you steered toward.

Manage your time. Before you begin writing, scan every question on the paper. Start with the question you are most confident about. If you get stuck on a part after a few minutes, leave space and move on. A student who spends 30 minutes on one difficult part and never attempts three other questions has traded 3 marks for 30 marks. That is bad physics and worse strategy.

The night before the examination: do not cram. Your brain consolidates knowledge during sleep. A student who revises until midnight and sleeps four hours performs worse than a student who stops at nine and sleeps eight hours. Rest is not laziness; it is preparation.

You now have five tools that most students never receive: a study method that works, a relationship with failure that is productive, a revision strategy that is honest, a discussion method that sharpens, and an examination approach that is strategic.

Pick up your pen, turn to Chapter 1, and begin. The physics is waiting, and it is better than its reputation.

PART ONE

FOUNDATIONS OF MOTION AND MEASUREMENT

Physics begins with motion and measurement. Every moving object; from a passing bus to a falling fruit, reflects principles that can only be understood when described carefully and measured properly. Since no measurement is perfectly exact, ideas such as units, dimensions, precision, and error are essential for reliable scientific work. This part strengthens your understanding of both measurement and motion, providing the foundation needed for deeper understanding of mechanics and the rest of Physics.

Chapter 1

MEASUREMENT

INTRODUCTION

The physics laboratory at Miono Secondary School had seen better decades. Of its twelve wooden benches, three wobbled so badly that students had learned to wedge folded exercise-book covers under the legs before every practical, a ritual so familiar it might as well have been listed as Step 1 in the apparatus setup. Two others had developed a worrying lean, like elderly relatives at a family gathering who had been standing too long.

When the school board finally approved funds for replacement benches, the relief in the science department was enormous. The headmaster, a practical man who believed in involving students in every aspect of school life, asked for volunteers during the July preparation week to measure the laboratory so that new benches could be ordered from a carpenter in Bagamoyo.

Kipanga's hand went up before the sentence was finished. This surprised absolutely no one. Kipanga was the kind of student who volunteered first and thought about the details afterwards, if he thought about them at all. He was tall, energetic, and carried himself with the unshakable confidence of someone who had never met a task he did not believe he could handle, a belief that his actual track record did remarkably little to support.

"I will measure the room, sir," **Kipanga** announced. *"I am very accurate."*

The headmaster, who taught History and had no particular reason to question this claim, handed Kipanga a notebook and a pen. He did not hand him a tape measure, because he assumed, quite reasonably, that a Form Five science student would know to find one. He assumed wrong.

Kipanga entered the empty laboratory, removed his shoes with the theatrical seriousness of a man preparing for surgery, and began walking heel-to-toe from one wall to the opposite wall. He counted each step aloud, tongue pressed between his teeth in concentration. When he reached the far wall, he wrote in the notebook: **LENGTH = 27 FEET**. He then repeated the process across the width, wrote **WIDTH = 14 FEET**, put his shoes back on, and reported to the headmaster with the satisfaction of a job completed to the highest scientific standard.

The measurements were sent to Fundi Baraka, a respected carpenter in Bagamoyo who had been building school furniture for fifteen years and had never received a complaint. Until now.

Three weeks later, a lorry arrived at the school gate carrying twelve brand-new laboratory benches. They were beautiful: sanded smooth, varnished to a warm honey glow, with sturdy legs and perfectly level surfaces. The entire Form Five class gathered to watch the installation. The first bench was carried inside and slid into position along the wall with a satisfying fit. Then the second. Then the third. By the sixth bench, the mood was celebratory. By the ninth, someone noticed a problem. By the eleventh, the problem was undeniable.

The twelfth bench would not fit. There was a gap of nearly fifteen centimetres between the last bench and the far wall on one side, and the benches along the width were packed so tightly that the stools could barely be tucked in.

Fundi Baraka, standing in the doorway with his arms folded, was baffled. He had built the benches exactly as specified: for a room twenty-seven feet long and fourteen feet wide. He had measured twenty-seven of his own feet along the timber for the length pieces, and fourteen for the width. His feet, unfortunately, were a full size and a half smaller than Kipanga's. The length was short. The width was wrong. The furniture, as far as the geometry of the room was concerned, belonged to a laboratory that existed nowhere on Earth.

Kipanga, who had been watching the installation with steadily decreasing confidence, attempted a defence. *"But I measured it! I counted very carefully!"*

This was the scene that greeted **Mr. Akilikubwa** when he walked into the laboratory that Tuesday morning for the first physics lesson of the new term. Twelve gleaming benches. One visible gap. One confused carpenter. One defensive student. And twenty-nine other students trying extremely hard not to laugh.

Mr. Akilikubwa was the kind of teacher who never raised his voice but somehow made every student pay attention, because his lessons had a habit of becoming stories you remembered long after the exam was over. He studied the room for a long moment. He looked at the gap. He looked at Kipanga. He looked at Kipanga's

feet. He looked at the carpenter's feet. The sequence of events assembled itself in his mind with the inevitability of a well-constructed proof.

"Kipanga," he said, with the calm of a man who has taught physics for twenty years and has learned that the universe always provides the perfect demonstration exactly when you need it, "*what unit did you use to measure this room?*"

"Feet, sir."

"Whose feet?"

A brief silence settled across the laboratory; the kind of silence that arrives when thirty-six people realise the answer at the same moment and one person still does not.

"My feet," said Kipanga. Then, more quietly: "...obviously."

Kipute, sitting in the second row with her notebook already open (she was the kind of student who arrived on the first day of term with questions from the Pre-Form Five programme still waiting to be answered), spoke without looking up. "*And the carpenter used his own feet. So you were speaking two different languages and neither of you knew.*"

"Exactly." Mr. Akilikubwa walked to the front. "*Kipanga counted something. But counting is not measuring. A measurement requires two things: a number and a unit that means exactly the same thing to everyone involved. Kipanga's feet are a personal invention. They mean something to Kipanga and absolutely nothing to Fundi Baraka. And now the school has beautiful furniture built for a room that does not exist.*"

He paused. "*But Kipute has spotted something even deeper. Without a standard unit, we cannot even describe the error. We cannot say who was wrong, or by how much, or how to fix it. There is no reference point. A measurement without a standard unit is a private language; it speaks to you alone.*"

He turned to face the whole class, and his voice settled into the tone they would come to recognise over the next two years as the sound of something important arriving.

"*This is not a lesson about benches. This is a lesson about everything you will study in this book. Every formula you will meet, from Newton's second law to Bernoulli's equation, is only as good as the measurements that feed it. Every number you write must carry a unit. Every unit must belong to an agreed system. And every measurement must be honest about its own uncertainty, because no measurement, no matter how careful, is perfectly exact. This is not a failure of the physicist. It is a property of the physical world itself.*"

He picked up a metre ruler from one of the new benches (which, despite their dimensional confusion, had perfectly smooth surfaces ideal for resting teaching aids) and held it where the whole class could see.

"*Welcome to Chapter 1. Before physics can describe anything, it must first learn how to measure. And that begins right here, in a room that is apparently twenty-seven Kipanga-feet long.*"

Kipanga sank slightly in his seat. The class laughed. And just like that, physics had begun.

PHYSICAL QUANTITIES: FUNDAMENTAL AND DERIVED

Before a single formula appears, before any equation is written or any graph is plotted, physics must answer one quiet but essential question: *what exactly are we measuring?* The answer to this question is the subject of this subtopic. It is not glamorous. It will not make anyone gasp. But every calculation in the next nine chapters depends on getting it right, so we will take our time and get it right.

What is a Physical Quantity?

Not everything in the world can be measured. Beauty cannot. Happiness cannot. The quality of your mother's cooking cannot (and if you tried, you would be in more trouble than Kipanga). But many things can be measured: the length of a table, the mass of a stone, the time taken for a bus to travel from Dar es Salaam to Dodoma, the temperature of boiling water, the speed of a football after a penalty kick.

A **physical quantity** is a property of matter or a phenomenon that can be measured and expressed as a number with a unit. The two parts of this definition are equally important. A physical quantity always has a **numerical value** (*how much*) and a **unit** (*of what*). Without both, the measurement is meaningless.

Consider the statement: "I ran 5." Five what? Five metres? Five kilometres? Five minutes? Without the unit, the number carries no information at all. It is as useless as a key without a lock. Physics insists, without exception, that every measurement must state both the number and the unit. This is not a formality. It is the

difference between communication and confusion, as Kipanga and Fundi Baraka discovered with twelve beautiful but useless benches.

Examples of physical quantities include: length, mass, time, temperature, velocity, force, pressure, energy, and electric current. Examples of things that are *not* physical quantities include: beauty, anger, intelligence, bravery, and the likelihood that Kipanga will volunteer for the next task without thinking it through. These cannot be measured on any agreed scale and expressed with a unit.

A common question worth settling now: *is temperature a physical quantity?* Yes, it is. It took humanity centuries to agree on how to measure it (and even today, different countries argue about Celsius versus Fahrenheit), but temperature can be expressed as a number with a unit, measured with a thermometer, and compared from one experiment to another. It qualifies fully.

Fundamental (Base) Physical Quantities

Some physical quantities are independent. They do not need any other quantity to define them. Length is length. Mass is mass. Time is time. You cannot break them down further into simpler physical quantities. These are called **fundamental physical quantities** (also known as **base physical quantities**).

A **fundamental physical quantity** is one that cannot be defined in terms of other physical quantities. It is an irreducible building block of measurement. The internationally agreed system of units recognises exactly seven such quantities:

Table: The Seven SI Base Quantities

No.	Fundamental Quantity	SI Unit	Symbol
1	Length	metre	<i>m</i>
2	Mass	kilogram	<i>kg</i>
3	Time	second	<i>s</i>
4	Electric current	ampere	<i>A</i>
5	Temperature	kelvin	<i>K</i>
6	Amount of substance	mole	<i>mol</i>
7	Luminous intensity	candela	<i>cd</i>

For A-level physics, the first five quantities (length, mass, time, electric current, and temperature) are the most frequently used. Amount of substance appears occasionally in thermodynamics and gas laws, while luminous intensity rarely features in the syllabus.

A critical distinction must be understood: “fundamental” refers to the *quantity itself*, not to the unit chosen to measure it. Length is a fundamental quantity whether it is measured in metres, feet, cubits, or Kipanga-feet. The quantity does not change; only the unit does. This distinction will become very important when we reach dimensional analysis later.

Derived Physical Quantities

Most physical quantities in physics are not fundamental. *They are built by combining two or more fundamental quantities through multiplication, division, or both.* These are called **derived physical quantities**.

The idea is simple: start with the building blocks (length, mass, time, ...) and assemble everything else. Watch how the construction works, step by step:

Velocity = length ÷ time

Acceleration = velocity $\div$ time = length $\div$ time $\div$ time = length $\div$ time²

Force = mass $\times$ acceleration = mass $\times$ length $\div$ time²

Pressure = force $\div$ area = mass $\times$ length $\div$ time² $\div$ length² = mass $\div$ (length $\times$ time²)

Energy = force $\times$ length = mass $\times$ length² $\div$ time²

Every derived quantity, no matter how complex, can be traced back to the seven fundamental quantities. The table below summarises the most important derived quantities you will encounter in this book.

Table: Key Derived Quantities in A-Level Physics

Quantity	How it is defined	Named SI Unit	In Base Units
Velocity	length / time	—	ms ⁻¹
Acceleration	velocity / time	—	ms ⁻²
Force	mass $\times$ acceleration	newton (N)	kgms ⁻²
Momentum	mass $\times$ velocity	—	kgms ⁻¹
Pressure	force / area	pascal (Pa)	kgm ⁻¹ s ⁻²
Energy / Work	force $\times$ length	joule (J)	kgm ² s ⁻²
Power	energy / time	watt (W)	kgm ² s ⁻³
Torque	force $\times$ distance	—	kgm ² s ⁻²
Surface tension	force / length	—	kg s ⁻²
Viscosity	stress / strain rate	—	kgm ⁻¹ s ⁻¹

Notice that some derived quantities have special named units (newton, pascal, joule, watt) while others are simply expressed in base units. The named units are shorthand: writing “1 N” is quicker than writing “1 kgms⁻²,” but they mean exactly the same thing. You must be comfortable working in both forms, because dimensional analysis requires the base-unit form.

With the vocabulary now established, let us see whether the ideas hold up when tested. The following examples will check whether the boundary between fundamental and derived is genuinely clear in your mind.

BINDER Example 1

Classify each of the following as a fundamental quantity or a derived quantity. Justify each answer.

(a) Mass (b) Velocity (c) Time (d) Force (e) Temperature (f) Pressure (g) Electric current (h) Energy (i) Acceleration (j) Density

Solution

(a) Mass — Fundamental. It is one of the seven SI base quantities. It cannot be expressed in terms of simpler physical quantities.

(b) Velocity — Derived. Velocity is defined as displacement divided by time. It depends on two fundamental quantities: length and time.

(c) Time — Fundamental. It is one of the seven SI base quantities. It stands on its own.

- (d) Force — Derived. Force equals mass times acceleration. Since acceleration itself depends on length and time, force depends on mass, length, and time.
- (e) Temperature — Fundamental. It is one of the seven SI base quantities, measured in kelvin.
- (f) Pressure — Derived. Pressure equals force divided by area. It depends on mass, length, and time.
- (g) Electric current — Fundamental. It is one of the seven SI base quantities, measured in amperes.
- (h) Energy — Derived. Energy equals force times displacement. It depends on mass, length, and time.
- (i) Acceleration — Derived. Acceleration is the rate of change of velocity, which depends on length and time.
- (j) Density — Derived. Density equals mass divided by volume. Since volume depends on length, density depends on mass and length.

Making Sense of the Answer: *The test is simple: can the quantity be broken down into simpler physical quantities? If yes, it is derived. If no, it is fundamental. This test works every time, regardless of how familiar or unfamiliar the quantity seems.*

Think Like a Physicist: *Do not be misled by the fact that a quantity has a named unit. Force is measured in newtons, but it is still a derived quantity because it depends on mass, length, and time. A named unit is a convenience, not a promotion to fundamental status.*

REAL Example 2

During a class discussion on physical quantities, Kipanga raises his hand and says:

Kipanga: *“Sir, I believe love is a physical quantity. When I am in love, my heart beats faster. Heart rate can be measured in beats per minute. Therefore, love has a number and a unit, so it is a physical quantity.”*

The class erupts in laughter. But Mr. Akilikubwa does not laugh. He pauses thoughtfully and says:

Mr. Akilikubwa: *“That is a more interesting argument than you realise, Kipanga. Let us examine it carefully.”*

Explain, using the definition of a physical quantity, why love does not qualify as a physical quantity, and identify the flaw in Kipanga’s reasoning.

Solution

Kipanga’s argument confuses the quantity itself with something it *causes*. Love is not the same thing as heart rate. A faster heart rate can also be caused by running, fear, drinking coffee, or seeing a spider. The fact that heart rate increases when a person is in love does not mean that love is heart rate.

A physical quantity must be a property that can be measured **directly and consistently** on an agreed scale with an agreed unit. Heart rate is a physical quantity (measured in beats per minute). But love itself cannot be measured directly, cannot be assigned a unique numerical value, and does not have a universally agreed unit. Two people experiencing the same intensity of emotion may have completely different heart rates, and the same person’s heart rate changes for many reasons unrelated to love.

Therefore, love is not a physical quantity. Kipanga measured an *effect*, not the thing itself. This is a surprisingly common error in science: confusing a cause with its observable consequences.

Making Sense of the Answer: *Kipanga’s reasoning sounds logical on the surface, which is exactly what makes it a valuable teaching moment. Many things in nature produce measurable effects without being measurable themselves. Anger increases blood pressure, but anger is not blood pressure. Fear produces sweat, but fear is not sweat. A physical quantity must be the thing being measured, not merely something that influences a measurable thing.*

Think Like a Physicist: *When someone claims a new physical quantity exists, apply the test rigorously: can it be assigned a numerical value, measured with an instrument, and expressed with an agreed unit? If any part of the test fails, it is not a physical quantity, no matter how real it feels.*

The vocabulary is now in place. Fundamental quantities are the bricks; derived quantities are the buildings constructed from them. Every physical quantity has a number and a unit, and without both, a measurement says nothing.

But knowing that units exist is not enough. The disaster in the laboratory taught us that units must be *standard*: agreed upon by everyone, everywhere, before a measurement can travel from one person to another without losing its meaning. In the next section, we meet the system that solved this problem for the entire world, and learn how to move fluently within it.

THE SI SYSTEM AND UNIT CONVERSION

In the introduction, we watched Kipanga's personal unit of measurement produce benches that belonged to a room that existed nowhere on Earth. The lesson was clear: a measurement can only travel from one person to another if the unit belongs to an **agreed system**. This subtopic introduces that system, explains the prefixes that make it practical for quantities ranging from the size of an atom to the distance between galaxies, and teaches the conversion technique that prevents expensive mistakes.

The International System of Units

Humanity took a surprisingly long time to agree on how to measure things. For most of history, every region used its own units. The cubit in ancient Egypt was the length of the Pharaoh's forearm. The foot in England was based on the King's foot. The problem was obvious: change the king, change the unit. Trade between regions required elaborate conversion tables, and even then, disputes were common. (Kipanga's bench problem was, in a sense, a re-enactment of several thousand years of measurement confusion, compressed into three weeks.)

The solution came in stages. In 1795, France introduced the metric system, based on natural standards rather than the dimensions of any particular ruler's body. In 1960, the international scientific community formalised this into the *Système International d'Unités* (International System of Units), abbreviated **SI**. Today, SI is the standard measurement system used in science, engineering, and commerce in nearly every country on Earth.

SI is built on the seven base quantities and their units that we met earlier. Every other unit in the system is derived from these seven. The strength of SI is that it is **universal**: a kilogram in Dar es Salaam is the same as a kilogram in Tokyo, Moscow, or on the International Space Station. This universality is not a convenience; it is a necessity.

How necessary? Consider the most expensive unit-conversion error in the history of science. In 1999, NASA launched the **Mars Climate Orbiter**, a spacecraft designed to study the Martian atmosphere. The mission cost \$327 million. As the spacecraft approached Mars, the navigation team at the Jet Propulsion Laboratory sent course-correction commands based on data received from the spacecraft's builder, Lockheed Martin. There was one problem: Lockheed Martin's software calculated thruster forces in *pound-force seconds* (an imperial unit), while NASA's navigation software expected *newton seconds* (the SI unit). Nobody noticed the mismatch. The spacecraft entered the Martian atmosphere at the wrong angle and was destroyed.

Three hundred and twenty-seven million dollars, years of engineering, and the work of hundreds of scientists, lost because two teams used different units and neither checked. The Mars Climate Orbiter is the most powerful argument ever made for using a single, agreed system of units. That system is SI.

For this entire book, we adopt a simple rule: **always convert all quantities to SI units before substituting into any formula, unless the question explicitly states otherwise**. This single habit will prevent more errors than any other piece of advice in this chapter.

SI Prefixes

Physics deals with quantities that span an extraordinary range. The radius of a hydrogen atom is about 0.00000000053 m. The distance from the Earth to the Sun is about 150,000,000,000 m. Writing these numbers in full is impractical and invites errors. SI solves this problem with **prefixes**: standard multipliers attached to a unit to indicate powers of ten.

Instead of writing 0.000000001 m, we write 1 nm (one nanometre). Instead of writing 1,000,000 W, we write 1 MW (one megawatt). The number stays manageable, and the power of ten is carried by the prefix.

Table: SI Prefixes Most Used in A-Level Physics

Prefix	Symbol	Meaning	Example
tera	T	10^{12}	1 THz = 10^{12} Hz
giga	G	10^9	1 GW = 10^9 W
mega	M	10^6	1 MJ = 10^6 J
kilo	k	10^3	1 km = 10^3 m
centi	c	10^{-2}	1 cm = 10^{-2} m
milli	m	10^{-3}	1 mm = 10^{-3} m
micro	μ	10^{-6}	1 μ m = 10^{-6} m
nano	n	10^{-9}	1 nm = 10^{-9} m
pico	p	10^{-12}	1 pF = 10^{-12} F

A critical warning: students consistently confuse micro (μ , 10^{-6}) with milli (m, 10^{-3}). These differ by a factor of 1000. Writing 5 mm when you mean 5 μ m means your answer is 1000 times too large. In electronics, capacitances are often given in microfarads (μ F) and lengths of wires in millimetres (mm). Mixing them up is one of the most common errors in A-level calculations.

A useful memory aid: arrange the prefixes from largest to smallest as a staircase. Each step down by three means dividing by 1000: kilo (10^3) $\rightarrow$ base unit (10^0) $\rightarrow$ milli (10^{-3}) $\rightarrow$ micro (10^{-6}) $\rightarrow$ nano (10^{-9}) $\rightarrow$ pico (10^{-12}). Every step is exactly a factor of 1000.

Unit Conversion

Unit conversion is the single most practical skill in this chapter. Every worked example in this book begins with converting given values into SI units. A reliable method, used consistently, eliminates conversion errors entirely.

The method is based on a golden rule: **multiply by a conversion factor that equals 1**. Since 1 km = 1000 m, the fraction 1000 m / 1 km equals 1. Multiplying any quantity by this fraction changes the unit without changing the value.

The three-step method:

Step 1: Write the given quantity with its unit.

Step 2: Multiply by one or more conversion factors, arranged so that the unit you want to eliminate appears on the opposite side of the fraction.

Step 3: Cancel units that appear on both top and bottom. The surviving unit is your answer.

For example, converting 72 km/h to m/s:

$$72 \frac{\text{km}}{\text{h}} \times \frac{1000 \text{ m}}{1 \text{ km}} \times \frac{1 \text{ h}}{3600 \text{ s}} = \frac{72 \times 1000 \text{ m}}{3600 \text{ s}} = 20 \text{ ms}^{-1}$$

The km cancels with km, the h cancels with h, and only m and s survive.

A critical point that demands special emphasis: area and volume conversions require squaring or cubing the conversion factor. This is where students make some of the most persistent errors in A-level physics.

Since $1 \text{ cm} = 10^{-2} \text{ m}$, it follows that:

$$1 \text{ cm}^2 = (10^{-2} \text{ m})^2 = 10^{-4} \text{ m}^2$$

$$1 \text{ cm}^3 = (10^{-2} \text{ m})^3 = 10^{-6} \text{ m}^3$$

Notice: 1 cm^2 is **not** 10^{-2} m^2 . It is 10^{-4} m^2 . Similarly, 1 cm^3 is **not** 10^{-3} m^3 . It is 10^{-6} m^3 . The exponent applies to the *entire* conversion factor, not just the number. Forgetting to square or cube is the single most common conversion error in A-level physics, and it appears in every chapter from here to Chapter 10.

Another frequently needed conversion: $1 \text{ litre} = 1000 \text{ cm}^3 = 10^{-3} \text{ m}^3$. This will appear in fluid mechanics, thermodynamics, and in a bill from a water supplier

With the method and the warning firmly in place, let us put the technique to work.

BINDER Example 3

Convert each of the following to SI units. Show full working.

- (a) 5 km (b) 3 hours (c) 250 cm^2 (d) 0.5 litres (e) 72 km/h

Solution

(a)

$$5 \text{ km} = 5 \times 1000 \text{ m} = 5000 \text{ m}$$

(b)

$$3 \text{ h} = 3 \times 3600 \text{ s} = 10\,800 \text{ s}$$

(c)

$$250 \text{ cm}^2 = 250 \times 10^{-4} \text{ m}^2 = 0.025 \text{ m}^2 = 2.5 \times 10^{-2} \text{ m}^2$$

(d)

$$0.5 \text{ litres} = 0.5 \times 10^{-3} \text{ m}^3 = 5 \times 10^{-4} \text{ m}^3$$

(e)

$$72 \frac{\text{km}}{\text{h}} = 72 \times \frac{1000 \text{ m}}{3600 \text{ s}} = 20 \text{ ms}^{-1}$$

Making Sense of the Answer: Part (c) is the one most students get wrong. The temptation is to write 250×10^{-2} instead of 250×10^{-4} . Always ask: am I converting a length, an area, or a volume? If area, square the factor. If volume, cube it.

Think Like a Physicist: The quick conversion $1 \text{ km/h} = 1/3.6 \text{ m/s}$ is useful to memorise. Divide by 3.6 to go from km/h to m/s. Multiply by 3.6 to go back.

BINDER Example 4

(a) Express 1 year in seconds.

(b) The mass of a proton is $1.67 \times 10^{-27} \text{ kg}$. Express this mass in micrograms (μg).

Solution

(a) $1 \text{ year} = 365 \text{ days}$ (ignoring leap years)

$$1 \text{ year} = 365 \times 24 \times 3600 \text{ s} = 31\,536\,000 \text{ s} \approx 3.15 \times 10^7 \text{ s}$$

(b) Since $1 \mu\text{g} = 10^{-6} \text{ g} = 10^{-6} \times 10^{-3} \text{ kg} = 10^{-9} \text{ kg}$:

$$m = \frac{1.67 \times 10^{-27} \text{ kg}}{10^{-9} \text{ kg}/\mu\text{g}} = 1.67 \times 10^{-18} \mu\text{g}$$

Making Sense of the Answer: A year contains roughly 31.5 million seconds, a number worth remembering for quick estimates. The proton mass in micrograms is an extraordinarily tiny number, which reflects just how small subatomic particles are compared to anything we can see or touch.

Think Like a Physicist: When converting between prefixed units, first convert both to the base SI unit, then convert to the target unit. Going directly from one prefix to another (say, from nanometres to micrometres) invites sign errors in the exponent.

REAL Example 5

Kipanga's family receives their monthly DAWASCO water bill. The bill charges 3,500 Tsh per cubic metre of water used. The household water meter, however, reads in litres. This month, the meter shows that the family used 12,400 litres. Kipanga's mother is furious: she believes the family is being overcharged because "12,400 is a very big number and 3,500 shillings per unit sounds expensive."

Kipanga: "Mum, I think they are converting litres to cubic metres and that makes the number smaller. Let me calculate."

Help Kipanga to convert 12,400 litres to cubic metres and hence calculate the total water bill.

Solution

Since 1 litre = 10^{-3} m^3 :

$$12400 \text{ litres} = 12400 \times 10^{-3} \text{ m}^3 = 12.4 \text{ m}^3$$

Total bill:

$$\text{Bill} = 12.4 \text{ m}^3 \times 3500 \text{ Tsh/m}^3 = 43400 \text{ Tsh}$$

The bill is correct. The family used 12.4 cubic metres of water, not 12,400 cubic metres. The large number in litres becomes a modest number in cubic metres because 1 m^3 contains 1000 litres.

Making Sense of the Answer: This is a real-life example of why unit conversion matters outside the physics classroom. Many billing disputes in Tanzania arise from confusion between litres and cubic metres. A household that uses 12.4 m^3 of water per month is typical for a family of five.

Think Like a Physicist: Whenever a number looks suspiciously large or small, check the units. A billing system that charges per cubic metre but measures in litres will always show a reading 1000 times larger than the billed volume. The physics is simple. The confusion is common.

HOT Example 6

A car of mass 1200 kg travels at 90 km/h.

- Convert the speed to ms^{-1} .
- Calculate the kinetic energy of the car.
- Express the kinetic energy in kJ and in MJ.

Solution

(a) Converting the speed:

$$v = 90 \frac{\text{km}}{\text{h}} = \frac{90}{3.6} \text{ ms}^{-1} = 25 \text{ ms}^{-1}$$

(b) Calculating kinetic energy:

Using:

$$\text{KE} = \frac{1}{2}mv^2$$

Substituting:

$$\text{KE} = \frac{1}{2} \times 1200 \text{ kg} \times (25)^2 \text{ m}^2\text{s}^{-2} = 375000 \text{ J}$$

(c) Expressing in kJ and MJ:

$$\text{KE} = \frac{375000 \text{ J}}{10^3 \text{ J/kJ}} = 375 \text{ kJ}$$

$$KE = \frac{375000 \text{ J}}{10^6 \text{ J/MJ}} = 0.375 \text{ MJ}$$

Making Sense of the Answer: 375 kJ is roughly the energy contained in a small chocolate bar. A car travelling at 90 km/h carries enough kinetic energy to be very dangerous in a collision, which is why speed limits exist. Doubling the speed would quadruple the kinetic energy to 1500 kJ, because kinetic energy depends on the square of velocity.

Think Like a Physicist: Notice that the unit conversion was done before substituting into the formula. If you had substituted 90 km/h directly into $KE = \frac{1}{2}mv^2$, you would have obtained a number in $\text{kg}\cdot\text{km}^2/\text{h}^2$, which is not joules. Converting first guarantees that the answer comes out in SI units automatically.

The SI system is now in your hands: seven base units, a clean set of prefixes, and a conversion method that works every time if followed carefully. The area and volume warning has been sounded loudly, and the DAWASCO bill has been settled.

But knowing how to convert units is only part of the story. In the next section, we discover a far more powerful use of units: they can tell you whether a formula is even *possible*. Welcome to dimensional analysis, the detective tool that catches wrong equations before they ever reach a calculator.

DIMENSIONAL ANALYSIS

If the previous subtopic taught us how to convert units, this section teaches us something even more powerful: how to use units to *think*. We are about to meet a tool that can verify whether a formula makes physical sense, derive relationships between quantities without solving a single differential equation, and convert measurements between entirely different unit systems. The tool is called **dimensional analysis**, and it is one of the most elegant techniques in all of physics.

Dimensional analysis is the method of analysing physical equations and relationships by examining the dimensions (fundamental nature) of the quantities involved. It is based on the simple but powerful principle that any physically meaningful equation must be consistent in its dimensions.

In this section, we will first establish what dimensions are and how they differ from units. Then we will explore the three uses of dimensional analysis, each with a full worked derivation. Finally, we will state honestly what the method cannot do. By the end, you will have a detective tool that catches wrong equations before they ever reach a calculator.

What are dimensions?

Earlier, we classified physical quantities as fundamental or derived. Then, we learned to express them using SI units. Now we go one level deeper.

Every physical quantity has a **unit** and a **dimension**. These are related but different. The **unit** tells you the *scale* of measurement chosen by humans: metres, feet, nanometres, or light-years for length. The **dimension** tells you the fundamental *nature* of the quantity, independent of any measurement system.

The dimension of a physical quantity is the expression that shows how the quantity is related to the fundamental (base) quantities. It describes *what kind of thing* the quantity is, stripped of all numerical values and units.

For example, velocity is defined as displacement divided by time. No matter how you measure it (in m/s, km/h, miles per hour, or any other unit), velocity is always *length per time*. That is its dimension. Change the unit system, and the numerical value changes, but the dimension stays exactly the same.

For A-level physics, we represent the dimensions of the five most important fundamental quantities using standard symbols:

- **M** for mass
- **L** for length
- **T** for time
- **I** for electric current
- **Θ** for temperature

The **dimensional formula** of a quantity is the expression that shows the powers to which the fundamental dimensions must be raised to represent that quantity. We write it using **square brackets**. For example, the dimensional formula of velocity is written as:

$$[\text{velocity}] = \text{LT}^{-1}$$

This tells us that velocity has the dimension of length (to the power 1) divided by time (to the power 1). The negative exponent indicates division.

How to find the dimensional formula of any derived quantity

The method is always the same: start from the *definition* of the quantity, replace each quantity with its fundamental dimensions, and simplify. Let us walk through several examples to build fluency.

Force is defined as mass $\times$ acceleration. Acceleration is velocity $\div$ time. Velocity is length $\div$ time. So:

$$\begin{aligned}\text{Force} &= \text{mass} \times \text{acceleration} = \text{mass} \times \frac{\text{velocity}}{\text{time}} = \text{mass} \times \frac{\text{length}}{\text{time}^2} \\ [\text{force}] &= \text{MLT}^{-2}\end{aligned}$$

Pressure is defined as force per unit area. Force has dimensions MLT^{-2} , and area has dimensions L^2 . So:

$$[\text{pressure}] = \frac{\text{MLT}^{-2}}{\text{L}^2} = \text{ML}^{-1}\text{T}^{-2}$$

Energy (or work) is defined as force $\times$ displacement:

$$[\text{energy}] = \text{MLT}^{-2} \times \text{L} = \text{ML}^2\text{T}^{-2}$$

Notice the pattern: every derived quantity, no matter how complex, reduces to a combination of M, L, and T (with I and Θ appearing in electrical and thermal quantities). This reduction is the foundation on which all of dimensional analysis is built.

One more point before we move on. Some quantities are **dimensionless**: they have no dimensions at all. Pure numbers like 2, π , and $1/2$ are dimensionless. Other examples are angles (measured in radians), strain (change in length divided by original length), and refractive index. Their dimensional formula is written as:

$$[\text{dimensionless quantity}] = \text{M}^0\text{L}^0\text{T}^0 = 1$$

Dimensionless quantities will play a quiet but important role in what follows, especially when we discuss the limitations of dimensional analysis.

Uses of Dimensional Analysis

Now that we can speak the language of dimensions, it is time to explore what this language can do through four applications of dimensional analysis. We begin with the most direct.

1) Finding the SI unit of a physical quantity

The simplest and most immediate application of dimensional analysis is this: once you know the dimensional formula of a quantity, you can write down its SI unit directly by replacing each dimension symbol with its corresponding SI base unit.

The rule is straightforward:

Replace M with kg, L with m, T with s, I with A, and Θ with K.

For example, we showed earlier that the dimensional formula of force is MLT^{-2} . Replacing the symbols:

$$\text{SI unit of force} = \text{kg} \cdot \text{m} \cdot \text{s}^{-2} = \text{kgms}^{-2}$$

This is exactly the newton (N), expressed in base units. The named unit is a shorthand; the base-unit form is what dimensional analysis produces.

Let us apply this to several quantities:

The dimensional formula of pressure is $\text{ML}^{-1}\text{T}^{-2}$. Therefore:

$$\text{SI unit of pressure} = \text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2} = \text{kgm}^{-1}\text{s}^{-2}$$

This is the pascal (Pa).

The dimensional formula of the coefficient of viscosity is $\text{ML}^{-1}\text{T}^{-1}$. Therefore:

$$\text{SI unit of viscosity} = \text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1} = \text{kgm}^{-1}\text{s}^{-1}$$

This is the pascal-second (Pas), the unit that will appear when we convert from poise in fourth use.

The dimensional formula of surface tension is MT^{-2} . Therefore:

$$\text{SI unit of surface tension} = \text{kg} \cdot \text{s}^{-2} = \text{kg s}^{-2}$$

This is equivalent to N/m (newtons per metre), confirming what we will meet in Chapter 10.

This technique is especially useful when you encounter an unfamiliar quantity in an examination. If the question gives you a formula involving the quantity, you can derive its dimensional formula, then immediately write down its SI unit without memorising it. The dimensional formula is the map; the SI unit is the destination

2) Checking the correctness of a formula

This is the simplest and most frequently used application of dimensional analysis. It is based on a principle so fundamental that it deserves to be stated clearly and remembered permanently.

The principle of dimensional homogeneity states that *in any physically valid equation, every term must have the same dimensions*. You cannot add metres to seconds, just as you cannot add apples to lorries. If the dimensions of the left side of an equation do not match the dimensions of the right side, the equation is certainly wrong.

The method is straightforward:

Step 1: Write down the equation to be checked.

Step 2: Find the dimensional formula of every term on both sides of the equation.

Step 3: Compare the dimensions of all terms. If they are all identical, the equation is dimensionally homogeneous and **could** be correct. If any term differs, the equation is definitely wrong.

Let us apply this to a well-known kinematic equation. Consider:

$$v^2 = u^2 + 2as$$

where v and u are velocities, a is acceleration, and s is displacement.

$$\text{Left side: } [v^2] = (LT^{-1})^2 = L^2T^{-2}$$

$$\text{Right side, first term: } [u^2] = L^2T^{-2}$$

$$\text{Right side, second term (2 is dimensionless): } [2as] = (LT^{-2}) \times L = L^2T^{-2}$$

All three terms have dimensions L^2T^{-2} . The equation is dimensionally homogeneous.

An important and honest warning: dimensional correctness does **not** guarantee physical correctness. A formula can have the right dimensions but incorrect due to the following reasons:

- the wrong numerical coefficient,
- a missing term, or
- be applied in the wrong context.

For instance, $v^2 = u^2 + 3as$ is also dimensionally homogeneous, but the coefficient 3 is wrong. Dimensional analysis catches equations that are **definitely wrong**; but it cannot confirm that an equation is definitely right.

3) Deriving relationships between physical quantities

This is the most powerful and most impressive application of dimensional analysis. Suppose you know that a certain physical quantity depends on several other quantities, but you do not know the exact formula connecting them. Dimensional analysis can determine the **form** of that formula, up to a dimensionless constant, without solving any differential equation or conducting any experiment.

The method:

Step 1: Identify the quantity you want to find and the quantities it depends on.

Step 2: Write the assumed relationship as a product of the dependent quantities raised to unknown powers:

$$Q = kR^aS^bU^c$$

where k is a dimensionless constant that cannot be found by this method.

Step 3: Replace every quantity with its dimensional formula.

Step 4: Equate the powers of each fundamental dimension (M, L, T) on both sides. This gives simultaneous equations.

Step 5: Solve the simultaneous equations for the unknown powers.

Step 6: Substitute back to obtain the formula.

Classic derivation: the period of a simple pendulum

Suppose we want to find the period T of a simple pendulum. Physical reasoning suggests that T might depend on the length l of the string and the acceleration due to gravity g . (We exclude the mass of the bob for now and will justify this exclusion at the end.)

Step 1: The quantity to find is T . It depends on l and g .

Step 2: Assume:

$$T = kl^a g^b$$

Step 3: The dimensions of each quantity are:

$$[T] = T, \quad [l] = L, \quad [g] = LT^{-2}$$

So the dimensional equation becomes:

$$\begin{aligned} M^0 L^0 T^1 &= L^a (LT^{-2})^b \\ M^0 L^0 T^1 &= M^0 L^{(a+b)} T^{-2b} \end{aligned}$$

Step 4: Equating powers of each dimension:

For M: $0 = 0$ (automatically satisfied)

For L: $0 = a + b$... (i)

For T: $1 = -2b$... (ii)

Step 5: From equation (ii): $b = -\frac{1}{2}$. Substituting into equation (i): $a = \frac{1}{2}$

Step 6: Substituting back:

$$T = kl^{1/2} g^{-1/2} = k \sqrt{\frac{l}{g}}$$

This is the result of dimensional analysis. It tells us the **form** of the formula. A full derivation from Newton's second law (or a careful experiment) reveals that $k = 2\pi$, giving:

$$T = 2\pi \sqrt{\frac{l}{g}}$$

Why was mass excluded? Because the period T has dimension T (time only), and l and g together can produce T without needing M. If we had included the mass m with an unknown power c , the equation for M would give $0 = c$, confirming that mass does not affect the period. Dimensional analysis does not just accept our assumption that mass is irrelevant; it *proves* it.

4) Converting units between systems

This is the most practical application of dimensional analysis in the laboratory. While SI is the standard for modern physics, many older reference tables, engineering handbooks, and published data still use the CGS system (centimetre-gram-second). A physicist must be able to convert numerical values between systems quickly and reliably.

The method is based on a simple idea: the **physical quantity itself does not change** when you change the unit system. Only its numerical value and its units change. A length of 5 metres is the same as 500 centimetres. The length has not changed; the number has changed because the unit changed.

Suppose a physical quantity has dimensional formula $M^a L^b T^c$. Let its numerical value be n_1 in System 1 (where the base units are M_1, L_1, T_1) and n_2 in System 2 (where the base units are M_2, L_2, T_2). Since the physical quantity is the same in both systems:

$$n_1 M_1^a L_1^b T_1^c = n_2 M_2^a L_2^b T_2^c$$

Rearranging for the numerical value in System 2:

$$n_2 = n_1 \left(\frac{M_1}{M_2} \right)^a \left(\frac{L_1}{L_2} \right)^b \left(\frac{T_1}{T_2} \right)^c$$

This is the general conversion formula. The ratios of base units are raised to the powers given by the dimensional formula. Once the dimensions are known, the conversion is entirely mechanical.

Example within the text: converting the coefficient of viscosity

In the CGS system, the unit of viscosity is the **poise** (P), defined as $1 \text{ g cm}^{-1} \text{ s}^{-1}$. We want to find how many SI units ($\text{kg m}^{-1} \text{ s}^{-1}$) one poise equals.

The dimensional formula of viscosity is $ML^{-1}T^{-1}$, so $a = 1, b = -1, c = -1$.

The base units in each system are:

CGS: $M_1 = 1 \text{ g}, L_1 = 1 \text{ cm}, T_1 = 1 \text{ s}$

SI: $M_2 = 1 \text{ kg}, L_2 = 1 \text{ m}, T_2 = 1 \text{ s}$

Substituting into the conversion formula:

$$\begin{aligned} n_2 &= 1 \times \left(\frac{1 \text{ g}}{1 \text{ kg}} \right)^1 \left(\frac{1 \text{ cm}}{1 \text{ m}} \right)^{-1} \left(\frac{1 \text{ s}}{1 \text{ s}} \right)^{-1} \\ &= 1 \times (10^{-3})^1 \times (10^{-2})^{-1} \times 1 \\ &= 10^{-3} \times 10^2 = 10^{-1} = 0.1 \end{aligned}$$

Therefore, $1 \text{ poise} = 0.1 \text{ kg m}^{-1} \text{ s}^{-1} = 0.1 \text{ Pas (pascal-second)}$.

This result is essential for Chapter 10 (Fluid Mechanics), where viscosity data from older sources is often given in poise or centipoise ($1 \text{ cP} = 10^{-3} \text{ Pas}$). With dimensional analysis, the conversion is systematic and error-free.

Limitations of Dimensional Analysis

Dimensional analysis is powerful, but it has clear boundaries. A responsible physicist uses the tool confidently while knowing exactly where it stops working. There are five important limitations:

1) It cannot determine dimensionless constants

The pendulum derivation gave $T = k \sqrt{\frac{l}{g}}$ but could not find $k = 2\pi$. The constant 2π has no dimensions, so it is invisible to dimensional analysis. Similarly, the kinetic energy formula $KE = \frac{1}{2}mv^2$ cannot be distinguished from $KE = mv^2$ or $KE = 3mv^2$ by dimensions alone. The factor of $\frac{1}{2}$ must come from a full derivation or from experiment.

2) It cannot distinguish between quantities with the same dimensions

Work and torque both have dimensions ML^2T^{-2} , yet they are physically different quantities. Speed and velocity both have dimensions LT^{-1} , yet one is a scalar and the other is a vector. Dimensional analysis treats them as identical because it sees only dimensions, not physical meaning.

3) It cannot handle trigonometric, logarithmic, or exponential functions

The arguments of $\sin$, $\cos$, $\log$, and exponential functions must all be dimensionless. Dimensional analysis cannot discover that a formula involves $\sin(\omega t)$ or $e^{-\lambda t}$ because it has no way of detecting these functions. A formula like $x = A \sin(\omega t + \phi)$ cannot be derived by dimensional analysis; only the amplitude dependence (A) can be found.

4) It does not guarantee uniqueness

Two wrong formulas can both be dimensionally correct. Dimensional analysis narrows the possibilities but does not always eliminate all incorrect options. A formula that passes the dimensional check is plausible, not proven.

5) It fails when there are too many unknowns

The method produces one equation for each fundamental dimension used (typically three: M, L, T). If the assumed relationship involves more unknown powers than available equations, the system is underdetermined and cannot be solved uniquely.

Despite these five limitations, dimensional analysis remains one of the most elegant and frequently used tools in physics. It catches errors cheaply, guides intuition reliably, and impresses examiners consistently. The honest physicist uses it as a *first check*, not the *final word*.

The theory is now complete: four uses, five limitations, each clearly defined. Let us put the detective to work with a set of worked examples that will build your fluency and confidence.

BINDER Example 7

Derive the dimensional formula for each of the following quantities:

- (a) Momentum (b) Work

Solution

- (a) Momentum = mass $\times$ velocity

$$[\text{momentum}] = M \times LT^{-1} = MLT^{-1}$$

- (b) Work = force $\times$ displacement

$$[\text{work}] = (ML^1T^{-2}) \times L = ML^2T^{-2}$$

Making Sense of the Answer: Each quantity has its own unique dimensional fingerprint. Momentum (MLT^{-1}) and force (MLT^{-2}) differ only in the power of T, which makes physical sense: force is the rate of change of momentum, so it has one extra factor of T^{-1} .

Think Like a Physicist: Always start from the definition of the quantity, not from a formula you have memorised. Definitions are more reliable starting points because they express exactly what the quantity means physically.

BINDER Example 8

Check whether the equation $s = ut + \frac{1}{2}at^2$ is dimensionally homogeneous. Show full working for every term.

Solution

Left side: $[s] = L$

First term on right: $[ut] = LT^{-1} \times T = L$

Second term on right ($\frac{1}{2}$ is dimensionless):

$$[\frac{1}{2}at^2] = LT^{-2} \times T^2 = L$$

All three terms have dimension L. The equation is dimensionally homogeneous.

Making Sense of the Answer: Notice how the time dimensions cancel perfectly in each term. In ut , one T^{-1} from velocity cancels with T from time. In $\frac{1}{2}at^2$, the T^{-2} from acceleration cancels with T^2 from time squared. These cancellations are not coincidence; they are required by the physics. Any term where the cancellation fails would signal a wrong formula.

Think Like a Physicist: If any single term has different dimensions from the others, the equation is guaranteed wrong. This makes dimensional checking one of the quickest error-detection tools in physics.

BINDER Example 9

The centripetal force F acting on a body moving in a circle is believed to depend on the mass m of the body, its speed v , and the radius r of the circular path. Use dimensional analysis to derive an expression for F .

Solution

Assume: $F = km^a v^b r^c$, where k is a dimensionless constant.

Dimensions of each quantity:

$$[F] = MLT^{-2}$$

$$[m] = M$$

$$[v] = LT^{-1}$$

$$[r] = L$$

Writing the dimensional equation:

$$MLT^{-2} = M^a(LT^{-1})^b L^c = M^a L^{b+c} T^{-b}$$

Equating powers:

$$\text{For } M: 1 = a$$

$$\text{For } T: -2 = -b \Rightarrow b = 2$$

$$\text{For } L: 1 = b + c \Rightarrow c = 1 - 2 = -1$$

Therefore:

$$F = km^1 v^2 r^{-1} = k \frac{mv^2}{r}$$

Making Sense of the Answer: The formula tells us that centripetal force increases with the square of speed (doubling the speed quadruples the required force) and decreases with radius (a tighter curve demands more force).

Think Like a Physicist: This is another case where dimensional analysis gives the complete formula with $k = 1$. Notice that three unknowns (a, b, c) required exactly three equations (from M, L, T). When the number of unknowns matches the number of equations, the method works perfectly.

REAL Example 10

After watching Mr. Akilikubwa derive the pendulum formula using dimensional analysis, Kipute raises her hand.

Kipute: “Sir, if dimensional analysis can derive $T = k\sqrt{\frac{l}{g}}$ without solving any differential equation, why do we still need experiments at all? Why not derive every formula this way?”

Mr. Akilikubwa: “That is exactly the question every student should ask after seeing this technique for the first time. The answer has three parts.”

Explain, using specific examples, why dimensional analysis alone is not sufficient to replace experiments and full theoretical derivations.

Solution

First, dimensional analysis cannot determine dimensionless constants. It gave $T = k\sqrt{\frac{l}{g}}$ but not $k = 2\pi$.

The constant 2π has no dimensions and is therefore invisible to the method. Only a full derivation from Newton’s second law or a careful experiment can supply it.

Second, dimensional analysis requires knowing in advance which variables the quantity depends on. For the pendulum, we assumed T depends only on l and g . But how did we know to exclude mass, air resistance, or the angle of swing? That knowledge came from experiment or physical reasoning external to dimensional analysis. If we had omitted a relevant variable, the method would have produced the wrong formula without warning.

Third, dimensional analysis assumes the relationship is a simple power law. Many physical relationships involve trigonometric functions, exponentials, or logarithms. The period of a pendulum at large angles involves a complex infinite series that dimensional analysis cannot predict.

Making Sense of the Answer: Dimensional analysis is a powerful first step, not the final word. It narrows the possibilities enormously, sometimes to a single formula up to a constant. But it always needs either experiment or theory to finish the job.

Think Like a Physicist: The greatest value of dimensional analysis is not replacing experiments but guiding them. If you know the formula must be $v = k \sqrt{\frac{T}{\mu}}$, you know exactly what to measure and what to plot. Dimensional analysis tells you where to look; experiment tells you what you find.

HOT Example 11

The speed v of a transverse wave on a stretched string depends on the tension T (force) and the linear mass density μ (mass per unit length). Use dimensional analysis to derive an expression for v . Verify using the known formula $v = \sqrt{\frac{T}{\mu}}$.

Solution

Assume: $v = kT^a\mu^b$

Dimensions:

$[v] = LT^{-1}$, $[T] = ML^1T^{-2}$ (force), $[\mu] = ML^{-1}$ (mass per length)

Dimensional equation:

$$LT^{-1} = (ML^1T^{-2})^a(ML^{-1})^b = M^{(a+b)}L^{(a-b)}T^{-2a}$$

Equating powers:

For M: $0 = a + b$

For L: $1 = a - b$

For T: $-1 = -2a$, so $a = 1/2$

From $a + b = 0$: $b = -1/2$

Therefore:

$$v = k \sqrt{\frac{T}{\mu}}$$

The known formula is $v = \sqrt{\frac{T}{\mu}}$, confirming $k = 1$.

Making Sense of the Answer: Greater tension means a faster wave. Greater linear density means a slower wave. Both match intuition. In this rare case, the dimensionless constant is exactly 1.

Think Like a Physicist: This formula appears later when studying waves in strings and columns. Knowing its origin from dimensional analysis makes it easier to remember and harder to misapply.

HOT Example 12

A student claims that the formula for kinetic energy is $E = mv^3$.

(a) Use dimensional analysis to show that this formula is wrong.

(b) Determine the correct power of v that kinetic energy must contain.

(c) State clearly what dimensional analysis alone cannot tell you about the formula for kinetic energy.

Solution

(a) Checking the student's claim:

Left side: $[E] = ML^2T^{-2}$

Right side: $[mv^3] = M(LT^{-1})^3 = ML^3T^{-3}$

Since $ML^2T^{-2} \neq ML^3T^{-3}$, the formula $E = mv^3$ is dimensionally inconsistent and therefore definitely wrong.

(b) Let $E = mv^n$ where n is unknown:

$$ML^2T^{-2} = M(LT^{-1})^n = ML^nT^{-n}$$

Equating powers of L : $2 = n$. Equating powers of T : $-2 = -n$, so $n = 2$.

Kinetic energy depends on v^2 , not v^3 .

(c) Dimensional analysis tells us $E = kmv^2$ for some dimensionless constant k . It cannot determine that $k = \frac{1}{2}$. The constant must come from the work-energy theorem or experiment.

Making Sense of the Answer: The student's formula fails on two counts: the power of L is wrong (3 instead of 2) and the power of T is wrong (-3 instead of -2). Either alone is enough to reject it.

Think Like a Physicist: This example shows both the power and the limitation of dimensional analysis in one problem. It proves a formula wrong, finds the correct power, but cannot find the coefficient. The most efficient first check in physics.

HOT Example 13

The gravitational constant G appears in Newton's law of universal gravitation, $F = \frac{Gm_1m_2}{r^2}$.

- Show that the dimensional formula of G is $M^{-1}L^3T^{-2}$.
- In the CGS system, the value of G is $6.67 \times 10^{-8} \text{ g}^{-1}\text{cm}^3\text{s}^{-2}$. Use the method of dimensions to convert this value to SI units.
- Express the SI unit of G in terms of newtons.

Solution

(a) From Newton's law of gravitation:

$$G = \frac{Fr^2}{m_1m_2}$$

Writing dimensions:

$$[G] = \frac{[F][r^2]}{[m_1][m_2]} = \frac{MLT^{-2} \times L^2}{M \times M} = \frac{ML^3T^{-2}}{M^2} = M^{-1}L^3T^{-2}$$

Hence:

$$[G] = M^{-1}L^3T^{-2}$$

(b) In CGS: $G = 6.67 \times 10^{-8} \text{ g}^{-1}\text{cm}^3\text{s}^{-2}$

The dimensions of G are $M^{-1}L^3T^{-2}$, so the unit of G transforms as:

$$G_{SI} = G_{CGS} \times \left(\frac{M_{CGS}}{M_{SI}}\right)^{-1} \times \left(\frac{L_{CGS}}{L_{SI}}\right)^3 \times \left(\frac{T_{CGS}}{T_{SI}}\right)^{-2}$$

The conversion factors are: $1\text{g} = 10^{-3}\text{kg}$, $1\text{cm} = 10^{-2}\text{m}$, $1\text{s} = 1\text{s}$.

$$G_{SI} = 6.67 \times 10^{-8} \times \left(\frac{1\text{g}}{1\text{kg}}\right)^{-1} \times \left(\frac{1\text{cm}}{1\text{m}}\right)^3 \times \left(\frac{1\text{s}}{1\text{s}}\right)^{-2}$$

$$G_{SI} = 6.67 \times 10^{-8} \times (10^{-3})^{-1} \times (10^{-2})^3 \times (1)^{-2}$$

$$G_{SI} = 6.67 \times 10^{-11} \text{ kg}^{-1}\text{m}^3\text{s}^{-2}$$

Therefore, $G = 6.67 \times 10^{-11} \text{ kg}^{-1}\text{m}^3\text{s}^{-2}$ in SI units.

(c) The SI unit $\text{kg}^{-1}\text{m}^3\text{s}^{-2}$ can be rewritten using the definition of the newton: $1\text{N} = 1\text{kgms}^{-2}$.

$$\text{kg}^{-1}\text{m}^3\text{s}^{-2} = \frac{\text{m}^3}{\text{kg}\text{s}^2} = \frac{\text{m}^3}{\text{kg}\text{s}^2} \times \frac{\text{kg}}{\text{kg}} = \frac{\text{kgm}^3}{\text{kg}^2\text{s}^2} = \frac{\text{kgms}^{-2} \times \text{m}^2}{\text{kg}^2} = \frac{\text{Nm}^2}{\text{kg}^2}$$

Therefore, the SI unit of G is $\text{Nm}^2\text{kg}^{-2}$.

Making Sense of the Answer: *The numerical value of G changed from 6.67×10^{-8} (CGS) to 6.67×10^{-11} (SI), a factor of 10^{-3} smaller. This makes sense: the SI unit of mass (kg) is 10^3 times larger than the gram, and since G has M^{-1} in its dimensions, switching to a larger mass unit decreases the number. The L^3 factor contributes 10^{-6} (since $1\text{cm} = 10^{-2}\text{m}$ and the cube gives 10^{-6}), and combined with 10^3 from the mass, the net factor is 10^{-3} . As a quick cross-check: the well-known SI value of G is indeed $6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$, confirming the conversion is correct.*

Think Like a Physicist: *When converting between unit systems, always start by writing the dimensional formula. Each base dimension (M, L, T) carries its own conversion factor raised to the appropriate power. This systematic approach prevents sign errors in the exponents, which is where most students make mistakes. Notice that a negative exponent in the dimensions (like M^{-1}) flips the conversion factor: switching to a larger unit of mass makes the number smaller, not larger.*

Dimensional analysis is now in your toolkit: four uses and five limitations, each understood through worked examples rather than memorised from a list. You can check formulas, derive relationships, and convert between systems. You also know, honestly, where the method stops and experiment must begin.

The next section takes us from the structure of equations to the honesty of numbers. Every measurement carries uncertainty, and pretending otherwise is not physics. Welcome to errors and uncertainties, where we learn to say what we know and admit what we do not.

ERRORS AND UNCERTAINTIES

Suppose Kipanga had used a proper metre ruler instead of his feet to measure the laboratory. Would his measurement have been perfect? Not a chance. The ruler itself has markings only to the nearest millimetre. His eye would have judged the position of the wall slightly differently each time he looked. The temperature in the room would have caused the ruler (and the wall) to expand or contract by tiny amounts. Even the most careful measurement, made with the best instrument, by the most skilled physicist, carries some degree of uncertainty.

This is not a failure. It is a fact of nature. And a physicist who ignores it is more dangerous than one who makes a large error, because at least the second physicist knows something went wrong. This section teaches you to measure honestly: to state what you know, to quantify what you do not, and to propagate uncertainty through calculations so that your final answer tells the truth about its own reliability.

Why Every Measurement Has Uncertainty

An **uncertainty** (also called an **error** in the technical sense) is an **unavoidable** deviation of a measured value from the true value of the quantity being measured. It arises not from carelessness but from the fundamental nature of the measurement process itself.

Three terms appear constantly in this section, and students confuse them so frequently that we must distinguish them right at the start:

An **error** (in the physics sense) is an unavoidable deviation from the true value, caused by the *limitations of the instrument or the measurement process*. It is present in every measurement, no matter how carefully performed. When a physicist says “the measurement has an error of 0.2 mm,” they are not confessing incompetence. They are reporting, with professional honesty, the range within which the true value is believed to lie.

A **mistake** is simply doing something wrong: misreading a scale, recording the wrong number, writing 2.54 when the scale clearly shows 2.45, or forgetting to subtract the zero reading. Mistakes are avoidable. They can and should be eliminated by careful technique, double-checking readings, and attentive record-keeping. They are not errors in the physics sense.

A **blunder** (sometimes called a **gross error** in older textbooks, though the name is misleading since it is avoidable) is a severe mistake that produces a reading wildly different from all the others in a set. For example, if five measurements of a pendulum period give 2.04, 2.06, 2.08, 2.05, and 3.71 seconds, the last value is almost certainly a blunder, perhaps caused by miscounting oscillations or starting the stopwatch at the wrong moment. *Blunders are identified by inspection: if one reading is dramatically different from the rest, it is discarded from the dataset before calculating the mean.* A blunder is not a type of experimental error; it is a human failure that should not contaminate the analysis.

With these three terms clearly separated, we can now focus entirely on genuine errors, the unavoidable uncertainties that remain even when every mistake and blunder has been eliminated.

Uncertainty arises from two fundamental sources:

- 1) **The instrument itself:** every measuring device has a smallest division, called its least count or resolution. A ruler marked in millimetres cannot reliably distinguish between 12.3 mm and 12.4 mm. A stopwatch that reads to 0.01 s cannot detect events shorter than that interval. The resolution sets a floor below which the instrument simply cannot see.
- 2) **The measurement process:** human judgement, environmental fluctuations (temperature, vibration, air currents), timing reflexes, and parallax all introduce variability that no instrument can remove entirely.

Types of Errors

Not all errors behave the same way. Understanding the difference between the two types is essential, because the strategy for dealing with each is completely different.

1. Systematic errors

A systematic error is an error that shifts all measurements in the same direction (always too high or always too low) by the same amount or by the same proportion. It is consistent and repeatable, which is precisely what makes it dangerous: the data looks reliable even though every value is wrong.

Common sources of systematic errors include:

Zero error: The instrument does not read zero when it should. A ruler whose end is worn down always gives readings that are slightly too short. A spring balance that reads 0.3 N when nothing is hanging from it adds 0.3 N to every subsequent measurement. A voltmeter that shows 0.02 V with no circuit connected shifts every voltage reading upward by that amount.

Calibration error: The instrument's scale is uniformly wrong. A thermometer that reads 102°C in steam at standard atmospheric pressure will give every subsequent temperature reading 2°C too high. The readings will be perfectly consistent with each other, perfectly precise, and all equally wrong.

Parallax error: Reading a scale from an angle rather than from directly in front. When the eye is not perpendicular to the scale, the apparent position of the pointer shifts because of the gap between the pointer and the scale markings. If you consistently read a mercury thermometer from slightly above, your eye sees the mercury level against a higher mark than the true reading. The error is always in the same direction.

To understand why parallax occurs, consider a pointer suspended a small distance above a ruler. Viewed from directly in front, the pointer appears to line up with the correct mark. Viewed from an angle, the pointer appears to align with a different mark because your line of sight passes through the pointer at an angle and hits the scale at the wrong position. The cure is simple: always *position your eye so that your line of sight is perpendicular to the scale*.

Reaction time: When timing with a stopwatch, the delay between seeing an event and pressing the button is roughly the same every time (typically 0.1 to 0.3 seconds for a human). This adds a consistent bias to every timing measurement. In a **pendulum experiment**, the error *appears twice, once when starting and once when stopping the watch, and the two delays may partially cancel or add depending on the situation*.

The effect on data: systematic errors shift the entire dataset in one direction. The readings may cluster tightly together (appearing precise), but their average is displaced from the true value. The data looks good but is inaccurate.

How to minimise: calibrate instruments before use, check for zero error and correct for it, use proper technique (eye level for meniscus readings, perpendicular viewing for scales), use instruments with higher resolution, and compare results with known standard values where possible. The most important point: **repetition does not reduce systematic errors**. If the ruler is 2 mm too short, measuring a hundred times gives a hundred wrong readings that all agree with each other.

2. Random errors

A random error is an error that varies unpredictably in both magnitude and direction from one measurement to the next. Sometimes the reading is slightly too high, sometimes slightly too low, with no consistent pattern.

Common sources of random errors include:

Environmental fluctuations: temperature drafts, vibrations from traffic or nearby machinery, air currents that deflect a pendulum bob slightly differently on each swing, and voltage fluctuations in the power supply during electrical measurements.

Human inconsistency: slight variations in how you align your eye with the scale, how you judge the exact moment a pendulum passes its rest position, or how firmly you close the jaws of a vernier calliper around an object.

Genuine variability: the quantity itself may fluctuate. Blood pressure changes slightly between heartbeats. The diameter of a wire varies slightly along its length. The period of a real pendulum is affected by tiny, unpredictable air disturbances.

The effect on data: random errors scatter readings above and below the true value. Individual readings are unreliable on their own, but the **mean of many readings** tends toward the true value because the positive and negative deviations tend to cancel.

How to minimise: repeat the measurement many times and calculate the mean. The more repetitions, the more the random errors cancel, and the closer the mean approaches the true value. Timing twenty oscillations of a pendulum instead of one effectively divides the random timing error by twenty.

The Distinction That Must Be Owned Permanently

This single distinction explains why a nurse takes a patient's blood pressure three times (to reduce random error) but also checks the cuff size before starting (to avoid systematic error from using a cuff that is too small or too large). The two strategies are completely different because the two problems are completely different:

Systematic errors **cannot** be reduced by repeating measurements. Every repetition carries the same shift. The cure is recalibration, better technique, or a better instrument.

Random errors **can** be reduced by repeating measurements. Each repetition adds information. The mean of ten readings is more reliable than a single reading, and the mean of a hundred is more reliable still.

A summary table may help anchor this permanently:

Feature	Systematic Error	Random Error
Direction	Always the same	Varies unpredictably
Effect on readings	Shifts all readings in one direction	Scatters readings above and below
Effect on mean	Displaces the mean from the true value	Mean approaches true value with more readings
Reduced by repetition?	No	Yes
Reduced by...?	Recalibration, better technique	More repetitions, averaging
Affects...	Accuracy	Precision

Expressing Errors: Absolute, Relative, and Percentage

Once we accept that every measurement has uncertainty, we need a language to express it precisely. Three quantities do this job, and each tells you something different.

Absolute error

Suppose we measure a quantity a several times and obtain values $a_1, a_2, a_3, \dots, a_n$. The best estimate of the true value is the arithmetic mean:

$$\bar{a} = \frac{a_1 + a_2 + a_3 + \dots + a_n}{n}$$

The absolute error of any single measurement a_i is the magnitude of its deviation from the mean:

$$\Delta a_i = |a_i - \bar{a}|$$

The mean absolute error is the average of all the individual absolute errors:

$$\Delta \bar{a} = \frac{|\Delta a_1| + |\Delta a_2| + \dots + |\Delta a_n|}{n}$$

Using the mean as the best estimate and the mean absolute error as the uncertainty determined from the deviations above, the result is expressed as:

$$a = \bar{a} \pm \Delta \bar{a}$$

This notation communicates two things at once: the best estimate ($\bar{a}$) and the range within which the true value is believed to lie (from $\bar{a} - \Delta \bar{a}$ to $\bar{a} + \Delta \bar{a}$). It is the standard way physicists report measurements.

Relative error and percentage error

Absolute error tells you the size of the uncertainty, but not how significant it is. Consider: a 1 mm error measuring a 2 mm wire diameter is catastrophic (50% of the measured value). The same 1 mm error measuring the 2 km Nyerere Bridge in Dar es Salaam is completely irrelevant (0.00005% of the measured value). The absolute error is identical; the significance is vastly different.

The relative error (also called fractional error) captures this significance:

$$\text{Relative error} = \frac{\Delta \bar{a}}{\bar{a}}$$

It is dimensionless and allows fair comparison between measurements of completely different quantities. A relative error of 0.01 in a length measurement and 0.01 in a mass measurement mean both measurements are equally reliable in proportional terms, even though their absolute errors have different units and magnitudes.

The percentage error is the relative error expressed as a percentage:

$$\text{Percentage error} = \frac{\Delta \bar{a}}{\bar{a}} \times 100\%$$

Errors in Derived Physical Quantities

In physics, we rarely measure the final quantity directly. We measure several quantities and combine them using a formula. For example, we find density by measuring mass and volume separately, then dividing. If each input measurement carries uncertainty, the calculated result also carries uncertainty. The question is: *how much?*

Three rules answer this question. Each rule is derived from first principles below, not merely stated, so that you understand why each rule takes the form it does.

Rule 1: Addition and Subtraction (*absolute errors add*)

If $x = a + b$ or $x = a - b$, then:

$$\Delta x = \Delta a + \Delta b$$

Derivation for addition: If the measured values are $(a \pm \Delta a)$ and $(b \pm \Delta b)$, then in the worst case both errors act in the same direction:

$$x + \Delta x = (a + \Delta a) + (b + \Delta b) = (a + b) + (\Delta a + \Delta b)$$

Since $x = a + b$, the maximum error is $\Delta x = \Delta a + \Delta b$.

Derivation for subtraction: This is the case students find surprising. If $x = a - b$, the worst case occurs when the errors push a and b in opposite directions, making the difference as large (or as small) as possible:

$$x + \Delta x = (a + \Delta a) - (b - \Delta b) = (a - b) + (\Delta a + \Delta b)$$

Since $x = a - b$, the maximum error is again $\Delta x = \Delta a + \Delta b$.

Notice the crucial point: even though the quantities are subtracted, the absolute errors still add. This is because the worst case for a difference is when a is too high and b is too low simultaneously (or the reverse), which pushes the errors in the same direction relative to the result.

Special warning: Subtracting two nearly equal quantities is extremely dangerous. Suppose $a = 50.0 \pm 0.5$ and $b = 49.0 \pm 0.5$. Each measurement has a modest 1% relative error. The difference is $x = 1.0$, but the absolute error is still $\Delta x = 0.5 + 0.5 = 1.0$. The relative error of the difference is $\frac{1.0}{1.0} = 100\%$. Two individually precise measurements have produced a result that is completely useless. This phenomenon, sometimes called catastrophic cancellation, is one of the most common sources of large errors in student experiments. Whenever your experiment involves subtracting two similar numbers, treat the result with extreme suspicion.

What to do about it: whenever possible, redesign the experiment to measure the small difference directly rather than obtaining it by subtracting two large numbers. For example, if you need the temperature rise of water during heating, do not measure the initial temperature and the final temperature with separate thermometers and subtract. Instead, use the same thermometer and read the change on its scale, or use a thermocouple designed to measure temperature differences directly. If subtraction cannot be avoided, use instruments with much higher precision for the individual measurements, so that the absolute errors are small compared to the expected difference. Whenever your experiment involves subtracting two similar numbers, treat the result with extreme suspicion and report its inflated uncertainty honestly.

Rule 2: Multiplication and Division (*relative errors add*)

If $x = ab$ or $x = a/b$, then:

$$\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

Derivation for multiplication: For $x = ab$, the measured value with errors is:

$$x + \Delta x = (a + \Delta a)(b + \Delta b) = ab + a\Delta b + b\Delta a + \Delta a \cdot \Delta b$$

Since Δa and Δb are both small compared to a and b , their product $\Delta a \cdot \Delta b$ is negligibly small. Dropping it:

$$\Delta x \approx a\Delta b + b\Delta a$$

Dividing both sides by $x = ab$:

$$\frac{\Delta x}{x} \approx \frac{a\Delta b + b\Delta a}{ab} = \frac{\Delta b}{b} + \frac{\Delta a}{a}$$

Derivation for division: For $x = \frac{a}{b}$, the maximum value of x occurs when a is at its largest and b is at its smallest:

$$x + \Delta x = \frac{a + \Delta a}{b - \Delta b}$$

The error is:

$$\Delta x = \frac{a + \Delta a}{b - \Delta b} - \frac{a}{b}$$

Finding a common denominator:

$$\Delta x = \frac{b(a + \Delta a) - a(b - \Delta b)}{b(b - \Delta b)} = \frac{ab + b\Delta a - ab + a\Delta b}{b(b - \Delta b)} = \frac{b\Delta a + a\Delta b}{b(b - \Delta b)}$$

Since Δb is small compared to b , the denominator $b(b - \Delta b) \approx b^2$:

$$\Delta x \approx \frac{b\Delta a + a\Delta b}{b^2} = \frac{\Delta a}{b} + \frac{a\Delta b}{b^2}$$

Dividing both sides by $x = \frac{a}{b}$:

$$\frac{\Delta x}{x} = \frac{\frac{\Delta a}{b}}{\frac{a}{b}} + \frac{\frac{a\Delta b}{b^2}}{\frac{a}{b}} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

The result is the same as for multiplication: the relative errors add. This is why a single rule covers both operations.

Rule 3: Powers (relative error multiplied by the power)

If $x = a^n$, then:

$$\frac{\Delta x}{x} = n \times \frac{\Delta a}{a}$$

This follows directly from Rule 2. If $x = a \cdot a \cdot a$ (three factors of a), then by Rule 2 the relative error is $\frac{\Delta a}{a} + \frac{\Delta a}{a} + \frac{\Delta a}{a} = 3 \left(\frac{\Delta a}{a} \right)$. For a general power n , the relative error is $n \left(\frac{\Delta a}{a} \right)$.

This is why squaring a measurement doubles its relative error, and cubing it triples it. A 2% error in measuring a radius becomes a 6% error in the calculated volume (since $V \propto r^3$). Students find this surprising the first time they see it, and important every time they use it afterwards.

The General Formula

For a quantity that depends on multiple variables raised to different powers, the results above combine into a single master formula. If:

$$x = a^n \times b^m$$

then:

$$\frac{\Delta x}{x} = n \times \frac{\Delta a}{a} + m \times \frac{\Delta b}{b}$$

where n and m are the magnitudes of the powers (always taken as positive, since we are calculating the maximum possible error). This is the formula you will use most often for error propagation in A-level physics.

Errors from Graphs

In practical examinations, you will often determine a physical quantity by plotting a graph and measuring its gradient. The uncertainty in the gradient must be estimated from the graph itself, using a systematic method.

The best-fit line is drawn so that data points are distributed as evenly as possible above and below the line. It is not forced through every single point (that would amplify random errors rather than averaging them out), and it is not forced through the origin unless there is a physical reason to do so.

Error bars are vertical (and sometimes horizontal) lines drawn at each data point, showing the uncertainty in that reading. A data point plotted at $y = 5.0 \pm 0.3$ has a vertical error bar extending from 4.7 to 5.3. The best-fit line should pass through as many error bars as possible, not necessarily through the points themselves.

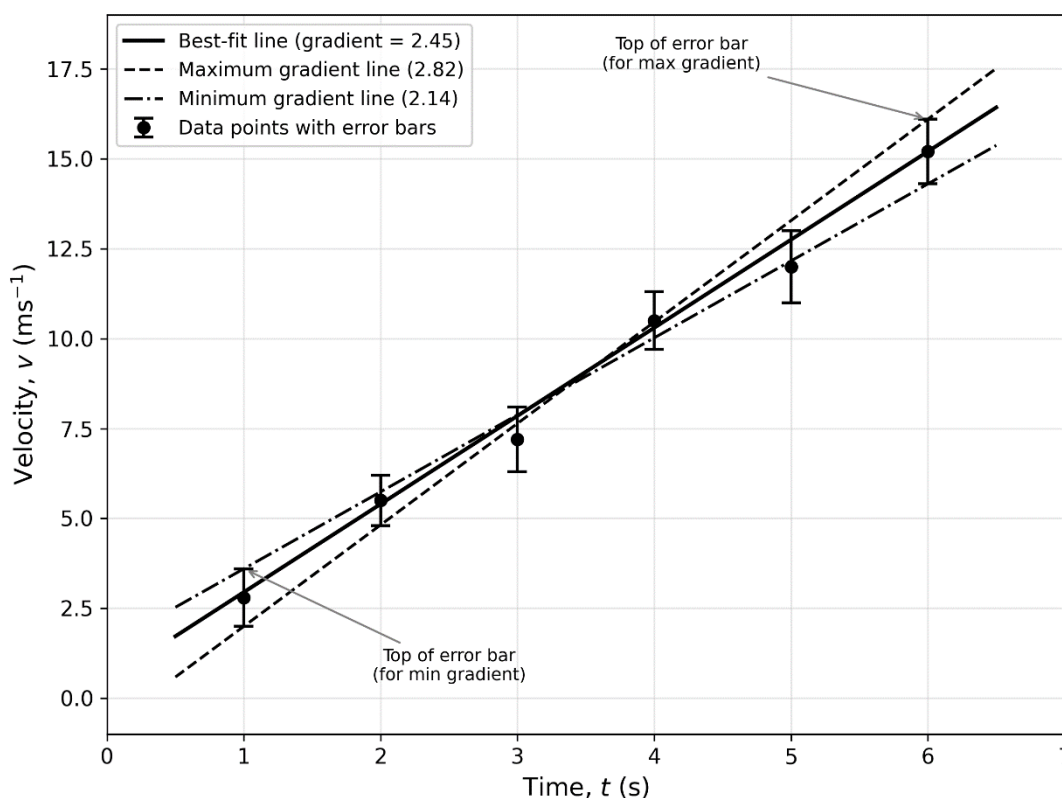


Figure: Determining the uncertainty in a gradient from a graph. Six data points are plotted with vertical error bars showing the uncertainty in each reading. The best-fit line (solid) is drawn through the data as evenly as possible. The maximum gradient line (dashed) passes from the bottom of the first error bar to the top of the last, giving the steepest plausible gradient. The minimum gradient line (dash-dot) passes from the top of the first error bar to the bottom of the last, giving the shallowest plausible gradient. The gradient uncertainty is calculated as $\Delta m = \frac{m_{\max} - m_{\min}}{2}$.

The maximum gradient line is the steepest straight line that still passes through (or close to) the error bars of the data points. **The minimum gradient line** is the shallowest such line.

The uncertainty in the gradient is:

$$\Delta m = \frac{m_{\max} - m_{\min}}{2}$$

where $m_{\max}$ and $m_{\min}$ are the gradients of the steepest and shallowest acceptable lines.

The gradient is then reported as:

$$m = m_{\text{best}} \pm \Delta m$$

The same approach applies to the y-intercept: read the intercept from each of the three lines and report the best-fit value with its uncertainty.

A practical tip for examinations: when drawing the maximum and minimum gradient lines, anchor one end at the top of the highest error bar and the other at the bottom of the lowest (for maximum gradient), or the reverse (for minimum gradient). This ensures you are using the full range of the data's uncertainty.

With the theory complete, let us build fluency through worked examples that progress from straightforward to demanding.

BINDER Example 14

A student measures the period of a simple pendulum five times and obtains the following values (in seconds): 2.04, 2.08, 2.06, 2.10, 2.02.

Find: (a) the mean value, (b) the mean absolute error, (c) the relative error, (d) the percentage error. Write the final result in standard notation.

Solution

(a) Mean value:

$$\bar{a} = \frac{(2.04 + 2.08 + 2.06 + 2.10 + 2.02)\text{s}}{5} = 2.06 \text{ s}$$

(b) Individual absolute errors:

$$|2.04 - 2.06| = 0.02 \text{ s}, \quad |2.08 - 2.06| = 0.02 \text{ s}, \quad |2.06 - 2.06| = 0.00 \text{ s}, \quad |2.10 - 2.06| = 0.04 \text{ s}, \\ |2.02 - 2.06| = 0.04 \text{ s}$$

Mean absolute error:

$$\Delta\bar{a} = \frac{(0.02 + 0.02 + 0.00 + 0.04 + 0.04)\text{s}}{5} = 0.024 \text{ s} \approx 0.02 \text{ s}$$

(c) Relative error:

$$\frac{\Delta\bar{a}}{\bar{a}} = \frac{0.02 \text{ s}}{2.06 \text{ s}} = 0.0097 \approx 0.01$$

(d) Percentage error:

$$0.0097 \times 100\% \approx 1.0\%$$

Final result in standard notation: $T = (2.06 \pm 0.02) \text{ s}$

Making Sense of the Answer: A 1% error in a pendulum period measurement is respectable for a school laboratory. The readings are closely clustered around the mean, indicating good technique and small random errors. The student should now use this period (with its uncertainty) to calculate g and propagate the error forward using the techniques from this subtopic.

Think Like a Physicist: Always round the uncertainty to one or two significant figures, then round the mean to match the last significant digit of the uncertainty. Writing $T = 2.06 \pm 0.024357 \text{ s}$ suggests a false precision in the uncertainty itself. The uncertainty is an estimate, not an exact number.

BINDER Example 15

Calculate the maximum possible error in:

- (a) The sum of two lengths measured as $(12.5 \pm 0.1) \text{ cm}$ and $(8.3 \pm 0.2) \text{ cm}$.
- (b) The product of a force $(25.0 \pm 0.5) \text{ N}$ and a displacement $(4.0 \pm 0.1) \text{ m}$.
- (c) The cube of a radius measured as $(3.0 \pm 0.1) \text{ cm}$.

Solution

(a) Addition (Rule 1: absolute errors add):

$$x = 12.5 \text{ cm} + 8.3 \text{ cm} = 20.8 \text{ cm} \\ \Delta x = 0.1 \text{ cm} + 0.2 \text{ cm} = 0.3 \text{ cm}$$

Result: $x = (20.8 \pm 0.3) \text{ cm}$

(b) Multiplication (Rule 2: relative errors add):

$$W = 25.0 \text{ N} \times 4.0 \text{ m} = 100 \text{ J} \\ \frac{\Delta W}{W} = \frac{0.5 \text{ N}}{25.0 \text{ N}} + \frac{0.1 \text{ m}}{4.0 \text{ m}} = 0.020 + 0.025 = 0.045 \\ \Delta W = 0.045 \times 100 \text{ J} = 4.5 \text{ J}$$

Result: $W = (100 \pm 5) \text{ J}$ (rounded)

(c) Powers (Rule 3: relative error multiplied by power):

$$r^3 = (3.0 \text{ cm})^3 = 27 \text{ cm}^3 \\ \frac{\Delta(r^3)}{r^3} = 3 \times \frac{0.1 \text{ cm}}{3.0 \text{ cm}} = 3 \times 0.033 = 0.10$$

$$\Delta(r^3) = 0.10 \times 27 \text{ cm}^3 = 2.7 \text{ cm}^3$$

Result: $r^3 = (27 \pm 3) \text{ cm}^3$

Making Sense of the Answer: In part (c), a 3.3% error in the radius became a 10% error in the cube. The power of 3 tripled the relative error. This is why volume measurements are inherently less precise than length measurements, even when the same ruler is used for both. Any quantity that depends on a high power of a measured variable will amplify that variable's uncertainty.

Think Like a Physicist: For addition and subtraction, work with absolute errors. For multiplication, division, and powers, work with relative errors. Mixing them up is one of the most common procedural errors in error propagation. A helpful memory aid: if the operation combines quantities by adding or subtracting, the absolute errors add. If it combines them by multiplying or dividing, the relative errors add. The error rule mirrors the operation.

BINDER Example 16

The acceleration due to gravity is determined using the formula $g = \frac{4\pi^2 l}{T^2}$. The percentage error in l is 1% and the percentage error in T is 2%. Find the percentage error in g . State which measurement contributes more to the error.

Solution

Since $g = 4\pi^2 l T^{-2}$, and $4\pi^2$ is an exact constant (no error):

$$\frac{\Delta g}{g} = 1 \times \frac{\Delta l}{l} + 2 \times \frac{\Delta T}{T}$$

The power of l is 1, so its percentage error enters with a multiplier of 1. The power of T is 2 (we use the magnitude), so its percentage error enters with a multiplier of 2.

$$\% \text{ error in } g = 1 \times 1\% + 2 \times 2\% = 1\% + 4\% = 5\%$$

The timing measurement dominates the error: T contributes 4% out of the total 5%, while l contributes only 1%.

Making Sense of the Answer: Because T is squared in the formula, its error is doubled before it enters the total. A 2% error in T becomes a 4% contribution to g . This is why, in a pendulum experiment, the most effective way to improve the accuracy of g is to improve the timing. Timing 20 oscillations instead of 1 divides the random timing error by 20, dramatically reducing the percentage error in T .

Think Like a Physicist: Always identify which measurement contributes the most error to the final result. Then improve that measurement first. Spending time and effort improving the less significant measurement is wasted work. In this case, buying a more precise ruler (to improve l) is far less effective than simply timing more oscillations (to improve T).

REAL Example 17

A nurse at Bugando Hospital measures a patient's temperature five times in quick succession and gets: 37.1°C, 37.3°C, 37.0°C, 37.2°C, 37.4°C. Kipute, observing during a school hospital visit, asks:

Kipute: "Is the thermometer broken? It gives a different answer every time."

Mr. Akilikubwa: "No, Kipute. The thermometer is working exactly as it should. In fact, a thermometer that gave exactly the same reading every single time would worry me more."

Explain why the readings differ, why the mean is more reliable than any single reading, and identify a situation where averaging would not improve the result.

Solution

The readings differ because of random errors: tiny fluctuations in blood flow near the skin surface, slight variations in thermometer placement between measurements, and natural biological variation in body temperature between breaths and heartbeats. These cause each reading to deviate slightly and unpredictably from the true body temperature.

The mean (37.2°C) is more reliable than any single reading because random errors are equally likely to push a reading above or below the true value. When many readings are averaged, the positive deviations tend to

cancel the negative deviations, and the mean converges toward the true temperature. Five readings give a better estimate than one; twenty would be better still.

However, if the thermometer had a systematic error (for example, if it consistently read 0.5°C too low because of a calibration fault), averaging would not help. Every single reading would be too low by the same amount, so the mean would also be too low by that amount. The nurse would need to recalibrate or replace the thermometer. Repetition cures random error; it cannot cure systematic error.

Making Sense of the Answer: *This is why hospitals calibrate their instruments regularly. A thermometer that consistently reads 0.5°C low could cause a doctor to miss a dangerous fever. The instrument looks precise (consistent readings) but is inaccurate (all readings displaced from the true value). Mr. Akilikubwa's comment that a perfectly consistent thermometer would worry him reflects this: perfect consistency in a real measurement is suspicious, because it may indicate that the instrument's resolution is too low to detect the natural random variation that should be present.*

Think Like a Physicist: *When you see scattered readings, ask yourself two questions. First: is this random error? If so, averaging will help. Second: could there also be a systematic error hiding beneath the scatter? Averaging handles the first problem but completely ignores the second. A good experimentalist addresses both.*

HOT Example 18

The density of a rectangular metal block is to be determined. The following measurements are taken:

Mass: $m = (50.0 \pm 0.1) \text{ g}$

Length: $l = (4.00 \pm 0.01) \text{ cm}$

Width: $w = (3.00 \pm 0.01) \text{ cm}$

Height: $h = (2.00 \pm 0.01) \text{ cm}$

Find the density with its absolute and percentage uncertainty. State the dominant source of error.

Solution

Volume:

$$V = l \times w \times h = 4.00 \text{ cm} \times 3.00 \text{ cm} \times 2.00 \text{ cm} = 24.0 \text{ cm}^3$$

Density:

$$\rho = \frac{m}{V} = \frac{50.0 \text{ g}}{24.0 \text{ cm}^3} = 2.083 \text{ gcm}^{-3}$$

Error propagation. Since $\rho = \frac{m}{l \times w \times h}$, the relative error is:

$$\frac{\Delta\rho}{\rho} = \frac{\Delta m}{m} + \frac{\Delta l}{l} + \frac{\Delta w}{w} + \frac{\Delta h}{h}$$

Substituting:

$$\begin{aligned} \frac{\Delta\rho}{\rho} &= \frac{0.1 \text{ g}}{50.0 \text{ g}} + \frac{0.01 \text{ cm}}{4.00 \text{ cm}} + \frac{0.01 \text{ cm}}{3.00 \text{ cm}} + \frac{0.01 \text{ cm}}{2.00 \text{ cm}} \\ &= 0.002 + 0.0025 + 0.0033 + 0.005 = 0.0128 \end{aligned}$$

Percentage error: $0.0128 \times 100\% = 1.3\%$

Absolute error in density:

$$\Delta\rho = 0.0128 \times 2.083 \text{ gcm}^{-3} = 0.027 \text{ gcm}^{-3}$$

Final result: $\rho = (2.08 \pm 0.03) \text{ gcm}^{-3}$

The dominant source of error is the height measurement, which contributes $\frac{0.005}{0.0128} = 39\%$ of the total relative error, despite using the same ruler as the other two dimensions.

This happens because 2.00 cm is the smallest dimension, so the same absolute error ($\pm 0.01 \text{ cm}$) produces the largest relative error (0.5%).

Making Sense of the Answer: The smallest dimension always dominates the error because relative error equals $\Delta x/x$, and the denominator is smallest for the shortest measurement. If you could improve only one measurement, improve the height: use a micrometer screw gauge instead of a ruler. Improving the length measurement (which already has only 0.25% relative error) would barely change the overall uncertainty.

Think Like a Physicist: Identifying and reporting the dominant error source in your final experimental report shows you understand not just how to calculate errors, but which measurement to improve. This is the difference between efficient experimentation and wasted effort.

HOT Example 19

From a velocity-time graph, a student draws a best-fit line and determines the gradient (acceleration) to be 12.5 ms^{-2} . The maximum gradient line gives 13.8 ms^{-2} and the minimum gradient line gives 11.4 ms^{-2} . Express the acceleration with its uncertainty in standard notation. Calculate the percentage uncertainty.

Solution

Uncertainty in gradient:

$$\Delta m = \frac{m_{\max} - m_{\min}}{2} = \frac{13.8 \text{ ms}^{-2} - 11.4 \text{ ms}^{-2}}{2} = 1.2 \text{ ms}^{-2}$$

Acceleration in standard notation:

$$a = (12.5 \pm 1.2) \text{ ms}^{-2}$$

Percentage uncertainty:

$$\frac{1.2 \text{ ms}^{-2}}{12.5 \text{ ms}^{-2}} \times 100\% = 9.6\%$$

Making Sense of the Answer: A 9.6% uncertainty from a graph is typical for school-level experiments with five or six data points. The gradient uncertainty reflects how much the data points scatter. More data points, measured more carefully, would produce a tighter scatter and a smaller difference between the maximum and minimum gradient lines.

Think Like a Physicist: Always draw the maximum and minimum gradient lines through the extremes of the error bars, not through arbitrary points you find convenient. The error bars are the data's honest statement about its own uncertainty; the gradient lines must respect them. If you skip the error bars, you are guessing at the uncertainty instead of measuring it.

Errors and uncertainties are now part of your vocabulary. You can classify errors as systematic or random, identify and discard blunders, express uncertainty in absolute, relative, and percentage form, propagate errors through any formula, and extract uncertainty from graphs. Every measurement you write from this point forward should carry its uncertainty, not as a confession of failure, but as a badge of scientific honesty.

In the next section, we sharpen two words that students use carelessly and examiners test mercilessly: accuracy and precision. They are not the same thing, and confusing them is the kind of error that costs marks quietly.

ACCURACY AND PRECISION

Two words. Six syllables. And more confusion than any other pair of terms in A-level physics. Students use them interchangeably. Textbooks sometimes blur the distinction. Examiners, however, do not. In every practical report, every error analysis, and every examination question that asks you to evaluate a set of measurements, these two words carry specific, different meanings. This section establishes those meanings clearly and permanently.

The Dartboard Analogy

The easiest way to understand the distinction is through an analogy that has nothing to do with physics and everything to do with darts.

Precision describes how closely repeated measurements agree with each other. A set of measurements is precise if the values are tightly clustered, regardless of where they fall relative to the true value.

Accuracy describes how close the measurements are to the true value. A set of measurements is accurate if their mean is close to the true value, regardless of how much the individual values scatter.

These are independent properties. A measurement can be one without the other, both, or neither. The four possible combinations are best visualised as a dartboard, where the bullseye represents the true value:

Precise AND accurate: All darts clustered tightly around the bullseye. The measurements agree with each other (precise) and their mean is very close to the true value (accurate). This is the ideal outcome, achieved by good technique, a well-calibrated instrument, and sufficient repetitions.

Precise but NOT accurate: All darts clustered tightly together, but the cluster is far from the bullseye. The measurements agree with each other (precise) but their mean is displaced from the true value (inaccurate). This is the signature of a systematic error. The instrument or technique is consistent but wrong. A thermometer that always reads 2°C too high produces readings that are beautifully precise and completely inaccurate.

Accurate but NOT precise: Darts scattered widely across the board, but their average position is the bullseye. The individual measurements scatter badly (imprecise), but their mean happens to land on or near the true value (accurate). This is the signature of large random errors with no systematic error. Averaging many readings would give a good result, but any single reading is unreliable.

Neither precise nor accurate: Darts scattered widely and not centred on the bullseye. Both random and systematic errors are large. The measurements neither agree with each other nor approach the true value. This is the worst outcome, indicating poor technique, a badly calibrated instrument, or both.

Connecting to Error Types

The dartboard analogy is not just a pretty picture. It connects directly to the error classification we learned earlier:

Systematic errors destroy accuracy. They shift all readings in the same direction, displacing the mean from the true value. The data may look precise (tight cluster) but the cluster is in the wrong place. This is why systematic errors are so dangerous: the data appears trustworthy when it is not.

Random errors destroy precision. They scatter readings above and below the true value, spreading the cluster. If no systematic error is present, the mean of many readings will still approach the true value (the centre of the scatter is correct), but any individual reading may be far off.

The practical consequence is direct:

To improve accuracy, you must identify and eliminate systematic errors: recalibrate instruments, correct for zero errors, use proper technique.

To improve precision, you must reduce random errors: take more readings, use instruments with finer resolution, control environmental conditions.

A measurement that is precise but inaccurate cannot be fixed by taking more readings. A hundred readings from a miscalibrated thermometer give a very precise wrong answer. Only recalibration fixes it.

A measurement that is accurate but imprecise can be improved by taking more readings. The scatter is random, so averaging reduces it. More data points give a tighter estimate of the true mean.

Resolution and Sensitivity

Two further terms appear in examination questions about measurement quality. They are related to precision but are not the same thing.

Resolution (also called **least count**) is the smallest change in a quantity that an instrument can detect. It equals the smallest division on the instrument's scale. A ruler marked in millimetres has a resolution of 1 mm. A micrometer screw gauge has a resolution of 0.01 mm. A digital stopwatch reading to 0.01 s has a resolution of 0.01 s.

Resolution sets the **minimum** uncertainty in a single reading. *You cannot claim an uncertainty smaller than the instrument's resolution, no matter how carefully you read it.*

Sensitivity is how much the instrument's output changes per unit change in the quantity being measured. A thermometer in which the mercury column moves 2 cm for every 1°C change is more sensitive than one in which it moves 0.5 cm per $^{\circ}\text{C}$. A galvanometer that deflects 10 divisions per milliamp is more sensitive than one that deflects 2 divisions per milliamp.

An important distinction: a sensitive instrument is not necessarily an accurate one. A highly sensitive thermometer may detect tiny temperature changes (good sensitivity) but still read 2°C too high on every reading (poor accuracy due to systematic error). Sensitivity tells you how finely the instrument responds; accuracy tells you whether it responds correctly.

With the vocabulary now sharp, let us test it.

BINDER Example 20

A student measures a length known to be exactly 25.0 cm. Six measurements are recorded: 24.8, 24.9, 25.1, 24.7, 25.0, 24.9 (all in cm).

A second student measures the same length and records: 26.3, 26.4, 26.3, 26.5, 26.4, 26.3 (all in cm).

For each set of measurements, determine whether the measurements are precise, accurate, both, or neither. Justify each conclusion.

Solution

First student:

$$\text{Mean} = \frac{(24.8 + 24.9 + 25.1 + 24.7 + 25.0 + 24.9) \text{ cm}}{6} = 24.9 \text{ cm}$$

The readings scatter from 24.7 to 25.1 cm, a spread of 0.4 cm. This is a moderate spread, so the precision is reasonable but not exceptional.

The mean (24.9 cm) is very close to the true value (25.0 cm), differing by only 0.1 cm. The measurements are accurate.

Conclusion: Student 1's measurements are **accurate but only moderately precise**. The scatter suggests random errors are present but no significant systematic error, since the mean is close to the true value.

Second student:

$$\text{Mean} = \frac{(26.3 + 26.4 + 26.3 + 26.5 + 26.4 + 26.3) \text{ cm}}{6} = 26.37 \text{ cm}$$

The readings scatter from 26.3 to 26.5 cm, a spread of only 0.2 cm. This is a tight cluster, indicating high precision.

However, the mean (26.37 cm) is 1.37 cm above the true value (25.0 cm). The measurements are inaccurate.

Conclusion: Second student's measurements are **precise but not accurate**. The tight clustering indicates small random errors, but the large displacement from the true value indicates a systematic error. The ruler may have a damaged or missing zero end, or the student is consistently misaligning the starting point.

Making Sense of the Answer: *Second student's data looks better at first glance because the readings are so consistent. But consistency without correctness is misleading. A set of tightly clustered wrong answers is more dangerous than a set of scattered readings whose mean is correct, because the tight clustering creates a false sense of confidence.*

Think Like a Physicist: *You cannot judge accuracy without knowing the true value. If the true value is unknown (as it usually is in a real experiment), precision is all you can assess from the data alone. This is why calibration against known standards is so important: it is the only way to check for systematic errors that precision alone cannot reveal.*

REAL Example 21

During a football practice session, Kipanga takes five penalty kicks, aiming for the top-right corner of the goal. All five shots land in the bottom-left corner. Kipute, watching from the sideline, makes an observation:

Kipute: *"You are very precise, Kipanga. All five shots went to exactly the same place."*

Kipanga: (offended) *"What do you mean, precise? I missed every single one!"*

Kipute: *"Exactly. You are precise but not accurate. Every shot is consistent with the others, but none of them went where you aimed."*

Mr. Akilikubwa, who has been watching quietly with a cup of tea, walks over.

Mr. Akilikubwa: *“Kipanga is right. And this is not just about football. This is exactly what happens when a measuring instrument has a systematic error. Imagine Kipanga is a thermometer. He consistently gives the same reading (bottom-left corner), and that reading is always wrong by the same amount and in the same direction (displaced from the top-right corner). Repeating the measurement does not help, because every repetition lands in the same wrong place.”*

He pauses. *“Now, if Kipanga’s shots had scattered randomly all over the goal, with some going left, some right, some high, some low, but the average landing position was the top-right corner, he would be accurate but imprecise. The individual shots would be unreliable, but the overall pattern would be centred on the target.”*

Kipanga: *“So how do I fix my kicks?”*

Mr. Akilikubwa: *“The same way you fix a systematic error. You do not take more shots hoping the problem will go away, because it will not. You change your technique. You re-aim. You recalibrate.”*

Explain, using this analogy, why repeating a measurement cannot correct a systematic error, and what must be done instead.

Solution

Kipanga’s five identical shots to the bottom-left corner represent a systematic error: every measurement is displaced from the true value by the same amount in the same direction. The displacement is consistent and repeatable.

Taking more penalty kicks (repeating the measurement) will not help, because every additional shot will also land in the bottom-left corner. The average of a hundred bottom-left shots is still the bottom-left corner. Repetition reduces random scatter, but there is no random scatter here. The problem is a consistent bias in technique.

To correct the error, Kipanga must change his technique: re-aim, adjust his body position, or alter his kicking angle. In measurement terms, this corresponds to recalibrating the instrument, correcting for zero error, or changing the experimental method. The cause of the systematic shift must be identified and removed. Only then will future measurements (or penalty kicks) land where they should.

Making Sense of the Answer: *The football analogy makes the abstract distinction between systematic and random errors concrete and memorable. Kipanga’s consistent miss is a systematic error. If his shots had scattered randomly around the target, the problem would be random error, fixable by averaging. The cure must match the disease: averaging cures scatter, recalibration cures shift.*

Think Like a Physicist: *In a real experiment, the hardest part is recognising that a systematic error exists, because the data looks precise and therefore trustworthy. The only way to detect a systematic error is to compare your result with a known standard or with an independent measurement using a different method. If both methods agree, you can be more confident that systematic errors are small. If they disagree, at least one method has a systematic problem.*

Accuracy and precision are now permanently separated in your mind. Precision is about agreement between readings. Accuracy is about agreement with truth. Systematic errors destroy accuracy. Random errors destroy precision. And Kipanga’s penalty kicks, while consistent, will need some serious recalibration before the inter-school tournament.

In the next section, we step briefly outside the laboratory to see why everything we have learned in this chapter matters in the real world, from the design of bridges to the dosing of medicines to the most expensive measurement error in the history of space exploration.

APPLICATIONS OF MEASUREMENT

Physics does not exist only inside textbooks. The principles of measurement, uncertainty, and error analysis that fill this chapter are at work every day in hospitals, construction sites, factories, and space agencies around the world. This section assembles a set of real-world applications that show why precision in measurement is not an academic exercise but a matter of safety, money, and sometimes survival.

1. Engineering and construction

Every structure you walk into, every bridge you cross, and every road you drive on was built from measurements. The consequences of getting those measurements wrong range from inconvenient to catastrophic.

In building construction, a small angular error in the foundation can amplify as the structure rises. A 1 mm misalignment at the base of a ten-storey building can become a 10 cm lean at the top. This is why surveyors use instruments with resolutions measured in fractions of a millimetre, and why every measurement is repeated and cross-checked.

In manufacturing, engine parts must fit together with tolerances measured in hundredths of a millimetre. A piston that is 0.1 mm too wide will seize inside the cylinder and destroy the engine. A piston that is 0.1 mm too narrow allows combustion gases to leak past, reducing power and efficiency. The entire performance of the engine depends on measurements that are invisible to the naked eye.

In Tanzania, the construction of major infrastructure projects such as the Standard Gauge Railway and the Julius Nyerere Hydropower Station required GPS-based measurement systems accurate to within centimetres over distances of hundreds of kilometres. Traditional surveying methods, while adequate for smaller projects, could not achieve the precision needed for structures of this scale.

2. Medicine and pharmacy

In medicine, measurement errors can be the difference between treatment and tragedy.

Drug dosage depends on precise measurement of the drug's mass or concentration and the patient's body weight. For drugs with a narrow therapeutic window (where the effective dose is close to the toxic dose), a 10% error can push the dose from therapeutic to dangerous. Digoxin, lithium, and warfarin are examples of drugs where dosage precision is critical. A pharmacist who measures carelessly risks poisoning the patient.

Blood pressure measurement is affected by systematic errors if the wrong cuff size is used. A cuff that is too small compresses the arm insufficiently, giving a falsely high reading. A doctor relying on this reading may prescribe unnecessary medication. The solution is simple: use the correct cuff size. But recognising the systematic error requires understanding what systematic errors are and how they behave, exactly the knowledge this chapter provides.

In laboratory diagnosis, clinical decisions depend on precise measurement of blood glucose levels, electrolyte concentrations, and cell counts. An error of a few percent in a blood glucose reading could lead to an incorrect diagnosis of diabetes or a missed case of hypoglycaemia.

3. Space exploration and the most expensive measurement error in history

Earlier, we briefly encountered the Mars Climate Orbiter. The story deserves its full telling here, because it is the most powerful argument for measurement discipline ever produced by human failure.

In December 1998, NASA launched the Mars Climate Orbiter, a spacecraft designed to study the Martian atmosphere and serve as a communications relay for future Mars missions. The mission cost 327 million US dollars. The spacecraft travelled for nine months across 670 million kilometres of space, functioning perfectly.

On 23 September 1999, the spacecraft began its approach to Mars. The navigation team at NASA's Jet Propulsion Laboratory sent course-correction commands based on trajectory data. The data was correct. The commands were correct. But there was one problem: the software written by the spacecraft's builder, Lockheed Martin, calculated thruster impulses in pound-force seconds (an imperial unit), while NASA's navigation software expected the data in newton seconds (the SI unit). One pound-force second equals approximately 4.45 newton seconds. Nobody checked. Nobody noticed the mismatch. For nine months.

The spacecraft entered the Martian atmosphere at an altitude of 57 km instead of the planned 226 km. At 57 km, the atmospheric friction was far too great. The spacecraft broke apart and was destroyed.

Three hundred and twenty-seven million dollars. Nine months of travel. Years of engineering. The work of hundreds of scientists and engineers. All lost because two teams used different units and neither performed a simple conversion check. The investigation board concluded that the root cause was a failure to verify units at the interface between two software systems. It was not a failure of physics. It was not a failure of engineering. It was a failure of measurement discipline.

The Mars Climate Orbiter has become the single most cited example in physics education of why units matter, why standard systems exist, and why every number must carry its unit. The lesson is simple: the

physics was perfect, and the spacecraft was destroyed anyway, because someone assumed the units would take care of themselves.

4. Scientific Research

At the frontier of physics, the margin between discovery and noise is often a matter of measurement precision.

The discovery of the Higgs boson at CERN in 2012 required measurements accurate to parts per billion. The signal that confirmed the particle's existence was only five standard deviations above background noise. Without rigorous error analysis, the signal would have been indistinguishable from a statistical fluctuation. The entire discovery, one of the most important in the history of physics, rested on the ability to measure, quantify uncertainty, and distinguish real effects from measurement noise.

In climate science, global temperature records depend on measurements taken at thousands of stations around the world over more than a century. A systematic error of even 0.1°C in how temperatures were recorded at certain stations can distort the apparent warming trend when data is accumulated over decades. Climate scientists spend enormous effort identifying and correcting these systematic biases, because the conclusions about global warming must rest on measurements that are both precise and accurate.

Measurement is not the most dramatic part of physics. It does not have the visual appeal of a rocket launch or the conceptual excitement of black holes. But it is the foundation on which every other part of physics stands. A formula is only as good as the measurements that feed it. A theory is only as convincing as the data that tests it. And a physicist who cannot measure honestly is not a physicist at all.

You now have the vocabulary, the tools, and (unlike Kipanga) the correct units. What remains is practice. The following miscellaneous worked examples will test everything from this chapter, combining concepts in the way examinations do. Pick up your pen, prepare your calculator, and let us begin.

MISCELLANEOUS WORKED EXAMPLES ON MEASUREMENT

Example 22

- (a) The viscous drag force on a sphere moving through a fluid is given by $F = 6\pi\eta rv$ where η is the coefficient of viscosity, r is the radius of the sphere, and v is its velocity. Use this equation to find the dimensional formula and SI unit of η .
- (b) A student claims that the formula for the period of a spring-mass system is $T = \frac{2\pi m}{k}$ where m is mass and k is the spring constant (dimensions MT^{-2}). Use dimensional analysis to show that this formula is incorrect, and find the correct form.

Solution

- (a) From $F = 6\pi\eta rv$, making η the subject: $\eta = \frac{F}{6\pi rv}$

$$[\eta] = \frac{[F]}{[r][v]} = \frac{\text{MLT}^{-2}}{\text{L} \times \text{LT}^{-1}} = \frac{\text{MLT}^{-2}}{\text{L}^2\text{T}^{-1}} = \text{ML}^{-1}\text{T}^{-1}$$

SI unit: $\text{kgm}^{-1}\text{s}^{-1}$ (also called Pas).

- (b) Checking the student's formula:

$$[T] = T$$

$$\left[\frac{m}{k}\right] = \frac{M}{\text{MT}^{-2}} = \text{T}^2$$

Since $T \neq \text{T}^2$, the formula $T = \frac{2\pi m}{k}$ is dimensionally incorrect.

For the correct form, let $T = 2\pi m^a k^b$:

$$T = M^a (\text{MT}^{-2})^b = M^{a+b} \text{T}^{-2b}$$

For M: $0 = a + b$

For T: $1 = -2b$, giving $b = -1/2$ and $a = 1/2$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Example 23

- (a) The density of mercury is 13.6gcm^{-3} . A student converts this to SI by writing $13.6 \times 10^{-3}\text{kgm}^{-3}$. Without performing the full conversion, explain why this answer must be wrong.
- (b) Perform the correct conversion and express the density in SI units.

Solution

- (a) Mercury is one of the densest liquids known. It is so heavy that iron floats on it. The density of air is approximately 1.2kgm^{-3} . The student's answer, 0.0136kgm^{-3} , is about 100 times less dense than air. This is physically absurd. A liquid that is less dense than air would float upward and disperse, not sit in a thermometer. The conversion must be wrong.
- (b) Density has dimensions ML^{-3} . Converting each unit:

$$13.6\text{gcm}^{-3} = 13.6 \times \frac{10^{-3}\text{kg}}{(10^{-2}\text{m})^3} = 13.6 \times \frac{10^{-3}\text{kg}}{10^{-6}\text{m}^3} = 13.6 \times 10^3\text{kgm}^{-3} = 13600\text{kgm}^{-3}$$

Example 24

- (a) The escape velocity from a planet depends on the planet's mass M , its radius R , and the gravitational constant G (dimensions $\text{M}^{-1}\text{L}^3\text{T}^{-2}$). Use dimensional analysis to derive an expression for the escape velocity.
- (b) The actual formula is $v = \sqrt{\frac{2GM}{R}}$. State what dimensional analysis was unable to determine.

Solution

- (a) Assume: $v = kG^aM^bR^c$

Dimensions: $[v] = \text{LT}^{-1}$, $[G] = \text{M}^{-1}\text{L}^3\text{T}^{-2}$, $[M] = \text{M}$, $[R] = \text{L}$

$$\text{LT}^{-1} = (\text{M}^{-1}\text{L}^3\text{T}^{-2})^a(\text{M})^b(\text{L})^c = \text{M}^{-a+b}\text{L}^{3a+c}\text{T}^{-2a}$$

For M : $0 = -a + b$, so $b = a$

For T : $-1 = -2a$, giving $a = 1/2$

For L : $1 = 3a + c = 3/2 + c$, giving $c = -1/2$

Therefore $b = 1/2$ and:

$$v = k\sqrt{\frac{GM}{R}}$$

- (b) Dimensional analysis could not determine the **dimensionless** constant k . The actual value is $k = \sqrt{2}$. This constant comes from the full derivation (using conservation of energy). Dimensional analysis tells us the structure; theory or experiment supplies the constant.

Example 25

A student measures the refractive index of a glass block seven times and obtains: 1.44, 1.50, 1.48, 1.45, 1.97, 1.52, 1.46.

- (a) Examine the data carefully. Calculate the mean value, mean absolute error, relative error, and percentage error. Write the result in standard notation.
- (b) Explain your treatment of any suspect reading.

Solution

- (a) Inspecting the data: six of the seven readings cluster between 1.44 and 1.52. The reading 1.97 is far outside this range. It differs from its neighbours by nearly 0.5, while the spread among the other six is only 0.08. This reading is almost certainly a blunder (perhaps caused by misreading the protractor or miscalculating the angle). It is **discarded** before further analysis.

Using the remaining six readings: 1.44, 1.50, 1.48, 1.45, 1.52, 1.46

Mean:

$$\bar{n} = \frac{1.44 + 1.50 + 1.48 + 1.45 + 1.52 + 1.46}{6} = 1.475$$

Individual absolute errors: $|1.44 - 1.475| = 0.035$, $|1.50 - 1.475| = 0.025$, $|1.48 - 1.475| = 0.005$, $|1.45 - 1.475| = 0.025$, $|1.52 - 1.475| = 0.045$, $|1.46 - 1.475| = 0.015$

Mean absolute error:

$$\Delta \bar{n} = \frac{0.035 + 0.025 + 0.005 + 0.025 + 0.045 + 0.015}{6} = 0.025$$

Relative error: $\frac{0.025}{1.475} = 0.017$

Percentage error: $0.017 \times 100\% = 1.7\%$

Result: $n = 1.48 \pm 0.03$

- (b) The reading 1.97 was discarded as a blunder. A blunder is a severe mistake that produces a reading wildly different from the rest of the dataset. Including it would have shifted the mean to 1.546 and inflated the error to 0.11, contaminating the result with a single human mistake rather than reflecting the genuine measurement uncertainty. Blunders are identified by inspection and removed before averaging.

Example 26

- (a) In the formula $g = \frac{4\pi^2 l}{T^2}$, explain why the percentage error in T contributes more to the total error than the percentage error in l , even when both are measured with equal precision.
- (b) In an experiment, $l = (0.800 \pm 0.002)\text{m}$ and $T = (1.796 \pm 0.005)\text{s}$. Calculate g with its absolute and percentage uncertainty. Identify the dominant source of error and suggest one improvement.

Solution

- (a) In $g = 4\pi^2 l T^{-2}$, the power of l is 1 and the power of T is 2. The percentage error in g is $\frac{\Delta g}{g} = 1 \times \frac{\Delta l}{l} + 2 \times \frac{\Delta T}{T}$. The factor of 2 multiplying $\frac{\Delta T}{T}$ doubles its contribution. Even with identical percentage precision in l and T , the timing error contributes twice as much because T is squared.
- (b)

$$g = \frac{4\pi^2 \times 0.800\text{m}}{(1.796\text{s})^2} = 9.791\text{ms}^{-2}$$

Percentage errors:

$$\frac{\Delta l}{l} = \frac{0.002\text{m}}{0.800\text{m}} = 0.25\%$$

$$\frac{\Delta T}{T} = \frac{0.005\text{s}}{1.796\text{s}} = 0.28\%$$

Total: $\frac{\Delta g}{g} = 1 \times 0.25\% + 2 \times 0.28\% = 0.25\% + 0.56\% = 0.81\%$

Absolute uncertainty: $\Delta g = 0.0081 \times 9.791\text{ms}^{-2} = 0.08\text{ms}^{-2}$

Result: $g = (9.79 \pm 0.08) \text{ms}^{-2}$

The dominant source is T (contributing 0.56% out of 0.81%).

To improve: time greater number of oscillations (e.g. 50 oscillations instead of 20), which divides more the timing error (by a further factor of 2.5 for 50 oscillations).

Example 27

The coefficient of viscosity η of a liquid is determined using Stokes' law:

$$\eta = \frac{2r^2(\rho_s - \rho_f)g}{9v}$$

The following measurements are taken: $r = (1.00 \pm 0.05)\text{mm}$, $\rho_s = (7800 \pm 50)\text{kgm}^{-3}$, $\rho_f = (1260 \pm 10)\text{kgm}^{-3}$, $v = (0.069 \pm 0.003)\text{ms}^{-1}$, $g = 9.8\text{ms}^{-2}$ (exact).

- Calculate η .
- Find the percentage error in η , showing the contribution from each measurement.
- Identify the dominant source of error and suggest how to reduce it.

Solution

(a)

$$\rho_s - \rho_f = 7800\text{kgm}^{-3} - 1260\text{kgm}^{-3} = 6540\text{kgm}^{-3}$$

$$\eta = \frac{2 \times (1.00 \times 10^{-3}\text{m})^2 \times 6540\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 0.069\text{ms}^{-1}} = 0.206\text{kgm}^{-1}\text{s}^{-1}$$

(b) This formula combines all three error rules:

Step 1 — Powers: r appears as r^2 , so: relative error from $r = 2 \times \frac{0.05\text{mm}}{1.00\text{mm}} = 0.1 = 10\%$

Step 2 — Subtraction: $(\rho_s - \rho_f)$ uses Rule 1. Absolute error: $\Delta(\rho_s - \rho_f) = 50 + 10 = 60\text{kgm}^{-3}$

Relative error: $\frac{60\text{kgm}^{-3}}{6540\text{kgm}^{-3}} = 0.92\%$

Step 3 — Division: combine all relative errors:

$$\frac{\Delta\eta}{\eta} = 10\% + 0.92\% + 4.35\% = 15.3\%$$

where the contribution from v : $\frac{0.003\text{ms}^{-1}}{0.069\text{ms}^{-1}} = 4.35\%$

Individual contributions: r : 10%, v : 4.35%, $(\rho_s - \rho_f)$: 0.92%.

- (c) The radius r dominates overwhelmingly (10% out of 15.3%). This happens because r is squared, doubling its relative error, **and** because the ball is small (1 mm), making any measurement uncertainty a large fraction of the value. To reduce it: use a larger ball (larger r means the same absolute error gives a smaller relative error) or use a micrometer screw gauge instead of a ruler to measure the diameter more precisely.

Example 28

Two forces act on an object in opposite directions: $F_1 = (25.0 \pm 0.5)\text{N}$ and $F_2 = (23.0 \pm 0.5)\text{N}$. The net force accelerates a mass $m = (2.0 \pm 0.1)\text{kg}$.

- Calculate the acceleration with its percentage uncertainty and give the final result.
- Explain why the result has such a large uncertainty even though each individual measurement seems precise.

Solution

(a) Net force: $F = F_1 - F_2 = 25.0\text{N} - 23.0\text{N} = 2.0\text{N}$

Absolute error in F (Rule 1 — subtraction): $\Delta F = 0.5\text{N} + 0.5\text{N} = 1\text{N}$

Percentage error in F : $\frac{1\text{N}}{2\text{N}} \times 100\% = 50\%$

Acceleration: $a = \frac{F}{m} = \frac{2.0\text{N}}{2.0\text{kg}} = 1\text{ms}^{-2}$

Percentage error in a (Rule 2 — division):

$$\frac{\Delta a}{a} = \frac{\Delta F}{F} + \frac{\Delta m}{m} = 50\% + \frac{0.1\text{kg}}{2.0\text{kg}} \times 100\% = 50\% + 5\% = 55\%$$

Absolute uncertainty: $\Delta a = 0.55 \times 1 \text{ ms}^{-2} = 0.6 \text{ ms}^{-2}$

Result: $a = (1 \pm 0.6) \text{ ms}^{-2}$

- (b) Each force measurement has only 2% relative error ($0.5/25.0$ and $0.5/23.0$). Individually, these are precise. But when two nearly equal quantities are subtracted, the result is small while the absolute errors add. The relative error of the difference (50%) is enormously larger than the relative errors of the individual forces (2%). This is catastrophic cancellation: the subtraction destroyed the precision of the original measurements. To improve this experiment, either use a force meter with much higher precision or redesign so that the net force is measured directly.

Example 29

- (a) A student plots a graph of T^2 (vertical axis) against l (horizontal axis) for a simple pendulum experiment. The best-fit gradient is $4.05 \text{ s}^2 \text{ m}^{-1}$, the maximum gradient is $4.35 \text{ s}^2 \text{ m}^{-1}$, and the minimum gradient is $3.78 \text{ s}^2 \text{ m}^{-1}$.

Given that $T^2 = \left(\frac{4\pi^2}{g}\right)l$, determine g with its absolute uncertainty and percentage uncertainty.

- (b) Explain why plotting T^2 against l (rather than T against l) is preferred in this experiment.

Solution

- (a) Gradient $m = \frac{4\pi^2}{g}$, so $g = \frac{4\pi^2}{m}$.

Best estimate: $g = \frac{4\pi^2}{4.05 \text{ s}^2 \text{ m}^{-1}} = 9.75 \text{ ms}^{-2}$

From maximum gradient: $g_{\min} = \frac{4\pi^2}{4.35 \text{ s}^2 \text{ m}^{-1}} = 9.08 \text{ ms}^{-2}$

From minimum gradient: $g_{\max} = \frac{4\pi^2}{3.78 \text{ s}^2 \text{ m}^{-1}} = 10.45 \text{ ms}^{-2}$

Uncertainty: $\Delta g = \frac{g_{\max} - g_{\min}}{2} = \frac{(10.45 \text{ ms}^{-2} - 9.08 \text{ ms}^{-2})}{2} = 0.69 \text{ ms}^{-2} \approx 0.7 \text{ ms}^{-2}$

Percentage uncertainty: $\left(\frac{0.7 \text{ ms}^{-2}}{9.75 \text{ ms}^{-2}}\right) \times 100\% = 7.2\%$

Result: $g = (9.8 \pm 0.7) \text{ ms}^{-2}$

- (b) The relationship $T = 2\pi\sqrt{\frac{l}{g}}$ is not linear: T is proportional to $\sqrt{l}$, not to l . Plotting T against l produces a curve, and fitting a straight line to a curve is unreliable. Squaring both sides gives $T^2 = \left(\frac{4\pi^2}{g}\right)l$, which is linear. A straight-line graph is much easier to draw accurately, and the gradient can be measured precisely using two well-separated points.

Example 30

During a break between lessons, Kipanga announces to the class:

Kipanga: “I have used dimensional analysis to prove that the amount of trouble I am in has dimensions ML^2T^{-2} . Trouble equals the energy I have wasted. Therefore, trouble is a physical quantity.”

Kipute: “That is not how it works, Kipanga.”

Mr. Akilikubwa: (overhearing from the doorway) “Go on, Kipanga. Show us your derivation.”

Kipanga: “The amount of trouble depends on three things: the mass of the object I broke (M), the distance it fell (L), and the time my mother takes to find out (T). So trouble $= kM^aL^bT^c$. Since trouble is like energy, it has dimensions ML^2T^{-2} . Therefore $a = 1$, $b = 2$, $c = -2$. Trouble $= kMd^2t^{-2}$.”

- (a) Identify the fundamental flaw in Kipanga’s argument.
- (b) Assuming the formula were valid, calculate the “trouble” when he drops a 2 kg vase from 1.5 m and his mother finds out in 3 s. Take $k = 1$.
- (c) Mr. Akilikubwa points out that the formula predicts the trouble decreases if Kipanga’s mother takes longer to find out. Explain whether this makes the formula more or less believable.

Solution

- (a) Kipanga assumes the conclusion before carrying out the analysis. He first declares that trouble has the dimensions of energy and then determines the powers that yield those dimensions. The argument therefore assumes what it is trying to prove and is logically circular. Dimensional analysis can relate the dimensions of known physical quantities, but it cannot establish that a non-physical quantity has particular dimensions. Trouble is not a physical quantity because it cannot be measured objectively on any agreed scale with an agreed unit.
- (b) $\text{Trouble} = 1 \times 2\text{kg} \times (1.5\text{m})^2 \times (3\text{s})^{-2} = \frac{2\text{kg} \times 2.25\text{m}^2}{9\text{s}^2} = 0.5\text{J}$
- (c) The formula predicts that the longer Kipanga's mother takes to discover the broken vase, the less trouble he is in (since $\text{trouble} \propto 1/t^2$). Whether delay increases or decreases the trouble depends on the mother, the vase, and the day of the week, but no one would seriously claim it follows a precise inverse-square law. The formula produces a prediction that cannot be tested on any agreed scale, illustrating that a dimensionally correct formula can be physically absurd. The algebra does not know about mothers.

If you followed all miscellaneous worked examples without breaking a sweat, you are ready for what comes next. The Digging Deeper exercise that follows will not be gentle. It will test every idea in this chapter, from the definition of a physical quantity to the propagation of errors through formulas that combine all three rules at once. Some questions will build your confidence. Others will stretch it. That is the point.

DIGGING DEEPER EXERCISE 1

EXERCISE 1A: BINDER QUESTIONS

Question 1

- (a) Explain why a physical quantity must have both a number and a unit. What information does a measurement carry without a unit?
- (b) Give one real-life example in which a missing or incorrect unit caused a practical problem.

Question 2

A student says: “Length is fundamental and velocity is derived, but I can measure both directly with instruments. So what makes one fundamental and the other derived?”

Explain clearly what determines whether a physical quantity is fundamental or derived.

Question 3

A student checks the equation $v^2 = u^2 + 2as$, finds it dimensionally correct, and concludes the equation must therefore be physically correct. Identify the flaw in this reasoning and support your answer with an example.

Question 4

State three uses of dimensional analysis. For each, give one specific example showing the method in action.

Question 5

State three limitations of dimensional analysis. For each, give one specific example demonstrating where the method fails.

Question 6

- (a) Distinguish between systematic errors and random errors.
- (b) Explain why the strategy for reducing each type is completely different. Give a practical example of each strategy.

Question 7

Explain what happens to the percentage error when two nearly equal measured quantities are subtracted.

Question 8

A student measures a standard mass known to be exactly 5.50 g. The readings are: 5.01, 5.02, 5.01, 5.02, 5.01 (all in grams).

- (a) Are these measurements precise? Justify.
- (b) Are they accurate? Justify.
- (c) What type of error is likely present?

Question 9

Explain why timing 20 complete oscillations of a pendulum and dividing by 20 gives a more precise value for the period than timing a single oscillation directly. Include a numerical estimate to support your argument.

Question 10

A student claims: “My ruler has millimetre markings, so the uncertainty in every length I measure is exactly 1 mm and nothing more.”

Identify two things wrong with this claim.

EXERCISE 1B: REAL QUESTIONS**Question 11**

A tailor in Kariakoo market uses a tape measure that has been stretched from years of heavy use. The first 5 cm of the tape has expanded and now covers what should be 5.3 cm.

- (a) What type of error does this introduce into every measurement the tailor makes?
- (b) If the tailor measures each customer three times and averages the readings, will the error reduce? Explain.

Question 12

A doctor at Mloganzila Hospital takes a patient's blood pressure three times in quick succession and records slightly different values each time. Explain why the readings differ and why the doctor averages them rather than trusting any single reading.

Question 13

Kipute and Kipanga both measure the length of the same laboratory table. Kipute gets 1.235 m on every attempt. Kipanga gets 1.20 m, 1.27 m, 1.22 m, 1.25 m, and 1.24 m, giving a mean of 1.236 m.

Kipanga: *"We got the same answer, so our measurements are equally good."*

Kipute: *"I disagree."*

- (a) Who is more precise? Justify.
- (b) Can you say with certainty which set is more accurate? Explain what additional information (if any) would be needed.
- (c) Explain why Kipanga is wrong to claim the measurements are equally good.

Question 14

In manufacturing, engineers specify tolerances (for example, a piston diameter of 10.000 ± 0.005 cm) rather than demanding exact dimensions.

- (a) Explain why demanding an exact dimension is impossible.
- (b) Explain why it is also unnecessary, provided the tolerance is met.

Question 15

A 5% error in a medicine dose can be life-threatening, but a 5% error in measuring the length of a classroom is harmless. Both are 5% errors. Explain what makes one dangerous and the other irrelevant.

Question 16

A student measures the period of a pendulum using a wristwatch.

- (a) Identify two possible sources of systematic error in this experiment.
- (b) Identify two possible sources of random error in this experiment.

Question 17

Explain why the relative error is more meaningful than the absolute error when comparing the quality of two different measurements. Support your answer with a specific example involving two measurements of different quantities.

Question 18

Kipanga measures his body temperature with a thermometer and gets 37.0°C on five consecutive attempts.

Kipanga: *"Five identical readings. My thermometer is both accurate and precise."*

Kipute: *"You can be sure about one of those. For the other, you need more information."*

- (a) Which property (accuracy or precision) can Kipanga confirm from his data alone? Explain.
- (b) What additional information or test would be needed to check the other property?

Question 19

A map of Tanzania is drawn to a scale of 1:1,000,000. A student measures the distance between Dar es Salaam and Dodoma on the map and makes an error of 1 mm.

- What is the corresponding error in the real-world distance?
- What does this tell you about the importance of precision in cartography?

EXERCISE 1C: HOT QUESTIONS**Question 20**

Express the SI unit of each of the following in terms of base units only:

- Pressure
- Energy
- Coefficient of viscosity (defined by $F = \eta A dv/dx$)
- Surface tension (defined as force per unit length)

Question 21

The terminal velocity of a sphere falling through a viscous fluid is given by:

$$v = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

where r is the radius of the sphere, ρ_s and ρ_f are the densities of the sphere and fluid, g is the acceleration due to gravity, and η is the coefficient of viscosity (dimensions $ML^{-1}T^{-1}$).

Verify that this equation is dimensionally homogeneous.

Question 22

The height h to which a liquid rises in a capillary tube depends on the surface tension γ (dimensions MT^{-2}), the density ρ of the liquid, the radius r of the tube, and the acceleration due to gravity g . Use dimensional analysis to derive an expression for h in terms of these quantities.

Question 23

A student uses dimensional analysis to derive the formula for gravitational potential energy and obtains $E = mgh^2$.

- Use dimensional analysis to show that this formula is wrong.
- Find the correct power of h .
- State clearly what dimensional analysis alone cannot tell you about the correct formula for gravitational potential energy.

Question 24

A ball is dropped from rest at height h and takes time t to reach the ground. Given that t depends only on h and g :

- Use dimensional analysis to express t in terms of h and g .
- A student measures $h = 5.0$ m and $t = 1.01$ s. Use these values and $g = 9.8ms^{-2}$ to determine the dimensionless constant in the formula and hence write the full formula.

Question 25

The density of a cylindrical wire is determined using $\rho = 4m/(\pi d^2 l)$. The measurements are: mass $m = (2.50 \pm 0.01)$ g, length $l = (50.0 \pm 0.1)$ cm, diameter $d = (0.80 \pm 0.01)$ mm.

- Convert all measurements to SI units and calculate ρ .
- Find the percentage error contribution from each of the three measurements.
- Find the total percentage error in ρ .

- (d) Explain why the diameter contributes the most error despite having the smallest absolute uncertainty.
- (e) The student wants the total percentage error below 3%. Which single measurement should they improve, and to what precision?

Question 26

The focal length of a converging lens is determined using $f = \frac{uv}{u-v}$ where $u = (30.0 \pm 0.5)$ cm and $v = (20.0 \pm 0.5)$ cm.

- (a) Calculate f .
- (b) Explain why this method produces such a large uncertainty in f even though u and v are each measured with only about 2% error.

Question 27

In an Atwood machine experiment, g is determined from:

$$g = \frac{2s(m_1 + m_2)}{(m_1 - m_2)t^2}$$

where $m_1 = (150.0 \pm 0.5)$ g, $m_2 = (130.0 \pm 0.5)$ g, $s = (0.800 \pm 0.002)$ m, and $t = (1.50 \pm 0.02)$ s.

- (a) Find the total percentage error in g .
- (b) Calculate g .
- (c) Explain why this method gives a less precise value of g than the simple pendulum method, even though all individual measurements appear precise.

Question 28

From a straight-line graph of distance s against time squared t^2 , the best-fit gradient is 4.85ms^{-2} , the maximum gradient is 5.20ms^{-2} , and the minimum gradient is 4.55ms^{-2} . The relationship is $s = \frac{1}{2}at^2$.

- (a) Determine the acceleration a from the best-fit gradient.
- (b) Find the uncertainty in a and express the result in standard notation.
- (c) Calculate the percentage uncertainty.

Question 29

Design a complete experiment to determine the density of an irregular stone using only a ruler, a measuring cylinder, water, and a balance. In your design, specify:

- (a) All measurements to be taken.
- (b) The instruments and their least counts.
- (c) Sources of error and how to minimise them.
- (d) The formula for density and how to estimate its uncertainty.

Question 30

A physics teacher wants students to determine g using a simple pendulum with a total percentage error below 1%. The pendulum length is $l = 1.000$ m, measured with a metre ruler (least count 1 mm). The total timing uncertainty from reaction time is 0.2 s.

- (a) Write an expression for the total percentage error in g in terms of $\Delta l/l$ and $\Delta T/T$, using the formula $g = 4\pi^2 l/T^2$.
- (b) Calculate the percentage error in l .
- (c) Hence determine the maximum allowable percentage error in T .
- (d) Determine the minimum number of oscillations that should be timed in a single run to achieve this target.

Chapter 2

MOTION IN STRAIGHT LINE

INTRODUCTION

A daladala moves, stops, jerks forward, and moves again. A boda-boda swerves, a football flies through the air, rain falls from the sky, and you walk from the school gate to your classroom. None of these events look like equations. Yet Physics insists that they all speak the same language; **motion**.

Most students meet motion as a collection of formulas. But your body already understands motion. You feel it when a bus starts suddenly. You predict it when you throw a ball. You fear it when a bicycle rushes downhill. Long before you meet formulas, you already live inside motion.

In this part, we rebuild the foundations you once met at O-level. Not to repeat them, but to understand them properly. We will move from everyday meaning to precise description, and only then to mathematics. By the time we reach formulas, they will feel like summaries of ideas you already own.

MOTION AND REFERENCE FRAMES

Is a person sitting in a bus at rest or in motion? The answer depends on who is watching. To another passenger, the person is at rest. To a person standing by the road, the same person is moving at the speed of the bus. Motion is therefore relative.

Physics resolves this confusion by introducing a **reference frame** which is *a chosen point of view from which motion is described*. Without stating the reference frame, statements about motion are incomplete. Whenever you describe motion, always ask: *From whose point of view is it being observed?*

For now, let us pause the theory and bring the ideas down to earth with a few carefully chosen worked examples.

BINDER Example 1

A student sits quietly inside a moving train, deeply absorbed in a book. One may argue that the student is both *at rest* and *in motion* at the same time. Justify this statement.

Solution

Since motion is not absolute but depends on the reference frame chosen, the student is **at rest relative to the train** because there is no change in position with respect to the seat, the floor, or other passengers. However, the student is **in motion relative to the ground** because the train itself is moving along the track.

Making Sense of the Answer: *The student looks still to someone inside the train, but to someone outside, the student is moving with the train. The difference comes from **who is observing**.*

Think Like a Physicist: *Before deciding whether something is at rest or in motion, always ask: “Relative to what?”*

REAL Example 2

Mr. Akilikubwa is teaching Motion and Reference Frames. While standing by the roadside with **Kipute** and **Kipanga**, they watch a daladala moving past them. At that moment, a passenger inside the daladala accidentally drops a coin.

Kipute says, *“From here, the coin seems to follow a curved path.”*

But Kipanga insists, *“To the passenger inside, the coin must fall straight down.”*

Explain why Kipute and Kipanga can both be correct.

Solution

Kipute and Kipanga are both correct because they are describing the motion of the coin from **different reference frames**.

- Kipute is standing on the ground, so she observes the coin as it falls while the daladala continues moving forward. To her, the coin has **two motions at the same time**:

- ✓ a forward motion (same as the daladala), and
- ✓ a downward motion (due to gravity).

The combination of these motions makes the coin appear to follow a curved path.

- The passenger inside the daladala is moving with the vehicle, so in that reference frame the coin has **no forward motion relative to the passenger**. It only moves downward due to gravity, so it appears to fall straight down.

Making Sense of the Answer: *The coin does the same motion, but it looks different depending on where you are observing from outside or inside the moving daladala.*

Think Like a Physicist: *Whenever an object is released from a moving vehicle, it still keeps the **horizontal velocity** of the vehicle at the moment of release.*

With these worked examples complete, we are ready to welcome the next subtopic and see how the story continues.

DISTANCE AND DISPLACEMENT

Distance answers the question: *How much ground did you cover in total?* It depends on the entire path taken and it ignores direction, so it is always a **scalar quantity** and can never be negative.

Displacement answers a deeper question: *Where is your final position relative to your starting point?* It depends only on the starting and ending points (not the path), and it carries direction, so it is a **vector quantity**. Displacement can be **positive, negative, or zero**, depending on the direction chosen.

Because of this, two journeys can have the **same distance** but **different displacements**, or even have a **large distance** with **zero displacement** (when you return to your starting point). In short, *distance measures the length of the journey, while displacement measures the actual change in position.*

To truly understand and enjoy these ideas, let us serve them in the form of worked examples.

BINDER Example 3

A student walks 60m east from the dormitory to the library, then 20m west to the water tap. Find:

- (a) Distance travelled
- (b) Displacement

Solution

- (a) Distance = $60 + 20 = 80\text{m}$
- (b) Displacement = $60 - 20 = 40\text{m east}$

Making Sense of the Answer: *The student walked 80m, but ended only 40m east of the starting point.*

Think Like a Physicist: *Distance is always **greater than or equal to** the magnitude of displacement, because distance counts the whole path.*

REAL Example 4

After school, **Kipanga** runs one complete lap around the circular football field and stops exactly where he started.

Determine his:

- (a) Distance travelled
- (b) Displacement

Solution

- (a) Distance travelled = the **circumference** of the circular field (one complete lap).
- (b) Displacement = **0**, because Kipanga ends at the same point where he started.

Making Sense of the Answer: *Kipanga ran a long distance, but his final position did not change. Distance counts the whole journey, while displacement depends only on the start and end points.*

Think Like a Physicist: *Whenever an object returns to its starting point, its displacement is zero, even though the distance travelled is not zero.*

Having given the worked examples their full say, we can now welcome the next subtopic and see what fresh ideas it brings along.

SPEED AND VELOCITY

Speed tells *how fast* an object moves. It is the rate of covering distance, so it is a scalar quantity and it is always positive or zero.

Velocity tells *how fast and in which direction* an object moves. It is the rate of change of displacement, so it is a vector quantity. Velocity can be positive, negative, or zero depending on the chosen direction.

Two cars may have the same **speed** but different **velocities** if they move in opposite directions. This is why speed is just a number, but velocity is a complete description of motion as it tells the *magnitude* and the *direction*.

In Physics, direction is never a decoration. It can change the final answer, especially in problems involving **vectors**, **relative motion**, and **acceleration**.

Before the ideas drift away, let us anchor them using a few worked examples.

BINDER Example 5

Two cars are moving along the same straight road. Car A moves at **20m/s north**, while Car B moves at **20m/s south**. Compare the **speeds** and **velocities** of the two cars.

Solution

- The **speeds** of both cars are the **same** because speed depends only on how fast they move:

$$\text{Speed} = 20\text{m/s}$$

- The **velocities** are **different** because velocity includes both magnitude and direction:

- ✓ Car A: **20m/s north**
- ✓ Car B: **20m/s south**

Making Sense of the Answer: *Speed ignores direction, but velocity depends on direction, so opposite directions give different velocities even if the speed is the same.*

Think Like a Physicist: *If two objects move with the same speed in opposite directions, their velocities have the same magnitude but opposite signs (if one direction is chosen as positive).*

REAL Example 6

One rainy afternoon, **Kipanga** stands still at the school gate, watching the rain fall **straight down**. A moment later, he remembers he left his exercise books at home, so he starts running fast toward home. Suddenly, the rain begins to hit his face as if it is coming **from the front**, not from above. Explain why the rain appears to fall **slanted** when Kipanga starts running.

Solution

When Kipanga is standing still, the rain has only a **vertical downward velocity** due to gravity, so it appears to fall straight down.

When he starts running forward, Kipanga gains a **forward velocity**. The rain is still falling downward, but relative to Kipanga, the rain now seems to have a **horizontal component** opposite to his motion. Combining the rain's downward motion and Kipanga's forward motion makes the rain appear to move **diagonally**, so it feels like it is coming from the front and hitting his face.

Making Sense of the Answer: *The rain has not changed its motion, but Kipanga's motion changes how he observes it; running forward makes the rain seem to "chase" him from the front.*

Think Like a Physicist: *Whenever two motions happen at the same time (forward motion of the observer and downward motion of rain), the observed direction becomes the **resultant** of the two velocities.*

The worked examples have said their piece; now the acceleration subtopic clears its throat and steps forward.

ACCELERATION

Many students think acceleration means “moving fast.” Others think it means “speeding up.” Both ideas are close, but not complete. In Physics, acceleration does not describe how fast you are moving; it describes *how quickly your velocity changes*.

To feel acceleration, you do not need a laboratory. You only need a daladala.

When the daladala is moving at a steady speed, your body feels normal. But when it suddenly starts, you feel pushed backward. When it suddenly stops, you feel pushed forward. That feeling is not speed, it is **acceleration**.

What Acceleration Really Means?

Acceleration is the *rate of change of velocity with time*.

This means acceleration happens when:

- velocity increases (speeding up)
- velocity decreases (slowing down)
- direction changes (even at constant speed!)

So, an object can have acceleration even if its speed stays the same as long as its direction is changing.

Acceleration is given by:

$$a = \frac{\Delta v}{\Delta t}$$

Where:

a = acceleration (m/s²)

Δv = change in velocity (m/s)

Δt = time taken (s)

For now, take a short breath and set theory aside, let us warm up with some carefully chosen worked examples.

BINDER Example 7

A car increases its velocity from 5m/s to 25m/s in 4s. Find its acceleration.

Solution

$$a = \frac{v - u}{t}$$

$$a = \frac{(25 - 5)\text{m/s}}{4\text{s}} = 5\text{m/s}^2$$

Therefore, the acceleration is 5m/s².

Making Sense of the Answer: *The car's velocity increases by 5m/s every second.*

Think Like a Physicist: *Acceleration tells you “how much velocity changes per second,” not how fast the object is moving.*

REAL Example 8

Mr. Akilikubwa asks **Kipute** and **Kipanga** to observe a boda-boda leaving the school gate. At first the boda-boda moves slowly, then within a few seconds it becomes very fast. What does the change in the boda-boda's motion tell you about its acceleration?

Solution

Since the boda-boda's velocity is **increasing with time**, its acceleration is **positive**.

Making Sense of the Answer: *The boda-boda is not only moving; it is **changing its velocity**, and that is what acceleration means.*

Think Like a Physicist: If velocity increases with time, acceleration is positive; if velocity decreases, acceleration is negative.

HOT Example 9

A bus is moving at 18m/s. The driver applies brakes and the bus comes to rest in 6s. Find the acceleration.

Solution

Given: $u = 18\text{m/s}$, $v = 0\text{m/s}$ (at rest), $t = 6\text{s}$

$$a = \frac{v - u}{t}$$

$$a = \frac{(0 - 18)\text{m/s}}{6\text{s}} = -3\text{m/s}^2$$

Therefore, the acceleration is -3m/s^2 .

Making Sense of the Answer: The negative sign shows the bus is slowing down. Its velocity decreases by 3m/s every second.

Think Like a Physicist: A negative acceleration does not mean “bad acceleration.” It simply means the acceleration is opposite to the direction of motion.

It is important for you to understand that:

In real life, acceleration can change every moment. A driver may press the accelerator gently, then strongly, then release it. But in many Physics problems, we simplify motion by assuming the acceleration is **constant**. This is called **uniform acceleration**.

Uniform acceleration does not mean the object is moving at a constant velocity. It means *the acceleration is constant, so the velocity changes by the same amount every second*.

That brings our worked examples to a satisfying close. The example meal has been enjoyed; now let us prepare our taste buds, because the next subtopic is about to serve its own delicious ideas.

EQUATIONS OF MOTION (SUVAT Equations)

Once you understand acceleration, Physics becomes more predictable. Why? Because acceleration connects three things that always matter in motion:

- 1) How fast you started moving (initial velocity, u)?
- 2) How fast you ended up moving (final velocity, v)?
- 3) How long the change took (time, t)?

When acceleration is constant (uniform acceleration), these quantities are linked by powerful formulas called the **equations of motion** (or **SUVAT equations**).

Meaning of SUVAT

These letters remind you of the five variables used:

- 1) **S** = displacement (m)
- 2) **U** = initial velocity (m/s)
- 3) **V** = final velocity (m/s)
- 4) **A** = acceleration (m/s^2)
- 5) **T** = time (s)

The Three Main Equations of Motion

When acceleration is constant, the following equations apply:

1. First equation:

$$v = u + at$$

2. Second equation:

$$s = ut + \frac{1}{2}at^2$$

3. Third equation:

$$v^2 = u^2 + 2as$$

Understand These Simple Tactics!

- If the question involves time, think of either: $v = u + at$ or $s = ut + \frac{1}{2}at^2$
- If the question does not include time, use: $v^2 = u^2 + 2as$
- If it asks about **average velocity**, remember:

$$\text{Average velocity} = \frac{u + v}{2}$$

And thus:

$$s = \left(\frac{u + v}{2} \right) t$$

Quick Reminder: Always pick the equation that matches the quantities given in the question.

Before the equations start arguing among themselves, let us invite a few worked examples to settle the matter!

BINDER Example 10

A car starts from rest and accelerates uniformly at 3m/s^2 for 5s. Find its final velocity.

Solution

“Starts from rest” means initial velocity is zero. Thus $u = 0\text{ m/s}$

Using the first equation of motion: $v = u + at$

Where: $u = 0\text{m/s}$, $a = 3\text{m/s}^2$, $t = 5\text{s}$

Substituting $v = (0\text{m/s}) + (3\text{m/s}^2)(5\text{s}) = 15\text{m/s}$

The final velocity of the car is 15m/s .

Making Sense of the Answer: Each second, the car gains 3m/s . After 5 seconds it has gained 15m/s .

Think Like a Physicist: If you start from rest, the final velocity after time t depends only on a and t ; that is $v = at$.

REAL Example 11

Kipute is late for class. She runs from the school gate with an initial velocity of 2m/s and accelerates uniformly at 1.5m/s^2 for 6s. Find her final velocity.

Solution

Again using: $v = u + at$

Where: $u = 2\text{m/s}$, $a = 1.5\text{m/s}^2$, $t = 6\text{s}$

Substituting $v = (2\text{ m/s}) + (1.5\text{m/s}^2)(6\text{ s}) = 11\text{m/s}$

Kipute’s final velocity is 11m/s .

Making Sense of the Answer: Kipute already had some velocity (2m/s), then she added another 9m/s due to acceleration.

Think Like a Physicist: Never ignore the initial velocity; it can change the entire answer.

HOT Example 12

A bus moving at 25m/s slows down uniformly and comes to rest after travelling 100m . Find its acceleration.

Solution

“Comes to rest” means the final velocity is zero. Thus $v = 0\text{m/s}$.

Using the third equation of motion: $v^2 = u^2 + 2as$.

Where: $u = 25\text{m/s}$, $s = 100\text{m}$, $v = 0\text{m/s}$.

Substituting $(0\text{m/s})^2 = (25\text{m/s})^2 + 2(a)(100\text{m})$; $a = -3.125\text{ m/s}^2$

The acceleration is -3.125m/s^2 .

Making Sense of the Answer: *The negative sign shows deceleration. The bus loses velocity steadily until it stops.*

Think Like a Physicist: *When time is missing, don't panic, use the equation that eliminates time.*

As the worked examples quietly leave the table, vertical motion under gravity arrives; not to overwhelm us, but to be understood and enjoyed!

VERTICAL MOTION UNDER GRAVITY

Motion can look very simple, yet it hides one of the most powerful ideas in Physics: near the Earth's surface, every object released in the air experiences the same downward pull. Whether you drop a mango from a tree, toss a stone upward, or jump off a step, the motion is controlled by a constant downward acceleration called the **acceleration due to gravity**. This is why a falling object speeds up as it descends and why an upward-moving object slows down until it briefly stops at the highest point before returning downward. Understanding this vertical motion is essential because it forms the entire foundation of **projectile motion**, where vertical and horizontal motions combine to produce curved paths.

Vertical motion under gravity is the motion of an object along a straight vertical line in which the only acceleration acting is due to gravity (air resistance is ignored).

This is going to be very helpful to you!

In dealing with motion under gravity it is important to understand the following facts and conventions about sign:

- ✓ The acceleration due to gravity (g) is always directed downward. Its value is 9.8m/s^2 which means that: each second, the vertical velocity changes by 9.8m/s .
- ✓ If an upward-thrown body is moving upward, the velocity and displacement will be positive while the acceleration due to gravity will be negative ($a = -g$).
- ✓ If the same body is moving downward, the velocity and displacement will be negative while the acceleration due to gravity will remain negative (g always acts downward).
- ✓ When an object is thrown upward, gravity acts downward (gravity opposes the motion), so the object loses velocity as it rises until it becomes momentarily at rest at highest point just before start returning downward. Thus, $\text{velocity} = 0\text{m/s}$ at the highest point.
- ✓ If a body returns to the same level it was thrown from, the motion is symmetric and hence:
 - Time going up = time coming down.
 - Speed at the same height going up = speed at the same height coming down.

The ingredients are ready; now let us see how they come together in worked examples.

BINDER Example 13

A stone is dropped from rest from the top of a building. Neglect air resistance.

(a) What is its initial velocity?

(b) What is its acceleration?

Solution

Tactical idea: "dropped from rest" $\Rightarrow u = 0\text{m/s}$

(a) Initial velocity: $u = 0\text{m/s}$ ("dropped from rest" means initial velocity is 0m/s).

(b) Acceleration: $a = -g = -9.8\text{m/s}^2$ (downward, negative)

Making Sense of the Answer: *The stone starts with no velocity, but gravity immediately makes it speed up downward.*

Think Like a Physicist: *In free fall, every second adds more velocity because gravity acts continuously.*

REAL Example 14

Kipute drops a small coin from the laboratory balcony. **Kipanga** notices the coin moves slowly at first, then faster and faster until it hits the ground. Explain why.

Solution

The coin speeds up because gravity provides a constant downward acceleration, g . As time increases, the downward velocity increases, so the coin falls faster and faster.

Making Sense of the Answer: *The coin is not being pushed by hands, but gravity is pulling it every moment.*

Think Like a Physicist: *Falling objects accelerate because the acceleration due to gravity acts downward, producing velocity changes each second.*

BINDER Example 15

A stone is dropped from rest and falls for 3s. Take $g = 10\text{m/s}^2$. Find its final velocity.

Solution

Using $v = u + at$

Where: $u = 0\text{m/s}$, $t = 3\text{s}$, $a = -10\text{m/s}^2$

Substituting $v = (0\text{m/s}) + (-10\text{m/s}^2)(3\text{s}) = v = -30\text{m/s}$ (negative means the stone is moving downward).

The final velocity is 30m/s (downward).

Making Sense of the Answer: *Each second the stone gains 10m/s , so after 3s it gains 30m/s .*

Think Like a Physicist: *If you start from rest, velocity after time t is simply gt .*

BINDER Example 16

A stone is dropped from rest from a height of 45m. Take $g = 10\text{m/s}^2$. Find the time taken to reach the ground.

Solution

Using $s = ut + \frac{1}{2}at^2$

Where: $u = 0\text{m/s}$, $s = -45\text{m}$, $a = -10\text{m/s}^2$

Substituting $-45\text{m} = (0\text{m/s})t - \frac{1}{2}(10\text{m/s}^2)t^2$

From which; $t = 3\text{s}$

The time taken is 3s.

Making Sense of the Answer: *The stone does not fall at a constant velocity. It starts from rest and gains velocity by 10m/s every second, so in the first second it covers a small distance, but in later seconds it covers much more. That is why it reaches the ground in 3s.*

Think Like a Physicist: *In vertical fall from rest, displacement depends on t^2 , so the fall accelerates quickly.*

HOT Example 17

A ball is thrown vertically upward from the ground with initial velocity 19.6m/s . Take $g = 9.8\text{m/s}^2$. Find the time taken to reach the highest point.

Solution

At the highest point, $v = 0\text{m/s}$

Using $v = u + at$

Where: $u = 19.6\text{m/s}$, $v = 0\text{m/s}$, $a = -9.8\text{m/s}^2$

Substituting $0\text{m/s} = 19.6\text{m/s} + (-9.8\text{m/s}^2)t$

From which; $t = 2\text{s}$

The time to the highest point is 2s.

Making Sense of the Answer: Gravity removes 9.8m/s from the upward velocity every second, so it takes 2 seconds to reduce 19.6m/s ($2 \times 9.8\text{m/s}$) to zero.

Think Like a Physicist: Time to the top is equal to $\frac{u}{g}$.

HOT Example 18

A ball is thrown vertically upward from the ground with initial velocity 24.5m/s. Take $g = 9.8\text{m/s}^2$. Find the maximum height reached.

Solution

Again, at the highest point, $v = 0\text{m/s}$.

Using $v^2 = u^2 + 2as$

Where $u = 24.5\text{m/s}$, $v = 0\text{m/s}$, $a = -9.8\text{m/s}^2$ and $s = H$ (maximum height)

Substituting $(0\text{m/s})^2 = (24.5\text{m/s})^2 + 2(-9.8\text{m/s}^2)H$

From which; $H = 30.625\text{m}$

The maximum height is 30.625m.

Making Sense of the Answer: The stone rises until gravity removes all its upward velocity at 30.625m above the ground.

Think Like a Physicist: Maximum height depends on the square of the initial velocity; that is $H = \frac{u^2}{2g}$.

HOT Example 19

A ball is thrown vertically upward from the ground with initial velocity 19.6m/s. Take $g = 9.8\text{m/s}^2$. Find the total time of flight before it returns to the ground.

Solution

Time to reach the highest point was found using $v = 0\text{m/s}$.

And by using $v = u + at$

Where: $u = 19.6\text{m/s}$, $a = -9.8\text{m/s}^2$

Then, by substituting $0\text{m/s} = 19.6\text{m/s} + (-9.8\text{m/s}^2)t$; $t = 2\text{s}$

The time to the highest point is 2s.

Since the motion is symmetric, the total time of flight, $T = 2t = 2 \times 2\text{s} = 4\text{s}$.

The time of flight is 4s.

Making Sense of the Answer: It takes 2 seconds to rise and 2 seconds to fall back to the ground, making a total of 4s.

Think Like a Physicist: The total time of flight is equal to $\frac{2u}{g}$.

HOT Example 20

A ball is thrown vertically upward from the ground with initial velocity 20m/s. Take $g = 9.8\text{m/s}^2$. Find the time when the ball is at a height of 5m on its way downward.

Solution

A given height is reached twice: once going up and once coming down. The larger time corresponds to the downward path.

Using $s = ut + \frac{1}{2}at^2$

Where: $u = 20\text{m/s}$, $a = -9.8\text{m/s}^2$, $s = 5\text{m}$

Substituting $5\text{m} = (20\text{m/s})t + \frac{1}{2}(-9.8\text{m/s}^2)t^2$

Rearrange to the quadratic equation:

$$4.9t^2 - 20t + 5 = 0$$

From which; $t = 0.27\text{s}$ or 3.81s

Since the ball is on its way downward, the suitable time is the larger one which is 3.81s .

The time at 5m on the way down is 3.81s .

Making Sense of the Answer: *The ball passes 5m quickly when rising, then returns to 5m much later while falling.*

Think Like a Physicist: *One displacement can correspond to two different times in vertical motion.*

That brings our subtopic-by-subtopic worked examples to a satisfying close. The plates are cleared! Now let us enjoy the full buffet, where all the ideas of this topic come together in miscellaneous worked examples.

MISCELLANEOUS WORKED EXAMPLES ON MOTION IN STRAIGHT LINE

Example 21

- (a) A daladala moves east. Kipanga walks inside it. Explain clearly what “Kipanga’s velocity relative to the ground” means, and why it depends on direction.
- (b) A daladala moves at 15m/s . Kipanga inside the daladala walks forward at 2m/s relative to the daladala.
 - (i) Find Kipanga’s speed relative to the ground.
 - (ii) If he instead walks toward the back at 2m/s , find his speed relative to the ground.

Solution

- (a) “Velocity relative to the ground” means **the speed and direction at which Kipanga’s position changes as measured by an observer standing on the ground**. It depends on direction because if Kipanga walks in the **same direction** as the daladala, his ground velocity increases (the velocities add). If he walks in the **opposite direction**, his ground velocity decreases (the velocities subtract). So direction changes the final velocity even if the walking speed is the same.
- (b) **For the same direction:**

Kipanga’s speed relative to ground = bus speed + Kipanga’s speed relative to bus

$$15\text{m/s} + 2\text{m/s} = 17\text{m/s}$$

- (i) Kipanga’s speed relative to the ground is 17m/s .

For the opposite direction:

Kipanga’s speed relative to ground = bus speed – Kipanga’s speed relative to bus

$$15\text{m/s} - 2\text{m/s} = 13\text{m/s}$$

- (ii) Kipanga’s speed relative to the ground is 13m/s .

Example 22

- (a) Explain why distance can be large even when displacement is zero.
- (b) A daladala cruised past a roadside checkpoint at a steady 25m/s . The officer signaled it to stop, but the driver kept going. The officer watched it pull away for 2s , then jumped onto a police bike. Starting from rest, he accelerated uniformly at 4m/s^2 , determined to catch the daladala, which continued at a constant 25m/s .
 - (i) How many seconds after the daladala passed the checkpoint does the police catch it?
 - (ii) How far from the checkpoint does the catch occur?

Solution

- (a) Distance measures the total path covered, while displacement depends only on the starting and final positions. So if you return to the starting point, displacement becomes zero even though distance is not.
- (b) Let t be the time (in seconds) measured from the instant the daladala passes the checkpoint to the moment it is caught by the police.

Daladala motion

The daladala moves at a constant velocity of 25m/s.

And for the body moving with constant velocity; displacement = velocity $\times$ time

So after time t , its distance from the checkpoint is: $s_d = 25t$

Police motion

The police starts chasing 2s later.

So by time t , the police has been moving for: $t_p = t - 2$

The police starts from rest and accelerates uniformly at 4m/s^2 .

Distance covered by the police in time t_p is given by: $s = ut + \frac{1}{2}at^2$

So $s_p = \frac{1}{2}a(t_p)^2$ ($u = 0$ and thus $ut_p = 0$)

$s_p = \frac{1}{2} \times 4(t - 2)^2$ (Substituting $t_p = t - 2$)

$s_p = 2(t - 2)^2$

Catching condition

The catch occurs when both are at the same position:

$$s_d = s_p$$

So:

$$2(t - 2)^2 = 25t$$

From which;

$$2t^2 - 33t + 8 = 0$$

Solving gives: $t = 16.25\text{s}$

(i) The police catches the daladala 12.5s after it passed the checkpoint.

Distance of the catch

$$s_d = 25t = 25\text{m/s} \times 16.25\text{s} = 406.3\text{m}$$

(ii) The catch occurs 406.3m from the checkpoint.

Be careful!

Many students mistakenly write $t_p = t + 2$ instead of $t_p = t - 2$.

That is wrong because t is measured from the moment the daladala passes the checkpoint. The police starts **2 seconds later**, so by time, t , the police has been moving for **less time**, not more.

Therefore:

$$\text{Police time} = \text{Total time} - \text{Delay} = t - 2$$

Hence, using $t + 2$ would mean the police started **before** the daladala passed the checkpoint, which contradicts the story and leads to a wrong model.

Example 23

- (a) A car moves at constant speed in a circle. Explain why it is accelerating even though the speed is constant.
- (b) Car A moves east at 18m/s, Car B moves west at 12m/s.
- Find their closing speed.
 - If they are 450m apart, find time to meet.

Solution

- (a) Velocity changes when direction changes. In circular motion the direction changes continuously, so velocity changes and acceleration exists.

(b) The two cars were moving in the opposite directions (one east and another west) and thus:

$$\text{Closing speed} = \text{relative speed of approach} = 18\text{m/s} + 12\text{m/s} = 30\text{m/s}$$

(i) The closing speed is 30m/s.

If the distance (in metres) travelled by car A up to a meeting point is x , then the distance travelled by car B to the same point will be $450 - x$.

Then, using: distance = speed $\times$ time;

$$x = 18t \dots \dots (i)$$

And;

$$450 - x = 12t \dots \dots (ii)$$

Substituting (i) to (ii) gives;

$$450 - 18t = 12t; t = 15\text{s}$$

Alternative solution

The time to the meeting point can be found more directly by using the fact that:

$$t = \frac{\text{Distance}}{\text{Closing speed}} = \frac{450\text{m}}{30\text{m/s}} = 15\text{s}.$$

The time to the meeting point is 15s.

Example 24

- (a) Explain why in real life average speed is not always the mean of the starting and final speeds.
 (b) A test driver sends a car straight forward along a road for 400m in 30s. He then immediately reverses along the same line and travels 300m back in 50s. Calculate:
 (i) the average **speed** of the car,
 (ii) the average **velocity** of the car.

Solution

- (a) Average speed is defined as total distance divided by total time. It depends on how the speed changes during the journey, not only on the starting and final speeds. The mean of the starting and final speeds, $(u + v)/2$, gives the average speed **only when the acceleration is uniform (constant) and the motion is along a straight line with no change of direction**. In most real motions the acceleration is not constant (or the motion involves stopping, changing direction, or different road conditions), so the speed does not increase or decrease evenly with time. In such cases, $(u + v)/2$ is not reliable, and the correct average speed must be found from total distance and total time.

- (b) **Total distance travelled**

The car moves 400m forward and 300m backward.

$$\text{Total distance} = 400\text{m} + 300\text{m} = 700\text{m}$$

Total time taken

$$\text{Total time} = 30\text{s} + 50\text{s} = 80\text{s}$$

- (i) Average speed

$$\text{Average speed} = \text{total distance} \div \text{total time}$$

$$\text{Average speed} = \frac{700\text{m}}{80\text{s}} = 8.75\text{m/s}$$

The average speed is 8.75m/s.

- (ii) Average velocity

Forward displacement = +400m

Backward displacement = -300m

Net displacement = 400m - 300m = 100m forward

$$\text{Average velocity} = \text{displacement} \div \text{total time}$$

$$\text{Average velocity} = \frac{100\text{m}}{80\text{s}} = 1.25\text{m/s}$$

The average velocity is 1.25m/s forward.

Interesting!

For the same motion, the magnitude of the average velocity is much smaller than the average speed. This is because the car covered a long total distance (700m), so its average speed is fairly large. However, it reversed for most of the journey, so its net displacement is only 100m from the starting point. Since average velocity depends on displacement (not total distance), its magnitude becomes much smaller.

Example 25

- (a) Kipute argues: “At the highest point, the ball stops, so acceleration must be zero.” Explain why this argument is incorrect.
- (b) A ball is projected vertically upward from the ground with initial velocity u and reaches a maximum height H .
- Show that the time to reach the highest point is $t_{\text{up}} = \frac{u}{g}$.
 - Show that the maximum height is $H = \frac{u^2}{2g}$.
 - For $u = 24.5\text{m/s}$, find H . Take $g = 9.8\text{m/s}^2$.

Solution

- (a) The argument is incorrect because stopping does not mean acceleration is zero. At the highest point the velocity becomes 0m/s momentarily, but gravity is still acting downward. Since gravity acts continuously, the acceleration remains downward with magnitude g .

- (b)
- At the highest point, $v = 0\text{m/s}$.

Using: $v = u + at$, where: $a = -g$:

$$0 = u + (-g)t_{\text{up}}$$

So $gt_{\text{up}} = u$

From which; $t_{\text{up}} = \frac{u}{g}$.

- (ii) Using: $v^2 = u^2 + 2as$, with $v = 0\text{m/s}$, $a = -g$, and displacement to the top, $s = H$:

$$0 = u^2 + 2(-g)H$$

$$0 = u^2 - 2gH$$

So $2gH = u^2$

From which; $H = \frac{u^2}{2g}$.

- (iii) Substituting $u = 24.5\text{m/s}$, $g = 9.8\text{m/s}^2$:

$$H = \frac{(24.5\text{m/s})^2}{2 \times 9.8\text{m/s}^2} = 30.625\text{m}$$

Thus, $H = 30.625\text{m}$.

Example 26

- (a) Explain why two stones of different masses fall with the same acceleration when air resistance is neglected.
- (b) Kipute says: “A ball thrown vertically upward with initial velocity 14.7m/s reaches the highest point after 3s. I used the value of g as 9.8m/s^2 .” Without reworking the whole problem, show that Kipute’s claim is incorrect.

Solution

- (a) Gravity gives the same acceleration, g to all bodies because acceleration due to gravity is independent of mass (when air resistance is ignored).
- (b) This can be done by calculating the value of g (constant) by using Kipute’s claimed values then comparing it with the given standard value.

The time to reach the highest point is given by $t_{\text{up}} = \frac{u}{g}$.

From which;

$$g = \frac{u}{t_{\text{up}}}$$

Substituting Kipute's claimed values;

$$g = \frac{14.7\text{m/s}}{3\text{s}} = 4.9\text{m/s}^2 \neq 9.8\text{m/s}^2$$

Since the calculated value of g by using Kipute's claimed values is not equal to the known value of acceleration due to gravity, Kipute's claim is incorrect.

Example 27

- (a) Explain why time going up equals time coming down when an object returns to the same height (air resistance neglected).
- (b) Ball A is thrown vertically upward from the ground with initial velocity of 24.5m/s . At the same instant, Ball B is dropped from rest from a point 40m above the ground. Take $g = 9.8\text{m/s}^2$.
- (i) Find the time when they meet.
- (ii) Find the height above the ground where they meet.

Solution

- (a) The motion is symmetric because the magnitude of acceleration due to gravity is constant. Consequently, the object loses velocity at the same rate while rising as it gains velocity while falling back to the same level.

- (b) Using: $s = ut + \frac{1}{2}at^2$

For ball A (thrown upward from ground):

$$u = 24.5\text{m/s}, s = h_A, a = -g = -9.8\text{m/s}^2$$

$$\text{Substituting } h_A = 24.5t - \frac{1}{2} \times 9.8t^2$$

From which:

$$h_A = 24.5t - 4.9t^2 \dots \dots (i)$$

For ball B (dropped from 40m):

If ball A has moved the distance of h_A , then at the meeting point, ball B will move the distance of $40 - h_A$.

$$\text{So: } u = 0\text{m/s}, s = -(40 - h_A), a = -g = -9.8\text{m/s}^2$$

$$\text{Substituting } -(40 - h_A) = 0 \times t - \frac{1}{2} \times 9.8t^2$$

From which:

$$40 - h_A = 4.9t^2 \text{ or } h_A = 40 - 4.9t^2 \dots \dots (ii)$$

Substituting (ii) to (i) gives:

$$40 - 4.9t^2 = 24.5t - 4.9t^2; t = 1.63\text{s}$$

- (i) The time of meeting is 1.63s .

Substituting the value of t in (ii) gives:

$$h_A = 40 - 4.9(1.63)^2 = 26.98\text{m}$$

The height above the ground where they meet is 26.98m .

Alternative solution

The question can be solved by using the **relative motion method** which is shorter.

$$\text{Relative acceleration, } a_R = a_A - a_B = -9.8\text{m/s}^2 - (-9.8\text{m/s}^2) = 0$$

And;

$$\text{Relative initial velocity, } u_R = u_A - u_B = 24.5\text{m/s} - 0\text{m/s} = 24.5\text{m/s}$$

Then, the equation $s = ut + \frac{1}{2}at^2$ becomes:

$$s = u_R t + \frac{1}{2}a_R t^2$$

Where: $u_R = 24.5\text{m/s}$, $s = 40\text{m}$, $a_R = 0$

Substituting $40 = 24.5t$; $t = 1.63\text{s}$

Also for ball A; $s = ut + \frac{1}{2}at^2$

Substituting $h_A = 24.5 \times 1.63 - \frac{1}{2} \times 9.8(1.63)^2 = 26.9\text{m}$

Example 28

- (a) Two cars move toward each other. Explain why their relative speed is the sum of their speeds.
 (b) Ball A is thrown vertically upward from the ground with initial velocity of 24.5m/s . One second later, ball B is thrown upward from the same point with initial velocity of 34.3m/s .
 Take $g = 9.8\text{m/s}^2$.
 (i) Find the time (measured from A's launch) when Ball B catches Ball A.
 (ii) Find the height where they meet.

Solution

- (a) If two cars move in opposite directions, the separation between them reduces by both distances each second, so relative speed adds.
 (b) Using: $s = ut + \frac{1}{2}at^2$

For ball A:

$$u = 24.5\text{m/s}, s = h_A, t_A = t, a = -g = -9.8\text{m/s}^2$$

$$\text{Substituting } h_A = 24.5t - \frac{1}{2} \times 9.8t^2$$

From which:

$$h_A = 24.5t - 4.9t^2 \dots \dots (i)$$

For ball B: thrown 1s later.

$$u = 34.3\text{m/s}, s = h_B, t_B = t - 1, a = -g = -9.8\text{m/s}^2$$

$$\text{Substituting } h_B = 34.3(t - 1) - \frac{1}{2} \times 9.8(t - 1)^2$$

From which:

$$h_B = -4.9t^2 + 44.1t - 39.2 \dots \dots (ii)$$

But at the catching point; $h_A = h_B$

It follows that:

$$24.5t - 4.9t^2 = -4.9t^2 + 44.1t - 39.2 \text{ or } 19.6t = 39.2; t = 2\text{s}$$

- (i) The time from A's launch is 2s (So B catches A exactly 1 second after B is launched.)
 (ii) Using $h_A = 24.5t - 4.9t^2$ (from (i))

$$\text{Substituting } h_A = (24.5 \times 2 - 4.9 \times 2^2)\text{m} = 29.4\text{m}$$

The height of meeting is 29.4m

Example 29

- (a) State one reason why distance is always greater than or equal to the magnitude of displacement.
 (b) Ball A is thrown vertically upward from the ground with initial velocity of 19.6m/s . One second later, ball B is thrown upward from the same point with velocity u_B . Find u_B so that both balls land at the same time. Take $g = 9.8\text{m/s}^2$

Solution

- (a) Distance counts the full path length, while displacement is the straight-line change from start to end, so distance cannot be smaller than displacement magnitude.
- (b) For a vertical throw returning to the same level, time of flight is $T = \frac{2u}{g}$.

So ball A time of flight:

$$T_A = \frac{2(19.6\text{m/s})}{9.8\text{m/s}^2} = 4\text{s}$$

Ball B is thrown **one second later**, so for both to land together, ball B must land at $t = 4\text{s}$ from A's launch. Therefore, ball B's time in the air must be $(4 - 1)\text{s}$ or 3s .

For ball B: $T_B = \frac{2u_B}{g}$

Substituting $\frac{2u_B}{9.8\text{m/s}^2} = 3\text{s}$

From which: $u_B = 14.7\text{m/s}$

Example 30

- (a) A ball is thrown vertically upward. At some moment it is still moving upward but slowing down. Explain how this is possible.
- (b) A boy wants to throw a ball from a ground so that it just reaches a roof 25m above the ground.
- Find the required initial velocity.
 - Find the time to reach the roof.

Solution

- (a) The ball continues moving upward because it has an initial upward velocity. However, gravity acts downward throughout the motion, opposing the upward motion. As a result, the upward velocity decreases each second until it becomes zero at the highest point.
- (b) "Just reaches" means it arrives at the top point of its motion (maximum height) at the roof, so $v = 0\text{m/s}$ there.

But the maximum height is given by the following equation:

$$H = \frac{u^2}{2g}$$

From which:

$$u = \sqrt{(2gH)} = \sqrt{(2 \times 9.8 \times 25)} = 22.1\text{m/s}$$

- (i) The required initial velocity is 22.1m/s .

$$t_{\text{up}} = \frac{u}{g} = \frac{22.1\text{m/s}}{9.8\text{m/s}^2} = 2.26\text{s}$$

- (ii) The time to reach the roof is 2.26s .

Example 31

- (a) In real life, does time going up always equal time coming down for vertical throw? Explain briefly.
- (b) A ball is thrown vertically upward with $u = 19.6\text{m/s}$.
- Find ideal time of flight.
 - State whether the real time of flight with air resistance is likely to be greater, smaller, or the same, and why.

Solution

- (a) Not always.

Reason

With **air resistance** in real life, symmetry breaks (time coming down becomes greater than time going up) because resistive force opposes motion both upward and downward, changing the acceleration magnitude differently in each phase.

(b)

- (i) Ideal time of flight: $T = \frac{2u}{g} = \frac{2(19.6)}{9.8} = 4\text{s}$.
- (ii) The real time of flight is more likely to be **greater than 4s** because air resistance reduces downward velocity, increasing the time of descent by a greater amount than it reduces the time of ascent.

Example 32

- (a) A mango and a stone are dropped at the same time from the same height. Many students expect the heavier stone to land first. In reality, they hit the ground together. Explain why.
- (b) A ball is dropped from 20m and rebounds to 12.8m. Take $g = 9.8\text{m/s}^2$.
- (i) Find velocity just before impact.
- (ii) Find velocity just after rebound.
- (iii) Find total time from release until it reaches the top of the rebound.

Solution

- (a) Near the Earth's surface, gravity gives the same acceleration to all objects regardless of their mass (ignoring air resistance). Although the stone is heavier, it does not receive a greater acceleration. Both objects start from rest and gain velocity at the same rate, so they reach the ground at the same time.
- (b) Using: $v^2 = u^2 + 2as$, with $u = 0$, $a = -9.8\text{m/s}^2$, and, $s = h = -20\text{m}$ (downward):

$$\text{Substituting } v^2 = 0^2 + 2 \times (-9.8\text{m/s}^2) \times (-20\text{m})$$

$$\text{From which: } v = \sqrt{(2 \times 9.8\text{m/s}^2 \times 20\text{m})} = 19.8\text{m/s}$$

- (i) The velocity just before impact is 19.8m/s.

After the rebound, the rebound velocity will be initial velocity for ascending to the maximum height (H) of 12.8m.

Thus using:

$$H = \frac{u^2}{2g}$$

From which:

$$u = \sqrt{(2gH)} = \sqrt{(2 \times 9.8\text{m/s}^2 \times 12.8\text{m})} = 15.8\text{m/s}$$

- (ii) The velocity just after rebound is 15.8m/s.

Time to reach the ground can be found by using:

$$s = ut + \frac{1}{2}at^2$$

Where: $u = 0$, $a = -g = 9.8\text{m/s}^2$, $s = -h = -20\text{m}$

$$\text{Substituting } -20 = 0 - \frac{1}{2} \times 9.8t^2; \mathbf{t = 2.02s}$$

After rebound, time to reach the maximum height, H (12.8m) can be found by using:

$$\mathbf{t_{up} = \frac{u}{g} = \frac{15.8\text{m/s}}{9.8\text{m/s}^2} = 1.61\text{s}}$$

$$\text{Total time} = 2.02\text{s} + 1.61\text{s} = 3.63\text{s}$$

- (iii) The total time from release until it reaches the top of the rebound is 3.63s.

Example 33

- (a) Explain why the acceleration of a ball thrown upward is still downward even when the ball is moving upward.

- (b) **Kipanga** was solving a vertical-motion problem. He calculated the time when a ball is at a height of 5m **on its way downward** and obtained 4.32s. When he presented his work, **Mr. Akilikubwa** marked the answer wrong and explained that Kipanga's mistake was taking the displacement as $s = -5\text{m}$ instead of $s = +5\text{m}$. Mr. Akilikubwa added that Kipanga had **correctly** used the acceleration due to gravity as $a = -9.8\text{m/s}^2$.
- Explain clearly why taking $s = -5\text{m}$ was inappropriate for a point that is 5m above the point of projection, even if the ball is moving downward at that moment.
 - Help Kipanga to calculate the correct time when the ball is at 5m on its way downward.
 - What does the time 4.32s really represent?

Solution

- (a) Gravity always acts downward, so the acceleration remains downward throughout the motion. The body rises only because it was given an initial upward velocity, not because it has an upward acceleration. As it moves upward, gravity steadily reduces the upward velocity until the velocity becomes zero at the highest point.
- (b)
- Displacement s depends on the **position of the ball relative to the starting point**, not on the direction the ball is moving at that instant. Since the ball is 5m above the point of projection, its displacement must be $s = +5\text{m}$ whether it is rising or falling.
The fact that the ball is moving downward is shown by the **sign of velocity**, not by changing the sign of displacement.
 -

Obtaining value of u from the incorrectly determined value of t

Using: $s = ut + \frac{1}{2}at^2$

Where: $s = -5\text{m}$, (As incorrectly used by Kipanga), $t = 4.32\text{s}$, $a = -9.8\text{m/s}^2$

Substituting Kipanga's values:

$$-5\text{m} = u(4.32\text{s}) + \frac{1}{2}(-9.8\text{m/s}^2)(4.32\text{s})^2$$

From which; $u = 20.01\text{m/s}$

So Kipanga's working implies the ball was thrown with initial velocity about 20.01m/s.

Determining correct value of t

Because the ball is still 5m above the point of projection: $s = +5\text{m}$

Again using $s = ut + \frac{1}{2}at^2$

Where: $u = 20.01\text{m/s}$, $a = -9.8\text{m/s}^2$, $s = 5\text{m}$

Substituting $5\text{m} = (20.01\text{m/s})t + \frac{1}{2}(-9.8\text{m/s}^2)t^2$

Rearrange to the quadratic equation:

$$4.9t^2 - 20.01t + 5 = 0$$

From which; $t = 0.27\text{s}$ or 3.82s

Since the ball is on its way downward, the suitable time is the larger one which is 3.82s.

The correct time is 3.82s.

- (iii) It represents the moment when the ball is **5m below** the point of projection on its way downward.

With these miscellaneous worked examples now complete, the picture should feel nicely connected. It is time to stretch our understanding a little further. Welcome to the Digging Deeper Exercise!

DIGGING DEEPER EXERCISE 2

EXERCISE 2A: BINDER QUESTIONS

Question 1

A boda-boda is “at rest” at the bus stand. State the values of:

- (a) its velocity
- (b) its acceleration

Give a brief explanation.

Question 2

A student says: “If acceleration is zero, the object must be at rest.” Is the statement correct? Explain.

Question 3

In straight-line motion, clearly distinguish **distance** from **displacement** using one example where distance is not equal to the magnitude of displacement.

Question 4

A car moves along a straight road and its displacement is negative. Explain what “negative displacement” means physically.

Question 5

A vehicle moves in a straight line with **constant velocity** of 12m/s. State:

- (a) its acceleration
- (b) the net force on it.

Question 6

A body has a constant acceleration of -3m/s^2 . Explain what the negative sign indicates. Give two motion situations that fit.

Question 7

State when **speed** equals the magnitude of **velocity**. State when speed can differ from the magnitude of **velocity**.

Question 8

An object moves in a straight line and its velocity changes uniformly from 4m/s to 10m/s in 3s.

- (a) State the sign of the acceleration.
- (b) Find the acceleration.

Question 9

A body moves in a straight line, comes momentarily to rest, then continues moving in the same direction.

- (a) State its velocity at that instant.
- (b) Is the acceleration necessarily zero at that instant? Explain.

Question 10

A car moves along a straight line. Its velocity is 20m/s at the start of a time interval and 20m/s at the end of the interval.

- (a) State the change in velocity.
- (b) What can you conclude about the *average acceleration* during that interval?

Question 11

An object moves 40m east, then 15m west along the same straight road.

- (a) Find the distance travelled.
- (b) Find the displacement.

Question 12

A runner starts from rest and reaches 8m/s in 4s with uniform acceleration.

- (a) Find the acceleration.
- (b) Interpret your answer in one sentence.

EXERCISE 2B: REAL QUESTIONS**Question 13**

Passengers in a bus notice that roadside trees appear to move backward as the bus moves forward. Explain why stationary objects outside appear to move.

Question 14

A student walks 200m east along a straight road to a shop, then walks 200m west back home. Her phone pedometer shows 400m, but her map app shows she ended where she started. Explain this difference.

Question 15

A boda-boda rider makes a straight out-and-back trip from a stand to a customer and returns to the same stand. He says, *“My average speed was not zero, but my average velocity for the whole trip was zero.”* Explain why both statements can be true.

Question 16

Two daladala drivers travel the same straight route. One drives steadily; the other repeatedly speeds up and slows down due to frequent stops. Passengers describe the second trip as *“more jerky,”* even if both arrive in similar time. Explain what this suggests about their motions.

Question 17

From a bridge, Kipute drops one stone and throws another stone straight downward at the same time. She observes the thrown stone reaches the ground first. Explain why (ignore air resistance).

Question 18

In real life, a dry leaf falls more slowly than a coin. Many people conclude that heavier objects fall faster. Explain what physics says about falling under gravity and why everyday observations can mislead.

Question 19

Students notice that a falling object gets faster and faster, yet physics says the acceleration due to gravity remains constant. Explain how constant acceleration can still produce increasing velocity.

Question 20

A student measures motion along a straight road and reports a negative velocity. The car was not reversing; it was moving forward on the road. Explain how velocity can be negative even though the car is moving “forward” in ordinary language.

Question 21

A ball is thrown straight upward and later comes back down. Observers notice that it slows down on the way up and speeds up on the way down. Explain how the direction of velocity and the direction of acceleration compare during the upward and downward motion.

Question 22

A student says: *“At the highest point of its motion, the ball has no velocity, so it has no acceleration.”* Was the student correct? Explain.

EXERCISE 2C: HOT QUESTIONS**Question 23**

A car moves along a straight road. It starts from rest, accelerates uniformly at 2.0m/s^2 for 8.0s, then continues at constant velocity for 10s, then decelerates uniformly at 4.0m/s^2 until it stops. Find:

- (a) the maximum velocity reached
- (b) the total distance travelled
- (c) the total time taken.

Question 24

Two bodabodas start from the same point at the same time. Bodaboda A moves with constant velocity 12m/s . Bodaboda B starts from rest and accelerates uniformly at 1.5m/s^2 . Find:

- (a) the time when B first catches A
- (b) the distance from the starting point where they meet.

Question 25

A ball is thrown vertically upward with velocity 24m/s . Take $g = 9.8\text{m/s}^2$. Find:

- (a) the time to reach the highest point
- (b) the maximum height reached
- (c) the total time of flight
- (d) Why the upward and downward times are equal.

Question 26

From a bridge, a stone is dropped. Exactly 1.5s later, a second stone is thrown vertically downward with velocity 20m/s from the same point. Take $g = 9.8\text{m/s}^2$. Find the time (measured from the first release) when the second stone catches the first.

Question 27

A car's velocity changes from $+18\text{m/s}$ to -12m/s in 5.0s with uniform acceleration. Find:

- (a) the acceleration
- (b) the displacement during the interval
- (c) the distance travelled
- (d) Why distance is not equal to the magnitude of displacement in this motion.

Question 28

A ball is thrown vertically upward from a balcony 30m above the ground and hits the ground after 4.0s. Take $g = 9.8\text{m/s}^2$. Find:

- (a) the initial velocity of the ball
- (b) the velocity just before impact.

Question 29

A stone is thrown vertically upward from the ground with velocity 25m/s . Take $g = 9.8\text{m/s}^2$. Find:

- (a) the two times when the stone is at a height of 20m
- (b) which time corresponds to upward motion
- (c) How velocity is used to select the correct time.

Question 30

A stone is dropped from a height of 80m at the same instant another stone is thrown vertically upward from the ground with velocity 30m/s . Take $g = 9.8\text{m/s}^2$. Find:

- (a) the time when they meet
- (b) the height above the ground where they meet
- (c) the velocity of each stone at the meeting instant.

Question 31

A ball is thrown vertically upward from a balcony 45m above the ground with velocity 10m/s. Take $g = 9.8\text{m/s}^2$. Find:

- (a) the time taken to reach the ground
- (b) the velocity just before impact.

Question 32

Ball A is thrown vertically upward from the ground with velocity 20m/s at the same instant Ball B is dropped from rest from a height of 30m. Take $g = 9.8\text{m/s}^2$. Find:

- (a) the time when they are at the same height
- (b) the height at which they meet.

Question 33

Ball A is thrown vertically upward with velocity 18m/s. Exactly 1.0s later, Ball B is thrown vertically upward from the same point with velocity 24m/s. Take $g = 9.8\text{m/s}^2$. Find:

- (a) the time (from A's launch) when B catches A
- (b) the height at which they meet
- (c) the velocity of each ball at the meeting instant.

Question 34

A ball is thrown vertically upward from a roof 25m above the ground with velocity 15m/s. Take $g = 9.8\text{m/s}^2$. Find:

- (a) the maximum height above the ground
- (b) the time taken to hit the ground
- (c) the velocity just before impact
- (d) If a student obtained a negative value for time, explain why this result is not physically acceptable.

Question 35

Stone A is dropped from a bridge at $t = 0$. Stone B is dropped from the same point at $t = 2.0\text{s}$. Take $g = 9.8\text{m/s}^2$. At $t = 6.0\text{s}$ (from the first release), find:

- (a) the separation between the stones
- (b) the relative velocity of A with respect to B
- (c) Explain why the separation is increasing at this instant at constant rate.

Chapter 3

NEWTON'S LAWS OF MOTION

INTRODUCTION

If you have ever tried to push a stalled motorcycle, drag a heavy desk across the floor, or suddenly brake a fast-moving bicycle, then you have already met Newton's Laws of Motion even if you did not know their names.

Every day, we interact with objects that move and objects that stubbornly refuse to move. Some start moving easily, others resist. Some stop quickly, others seem unwilling to stop at all. Yet we often describe these experiences using casual words such as "*heavy*", "*fast*", or "*hard to push*". These words are useful in daily conversation, but they are not good enough for physics. Physics enjoys asking uncomfortable questions.

After completing **Chapter 2**, a curious student should naturally begin to ask deeper questions such as:

- *Why does a body at rest require effort to start moving?*
- *Why does the same push affect different objects differently?*
- *Why do we feel thrown forward when a bus suddenly stops?*

These questions are no longer about **how** objects move, but about **why** they move the way they do. The answers are found in a set of simple but powerful ideas known as **Newton's Laws of Motion**.

Newton's Laws do not describe rare or special situations. They describe ordinary motion; walking, pushing, braking, jumping, and falling. Yet from these everyday actions, they explain an astonishing range of physical phenomena, from the motion of vehicles on the road to the motion of rockets in space.

Because of this, Newton's Laws form the **foundation of mechanics**. Everything that follows in A-Level Physics builds upon them. A weak understanding here makes later topics unnecessarily difficult.

There is, however, a common danger. Many students memorize Newton's Laws as statements, but struggle to use them correctly. This happens when the laws are learned as words rather than as tools for thinking.

To avoid this, this chapter will not rush directly into formulas or laws. Instead, it will focus first on the central idea that gives meaning to all three laws. That idea is **force**.

If Newton's Laws were a complete meal, then force would be the ingredient that gives the meal its taste. Once force is understood clearly, Newton's Laws become natural and logical rather than mysterious.

With this in mind, we now turn our attention to force.

FORCE

In Chapter 2, we described motion using quantities such as displacement, velocity, and acceleration. We answered questions like *how fast*, *how far*, and *how motion changes with time*. However, one important question was left unanswered: ***Why does motion change in the first place?***

For example:

- *Why does a body at rest start moving when pushed?*
- *Why does a moving object sometimes slow down even when no one is touching it?*
- *Why does the same push affect a light object more than a heavy one?*

To answer such questions, physics introduces the idea of **force**.

Why Do We Need the Idea of Force?

A ball lying on the ground remains at rest for a long time. When kicked, it suddenly starts moving. After some distance, it slows down and eventually stops.

Describing this motion using velocity and acceleration is possible, but those quantities alone do not explain the **cause** of the motion or its change. Objects do not begin to move, stop, or change direction by themselves.

Experience tells us that motion changes only when something acts on the object. This "something" is what physics calls a **force**.

So at this stage, it is important not to treat force as an abstract formula. Instead, it should be understood as the **physical reason** behind changes in motion.

At this point, a simple but very important question arises: *What a force is, and what it is not?*

A force can be described as a **push or a pull** resulting from the interaction between two bodies.

However, this description must be used carefully.

A force:

- is not a property that an object possesses,
- cannot exist without another body involved,
- is always associated with an interaction.

For example:

- When you push a wall, the force exists because your hand and the wall interact.
- When the Earth attracts a stone, the force exists because of the interaction between the stone and the Earth.

An isolated object, completely alone, cannot experience a force. This idea will become important later when discussing interacting systems and Newton's third law.

Avoid This Misconception

A very common misconception is that force is required to keep an object moving. Everyday experience seems to support this idea: when a person stops pushing a box, the box slows down and stops.

However, this observation does not tell the full story, how?

From Chapter 2, we learned that:

- Acceleration is the rate of change of velocity.
- Motion with constant velocity corresponds to zero acceleration.

Now we connect this to force as follows:

When the velocity of an object changes; either in magnitude or direction, the object is accelerating. Such a change does not occur without a cause. That cause is the presence of a **net force** acting on the object.

If the net force acting on a body is zero, then its acceleration is zero. In that case, the body either remains at rest or continues to move with constant velocity. This does not mean that no forces act on the body. It means that the forces acting on it **balance each other**.

Therefore:

- Force is linked to **change of motion**, not motion itself.
- An object can move with constant velocity even while forces are acting on it.

This idea is central to Newton's laws and must be clearly understood before moving forward.

Multiple Forces Acting on a Body

In real situations, a body is rarely acted upon by only one force. Usually, several forces act at the same time. For example:

- A book resting on a table is pulled downward by the Earth and pushed upward by the table.
- A moving vehicle experiences a driving force forward and resistive forces backward.

What determines the motion of the body is not any single force, but the **combined effect of all forces acting on it**. This combined effect is called the **net force** or **resultant force**.

- If the net force is zero, the motion does not change which implies that acceleration is zero.
- If the net force is not zero, the motion changes and acceleration occurs in the direction of the net force.

This idea will appear repeatedly throughout this chapter and must be kept in mind at all times.

With the ideas now simmering nicely, let us serve them properly through a few worked examples and enjoy the flavour of physics in action.

BINDER Example 1

A stone rests on the ground and remains at rest for a long time. Explain, using force ideas, why the stone does not start moving on its own.

Solution

The stone is acted upon by two main forces: its weight acting downward and the support force from the ground acting upward. These two forces act in opposite directions and balance each other. Since the resultant force on the stone is zero, there is no acceleration. With zero acceleration, the stone remains at rest.

Making Sense of the Answer: *If one force were stronger than the other, the stone would start moving upward or downward. Since it stays at rest, the forces must be balanced.*

Think Like a Physicist: *Rest or constant velocity means the resultant force on the body is zero.*

REAL Example 2

Kipute pushes a heavy wooden crate along a straight path. At first, the crate speeds up. After some time, she continues pushing with the same effort, but the crate moves at constant velocity. Explain this observation using force ideas.

Solution

At the beginning, Kipute's pushing force is greater than the forces opposing the motion, so the net force is forward. This produces acceleration, causing the crate to speed up. Later, when the crate moves at constant velocity, its acceleration is zero. Zero acceleration means the resultant force is zero. Therefore, the resistive forces must have increased until they balance Kipute's pushing force.

Making Sense of the Answer: *It feels harder to start moving a heavy object than to keep it moving steadily, which agrees with the idea that acceleration requires a non-zero resultant force, but constant velocity does not.*

Think Like a Physicist: *Constant velocity means forces are balanced, not that no force is applied.*

HOT Example 3

Three forces act on a body moving along a straight line. Two of the forces have magnitudes of 18N and 6N and act in the same direction. The third force has magnitude 10N and acts in the opposite direction.

- (a) Determine the net force acting on the body.
- (b) State the effect of this net force on the motion of the body.

Solution

(a) Forces acting in one direction: $18\text{N} + 6\text{N} = 24\text{N}$

Opposing force: 10N

Net force = $24\text{N} - 10\text{N} = 14\text{N}$, acting in the direction of the 18N and 6N forces.

(b) Since the net force is not zero, the body accelerates in the direction of the net force. Its velocity increases in that direction.

Making Sense of the Answer: *The larger combined force must dominate, but part of it is cancelled by the opposing force, leaving a smaller resultant force of 14N.*

Think Like a Physicist: *You can decide whether motion changes by finding the net force, even before knowing the mass.*

The worked examples have done their part. Now, with curiosity switched on, let us enjoy the next subtopic as it unfolds.

INERTIA AND NEWTON'S FIRST LAW OF MOTION

Inertia

So far, we are familiar with the idea of force as the physical reason behind changes in motion. We also understand that is the **resultant force** that matters, not individual forces acting alone.

We now take a crucial step forward by examining what happens **when the resultant force on a body is zero**. Surprisingly, this simple situation explains a large part of everyday motion and introduces one of the most fundamental ideas in physics: **inertia**.

To understand inertia, consider the following observations:

First observation: Consider a book resting on a table. It remains at rest unless someone pushes it.

Second observation: Consider the same book sliding smoothly on a polished surface. It continues moving unless something stops it.

These observations suggest a common idea: *Bodies resist changes in their state of motion.*

This resistance to change is what we call **inertia**. Inertia is the tendency of a body to remain at rest if it is at rest, or continue moving with constant velocity if it is already in motion, unless acted upon by a **resultant force**. It is not a force. It is a property of matter.

Mass as a Measure of Inertia

Not all bodies resist change in the same way. A light object is easy to start moving or stop. A heavy object is much harder to do so.

This tells us that inertia depends on the **mass** of the body. Therefore, we can conclude that:

- A body with large mass has large inertia.
- A body with small mass has small inertia.

So, *mass is a measure of inertia.*

This idea explains why the same force produces different accelerations on different bodies.

Newton's First Law of Motion

The ideas of inertia and resultant force are formally stated in **Newton's first law of motion** as follows:

A body remains at rest or continues to move with constant velocity in a straight line unless acted upon by a resultant force.

This law does **not** say that forces are absent. It says that **the resultant force is zero**. Therefore:

- zero resultant force implies that acceleration is zero,
- no acceleration implies that velocity remains constant (or zero).

Newton's first law is sometimes called the **law of inertia** because it describes how inertia governs motion when forces balance. The law is important because:

- It corrects the common belief that force is needed to maintain motion.
- It explains what happens when forces balance.
- It provides the reference case for all other force situations.

For now, take a short breath and set theory aside, let us warm up with some carefully chosen worked examples.

BINDER Example 4

A puck slides along a smooth horizontal surface with constant velocity. Explain what this motion tells you about the forces acting on the puck.

Solution

Constant velocity means zero acceleration. Zero acceleration means the resultant force on the puck is zero. This implies that any forces acting on the puck balance each other, so there is no unbalanced force causing a change in velocity.

Making Sense of the Answer: *If there were a forward or backward resultant force, the puck would speed up or slow down. Since its velocity remains constant, no such force exists.*

Think Like a Physicist: *Constant velocity always means zero resultant force.*

REAL Example 5

Kipanga is standing inside a bus that is moving with constant velocity along a straight road. Suddenly, the bus stops abruptly, and Kipanga is thrown forward. Explain this observation using inertia.

Solution

Before the bus stops, Kipanga and the bus are moving together with the same velocity. When the bus stops, an external force acts on the bus, but Kipanga's body tends to continue moving forward due to inertia. Since no immediate force stops his body at the same time, he is thrown forward.

Making Sense of the Answer: *Kipanga's body tries to keep its original motion even though the bus has stopped.*

Think Like a Physicist: *Inertia makes a body resist changes in velocity.*

HOT Example 6

A block rests on a horizontal surface. Several forces act on it, but the block remains at rest. What can you conclude about the resultant force acting on the block? Explain.

Solution

Resultant force acting on block is zero.

Explanation

Since the block remains at rest, its velocity is constant and equal to zero. Constant velocity means zero acceleration. According to Newton's first law, this implies that the resultant force acting on the block is zero. Therefore, the forces acting on the block must balance.

Making Sense of the Answer: *If the forces did not balance, the block would start moving in the direction of the unbalanced force.*

Think Like a Physicist: *Rest is simply constant velocity equal to zero.*

That brings our worked examples to a comfortable stop; the second law is already peeking around the corner, curious to be explored!

NEWTON'S SECOND LAW OF MOTION

From Balanced Forces to Unbalanced Forces

In Newton's first law of motion, we examined situations in which forces balance. We saw that when the resultant force on a body is zero, the body either remains at rest or continues moving with constant velocity. In simple terms:

If resultant force = 0N, then acceleration = 0m/s².

We now take the next step. Most real-life motion is not perfectly balanced. Vehicles start, stop, and turn. Balls speed up or slow down. A crate begins to move when pushed hard enough. All of these happen because the forces are not balancing.

This leads to a more general and more interesting case: *When the resultant force is not zero, what exactly happens to its motion?*

Newton's second law answers this question with both meaning and mathematics. It can be stated as follows:

The acceleration of a body is directly proportional to the resultant force acting on it and inversely proportional to its mass, and occurs in the direction of the resultant force.

This statement contains four key ideas:

- 1) Resultant force causes acceleration.
- 2) For a fixed mass, a greater resultant force produces greater acceleration.
- 3) For a fixed resultant force, a larger mass experiences smaller acceleration.

- 4) The direction of acceleration is the same as the direction of the resultant force.

These ideas form the foundation of all force–motion analysis in mechanics.

Mathematical Form of the Law

For motion involving constant mass, Newton's second law can be written in the simple mathematical form:

$$F = ma$$

Where:

F is the resultant force acting on the body in N,

m is the mass of the body in kg,

a is the acceleration produced in m/s².

This equation does not introduce a new idea; it simply expresses, in mathematical form, the physical meaning of the law. This form is the special case (when the mass of the body remains constant) of general form of Newton's second law which can be stated as follows:

The resultant force acting on a body is equal to the rate of change of its momentum and acts in the same direction as that change.

Mathematically:

$$F = \frac{dp}{dt}$$

Where:

F is the resultant force in N.

p is the momentum of the body in kgm/s,

t is time in s.

This general form is more powerful than $F = ma$ because it is valid even in cases where *the mass changes* or where *force acts over a short time interval*. In other words; $F = ma$ is contained in $F = dp/dt$ as shown below:

Using $p = mv$

It follows that:

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = v \frac{dm}{dt} + m \frac{dv}{dt}$$

So mathematically, the second law can be written as:

$$F = v \frac{dm}{dt} + m \frac{dv}{dt}$$

If mass is constant:

$$\frac{dm}{dt} = 0$$

The above equation reduces to:

$$F = m \frac{dv}{dt}$$

But;

$$\frac{dv}{dt} = a$$

Hence;

$$F = ma$$

Be careful!

Newton's second law does **not** say that force creates motion. Instead, it says that force creates **changes in motion**. In other words, it says that force produces acceleration (change in velocity), not velocity.

In closing the discussion of Newton's second law, we emphasize that *the direction of a body's acceleration is always the same as the direction of the resultant force acting on it*. From this, the following three important conclusions can be drawn:

- 1) If the resultant force acts in the same direction as the velocity of the body, the body accelerates and its velocity increases.
- 2) If the resultant force acts in the opposite direction to the velocity of the body, the body decelerates and its velocity decreases.
- 3) If the resultant force acts perpendicular to the velocity of the body, the velocity of the body remains constant but its direction of motion changes.

This final case prepares the ground for motion in two dimensions, particularly circular motion, where acceleration is always perpendicular to velocity.

To truly understand and enjoy these ideas, let us serve them in the form of worked examples.

BINDER Example 7

A body of mass 3kg moves along a straight line. Two forces act on it: a driving force of 18N forward and a resistive force of 6N backward. At a certain instant its velocity is 4m/s forward.

- (a) Find the acceleration of the body.
- (b) Find its velocity after 5s.

Solution

$$\text{Resultant force} = 18\text{N} - 6\text{N} = 12\text{N}$$

$$\text{Using } F = ma$$

$$12\text{N} = 3\text{kg} \times a; a = 4\text{m/s}^2$$

$$\text{Using } v = u + at$$

$$v = 4\text{m/s} + (4\text{m/s}^2)(5\text{s}); v = 24\text{m/s}$$

- (a) The acceleration is 4m/s^2 forward.
- (b) The velocity after 5s is 24m/s forward.

Making Sense of the Answer: A forward resultant force should increase velocity. A large increase is expected after 5s of steady acceleration.

Think Like a Physicist: Always find the resultant force first; then acceleration follows from F/m .

BINDER Example 8

A body of mass 2kg moves along a straight line. A driving force of 10N acts forward while a frictional force of 2N acts backward. Its velocity changes from 3m/s to 13m/s in a time t . Find the value of t .

Solution

$$\text{Resultant force} = 10\text{N} - 2\text{N} = 8\text{N}$$

$$\text{Using } F = ma$$

$$8\text{N} = 2\text{kg} \times a; a = 4\text{m/s}^2$$

$$\text{Using } v = u + at$$

$$13\text{m/s} = 3\text{m/s} + (4\text{m/s}^2)t$$

$$10\text{m/s} = (4\text{m/s}^2)t; t = 2.5\text{s}$$

The time taken is 2.5s.

Making Sense of the Answer: An acceleration of 4m/s^2 adds 4m/s each second, so gaining 10m/s should take 2.5s.

Think Like a Physicist: If resultant force is constant, acceleration is constant, so velocity changes linearly with time.

BINDER Example 9

A body of mass 5kg moves along a straight line with constant velocity. Two forces act along the line: a driving force forward and a resistive force backward. State the relationship between the two forces and explain.

Solution**The relationship:**

The driving force is equal in magnitude to the resistive force.

Explanation:

Constant velocity means acceleration is 0m/s^2 . By Newton's second law, resultant force = mass $\times$ acceleration, so resultant force = 0N . Therefore, the forces along the line must balance, meaning the forward and backward forces are equal in magnitude.

Making Sense of the Answer: *If one force were larger, the velocity would not remain constant; it would increase or decrease.*

Think Like a Physicist: *Constant velocity does not mean no forces; it means zero resultant force.*

REAL Example 10

Kipute rides a bicycle along a straight road. While she pedals, the bicycle gains velocity. When she stops pedalling, the bicycle continues moving but its velocity gradually decreases until it stops. Use Newton's second law to explain both phases of motion.

Solution

During pedalling, the driving force from the chain and wheel-ground interaction exceeds resistive forces (friction and air resistance), giving a forward resultant force. A forward resultant force produces forward acceleration, so velocity increases. When she stops pedalling, the driving force drops greatly while friction and air resistance remain backward. The resultant force becomes backward, giving backward acceleration (deceleration), so velocity decreases.

Making Sense of the Answer: *This matches everyday riding: you gain velocity when pushing forward, and you lose velocity when the forward push disappears.*

Think Like a Physicist: *Speeding up or slowing down depends on whether the resultant force is in the same direction as, or opposite to, the velocity.*

REAL Example 11

Mr. Akilikubwa pushes a loaded trolley with a nearly steady push, but the trolley's velocity increases slowly at first and then increases more noticeably after it has been moving for some time. Explain why the acceleration may increase even when the applied force is nearly constant.

Solution**Reason:**

The resistive force can decrease after motion begins, increasing the resultant force and therefore increasing acceleration.

Explanation:

At the start, frictional effects (especially static friction or higher starting resistance) can be larger. Once the trolley is moving, the resistive force may reduce slightly (rolling becomes smoother, wheels align, surfaces settle). If the applied force stays nearly constant but the resistive force decreases, the resultant force increases. By Newton's second law, a larger resultant force produces a larger acceleration.

Making Sense of the Answer: *Many objects feel hardest to start moving; once rolling smoothly, the same push produces more noticeable change in motion.*

Think Like a Physicist: *Acceleration changes when the resultant force changes, even if the applied force remains the same.*

HOT Example 12

A body of mass 4kg moves along a straight line with velocity 12m/s. A constant resistive force of 10N acts opposite to its motion. At the same time, a driving force of 26N acts in the direction of motion for 3s, after which the driving force is removed but the resistive force continues.

- (a) Find the acceleration during the first 3s.
- (b) Find the velocity after 3s.
- (c) After the driving force is removed, find the additional time taken for the body to come to rest.

Solution

(a) **First 3s:**

$$\text{Resultant force} = 26\text{N} - 10\text{N} = 16\text{N}$$

$$\text{Using } F = ma$$

$$16\text{N} = 4\text{kg} \times a, a = 4\text{m/s}^2$$

The acceleration during the first 3s is 4m/s².

(b) Using $v = u + at$

$$v = 12\text{m/s} + (4\text{m/s}^2)(3\text{s}), v = 24\text{m/s}$$

The velocity after 3s is 24m/s.

(c) After 3s, only resistive force acts opposite to the motion direction. So resultant force = -10N.

$$\text{Using } F = ma$$

$$-10\text{N} = 4\text{kg} \times a; a = -2.5\text{m/s}^2$$

Now use $v = u + at$ with $v = 0\text{m/s}$, $u = 24\text{m/s}$

$$0\text{m/s} = 24\text{m/s} + (-2.5\text{m/s}^2)t$$

$$2.5t = 24; t = 9.6\text{s}$$

The additional time to come to rest is 9.6s.

Making Sense of the Answer: While the driving force acts, resultant force is forward so velocity increases. After it is removed, only motion resistance remains so the body must slow down to rest.

Think Like a Physicist: When forces change in stages, analyze the motion in stages as well, using the final velocity of one stage as the initial velocity of the next.

HOT Example 13

A body of mass 2kg moves along a straight line. A constant driving force of 14N acts forward. The resistive force is not constant; it increases with velocity and is given by $R = 2v$, where R is in N and v is in m/s. At a certain instant, the velocity is 4m/s.

- (a) Find the resistive force at that instant.
- (b) Find the resultant force at that instant.
- (c) Find the acceleration at that instant.

Solution

$$(a) R = 2v = 2(4\text{m/s}) = 8\text{N}$$

$$(b) \text{Resultant force} = 14\text{N} - 8\text{N} = 6\text{N}$$

$$(c) \text{Using } F = ma$$

$$6\text{N} = 2\text{kg} \times a; a = 3\text{m/s}^2$$

The acceleration is 3m/s².

Making Sense of the Answer: Higher velocity gives higher motion resistance, so the resultant force becomes smaller than the driving force, giving a moderate acceleration.

Think Like a Physicist: If resistive force depends on velocity, acceleration is not constant; evaluate forces at the instant asked.

HOT Example 14

A body moves along a straight line with constant velocity 15m/s. A constant resultant force of 12N then acts opposite to its motion. After 5s, the body starts to reverse its direction of motion. Find the mass of the body.

Solution

Since velocity varies continuously, the instant at which the body reverses direction is the instant at which its velocity becomes zero. Therefore, the body's velocity is momentarily zero at $t = 5s$.

Using $v = u + at$; where: $v = 0m/s, u = 15m/s, t = 5s$

$$0m/s = 15m/s + a(5s); a = -3m/s^2$$

Now use $F = ma$; where: $F = -12N, a = -3m/s^2$

$$-12N = m(-3m/s^2); m = 4kg$$

The mass of the body is 4kg.

Making Sense of the Answer: A larger mass would resist the change more and would take longer to stop and reverse under the same force, so 4kg is reasonable.

Think Like a Physicist: "Reverses direction" usually means the velocity becomes zero at the turning instant, then changes sign.

HOT Example 15

A force acts on a body and changes its momentum from 6kgm/s to 18kgm/s in 0.4s along a straight line. Assume the force is constant during this time.

(a) Find the resultant force.

(b) If the mass of the body is 3kg, find the change in velocity during this time.

Solution

(a) Using $F = \frac{\Delta p}{\Delta t}$

Change in momentum, $\Delta p = 18kgm/s - 6kgm/s = 12kgm/s$

Time, $\Delta t = 0.4s$

$$F = \frac{\Delta p}{\Delta t} = \frac{12kgm/s}{0.4s}; F = 30N$$

The resultant force is 30N.

(b) Using $p = mv$, change in momentum $\Delta p = m\Delta v$

$$12kgm/s = 3kg \times \Delta v; \Delta v = 4m/s$$

The change in velocity is 4m/s.

Making Sense of the Answer: Even when a force acts for a very short time, a sufficiently large force can produce a significant change in momentum.

Think Like a Physicist: When a force acts for a short time, it is often more efficient to use the momentum form of Newton's second law rather than $F = ma$.

HOT Example 16

A body of mass 6kg moves along a straight line with velocity 18m/s. A constant resistive force of 9N acts opposite to the motion until the body comes to rest.

(a) Find the acceleration of the body.

(b) Find the time taken to come to rest.

(c) Find the distance travelled before coming to rest.

Solution(a) Resultant force = -9N Using $F = ma$

$$-9\text{N} = 6\text{kg} \times a; a = -1.5\text{m/s}^2$$

The acceleration is -1.5m/s^2 .(b) Using $v = u + at$; where: $v = 0\text{m/s}$, $u = 18\text{m/s}$, $a = -1.5\text{m/s}^2$

$$0\text{m/s} = 18\text{m/s} + (-1.5\text{m/s}^2)t$$

$$1.5t = 18; t = 12\text{s}$$

The time taken to come to rest is 12s.

(c) Using $s = ut + \frac{1}{2}at^2$

$$s = (18\text{m/s})(12\text{s}) + \frac{1}{2}(-1.5\text{m/s}^2)(12\text{s})^2$$

$$s = 216\text{m} - 0.75(144)\text{m}$$

$$s = 216\text{m} - 108\text{m}$$

$$s = 108\text{m}$$

The stopping distance is 108m.

Making Sense of the Answer: A small resistive force gives a small deceleration, so the body takes a long time and a long distance to stop.

Think Like a Physicist: Once you have acceleration, use kinematics to connect velocity change to time and distance.

The worked examples have filled the plate nicely; now it is time to enjoy the third law and see what new ideas it brings to the table.

NEWTON'S THIRD LAW OF MOTION

Newton's second law tells us what a resultant force does to a body: it produces acceleration and changes velocity. But it does not answer a deeper question: *Where do forces come from?*

In real life, a force never appears from nowhere. Whenever a force acts, it comes from an interaction between two bodies. Newton's third law tells us what is always true about forces that arise from interactions. It states that:

When two bodies interact, each body exerts a force on the other. These two forces are equal in magnitude, opposite in direction, and act on different bodies.

This is often written in short form as:

To every action there is an equal and opposite reaction.

Be careful with that short form. The key idea is not the words "action" and "reaction." The key idea is: *The pair of forces act on different bodies.*

Avoid the Most Common Misconception

Many students think Newton's third law means forces always cancel. That is not correct.

Newton's third law pairs do not cancel because they act on different bodies. Forces cancel only when they act on the same body and are opposite.

So always separate these two ideas:

- 1) Forces **on one body** can cancel (balanced forces).
- 2) Action–reaction forces do not cancel because they act on two different bodies.

How to Recognise a Third Law Pair

A Newton's third law force pair always has the following essential properties:

1. They are the same type of force (for example, both gravitational, both normal, or both frictional).
2. They have equal magnitudes.
3. They act in opposite directions.
4. They act on different bodies (never on the same object).
5. They arise from the same interaction between the two bodies.

The most reliable way to identify a third law pair is to describe the forces using clear, complete language:

- Force on A by B
- Force on B by A

These two forces form a Newton's third law pair. For example:

- Force on the book by the table (upward)
- Force on the table by the book (downward)

Newton's Third Law and Motion

Newton's third law does not directly tell you whether a body accelerates. Acceleration depends on the resultant force on that body (second law). Third law tells you how interaction forces come in pairs.

A body can still accelerate even though it is part of a third law pair, because the partner force acts on the other body, not on it. This is one reason Newton's third law becomes very powerful when studying systems such as:

1. Walking and running.
2. Vehicles moving on a road.
3. Pushing objects.
4. Recoil.
5. Two bodies connected by a string.
6. Collisions.

Later, when we study momentum and collisions, Newton's third law becomes the bridge between forces and changes in momentum during interactions. *Since the forces during an interaction are equal in magnitude and opposite in direction, the momentum changes of the two interacting bodies occur in opposite directions.* We will examine this relationship in detail when we study collisions.

For now, take a short breath and set momentum and collisions aside, let us warm up with some carefully chosen worked examples.

BINDER Example 17

A student presses a palm strongly against a wall. The hand feels pain, even though the wall does not move. Explain why the hand feels a force from the wall.

Solution

Reason:

The wall exerts a force on the hand equal in magnitude and opposite in direction to the force the hand exerts on the wall.

Explanation:

The hand and the wall interact. When the hand pushes the wall, the wall pushes back on the hand. By Newton's third law, these forces are equal and opposite and act on different bodies. The force from the wall acts on the hand, producing pressure on the skin, which is why pain is felt.

Making Sense of the Answer: *If the wall did not push back, the hand would go through the wall, which is not realistic for a rigid wall.*

Think Like a Physicist: *If you can name "force on A by B," the partner force is automatically "force on B by A."*

REAL Example 18

Kipanga is running on a dry road. Each time his foot pushes backward on the ground, his body moves forward. Explain how Newton's third law helps to explain forward motion in running.

Solution

Kipanga moves forward because the ground pushes him forward.

Explanation:

When Kipanga's foot pushes backward on the ground, the ground exerts an equal and opposite force forward on his foot. That forward force is transmitted to his body, producing a forward resultant force and therefore forward acceleration. Without the forward push from the ground, his velocity could not increase forward.

Making Sense of the Answer: *On a slippery surface, the ground cannot provide a strong forward force, so it becomes hard to run forward effectively.*

Think Like a Physicist: *You do not "move forward by pushing yourself forward"; you move forward by pushing something backward and receiving an equal forward force.*

HOT Example 19

Two students, Kipute and Kipanga, stand facing each other on a smooth surface. Kipute exerts a horizontal force of 60N on Kipanga.

- State the magnitude and direction of the force exerted by Kipanga on Kipute.
- If Kipute has mass 50kg and Kipanga has mass 75kg, find their accelerations immediately after the push.

Solution

- By Newton's third law, the force exerted by Kipanga on Kipute is equal in magnitude and opposite in direction to the force exerted by Kipute on Kipanga.

So the force on Kipute by Kipanga is 60N opposite to the direction of the force on Kipanga by Kipute.

- For Kipute: resultant force = 60N (in her backward direction)

Using $F = ma$

$$60\text{N} = 50\text{kg} \times a; a = 1.2\text{m/s}^2$$

For Kipanga: resultant force = 60N (in his forward direction)

Using $F = ma$

$$60\text{N} = 75\text{kg} \times a; a = 0.8\text{m/s}^2$$

Thus Kipute's acceleration is 1.2m/s^2 and Kipanga's acceleration is 0.8m/s^2 , in opposite directions.

Making Sense of the Answer: *The forces are equal, but the accelerations are different because the masses are different. The smaller mass gets the larger acceleration.*

Think Like a Physicist: *Equal and opposite forces do not mean equal accelerations; acceleration depends on mass.*

With the regular worked examples now complete, we can step back and enjoy the bigger picture; a set of miscellaneous worked examples where all the ideas meet at the same table.

MISCELLANEOUS WORKED EXAMPLES ON NEWTON'S LAWS OF MOTION**Example 20**

- A cyclist moving at constant velocity on a level road suddenly stops pedalling. Why the cyclist begins to slow down.
- A 0.45kg ball is initially at rest. A player kicks it so that it leaves with velocity 18m/s. The contact time is 0.030s. Calculate:
 - the change in momentum of the ball,
 - the average resultant force on the ball during contact.

Solution

- When pedalling stops, the driving force reduces greatly (or becomes zero) while resistive forces (air resistance and friction) still act backward. The resultant force becomes backward, so the cyclist decelerates.
- The solution of each part is as follows:

$$(i) \quad \Delta p = m(v - u) = 0.45\text{kg}(18\text{m/s} - 0\text{m/s}) = 8.1\text{kgm/s}$$

The change in momentum is 8.1kgm/s.

$$(ii) \quad \text{Average resultant force } F = \frac{\Delta p}{t} = \frac{8.1\text{kgm/s}}{0.03\text{s}} = 270\text{N}$$

The average resultant force is 270N.

Example 21

(a) Kipute says: "If I throw a ball upward, there must be an upward force acting on it as it rises."

Is Kipute correct? Explain.

(b) A crate of mass 40kg is on a rough floor. The maximum frictional force available is 180N. A worker pulls the crate with a horizontal force of 220N. When slipping begins, the frictional force becomes constant at 160N.

- (i) Explain why the crate is able to move despite the presence of a large frictional force.
- (ii) Calculate the acceleration immediately after it starts moving.
- (iii) Calculate the velocity after 2.5s from the start of motion.

Solution

(a) Kipute is **not** correct.

Explanation

After the ball leaves the hand, there is no longer any upward push from the hand. The only significant force is weight acting downward due to gravity. The ball continues rising due to its upward velocity, but it has downward acceleration because the resultant force is downward.

(b) The solution of each part is as follows:

(i)

Reason:

The applied pulling force exceeds the maximum frictional force available.

Explanation:

Before the crate starts moving, friction is static and can increase only up to its maximum value of 180N. Since the applied horizontal force is 220N, which is greater than the maximum frictional force, the forces cannot balance. A forward resultant force therefore acts on the crate, causing it to start moving. Once slipping begins, the frictional force reduces to a constant value of 160N, allowing continued motion.

(ii) Once moving, friction = 160N, so resultant force $F = 220\text{N} - 160\text{N} = 60\text{N}$ forward

$$a = \frac{F}{m} = \frac{60\text{N}}{40\text{kg}} = 1.5\text{m/s}^2$$

The acceleration is 1.5m/s^2 .

(iii) $v = u + at = 0\text{m/s} + (1.5\text{m/s}^2)(2.5\text{s}) = 3.75\text{m/s}$

The velocity is 3.75m/s .

Example 22

(a) A bus moving forward at constant velocity suddenly enters a muddy section of road and begins to slow down, even though the driver has not pressed the brakes. Explain why this does not contradict Newton's first law.

(b) A 1200 kg car is moving forward at 24m/s on a straight level road. At the instant the driver removes the foot from the accelerator, the driving force becomes zero. The total resistive force on the car depends on velocity and is given by:

$$R = 600 + 40v$$

(where R is in Newton when v is in m/s)

Assume that during this motion the car continues in a straight line and the only horizontal force on the car is this resistive force acting opposite to the motion.

Find:

(i) The initial deceleration at $v = 24\text{ m/s}$.

- (ii) The velocity after 5.0 s (treat the deceleration as constant and equal to the initial value).
- (iii) The distance covered in the 5.0 s
- (iv) Explain briefly why treating the deceleration as constant is an approximation, and state whether the true velocity after 5.0 s would be greater or smaller than your answer.

Solution

- (a) Newton's First Law states that if the resultant force on a body is zero, its velocity remains constant. In the muddy section of road, the resistive forces (friction and drag) increase, so the resultant force on the bus is no longer zero. The backward resultant force produces a backward acceleration (deceleration), causing the bus to slow down. This behaviour therefore agrees with Newton's First law rather than contradicting it.

(b)

Given:

$$\text{Resistive force: } R = 600 + 40v$$

- (i) Initial deceleration at $v = 24 \text{ m/s}$

$$R = 600 + 40(24)$$

$$R = 600 + 960$$

$$R = 1560 \text{ N (acting backward)}$$

Since the resistive force oppose the motion, it is negative.

Thus, the resultant force on the car is: $F = -1560 \text{ N}$

Using Newton's second law: $F = ma$

$$a = F / m$$

$$a = \frac{-1560 \text{ N}}{1200 \text{ kg}} ; a = -1.30 \text{ m/s}^2$$

So the initial deceleration is 1.30 m/s^2 . (Negative sign is omitted in the final answer because it has been contained in the word deceleration).

- (ii) Velocity after 5.0 s

Using: $v = u + at$

$$v = 24 \text{ m/s} + (-1.3 \text{ m/s}^2)(5 \text{ s})$$

$$v = 17.5 \text{ m/s}$$

The velocity is 17.5 m/s .

- (iii) Distance covered in 5.0 s

Using: $s = ut + \frac{1}{2}at^2$

$$s = 24 \text{ m/s}(5 \text{ s}) + \frac{1}{2}(-1.3 \text{ m/s}^2)(5 \text{ s})^2$$

$$s = 103.75 \text{ m}$$

The distance covered is 103.75 m .

- (iv) The deceleration is not truly constant because the resistive force depends on velocity. As the car slows down, v decreases, so the resistive force becomes smaller and the deceleration decreases with time. Treating the deceleration as constant therefore overestimates the slowing effect. Hence, the true velocity after 5.0 s would be **greater than 17.5 m/s** .

Example 23

- (a) A passenger standing in a daladala feels as if a "force pushes them forward" when the driver suddenly applies the brakes. Explain why this feeling occurs, and state the actual direction of the resultant force on the passenger at the beginning of braking.

- (b) A 1500kg car is moving forward with velocity 28m/s on a straight level road. The driver applies the brakes and the car comes to rest in 7.0 s. During braking, the total resistive force on the car may be treated as constant. Find:
- the acceleration of the car,
 - the resultant force acting on the car,
 - the distance travelled while braking,
 - the change in momentum of the car.
 - Use your results to explain why doubling the braking time reduces the average braking force.

Solution

(a)

Reason:

The feeling occurs because of **inertia**. When the daladala begins to slow down, the passenger's body tends to maintain its original forward velocity.

Explanation:

At the instant braking starts, the daladala experiences a backward resultant force and begins to decelerate. The passenger, however, initially continues moving forward at the original velocity due to inertia. Relative to the decelerating daladala, this makes the passenger appear to move forward, creating the sensation of being "pushed" forward. In reality, no forward force acts on the passenger; instead, a **backward resultant force** must act on the passenger to reduce their forward velocity and bring them to rest with the vehicle.

(b) The solution of each part is as follows:

(i) Acceleration

$$a = \frac{v - u}{t}$$

Substituting given values:

$$a = \frac{(0 \text{ m/s} - 28 \text{ m/s})}{(7.0 \text{ s})} = -4.0 \text{ m/s}^2$$

So the car's acceleration is -4.0 m/s^2 .

(ii) Resultant force

$$F = ma$$

Substituting:

$$F = (1500 \text{ kg})(-4.0 \text{ m/s}^2) = -6000\text{N}$$

So the resultant force is 6000N backward.

(iii) Braking distance

$$s = ut + \frac{1}{2}at^2$$

Substituting:

$$s = (28 \text{ m/s})(7.0 \text{ s}) + \frac{1}{2}(-4.0 \text{ m/s}^2)(7.0 \text{ s})^2 = 98\text{m}$$

The braking distance is 98m.

(iv) Change in momentum

$$\Delta p = m(v - u)$$

$$\Delta p = (1500 \text{ kg})(0 \text{ m/s} - 28 \text{ m/s})$$

$$\Delta p = (1500 \text{ kg})(-28 \text{ m/s}) = -42000 \text{ kgm/s}$$

So the momentum **decreases** by 42000 kgm/s.

Alternative solution

Resultant force can be given by:

$$F = \frac{\Delta p}{\Delta t}$$

From which, the change in momentum can be given as:

$$\Delta p = F\Delta t$$

Where: $F = -6000\text{N}$, $\Delta t = 7\text{s}$

Substituting:

$$\Delta p = -6000\text{N} \times 7\text{s} = -42000\text{Ns} = -42000 \text{ kg m/s (the negative sign implies the decrease in momentum)}$$

(v) Explanation about doubling braking time:

The change in momentum needed to stop the car is fixed: it is $\Delta p = -42000 \text{ kg m/s}$. The average resultant braking force, however, depends on how quickly this momentum change occurs:

$$\text{Average force} = \frac{\text{change in momentum}}{\text{time}}$$

If the stopping time is doubled while the same momentum change is required, then average force becomes half, because the same Δp is spread over a larger time. This is why increasing braking time reduces the average force experienced.

Example 24

- (a) A driver says: “*Even when my engine is very powerful, the car does not accelerate much at high velocity, and may eventually continue moving with constant velocity.*” Explain why this statement is physically reasonable.
- (b) A car of mass 1000kg moves along a straight, level road. The engine provides a constant driving force of 2500N forward. The total resistive force opposing the motion depends on the velocity and is given by:

$$R = 500 + 40v$$

(where R is in Newton when v is in m/s)

Assume that the motion remains in a straight line and that these are the only horizontal forces acting on the car. Find:

- (i) the acceleration of the car at $v = 10\text{m/s}$,
- (ii) the resistive force when the acceleration becomes zero,
- (iii) the velocity at which the car moves with constant velocity.

Solution

(a)

Reason:

Because the acceleration of the car depends on the **resultant force**, not on the engine force alone.

Explanation:

As the car's velocity increases, resistive forces such as air resistance increase. Although the engine continues to provide a large forward force, the increasing resistive force reduces the resultant forward force acting on the car. According to Newton's second law, a smaller resultant force produces a smaller acceleration.

At sufficiently high velocity, the resistive force may become equal to the engine force, making the resultant force zero and causing the acceleration to fall to zero, leading to the constant velocity.

(b) The solution of each part is as follows:

- (i) Using: Resultant force $F = \text{Driving force} - \text{Resistive force}$

Where:

Driving force = 2500N (forward)

Resistive force $R = 500 + 40v$ (backward); with $v = 10\text{m/s}$, $R = 500 + 40 \times 10 = 900\text{N}$

It follows that:

$$F = 2500\text{N} - 900\text{N} = 1600\text{N (forward)}$$

Then using: $F = ma$ or $a = \frac{F}{m}$

Substituting:

$$a = \frac{1600\text{N}}{1000\text{kg}}$$

$$a = 1.6\text{m/s}^2$$

So the acceleration when $v = 10\text{m/s}$ is 1.6m/s^2 .

(ii) The acceleration becomes zero, when the resultant force is zero.

Now from: Resultant force $F = \text{Driving force} - \text{Resistive force}$

When $F = 0$; Resistive force = Driving force

But: Driving force = 2500N

Hence, the resistive force when the acceleration becomes zero is 2500N .

(iii) Constant velocity implies zero acceleration and from (ii) above, acceleration is zero when resistive force, R is 2500N .

Using the resistive force expression:

$$R = 500 + 40v$$

Substituting:

$$2500 = 500 + 40v$$

From which: $v = 50\text{m/s}$

So the constant velocity is 50m/s .

Example 25

(a) A driver says: *"If my car is moving forward, the resultant force on it must be forward."* Explain why this statement is physically incorrect.

(b) Mr. Akilikubwa is driving a 1500kg car to school along a straight, level road at a steady velocity of 22m/s . As he approaches the school gate, a goat suddenly starts crossing the road ahead. However, being briefly absorbed in thought about the Physics lesson he is about to teach, he delays applying the brakes for 0.60s .

During this reaction time, the car continues to move forward with the same velocity. After the delay, he applies the brakes, and from that moment the resultant force acting on the car is constant and equal to 4500N , acting backward, until the car comes to rest. Find:

- (i) the deceleration during braking,
- (ii) the time taken to stop after braking begins,
- (iii) the braking distance,
- (iv) the total stopping distance from the moment Mr. Akilikubwa first sees the goat.
- (v) Explain, using your results, why reaction time can be as important as braking strength.

Solution

(a)

Reason:

Because the direction of the resultant force is the direction of **acceleration**, not necessarily the direction of **velocity**.

Explanation:

A body can move forward while slowing down. In that case, its velocity is forward but its acceleration is backward. Since the resultant force is in the direction of acceleration (Newton's second law), the resultant force can be backward even while the car is still moving forward. Therefore, forward motion does not require a forward resultant force.

(b) The solution of each part is as follows:

- (i) Using Newton's second law:

$$F = ma \text{ or } a = \frac{F}{m}$$

Substituting:

$$a = \frac{-4500\text{N}}{1500\text{kg}}$$

$$a = -3.0\text{m/s}^2$$

So the deceleration is 3.0m/s^2 .

- (ii) During braking:
- $u = 22\text{m/s}$
- ,
- $v = 0\text{m/s}$
- ,
- $a = -3.0\text{m/s}^2$

Using: $v = u + at$

$$0\text{m/s} = 22\text{m/s} + (-3.0\text{m/s}^2)t$$

From which: $t = 7.33\text{s}$ So the braking time is 7.33s .

- (iii) Using:
- $s = ut + \frac{1}{2}at^2$

$$s = (22\text{m/s})(7.33\text{s}) + \frac{1}{2}(-3.0\text{m/s}^2)(7.33\text{s})^2 = 80.66\text{m}$$

The braking distance is 80.66m .

- (iv) Total stopping distance = Reaction distance + Braking distance

But during reaction, acceleration is zero, so velocity remains 22m/s .So reaction distance = velocity $\times$ reaction time = $(22\text{m/s}) \times (0.60\text{s}) = 13.2\text{m}$ Total stopping distance = $13.2\text{m} + 80.66\text{m} = 93.86\text{m}$ So the total stopping distance is 93.86m .

- (v) From the results, the car travels
- 13.2m**
- before braking even begins. This distance depends only on reaction time and initial velocity, not on braking force. Even if the brakes are very strong, a long reaction time adds a significant distance to the stopping distance. Therefore, improving reaction time (alertness, attention) can reduce total stopping distance just as meaningfully as improving braking strength.

Example 26

- (a) Why is it safer to land on sand than on concrete when falling from the same height, even though the person comes to rest in both cases?
- (b) A 70 kg person running forward at 6.0m/s falls and is brought to rest.
Case I: stopping time is 0.10s .
Case II: stopping time is 0.40s .

Calculate:

- (i) the change in momentum in each case,
 (ii) the average resultant force magnitude in each case,
 (iii) the ratio of the forces.
 (iv) Comment on what the results imply about injury risk.

Solution

- (a) The momentum change needed to come to rest is the same in both cases. However, soft ground deforms during impact, increasing the stopping time. Since the average resultant force equals the change in momentum divided by the stopping time, a longer stopping time results into a smaller force during impact.
- (b) The solution of each part is as follows:
- (i) $\Delta p = m(v - u) = 70(0 - 6.0) = -420 \text{ kgm/s}$
 So in each case, the momentum decreases by 420kgm/s .
- (ii) Using average force, $F = \frac{\Delta p}{\Delta t}$

$$\text{Case I: } F_1 = \frac{420\text{kgm/s}}{0.10\text{s}} = 4200\text{N}$$

$$\text{Case II: } F_2 = \frac{420\text{kgm/s}}{0.40\text{s}} = 1050\text{N}$$

$$(iii) \quad \frac{F_1}{F_2} = \frac{4200}{1050} = 4$$

(iv) The shorter stopping time produces four times the average force, increasing injury risk.

Example 27

(a) **Kipute** and **Kipanga** are revising Newton's laws of motion after class. Kipanga says:

"If an object is moving in a circle at constant speed, then there is no acceleration, because acceleration only occurs when speed changes."

However, Kipute disagrees. Who is correct? Explain why.

(b) A ball of mass 0.30kg is moving horizontally at 8.0m/s. It strikes a wall and rebounds along the same line with velocity 6.0m/s in the opposite direction. The contact time is 0.040s. Calculate:

- the change in momentum of the ball,
- the average resultant force on the ball during contact and its direction.
- Using your answers in (i) and (ii), explain why the force on the ball during contact is opposite to its initial direction of motion, even though the ball rebounds with a smaller velocity than it had before impact.

Solution

(a) **Correct student:** Kipute

Reason:

Acceleration depends on the change of **velocity**, not only on the change of speed.

Explanation:

Velocity is a vector quantity, so it changes when either its magnitude (speed) or its direction changes. In circular motion, even though the speed remains constant, the direction of motion changes continuously. Because the direction of velocity is changing at every instant, the velocity is changing, and therefore the object has acceleration.

(b) The solution of each part is as follows:

(i) Take the initial direction as positive: $u = +8.0\text{m/s}$, $v = -6.0\text{m/s}$

It follows that:

$$\Delta p = m(v - u) = 0.3\text{kg}(-6\text{m/s} - 8\text{m/s}) = 0.3\text{kg}(-14\text{m/s}) = -4.2\text{kgm/s}$$

The change in momentum is -4.2kgm/s ; (the negative sign shows that the change in momentum is directed opposite to the original direction of motion of the ball).

$$(ii) \quad \text{Average force } F = \frac{\Delta p}{t} = \frac{-4.2\text{kgm/s}}{0.04\text{s}} = -105\text{N}$$

The magnitude of the average resultant force is 105N, and its direction is opposite to the ball's initial direction of motion.

(iii)

Reason:

The direction of the resultant force depends on the direction of the change in momentum, not on the final velocity alone.

Explanation:

Before impact, the ball's momentum is in the direction of its initial motion. After rebounding, its momentum is in the opposite direction. Therefore, the change in momentum is directed opposite to the initial motion. Since the average resultant force acts in the direction of the change in momentum, the force during contact must be opposite to the initial direction of motion, even though the rebound velocity is smaller.

If the ideas now feel familiar and connected, you are ready! Let us understand and enjoy them even more in the Digging Deeper Exercise in the next page

DIGGING DEEPER EXERCISE 3

EXERCISE 3A: BINDER QUESTIONS

Question 1

A body is observed to move in a straight line with constant velocity for several seconds. Explain what must be true about the vector sum of forces during this interval, and why this conclusion does not depend on the magnitude of the velocity.

Question 2

Two bodies move along the same straight line:

Body A is at rest.

Body B moves with constant velocity 8m/s.

Explain why physics imposes the same force-condition on both situations, even though one involves motion and the other does not.

Question 3

Everyday experience suggests “motion needs force.” Identify precisely which physical quantity actually needs a resultant force to change, and explain why the everyday statement seems true on Earth.

Question 4

Give a careful explanation of why “no force acts” and “resultant force is zero” are not the same statement, and explain which one matches real motion on Earth more often.

Question 5

A student says: “*Since Newton’s third law suggests that forces come in equal and opposite pairs, acceleration should be impossible.*” Explain the exact error in this reasoning without using any numerical example.

Question 6

Two students push each other on a smooth surface and move apart. Without using action–reaction language as a shortcut, explain why both can accelerate even though they interact with equal and opposite forces.

Question 7

A body moves upward while slowing down. Many learners say “*the acceleration is upward because it is moving upward.*” Explain why that statement is wrong and how to decide the acceleration direction correctly.

Question 8

A student insists that “*heavier objects always fall faster because they have larger weight.*” Explain the missing idea that repairs the argument and show how Newton’s second-law thinking resolves the confusion.

Question 9

On a rough horizontal surface, a box is pushed and moves at constant velocity. Explain what this implies about the horizontal forces, and why this does not imply that friction is absent.

Question 10

A student treats the normal reaction as an “automatic equal and opposite” force to weight. Explain why this is not generally correct and describe one condition that must be true for $N = mg$ to hold.

Question 11

Two objects collide and exert forces on each other for the same time interval. Explain how they can experience equal and opposite forces yet undergo different accelerations and different changes in velocity.

Question 12

In a braking event, explain why increasing stopping time reduces injury risk, and identify which Newton-law idea links force size to how rapidly velocity changes.

EXERCISE 3B: REAL QUESTIONS**Question 13**

Kipute is standing in a daladala. The driver brakes suddenly and she lurches forward even though nobody pushes her. Explain this observation.

Question 14

Kipanga rides a bicycle on a straight road. When he stops pedalling, the bicycle does not stop immediately but gradually slows down. Explain what this reveals about the forces acting during the motion.

Question 15

Mr. Akilikubwa places a phone on the dashboard of a moving car. When the car accelerates forward, the phone sometimes slides backward. Explain why the phone does not always move together with the car.

Question 16

Kipute pushes a heavy desk across a classroom floor. At first it does not move, then it suddenly starts moving. Explain why the behaviour changes.

Question 17

A bag of cement is loaded in the back of a pickup. When the pickup speeds up, the bag tends to slide backward relative to the vehicle. Explain why this relative motion occurs.

Question 18

Kipute is inside a lift. As the lift starts moving upward, she feels heavier for a moment. As it slows down near the top, she feels lighter. Explain these sensations.

Question 19

A bodaboda rider carries a passenger and tries to accelerate away from rest. Compared with riding alone, the motorcycle picks up speed more slowly. Explain why.

Question 20

A student jumps down from a low platform and bends their knees deeply on landing. Explain why this reduces the risk of injury.

Question 21

A book rests on a table. Kipanga says: *“Because the book is not moving, there are no forces acting on it.”* Explain why this statement is incorrect.

Question 22

Kipanga rides a bicycle through a muddy path and then on a smooth tarmac road. He notices that the same pedalling effort gives different accelerations. Explain this difference.

Question 23

A bodaboda suddenly stops but the passenger may fall if not holding tightly. Explain this effect.

Question 24

Two identical plastic bottles are pushed along the floor with the same initial velocity. One is empty and one is filled with sand. The filled bottle travels a shorter distance before stopping. Explain why this happens in real conditions.

Question 25

A car hits a pothole and passengers feel a sudden upward jolt. Explain why the forces on the passengers change suddenly.

EXERCISE 3C: HOT QUESTIONS**Question 26**

A lift moves vertically. At a certain instant, the lift is moving upward but slowing down at 2.0m/s^2 . A passenger of mass 70kg stands on a scale.

- (a) Determine the scale reading at that instant.
- (b) State whether the passenger feels heavier, lighter, or normal, and justify using the result.

(Take $g=9.8\text{m/s}^2$)

Question 27

A block of mass 5.0kg rests on a rough horizontal surface. The coefficient of static friction is 0.40 and the coefficient of kinetic friction is 0.30 .

A horizontal force P is applied and gradually increased.

- (a) Determine the maximum value of P for which the block remains at rest.
- (b) Immediately after the block just begins to move, determine its acceleration if P remains unchanged.

(Take $g=9.8\text{m/s}^2$)

Question 28

A car of mass 1000kg travels on a straight horizontal road at 25m/s . The driver applies the brakes, producing a constant braking force of magnitude F . The car comes to rest in 100m .

- (a) Determine F .
- (b) Determine the stopping time.
- (c) State whether increasing the mass of the car (with the same braking force) would increase, decrease, or leave unchanged the stopping distance. Explain.

Question 29

Kipute and Kipanga stand on frictionless ice facing each other. Kipute has mass 50kg and Kipanga has mass 75kg . They push each other apart and Kipute moves away at 4m/s .

- (a) Determine Kipanga's velocity immediately after the push.
- (b) Determine the ratio of their accelerations during the push.

Question 30

A ball of mass 0.20kg strikes a wall normally with velocity 6.0m/s and rebounds with velocity 4.0m/s . The contact time with the wall is 0.020s .

- (a) Determine the average force exerted by the wall on the ball.
- (b) State the direction of this force clearly.

EXAMINATION QUESTIONS FOR PART ONE

Question 1

- (a) A student claims: “*Weight is a fundamental physical quantity because every object has weight and it can be measured directly on a scale.*” Explain why this claim is incorrect.
- (b) Work and torque both have the same dimensional formula ML^2T^{-2} , yet they are physically different quantities. Explain how two quantities can share the same dimensions but differ physically. Give one other pair of quantities with this property.
- (c) The frequency f of a vibrating string depends on its length l , the tension T (force), and the linear mass density μ (mass per unit length). Use dimensional analysis to derive an expression for f in terms of l , T , and μ .

Question 2

- (a) A student says: “*The number of students in a classroom is a physical quantity because it has a numerical value.*” Explain clearly why this reasoning is flawed and why the number of students is not a physical quantity.
- (b) Angles measured in radians are dimensionless, yet they represent a physical rotation. Explain why angles are treated as dimensionless in dimensional analysis and state one consequence this has for deriving formulas that contain angles.
- (c) The Bernoulli equation for fluid flow is $P + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$, where P is pressure, ρ is density, v is velocity, g is acceleration due to gravity, and h is height. Verify that the equation is dimensionally homogeneous.

Question 3

- (a) A student repeats a measurement twenty times to reduce the error. Explain why this strategy reduces random errors but has no effect on systematic errors.
- (b) Explain the physical mechanism by which parallax error arises when reading a scale. Describe how it can be eliminated.
- (c) A student times 50 complete oscillations of a simple pendulum three times and obtains: 98.4 s, 98.0 s, 98.8 s. The pendulum length is $l = (0.950 \pm 0.002)$ m.
- (d) Calculate the period T with its uncertainty.
- (i) Calculate g using $g = 4\pi^2 l/T^2$ with its percentage and absolute uncertainty.
- (ii) Identify the dominant source of error.

Question 4

- (a) Explain why the international scientific community adopted a single system of units (SI) rather than allowing each country to use its own system.
- (b) In 1999, the Mars Climate Orbiter spacecraft was destroyed during its approach to Mars. The root cause was a unit mismatch between two engineering teams. What lesson does this incident teach about measurement discipline?
- (c) Convert a pressure of 760 mmHg into Nm^{-2} by using the method of dimensions. The density of mercury is $13.6gcm^{-3}$.

Question 5

- (a) Explain why averaging repeated measurements reduces random error. State clearly the condition under which this strategy works.
- (b) Seven measurements of the thickness of a glass plate are recorded (in mm): 3.52, 3.48, 3.55, 3.50, 3.53, 4.21, 3.49. Examine the data carefully and explain your treatment of any suspect reading.

- (c) Using the data from (b), after any necessary treatment, calculate the mean thickness with its absolute and percentage uncertainty. Express the result in standard notation.

Question 6

- (a) Explain why the percentage error in a quantity raised to a power n is n times the percentage error in the original measurement.
- (b) The period of a compound pendulum is given by:

$$T = 2\pi \sqrt{\frac{k^2 + h^2}{gh}}$$

where k is the radius of gyration and h is the distance from the pivot to the centre of mass. The measurements are $k = (0.200 \pm 0.001)$ m, $h = (0.300 \pm 0.001)$ m, and $g = 9.8\text{ms}^{-2}$ (exact). Calculate the percentage error in T .

Question 7

- (a) Explain what is meant by catastrophic cancellation in measurement. State the condition under which it occurs and give one practical example.
- (b) Explain how an experiment can be redesigned to avoid catastrophic cancellation. Give one specific example.
- (c) The difference in gravitational potential energy between two points at distances r_1 and r_2 from the centre of the Earth is:

$$\Delta\text{PE} = \text{GMm} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

Given $r_1 = (6400 \pm 10)$ km, $r_2 = (6800 \pm 10)$ km, and GMm is exact, find the percentage error in ΔPE .

Question 8

- (a) Distinguish between the resolution of a measuring instrument and its sensitivity. Explain why a highly sensitive instrument is not necessarily accurate.
- (b) A student says: “*I obtained the textbook value of $g = 9.8\text{ms}^{-2}$, so my experiment has zero error.*” Identify two things wrong with this reasoning.
- (c) In an experiment, the refractive index of glass is determined from $n = \sin i / \sin r$ where $i = (45.0 \pm 0.5)^\circ$ and $r = (28.0 \pm 0.5)^\circ$. Calculate the percentage uncertainty of n and its value in the standard notation.

(Hint: for a small change $\Delta\theta$ in an angle, the change in its sine is approximately $\Delta(\sin\theta) \approx \cos\theta \times \Delta\theta$, where $\Delta\theta$ must be in radians.)

Question 9

- (a) In a pendulum experiment, explain why plotting T^2 against l is preferred over plotting T against l .
- (b) A student draws a best-fit line through all the data points on a graph and forces it through the origin. Explain two things wrong with this approach.
- (c) In a simple pendulum experiment, a student measures the period by timing 20 oscillations for six different lengths. The results are:

l (m)	0.40	0.50	0.60	0.70	0.80	0.90
T^2 (s^2)	1.58	2.05	2.48	2.78	3.22	3.60

The best-fit gradient is $4.05\text{s}^2\text{m}^{-1}$, the maximum gradient is $4.32\text{s}^2\text{m}^{-1}$, and the minimum gradient is $3.80\text{s}^2\text{m}^{-1}$. Using the relationship $T^2 = (4\pi^2/g)l$, determine g with its absolute and percentage uncertainty.

Question 10

- (a) A student writes: “The mass of the Earth is 6×10^{24} .” Explain what is wrong with this statement.

- (b) A student measures the period of a simple pendulum by timing 20 oscillations and obtains $T = (2.01 \pm 0.01)$ s. The pendulum length is $l = (1.000 \pm 0.002)$ m. Using this value of g , the student then calculates the escape velocity from the Earth's surface using $v_e = \sqrt{2gR}$ where $R = 6.4 \times 10^6$ m (exact). Find percentage error in g and v_e .

Question 11

- (a) Distinguish between accuracy and precision using a specific example from a school laboratory.
- (b) A student claims: *"Random errors always cancel out when you take the mean, so the mean is always equal to the true value."* Explain why this is an oversimplification and state the conditions under which the claim approximately holds.
- (c) A student determines g using a simple pendulum of length $l = (0.900 \pm 0.002)$ m. They time 30 complete oscillations three times and record: 57.24 s, 57.48 s, 57.12 s.
- (i) Calculate the period T with its uncertainty.
- (ii) Calculate g with its percentage and absolute uncertainty.
- (iii) The accepted value of g at the location is 9.81 ms^{-2} . Comment on whether the student's result is consistent with the accepted value.

Question 12

- (a) A vehicle travels along a straight horizontal road for several minutes while its velocity remains constant. During this interval, fuel consumption continues and engine sound remains noticeable. Give a physical account of the force situation consistent with this observation.
- (b) When the same vehicle enters dense traffic, passengers experience repeated forward and backward jolts even though the average velocity of the journey is not high. Account for these sensations using principles of motion.
- (c) A car of mass 1400 kg moving at 28 m/s on a level road is brought uniformly to rest after travelling 84 m. Determine:
- (i) the braking force acting on the car and
- (ii) the time taken to stop.

Question 13

- (a) A small object released inside a smoothly moving train appears to fall vertically to a passenger inside, while an observer beside the track describes its path differently. Account for this difference without assuming either observation is incorrect.
- (b) A cyclist maintains nearly the same pedalling effort but notices increasing difficulty in sustaining the same velocity when facing a strong headwind. Explain.
- (c) A body of mass 3.0 kg moves along a straight line under a constant driving force of 18 N. A resistive force $R = 3v$ acts simultaneously, where R is in Newton and v in m/s. Determine the acceleration at the instant when $v = 4$ m/s.

Question 14

- (a) A puck set in motion on a nearly frictionless horizontal surface continues moving for a long time after the initial push has ceased. Account for this observation.
- (b) A passenger in a lift experiences brief changes in apparent weight when the lift starts moving upward and again when it slows near the top. Give an account of these sensations.
- (c) A lift of mass 800 kg carries a passenger of mass 70 kg. At a certain instant, the lift is moving upward but slowing uniformly at 1.5 m/s^2 . Determine the normal reaction on the passenger.

(Take $g = 9.8 \text{ m/s}^2$)

Question 15

- (a) Two skaters initially at rest on frictionless ice push apart. The lighter skater is observed to gain velocity more rapidly. Account for this observation.
- (b) A runner accelerates forward while pushing backward on the ground. Give a physical explanation of how forward motion arises in this situation.
- (c) On frictionless ice, Kipute (55 kg) and Kipanga (75 kg) are initially at rest. Kipute pushes Kipanga for 0.40 s. Immediately after the interaction, Kipute moves backward at 2.5 m/s. Determine:

- (i) Kipanga's velocity and
- (ii) the magnitude of the force exerted during the push.

Question 16

- (a) A ball projected vertically upward eventually returns to its starting level. Give a physical account of this motion.
- (b) Athletes landing on sand often bend their knees, whereas landing stiffly on rigid ground frequently causes injury. Account for this difference.
- (c) A ball is projected vertically upward from the ground at 26m/s. At the same instant, another ball is released from rest from a point 35m directly above the ground. Determine:
 - (i) the time and
 - (ii) height at which the two balls are at the same level.

(Take $g = 9.8\text{m/s}^2$)

Question 17

- (a) A ball is released from a moving vehicle travelling along a straight road. Relative to the ground, the ball lands some distance ahead of the point vertically below where it was released. Explain this observation.
- (b) A runner on a smooth surface finds it difficult to accelerate compared with running on rough ground. Explain this observation.
- (c) A body of mass 5.0kg initially at rest is acted upon by a constant horizontal force of 20N while a resistive force of 4N opposes the motion. Calculate the velocity attained after 5.0s and the displacement during this time.

Question 18

- (a) During circular motion at constant speed, an object is still described as accelerating. Explain how this statement is physically consistent.
- (b) When a car rounds a bend at steady speed, passengers tend to lean sideways. Discuss the physical reason for this behaviour.
- (c) A car of mass 1200kg travels at 18m/s along a straight horizontal road. A constant braking force brings it to rest in 9.0s. Calculate the braking force and obtain the stopping distance.

Question 19

- (a) A vehicle moves along a straight horizontal road while its engine power remains roughly constant, yet its velocity gradually approaches a steady value instead of increasing indefinitely. Explain physically why continuous acceleration does not persist.
- (b) A crate is pushed across a warehouse floor. Initially it resists motion strongly, but once moving it slides more easily even though the applied push is unchanged. Discuss the physical reasons behind this behaviour.
- (c) A car of mass 1200kg accelerates uniformly from 12m/s to 30m/s while travelling 270m along a straight level road. Calculate the resultant force acting on the car.

Question 20

- (a) Two observers describe the same moving object but assign different velocities to it. Clarify how both descriptions can remain physically valid.
- (b) A passenger standing in a bus notices that when the driver suddenly changes lanes, unsecured objects inside the bus shift sideways. Give a physical explanation for this effect.
- (c) A ball is projected vertically upward from the edge of a 40m-high cliff with velocity 18m/s. Calculate the time taken to reach the ground below.

(Take $g = 9.8\text{m/s}^2$)

Question 21

- (a) A student claims that whenever a body is moving, some net force must be acting in the direction of its motion. Explain why this claim is not generally valid.
- (b) A box is pushed along a rough floor at constant velocity by a person whose push remains steady. Explain what must be true about the horizontal forces during the motion.
- (c) A 0.25kg ball strikes a wall normally with velocity 10m/s and rebounds with velocity 8m/s in the opposite direction. The contact time is 0.030s. Calculate:
 - (i) the change in momentum of the ball and

- (ii) the average force exerted on the ball by the wall.

Question 22

- (a) A body can be accelerating even when its velocity magnitude remains constant. Explain how this can occur.
- (b) A passenger standing in a daladala feels a sideways “pull” when the vehicle turns suddenly at a junction. Explain why this sensation is experienced.
- (c) A stone is projected vertically upward from the ground with velocity 24m/s. Find:
- (i) the time when it is at a height of 20m on its way downward and
- (ii) its velocity at that instant.

(Take $g = 9.8\text{m/s}^2$)

Question 23

- (a) Two forces can be equal in magnitude and opposite in direction yet fail to cancel. Explain why this can happen.
- (b) When walking on a slippery surface, a person struggles to move forward effectively. Explain this observation using force ideas.
- (c) On a smooth horizontal surface, two blocks A and B of masses 3kg and 5kg are in contact. A horizontal force of 16N is applied to block A, pushing both blocks together. Neglect friction. Calculate:
- (i) the common acceleration and
- (ii) the force exerted by A on B.

Question 24

- (a) A body may momentarily have zero velocity without being at rest for any finite time. Explain this statement.
- (b) A passenger inside a bus feels pressed sideways when the bus negotiates a roundabout at nearly constant velocity magnitude. Explain the sensation.
- (c) A stone is projected vertically upward from ground level with velocity 22m/s. Calculate the time interval during which the stone is above a height of 15m. (Take $g = 9.8\text{m/s}^2$)

Question 25

- (a) Explain why a constant resultant force does not always imply constant acceleration for every possible situation.
- (b) A driver applies the brakes on a wet road and notices the stopping distance becomes much longer than on a dry road at the same initial velocity. Explain why.
- (c) A car of mass 1200kg is moving at 20m/s on a straight road. The driver notices danger and reacts for 0.70s before braking begins. During braking, the resultant retarding force is constant at 3600N until the car stops. Calculate:
- (i) the braking time after braking begins,
- (ii) the braking distance,
- (iii) the total stopping distance from the instant the danger is noticed.

Question 26

- (a) Explain why two bodies in collision experience equal and opposite forces yet can undergo different accelerations.
- (b) A person in a moving vehicle is safer wearing a seatbelt during sudden braking. Explain why this is physically expected.
- (c) A ball is projected vertically upward from the ground with velocity 28m/s. Exactly 1.2s later, a second ball is projected vertically upward from the same point with velocity 34m/s. Calculate:
- (i) the time after the first launch when the two balls meet,
- (ii) the height above the ground at which they meet.

(Take $g = 9.8\text{m/s}^2$)

Question 27

- (a) A student claims that *if two forces are equal in magnitude and opposite in direction, then they must form a Newton’s third-law pair*. Explain why this is not necessarily true.
- (b) In many road accidents, two drivers claim they were travelling at the “same velocity” because their speedometers showed the same reading. Explain why this claim can still be wrong in physics terms.

- (c) A car of mass 1000kg moves along a straight road with velocity 12m/s. The engine provides a constant driving force of 2200N. The resistive force is given by $R = 400 + 50v$, where R is in N and v is in m/s. Calculate:
- the acceleration at $v = 12\text{m/s}$,
 - the velocity at which acceleration becomes zero.

Question 28

- (a) Two different force descriptions are proposed for a moving body: “*forces are absent*” and “*forces balance*.” Explain why only one of these descriptions matches ordinary motion on Earth most of the time.
- (b) A form five student claims that in any interaction, the “bigger force” must act on the object that accelerates more. Explain why this claim is not compatible with Newton’s third law.
- (c) A motorcycle and a car move along the same straight road. At a certain instant the motorcycle has velocity 18m/s and the car has velocity 30m/s in the same direction. The motorcycle begins to accelerate uniformly at 2m/s^2 at that instant, while the car begins to decelerate uniformly at 3m/s^2 at the same instant. Find:
- the time after that instant when their velocities become equal,
 - the separation between them at that instant if they were at the same point initially.

Question 29

- (a) Kipute argues that *if two opposing forces act on a body and the body moves, then those forces cannot be equal in magnitude*. Explain why her reasoning can fail even when the body is moving.
- (b) A diver steps off a platform into water. The diver comes to rest, yet serious injury is uncommon when the water is deep. Explain the physical reason using force and motion ideas.
- (c) A stone is projected vertically upward from ground level with velocity 28m/s. Calculate the time interval during which the stone is below a height of 18m. (Take $g = 9.8\text{m/s}^2$)

PART TWO

PARTICLE MECHANICS

Having established how motion is described and measured, we now turn to the causes of motion. Particle mechanics examines how forces influence the movement of bodies treated as point masses, allowing us to apply Newton's laws to situations such as equilibrium, collisions, projectiles, circular motion, and gravitational interaction. These ideas form the core of classical mechanics and provide essential tools for analysing motion quantitatively. Mastering this part will deepen your understanding of how forces govern motion and prepare you for more advanced mechanical systems ahead.

Chapter 4

EQUILIBRANT FORCES

INTRODUCTION

In the previous chapters, motion was the star of the show: buses accelerating, balls flying, bicycles braking, and students occasionally running when the emergency assembly bell suddenly rings and everyone remembers that walking is no longer fast enough. But Physics has another equally interesting side: situations where *nothing seems to happen*. A book sits quietly on a table, a signboard hangs without drama, and you may be sitting perfectly still in class while appearing deeply attentive. Yet behind this peaceful appearance, forces are actually having a silent tug-of-war.

When forces balance each other, motion does not change. Not because forces disappeared, but because they cancel perfectly. At O-Level you met balanced forces in simple situations. Now at A-Level, we sharpen that idea with a powerful concept; the **equilibrant force**: *the single force that would exactly oppose all others and bring a system into equilibrium*. Engineers use it when designing bridges, pilots rely on it unknowingly during steady flight, and even your body depends on it just to stand upright without falling like an unplugged robot.

Do not worry if this sounds serious already; equilibrium may sound calm, but the Physics behind it is lively, elegant, and sometimes surprisingly funny. By the end of this chapter, you will not only understand why objects stay still or move steadily, but you will also begin to see balance everywhere; from classroom desks to skyscrapers... and perhaps even in your study habits (which, hopefully, will now include balanced revision!).

Welcome to the first chapter of **Particle Mechanics** where stillness finally gets the attention it deserves.

From Balanced Forces to the “Balance-Maker”: Meet the Equilibrant

In O-level Physics, you learnt the comforting idea of **balanced forces**: *when forces cancel, the object does not accelerate*. A book on a table stays put because the downward weight is balanced by the upward normal reaction. A picture frame hangs peacefully because the tension in the string balances its weight. So far, so good; O-level teaches you how to *recognise* equilibrium.

Now A-level takes that familiar idea and upgrades it into a **tool**. Instead of only saying “the forces are balanced”, we begin asking: *What single force would I need to add to make the body perfectly balanced?* That single “balance-maker” is called the **equilibrant force**. It is not a new mysterious type of force; it is simply the force that would exactly cancel the **resultant** (net force) of all the other forces.

Concisely: **The equilibrant force is the single force equal in magnitude and opposite in direction to the resultant of all other forces acting on a body.**

If the forces already balance, the resultant is zero and the equilibrant is also zero, meaning there is no extra force needed. But if the forces do not balance, then the equilibrant tells you exactly what to add (magnitude and direction) to restore equilibrium. This is why equilibrant forces matter in A-level Particle Mechanics: they help us analyse real situations like signboards, ladders, cables, cranes, bridges, and any object that must stay steady while forces pull in different directions including you, sitting on a chair, pretending you are not tired, while gravity quietly keeps pulling you downward.

RESULTANT FORCE VERSUS EQUILIBRANT FORCE

Before we talk about *equilibrium*, we must be fluent in the two “opposite twins” of force analysis: the **resultant** and the **equilibrant**. They are not enemies; they are simply two ways of describing the same situation, depending on whether you are *adding forces* or *cancelling forces*.

Resultant force (net force)

The **resultant force** is the single force that has the same effect as all the forces acting together on a body.

$$\text{Resultant force, } \mathbf{F_R} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \dots$$

Meaning: if you replace all the separate forces by one force $\mathbf{F_R}$, the body would accelerate in the same way.

Key idea:

- If $\mathbf{F_R} \neq 0$, the body accelerates in the direction of $\mathbf{F_R}$.

- If $F_R = 0$, the body has zero acceleration (it stays at **rest** or moves with **constant velocity**).

So equilibrium is simply the case when $F_R = 0$.

Equilibrant force

The **equilibrant force** is the single force that would bring the body into equilibrium by cancelling the resultant of all the other forces.

$$\text{Equilibrant force, } F_E = -F_R$$

This means:

- F_E has the same magnitude as F_R .
- F_E acts in the opposite direction to F_R .

So if you apply the equilibrant, the new resultant becomes zero:

$$F_R + F_E = 0$$

Remember this:

- Resultant tells you: “What is the overall push?”
- Equilibrant tells you: “What single push would cancel that overall push?”

To have better understanding on this; suppose two forces act on a body:

- 12 N to the right
- 7 N to the left

Then, resultant force: $F_R = 12\text{N} - 7\text{N} = 5\text{N}$ to the right

And equilibrant force: $F_E = 5\text{N}$ to the left

So the equilibrant is the force that would make the body balanced.

For now, give theory a short holiday and let us sharpen our understanding with two worked examples.

BINDER example 1

A lamp is suspended from the ceiling by a vertical cord. The tension in the cord is 25 N upward while the weight of the lamp is 25 N downward.

- Determine the resultant force acting on the lamp.
- State the equilibrant force.

Solution

Upward force = 25N

Downward force = 25N

Resultant, $F_R = 25\text{N} - 25\text{N}$; $F_R = 0\text{N}$

- So the resultant force is 0N (the lamp is in equilibrium).
- Since the resultant force is zero, no additional force is required to balance the system.
Equilibrant force = 0N.

Making Sense of the Answer: If forces already balance, the system is already in equilibrium. The equilibrant simply tells us that nothing extra is needed; balance has already been achieved.

Thinking Like a Physicist: Do not assume an equilibrant must always exist physically. Sometimes it is only a conceptual force used to check whether equilibrium already exists.

REAL example 2

Kipute is sitting quietly on a chair revising Physics. She says: “I am at rest because the upward force from the chair balances my weight.”

Kipanga disagrees: “If gravity is pulling you downward, there must be a stronger downward force. Otherwise you would float!”

Explain who is correct and clarify the roles of resultant and equilibrant forces in this situation.

Solution

Kipute is correct.

Explanation:

When she sits at rest: Weight (W) acts downward. Normal reaction (R) from the chair acts upward. These two forces are equal in magnitude and opposite in direction, so the resultant force is zero. This means there is no acceleration and the system is in equilibrium. Kipanga's idea is incorrect because equilibrium does not require a stronger force but only equal and opposite forces.

Clarification:

The upward reaction from the chair acts as the equilibrant force to her weight. It balances the downward gravitational force that would otherwise cause downward acceleration.

Making Sense of the Answer: *Objects at rest are not force-free. Usually, forces are present but balanced. Sitting comfortably is actually a perfect demonstration of equilibrium though it may not feel like Physics at the time.*

Thinking Like a Physicist: *Whenever you see something stationary, ask:*

- *What forces are acting?*
- *Which ones balance each other?*

Having given the worked examples their full say, we can now welcome the next subtopic and see what fresh ideas it brings along.

THE TWO EQUILIBRIUM EQUATIONS

Up to now, we have talked about equilibrium in a simple way: if the resultant force on a body is zero, the body either remains at rest or continues moving with constant velocity. That idea works perfectly when all forces act along one straight line. But real life is rarely that cooperative. Forces often act in different directions: a hanging sign pulls downward while its cables pull sideways and upward, a ladder leans against a wall, a crane cable pulls at an angle, or even your school bag straps pull in directions you did not plan when you overloaded the bag.

To deal with such situations, Physics does not panic. Instead, it separates forces into two perpendicular components (directions): horizontal component (x-direction) and vertical component (y-direction). This process is called **resolution of forces**. To have better understanding on this consider the following two cases.

Case 1: Angle measured from the horizontal

In most cases, angles are measured from the horizontal, as shown in the figure below.

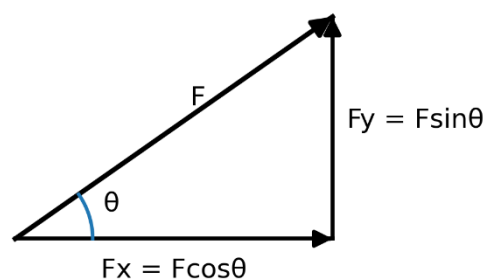


Figure: A force resolved into horizontal and vertical components when the angle is measured from the horizontal.

From the diagram, a force F act at angle θ above the horizontal (**measured from positive x-axis**).

From right-triangle geometry:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{F_x}{F}$$

From which:

$$F_x = F \cos \theta = \text{Horizontal component}$$

Analogously:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{F_y}{F}$$

From which:

$$F_y = F \sin \theta = \text{Vertical component}$$

Case 2: Angle Measured from the Vertical

In some situations, angles are measured from the vertical, as shown in the figure below.

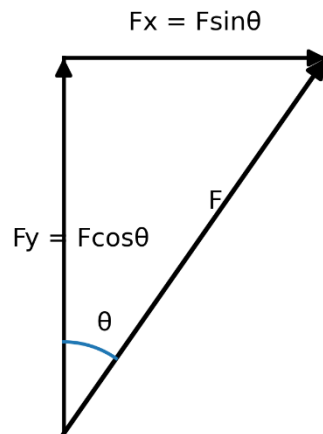


Figure: Resolution of a force when the angle is measured from the vertical; note that the component expressions reverse.

In this case:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{F_x}{F}$$

From which:

$$F_x = F \sin \theta = \text{Horizontal component}$$

Similarly:

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{F_y}{F}$$

From which:

$$F_y = F \cos \theta = \text{Vertical component}$$

Important Alert!

In the second case, *the components reverse*.

This happens because the horizontal component is now opposite to the angle, while the vertical component is adjacent to it. So, always **remember to check which axis the angle is measured from**.

It is also worth noting that, from geometry (Pythagoras' theorem), the magnitude of a force is related to its components in both cases by:

$$F^2 = (F_x)^2 + (F_y)^2 = (F \cos \theta)^2 + (F \sin \theta)^2$$

Hence:

$$F = \sqrt{(F_x)^2 + (F_y)^2} \text{ or } F = \sqrt{(F \cos \theta)^2 + (F \sin \theta)^2}$$

This confirms that the resultant magnitude is independent of whether the angle is measured from the horizontal or the vertical.

Before the ideas start colliding in our heads, let us calm them down with a few simple worked examples.

BINDER Example 3

A force of 50N acts on a body at an angle of 30° above the horizontal. Calculate the horizontal and vertical components of this force.

Solution

Since the angle was measured from horizontal:

$$\text{Horizontal component, } F_x = F \cos \theta = 50 \cos 30^\circ = 43.3 \text{ N}$$

$$\text{Vertical component, } F_y = F \sin \theta = 50 \sin 30^\circ = 25 \text{ N}$$

Making Sense of the Answer: The horizontal component is larger because the force leans more toward the horizontal. If the angle were 60° , the vertical component would dominate.

Thinking Like a Physicist: Component analysis allows us to treat two-dimensional force problems as two separate one-dimensional problems. This simplification is one of Physics' most elegant problem-solving techniques.

BINDER Example 4

A hanging sign is supported by a cable. The cable pulls on the sign with a force of 120N and makes an angle of 40° with the vertical. Without worrying yet about the detailed theory of tension, resolve this force into:

- (a) the horizontal component, and
- (b) the vertical component.

Solution

Since the angle was measured from vertical:

$$\text{Horizontal component, } F_x = F \sin \theta = 120 \sin 40^\circ = 77.1 \text{ N}$$

$$\text{Vertical component, } F_y = F \cos \theta = 120 \cos 40^\circ = 91.9 \text{ N}$$

Making Sense of the Answer: Notice how the component equations reversed compared to Example 3. This happens because now the vertical component is adjacent to the angle (cosine) while the horizontal is opposite (sine).

Thinking Like a Physicist: The physics doesn't change with how we measure angles, but the mathematics adjusts accordingly. Staying alert to reference axes prevents sign errors and incorrect component calculations.

The worked examples have done their job nicely; now let us invite the next concept to step forward and show us its charm.

Fundamental Condition for Equilibrium in Two Dimensions

From Newton's second law:

$$\text{Resultant force} = \Sigma F = ma; \text{ where } \Sigma \text{ stands for the summation of.}$$

When a resultant force (ΣF) acts diagonally, it has components in both horizontal and vertical directions. If these components are unbalanced, the body will experience acceleration in both directions, leading to **motion in two dimensions**.

For horizontal motion:

$$\Sigma F_x = ma_x$$

For vertical motion:

$$\Sigma F_y = ma_y$$

But for equilibrium, acceleration is zero. Thus:

$$a_x = 0 \text{ and } a_y = 0$$

Hence:

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

The final result is the set of **two equilibrium equations for forces acting in two dimensions**.

The equations mean that for equilibrium to be maintained:

- The algebraic sum of horizontal components of all forces must be zero.
- The algebraic sum of vertical components of all forces must be zero.

When either of these conditions fails, equilibrium is broken. If:

- $\Sigma F_x \neq 0$; horizontal acceleration occurs resulting to sideway motion.
- $\Sigma F_y \neq 0$; vertical acceleration occurs resulting to vertical (upward or downward) motion.

Equilibrium is therefore a multidirectional condition.

You have to remember that:

A body could move with constant velocity and still satisfy: $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

To truly understand and enjoy these ideas, let us serve them in the form of worked example.

BINDER Example 5

Three coplanar forces act on a particle in equilibrium: 10N eastward, 6N northward, and a third force F at an angle. If the particle is in equilibrium, find the magnitude and direction of the third force F.

Solution

For equilibrium: $\Sigma F_x = 0$ and $\Sigma F_y = 0$

Taking east as positive x-direction and north as positive y-direction.

Applying horizontal equilibrium equation: $\Sigma F_x = 0$

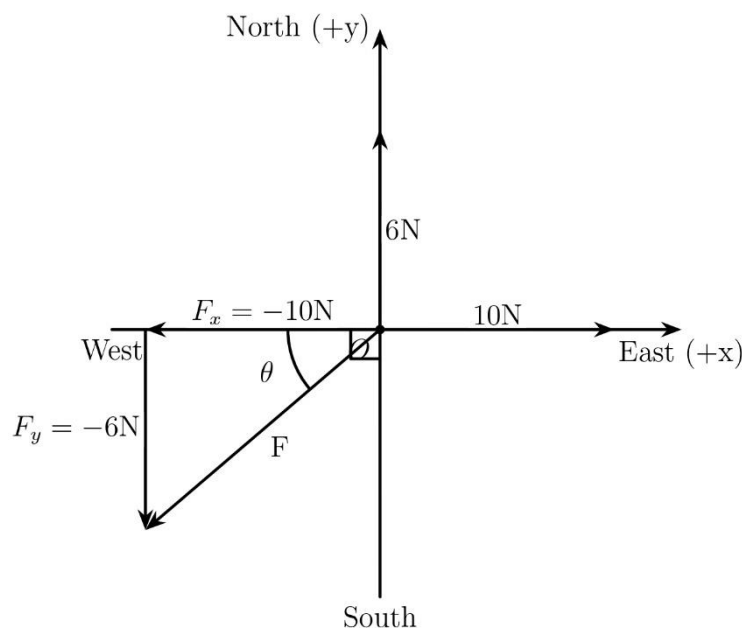
$$F_x + 10 = 0; F_x = -10\text{N} = \text{Horizontal component of the third force}$$

*The negative sign means the horizontal component acts **westward**.*

Applying vertical component equilibrium equation: $\Sigma F_y = 0$

$$F_y + 6 = 0; F_y = -6\text{N} = \text{Vertical component of the third force}$$

*Here, the negative sign means the vertical component acts **southward**.*



From the diagram:

$$\tan\theta = \frac{F_y}{F_x} = \frac{-6}{-10} = 0.6; \theta = \tan^{-1} 0.6 = 31^\circ$$

$$\text{Also, using } F = \sqrt{(F_x)^2 + (F_y)^2} = \sqrt{(-10)^2 + (-6)^2} = 11.7\text{N}$$

The magnitude is 11.7N.

The direction is 31° south of west.

Making Sense of the Answer: *The third force must "cancel" the combined effect of the other two. Since we have eastward and northward forces, the equilibrant must pull westward (to balance eastward force) and southward (to balance northward force).*

Thinking Like a Physicist: *Equilibrium means the vector sum is zero. Treating x and y directions separately transforms a vector problem into two simple algebraic equations.*

With this worked example now neatly packed away, let us move on and meet the next subtopic; it has been waiting patiently to join the conversation.

FREE-BODY DIAGRAMS (FBDs)

Before applying equilibrium equations, we need a clear picture of what forces are actually acting on a body. In real situations, forces often come from different sources: gravity pulls downward, surfaces push upward, strings pull at angles, and friction resists motion. To analyse such situations properly, Physics isolates the body from its surroundings and represents all the forces acting on it in a simple sketch called a **free-body diagram (FBD)**. This diagram acts as the bridge between the physical situation and the mathematical equations used to analyse equilibrium. It is a bit like cooking **pilau** without the spices; you may still call it pilau, but everyone knows something important is missing. In the same way, solving equilibrium problems without a free-body diagram often produces an answer, but it often lacks balance, clarity, and sometimes correctness.

Concisely, a free-body diagram *is a diagram of a body isolated from its surroundings, showing **only the forces acting on it***. It is the bridge between the real situation and the equilibrium equations: $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

Is a Free-Body Diagram Always Necessary?

It is not always necessary to draw a free-body diagram before solving equilibrium problems, especially in very simple situations where the forces are obvious and act along a single straight line. In such cases, the equilibrium equations can sometimes be written directly without first sketching a diagram.

However, as soon as forces act in different directions, particularly at angles, drawing a free-body diagram becomes extremely important. The diagram helps to:

- Identify all forces acting on the body.
- Show their correct directions.
- Avoid missing or inventing forces.
- Guide the correct resolution of forces into components.

Without a proper free-body diagram, mistakes often occur not in algebra but in identifying forces and their directions.

For this reason, although not always compulsory, drawing a free-body diagram is considered best practice in A-Level Physics and is strongly recommended before applying the equilibrium equations.

How to Draw a Free-Body Diagram (Step-by-Step)

Step 1: Choose the body

Decide exactly what you are analysing (a block, a ring, a hanging load, a trolley, etc.). Treat it as a **particle** (a point).

Step 2: Isolate it

Imagine the body is cut free from contact surfaces, strings, and supports. The surroundings are removed, but the **forces they exert remain**.

Step 3: Replace each interaction by a force arrow

Draw arrows starting from the particle. Each arrow represents a force on the body.

Step 4: Label every force

Use standard symbols (**W** for weight, **R** for normal reaction, **T** for tension, **f** for friction force). If a force is at an angle, show the angle clearly.

Step 5: Choose axes

Usually: **x** for horizontal, **y** for vertical. Choose positive directions. If an incline is involved, you may choose axes parallel and perpendicular to the plane (later in the chapter).

What to Include (and What NOT to Include) in FBDs

A correct free-body diagram includes only the relevant forces acting on the body and nothing more. The following guidelines help ensure clarity and accuracy.

Include:

- Forces **acting on** the body (pushes, pulls, weight, contact forces).
- Directions of forces.
- Angles where needed.

Do NOT include:

- Motion arrows (velocity) as if they were forces.
- Forces exerted **by** the body on other objects (unless that other object is your chosen body).
- Extra forces invented to “make it balance.”

Always obey the following **rule**: *Every force must have a clearly identifiable source (agent). If you cannot say what is causing the force, you should question whether it really exists.*

Common FBD Mistakes

Errors in equilibrium analysis frequently originate from incorrect free-body diagrams rather than algebraic mistakes. The following are common pitfalls students should avoid.

- **Mixing action and reaction on the same diagram**

Example: If you draw forces on a block resting on a table, include the weight of the block and the upward reaction from the table. Do not include the force the block exerts on the table, because that force acts on the table, not on the block.

So, always draw only forces acting on the object you are analysing, not the forces it exerts on other bodies.

- **Friction drawn in the wrong direction**

Friction opposes motion: it acts opposite to actual movement or to the direction an object is trying to move.

- **Assuming the Normal Reaction Is Always Vertical**

The normal reaction force acts perpendicular (at right angle) to the contact surface and is therefore **vertical only when the surface is horizontal** (the normal reaction is not necessarily vertical).

- **Leaving out one force**

In equilibrium problems, missing a force is the fastest way to destroy the conditions $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

Before these ideas start colliding in our heads, let us calm them down with a few useful worked examples.

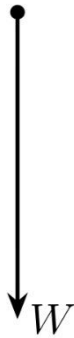
BINDER Example 6

A small metal ball of mass 0.50kg is released and allowed to fall freely through the air (air resistance neglected).

- Draw the free-body diagram of the ball.
- State the resultant force on the ball.
- Show that the acceleration of the ball is equal to the acceleration due to gravity, g .

Solution

-



- Resultant force is downward and equals the weight, $W = 0.5\text{N} \times 9.8\text{N/kg} = 4.9\text{N}$ downward.
- Using Newton's second law in the vertical direction:

$$\Sigma F_y = ma_y$$

$$W = ma_y$$

But $W = mg$

$$mg = ma_y$$

So: $a_y = g$

Making Sense of the Answer: The ball accelerates because there is no other force to cancel its weight.

Thinking Like a Physicist: If you cannot name the agent producing a force (string, surface, air), do not draw it.

BINDER Example 7

A particle is acted on by two horizontal forces: 20N to the right and 12N to the left.

- Draw the free-body diagram.
- Find the resultant force.
- A student adds an extra 8N force to the left on the diagram and says, “*The particle must be in equilibrium.*” Explain what is wrong with this thinking.
- State the equilibrant force for the given forces.

Solution

- FBD



- Taking right as positive:

$$F_R = 20\text{N} - 12\text{N} = 8\text{N}$$

The resultant force is 8N to the right.

- The extra 8N force has no physical source (no agent). Forces cannot be added simply because we want equilibrium; they must come from a real interaction (a string, a push, contact, etc.). The particle is not automatically in equilibrium; it accelerates if the resultant is not zero.

(d) The equilibrant is the single force that would cancel the resultant: $F_E = 8\text{N}$ to the left

Making Sense of the Answer: *Equilibrium is a condition produced by real forces, not a wish added to the diagram.*

Thinking Like a Physicist: *Never “repair” a diagram by inventing forces. First ask: what object could apply that force?*

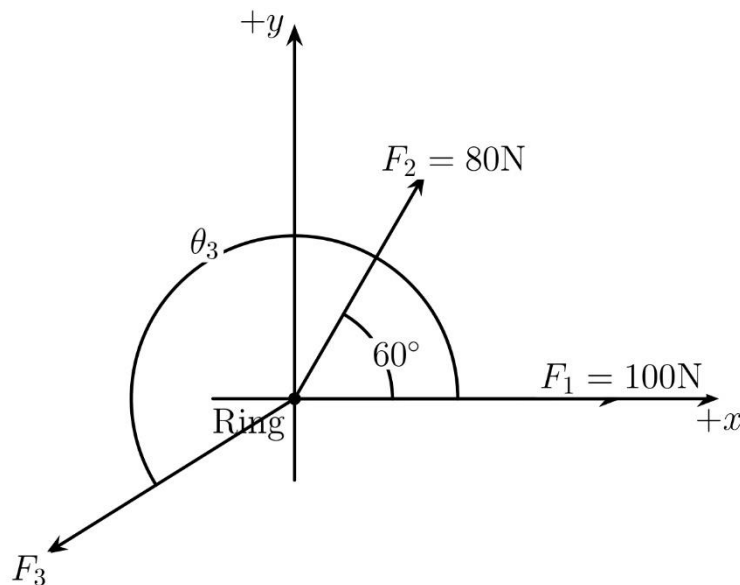
BINDER Example 8

Three coplanar forces act on a small ring at rest at a junction of light cords. Force $F_1 = 100\text{N}$ acts horizontally to the right. Force $F_2 = 80\text{N}$ acts at 60° above the horizontal (measured counterclockwise from the positive x-axis). Force F_3 has unknown magnitude and unknown direction.

- Determine the **magnitude** of F_3 required for equilibrium.
- Determine the **direction** of F_3 (measured counterclockwise from the positive x-axis).

Solution

Since F_1 and F_2 together act generally northeast, the equilibrant force F_3 must act generally southwest. So, the free body diagram for the example 8 will be as follows:



Resolving F_1 and F_2 into components

$F_1 = 100\text{N}$ to the right (0° from the positive x – axis)

- $(F_1)_x = 100\text{N}$ (By using formula it is $100\text{N}\cos 0^\circ = 100\text{N}$)
- $(F_1)_y = 0$ (By using formula it is $100\text{N}\sin 0^\circ = 0\text{N}$)

$F_2 = 80\text{N}$ at 60°

- $(F_2)_x = 80\cos 60 = 40\text{N}$
- $(F_2)_y = 80\sin 60 = 69.3\text{N}$

The equilibrium condition: $\Sigma F_x = 0$ and $\Sigma F_y = 0$

Then;

$$\Sigma F_x = 100\text{N} + 40\text{N} + (F_3)_x = 0; (F_3)_x = -140\text{N}$$

And;

$$\Sigma F_y = 0\text{N} + 69.3\text{N} + (F_3)_y = 0; (F_3)_y = -69.3\text{N}$$

The magnitude of F_3 is given by the following relationship:

$$F_3 = \sqrt{((F_3)_x)^2 + ((F_3)_y)^2} = \sqrt{(-140)^2 + (-69.3)^2} = 156.21\text{N}$$

(a) The magnitude is 156.21N.

From geometry of free body diagram;

$$\tan(\theta_3 - 180) = \frac{(F_3)_y}{(F_3)_x} = \frac{-69.3}{-140} = 0.495$$

$$\theta_3 - 180 = \tan^{-1} 0.495 = 26.33^\circ; \theta_3 = 26.33^\circ + 180^\circ = 206.33^\circ$$

(b) The direction of F_3 is 206.33° from the positive x-axis (counterclockwise).

Making Sense of the Answer: Since F_1 and F_2 together act generally northeast, the equilibrant F_3 must act generally southwest to maintain equilibrium. Its magnitude simply matches the resultant of F_1 and F_2 , but its direction is opposite.

Thinking Like a Physicist: Before calculating, always predict the direction. If the combined forces point northeast, the equilibrant must point southwest. This quick check helps prevent sign and angle mistakes.

As the worked examples quietly leave the table, the next subtopic arrives; not to overwhelm us, but to be understood and enjoyed!

STANDARD FORCE MODELS

In equilibrium problems, correctly identifying the forces acting on a body is often more important than the mathematics that follows. Fortunately, many physical situations involve a small number of common force types that appear repeatedly. Recognising these standard force models allows you to draw accurate free-body diagrams quickly and apply the equilibrium equations with confidence. Among the most frequently encountered forces are **weight (W)**, **normal reaction (R)**, **tension (T)**, and **friction**. Each has characteristic origins, directions, and physical interpretations, and mastering them forms an essential foundation for solving problems in Particle Mechanics.

Understanding these force models not only improves diagram accuracy but also prevents common mistakes such as inventing forces, misdirecting forces, or overlooking important interactions. Once these forces are correctly identified, the equilibrium equations: $\Sigma F_x = 0$ and $\Sigma F_y = 0$ become straightforward to apply.

And just as a well-prepared free-body diagram was compared to the essential spices in **pilau**, these standard force models are the actual ingredients themselves. Without recognising them properly, you may still attempt the analysis, but the final result can feel as disappointing as pilau cooked without the right spices; it may fill the stomach, but it certainly does not excite the taste buds, much like solving Physics problems without properly identifying the forces.

Weight, W

Weight is the gravitational force exerted on a body due to the Earth's gravitational field. It is one of the most fundamental forces encountered in Particle Mechanics because it acts on virtually every object near the Earth's surface.

The magnitude of the weight is given by: $W = mg$

Where:

m = mass of the body

g = acceleration due to gravity (approximately 9.8 m/s^2 near the Earth's surface).

Recognising weight correctly in free-body diagrams is essential. It is often the first force to identify, and errors in its direction or magnitude can lead to incorrect equilibrium analysis.

It is very important to understand that:

Weight always acts *vertically downward*, towards the centre of the Earth, regardless of the body's orientation or motion. *Even when an object rests on an inclined plane, hangs from a string, or moves through the air, its weight still acts vertically downward.*

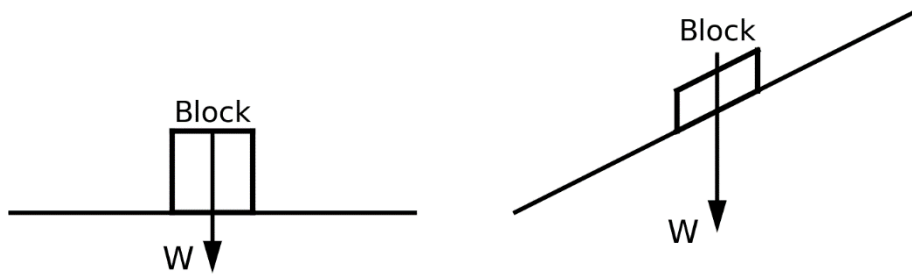


Figure: The weight of a body always acts vertically downward towards the centre of the Earth, regardless of whether the body rests on a horizontal surface, an inclined plane, or is suspended in space. Surface orientation does not affect the direction of weight.

Components of Weight Parallel and Normal to an Inclined Plane

When a body rests on an inclined plane, its weight still acts vertically downward toward the centre of the Earth. However, because the surface is tilted, it is often more convenient to analyse this single gravitational force by resolving it into two components: one **parallel to the plane** and one **perpendicular (normal) to the plane**. This approach simplifies equilibrium analysis because each component directly relates to the forces (friction and normal reaction) that usually oppose them: friction acts parallel to the surface (opposite to the weight's parallel component), while the normal reaction acts perpendicular to it (opposite to the weight's perpendicular component).

To have better understanding of this, consider a body of weight, $\mathbf{W} = m\mathbf{g}$ placed on a plane inclined at an angle θ to the horizontal. Resolving the weight into components relative to the plane gives:

- **Component parallel to the plane** = $W\sin\theta = mg\sin\theta$
- **Component perpendicular to the plane** = $W\cos\theta = mg\cos\theta$

These expressions arise from simple right-triangle geometry formed when the vertical weight vector is resolved relative to the inclined surface as shown in the figure.

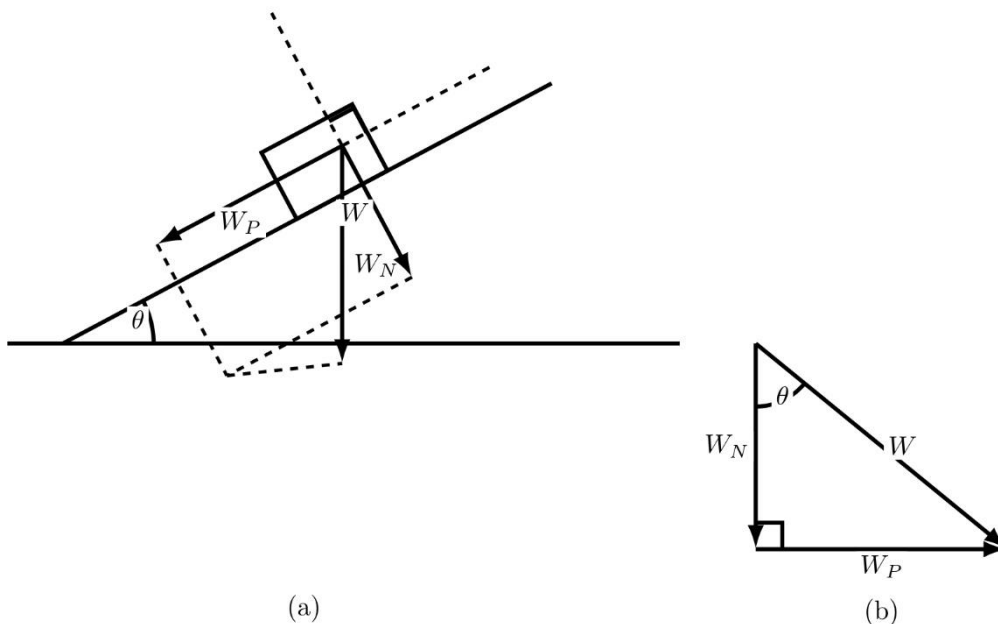


Figure: (a) Weight (W) of a body on an inclined plane resolved into components parallel to the plane (W_P) and normal (perpendicular) to the plane (W_N). (b) Component triangle showing the geometric relationship between the weight and its parallel and normal components.

From the figure it is clearly understood that when a body rests on an inclined plane, its weight $W = mg$ acts vertically downward. For analysis relative to the plane, we resolved this single force into two mutually perpendicular components:

- 1) W_P : component parallel to the plane (down the slope).
- 2) W_N : component normal (perpendicular) to the plane (into the plane).

From diagram (a), the three vectors form a right-angled triangle: the hypotenuse is W , and other sides are W_N and W_P .

Before we proceed, let us first familiarise ourselves with the key geometric features of the diagram.

1. The plane makes an angle θ with the horizontal.
 2. The normal line is perpendicular to the plane, so the normal is tilted by θ from the vertical.
 3. Since W is vertical, the angle between W and the normal direction is θ .
- So, in the component triangle (diagram b), θ is the angle between W and W_N .

Now, in the triangle:

Hypotenuse = W , Adjacent to θ = W_N , Opposite to θ = W_P

Therefore:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{W_N}{W}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{W_P}{W}$$

Hence:

$$W_N = W\cos\theta = mg\cos\theta = \text{Normal (perpendicular) component of the weight}$$

$$W_P = W\sin\theta = mg\sin\theta = \text{Parallel component of the weight}$$

An important conceptual point is that resolving weight does **not create new forces**. The weight remains a single gravitational force acting vertically downward. The components are simply a convenient way of describing how that force influences motion relative to the plane.

Understanding these components is essential in Particle Mechanics because many real systems: vehicles on slopes, ladders against walls, cables supporting loads, or objects resting on ramps; depend on correctly analysing forces relative to inclined surfaces. Once the weight is resolved appropriately, applying the equilibrium equations: $\Sigma F_{\text{parallel}} = 0$ and $\Sigma F_{\text{perpendicular}} = 0$ becomes straightforward and systematic.

In practice, *whenever an inclined plane appears in a problem, resolving the weight parallel and perpendicular to the plane is usually the first and most important step toward a clear and accurate solution.*

Normal Reaction, R (or R_n)

The **normal reaction** is the force exerted by a surface on a body in contact with it. It arises due to the interaction between the surfaces and acts to prevent the two bodies from passing through each other and thus they do not occupy the same space. In equilibrium problems, the normal reaction is one of the most frequently encountered contact forces.

The direction of the normal reaction is always perpendicular (**normal**) to the surface of contact. This is an important point: *the normal reaction is not necessarily vertical*. It becomes vertical **only** when the surface itself is horizontal. When a body rests on an inclined plane, for example, the normal reaction acts perpendicular to that plane rather than vertically upward.

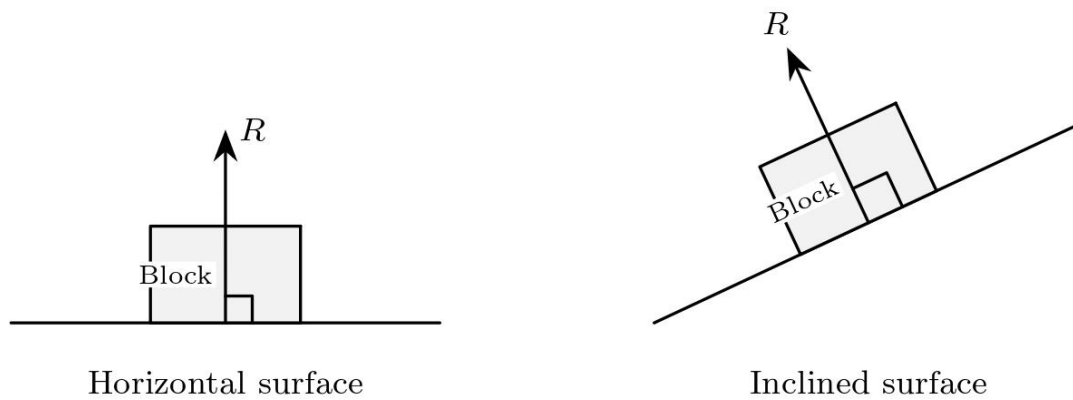


Figure: The normal reaction always acts perpendicular to the surface of contact. Its direction depends on the orientation of the surface and is not always vertical.

Correct identification of the normal reaction in free-body diagrams is essential for accurate equilibrium analysis. Confusing its direction with that of weight is a common mistake, especially in problems involving inclined surfaces.

It should also be remembered that the normal reaction is a responsive force. It adjusts its magnitude according to the situation. In some cases, it may equal the weight of the body, but this is not a general rule; additional forces acting on the body can increase or decrease its value. **For example,** if a downward push is applied in addition to weight, the normal reaction increases; if a lifting force acts upward, the normal reaction decreases.

The following points summarise the key features of the normal reaction:

1. It always acts perpendicular to the surface of contact.
2. It exists only when there is physical contact between the body and the surface.
3. Its magnitude is not necessarily equal to the weight of the body. Equality occurs only in special cases, such as when a body rests on a horizontal surface with no other vertical forces acting.
4. For a body of mass, m resting on an inclined plane at an angle θ to the horizontal, **the normal reaction is $R = mg\cos\theta$** (normal component of weight) provided that no other forces act in that perpendicular direction.

Tension, T

In Particle Mechanics, **tension** is one of the most common forces in equilibrium problems. Tension appears whenever a body is pulled, supported, or suspended by a string, rope, cable, chain, or wire. From hanging signboards and ceiling lamps to suspension bridges, cranes, elevators, and even the strap of your school bag, tension forces are quietly working to maintain balance.

Concisely, tension is the **pulling force** transmitted through a **stretched** string, rope, cable, or wire.

A key feature of tension is its direction. Unlike the normal reaction, which pushes perpendicular to a surface, tension always acts **along the length** of the string or cable and pulls **away from the body** to which it is attached. Therefore, when representing tension in a free-body diagram, always draw the arrow pointing **away from (NOT toward) the body along the string or cable** (see the figure). This reflects the physical fact that a string can only pull, not push. *Drawing the arrow toward the body may suggest compression rather than tension and can lead to incorrect interpretation of the forces acting on the system.*

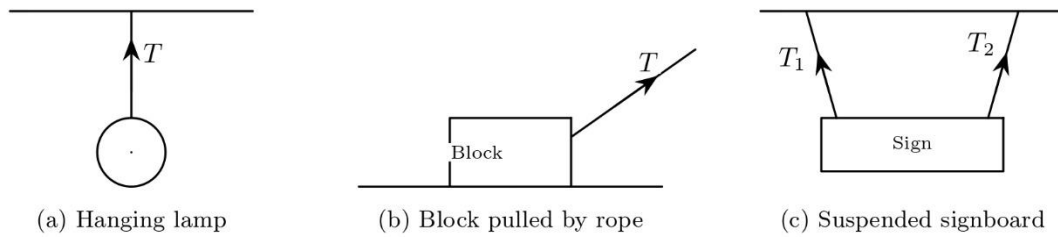


Figure: In all cases, tension acts along the string or cable and always pulls away from the body.

Modelling assumptions used in A-Level mechanics

To make problems solvable, we often use idealised models:

- **Light (massless) string:**

The string's weight is negligible, so we ignore its weight in the analysis.

- **Inextensible string:**

The string does not stretch, so its length remains constant.

- **Smooth pulley (if present):**

No friction between string and pulley, so the tension remains the same on both sides of the pulley.

Under these ideal conditions, the tension has the same magnitude throughout a continuous string.

Important clarifications about tension

- **Tension is not automatically equal to weight.**

For a hanging object at rest, tension equals weight only if the object is supported by a single vertical string and no other vertical forces act.

- **More than one string means tensions can share the load.**

For example, if a body is supported by two strings at angles, each string provides a tension that contributes a **vertical component** supporting the weight.

- **Tension acts at the point of attachment.**

In particle mechanics we often draw it as acting on the particle, but the physical meaning is “**the string pulls the body at the attachment point.**”

- **Tension exists only when the string is taut**

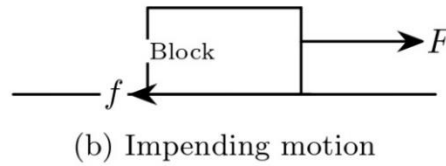
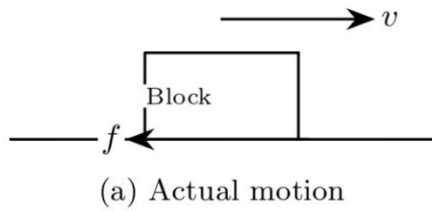
If the string becomes slack, it can no longer exert a pulling force; therefore, the tension effectively becomes zero.

Friction Force, f

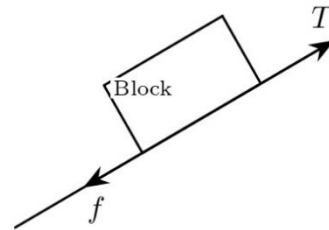
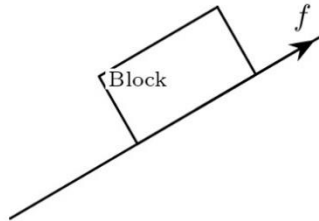
In Particle Mechanics, **friction** is another very common force encountered in many physical situations, including both equilibrium and motion. From walking without slipping and vehicles braking safely to ladders resting against walls and objects remaining stationary on inclined surfaces, friction plays an important role in maintaining stability, control, and balance in everyday life.

Friction is a contact force that arises when two surfaces touch and there is relative motion, or a tendency for relative motion, between them. It acts along the surface of contact and always opposes the direction in which motion occurs or would occur.

A crucial feature of friction is its direction. **Friction always acts parallel to the surface and opposite to the direction of relative motion or the tendency to move.** Therefore, when representing friction in a free-body diagram, always draw the friction force parallel to the surface and opposite to the direction of motion or impending motion (see the figure).



Friction on a horizontal surface: Friction acts along the surface of contact and always opposes either the actual motion of the body or its tendency to move.



Friction on an inclined plane: Friction acts parallel to the plane and opposes the tendency of motion, acting up the plane if the body tends to slide down and down the plane if the body is pulled upward.

Two main types of friction are commonly considered in Particle Mechanics:

1. Static friction

Static friction acts when two surfaces are in contact but there is **no relative motion** between them. Its role is to prevent motion from starting.

An important property of static friction is that its magnitude adjusts itself as needed (up to a limiting value) to prevent motion. This is why a gently pushed object may remain at rest as friction increases just enough to maintain equilibrium.

The adjustment obeys the following condition:

$$f \leq \mu R$$

Where:

f is frictional force.

μ is coefficient of static friction.

R is normal reaction between the surfaces.

It is worth to understand that when static friction reaches its maximum possible value just before motion begins, it is called **limiting friction** and the body is said to be at **limiting equilibrium**.

At this point:

$$f = \text{limiting friction} = \mu R$$

Beyond this point, motion starts and static friction is no longer existing.

2. Kinetic (sliding) friction

Once motion begins, static friction no longer operates because the surfaces are no longer at rest relative to each other. However, a resisting force still exists that opposes the motion. This force is known as **kinetic (or sliding) friction**. Its magnitude is usually slightly smaller than the maximum static friction (limiting friction), which explains why an object often moves more easily once it has started sliding.

Important clarifications about friction**1. Friction always acts parallel to the contact surface and never perpendicular to it**

Although friction is related to the normal reaction (the perpendicular contact force), friction itself does not act perpendicular to the surface. Instead, it always acts parallel to the surface of contact.

2. Friction is not always present

In many mechanics models, surfaces may be treated as smooth (frictionless). In such cases, friction is assumed to be zero to simplify analysis.

3. Friction does not always equal μR

The relation $f = \mu R$ applies only at limiting equilibrium. In many situations, friction is less than μR .

4. Friction opposes motion or attempted motion

It does not necessarily oppose an applied force directly; it opposes the resulting motion tendency.

5. Friction exists only when surfaces are in contact

If contact is lost, friction immediately becomes zero.

Do not invent friction: A conceptual warning

Friction should never be invented simply to “make forces balance.” It must have a physical cause: contact between surfaces with a tendency to slide. If no such tendency exists, friction is zero. For example: *A book resting on a horizontal table experiences only its weight downward and the normal reaction upward. Since there is no horizontal force tending to cause motion, no friction is required, and the frictional force is therefore zero.*

Combining All Four Forces: W, R, T, and f

Up to this point, we have examined the standard forces of Particle Mechanics one at a time:

- **Weight (W)** acting vertically downward due to gravity,
- **Normal reaction (R)** arising from surface contact and acting perpendicular to that surface,
- **Tension (T)** transmitted through strings, ropes, or cables and always pulling away from the body, and
- **Friction (f)** acting parallel to the surface and opposing relative motion or the tendency of motion.

Individually, each force is straightforward. But real physical situations rarely present forces one by one. A block may rest on a rough inclined plane while attached to a string, a ladder may lean against a wall with friction preventing slipping, or a suspended load may simultaneously experience tension, weight, and contact forces. Understanding equilibrium therefore requires seeing **how these forces work together**.

This integration marks an important step: you are no longer just recognising forces; you are analysing complete physical systems.

A unified view of equilibrium

When all four forces appear together, three fundamental direction rules must always be remembered:

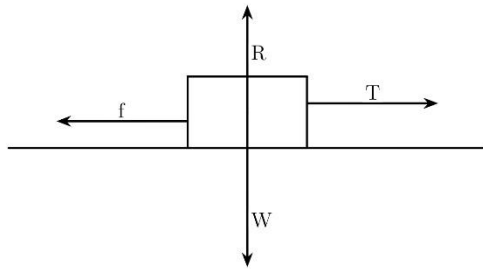
Rule 1: Weight (W) always acts vertically downward, regardless of the surface orientation.

Rule 2: Normal reaction (R) always acts perpendicular to the contact surface.

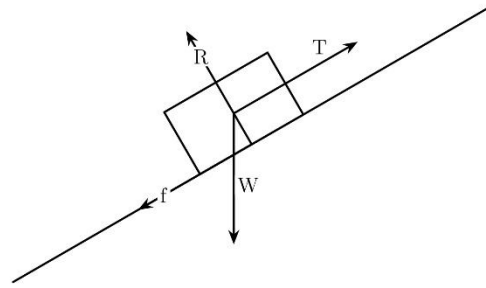
Rule 3: Tension (T) always acts along the string or cable, pulling away from the body.

Rule 4: Friction (f) always acts parallel to the surface and opposes actual or impending motion.

These directional rules are not optional; they come directly from the physical origin of each force. Correct free-body diagrams depend on respecting these directions.



(a) Horizontal surface



(b) Inclined plane

Forces on a block on horizontal and inclined surfaces: Weight (W), normal reaction (R), tension (T), and friction (f) acting on a body.

Have you noticed this interesting fact?

Weight comes from gravity,

Reaction comes from contact,

Tension comes from connection,

Friction comes from surface interaction.

Different physical origins; yet all cooperate to maintain equilibrium.

Physics often looks complicated only until the forces are identified correctly. Once the forces are clear, equilibrium analysis becomes systematic rather than mysterious.

Why this integration matters?

Combining W , R , T , and f allows us to analyse many practical situations, including:

1. Bodies on rough inclined planes supported by strings,
2. Ladders resting against walls,
3. Suspended structures in contact with surfaces,
4. Engineering systems where stability depends on multiple interacting forces.

These are not artificial textbook problems; they reflect how structures remain safe and stable in everyday life.

Congratulation!

You have now completed one of the most important subtopics in Particle Mechanics. Mastery of free-body diagrams, the two equilibrium equations ($\Sigma F_x = 0$ and $\Sigma F_y = 0$), and the standard force models (weight, normal reaction, tension, and friction) gives you powerful tools for analysing forces with confidence. These ideas will not only make Particle Mechanics clearer and more enjoyable, but will also support your understanding across many other areas of Physics where balance, motion, and interaction are involved.

From this stage onward, equilibrium problems become less about memorising formulas and more about thinking clearly. And that, ultimately, is the goal of Advanced Physics: not just solving problems, but understanding why systems remain balanced in the first place.

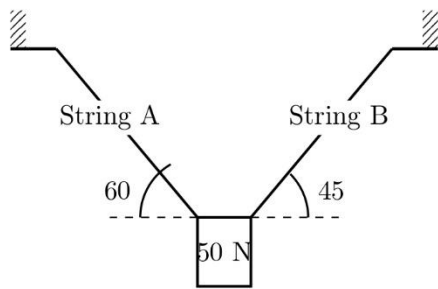
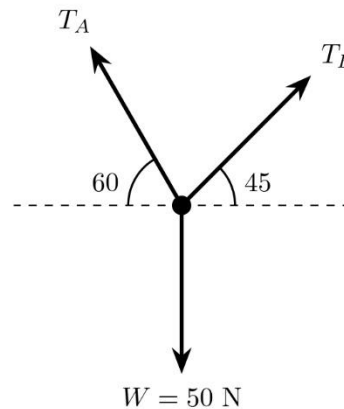
With the ideas now simmering nicely, let us serve them properly through a few worked examples and enjoy the flavour of physics in action.

BINDER Example 9

A 50N weight is suspended at rest from a junction by two light strings. **String A** is inclined at 60° to the horizontal and **string B** is inclined at 45° to the horizontal (both above the horizontal). Calculate the tension in each string.

Solution

To be at rest (equilibrium), the two strings must pull upward and outward from opposite sides of the load as shown in the **situation diagram**.

Situation Diagram**Free Body Diagram**

Resolving each force to horizontal and vertical component (refer to the free body diagram):

For string A:

$$\theta = \text{angle measured from positive x - axis} = 180^\circ - 60^\circ = 120^\circ$$

$$\text{Horizontal component} = (T_A)_x = T_A \cos 120^\circ = -0.5T_A$$

$$\text{Vertical component} = (T_A)_y = T_A \sin 120^\circ$$

For string B:

$$\theta = \text{angle measured from positive x - axis} = 45^\circ$$

$$\text{Horizontal component} = (T_B)_x = T_B \cos 45^\circ$$

$$\text{Vertical component} = (T_B)_y = T_B \sin 45^\circ$$

For weight:

Weight always acts vertically downward; therefore, its horizontal component is zero, while its vertical component is equal in magnitude to the weight but directed downward (**-50N**).

This can also be shown using the **standard resolution method** as follows:

$$\theta = \text{angle measured from positive x - axis} = 270^\circ$$

$$\text{Horizontal component} = W_x = W \cos 270^\circ = 0$$

$$\text{Vertical component} = W_y = W \sin 270^\circ = -W = -50\text{N}$$

$$\text{From equilibrium equations: } \Sigma F_x = 0 \text{ and } \Sigma F_y = 0$$

Then;

$$\Sigma F_x = -0.5T_A + T_B \cos 45^\circ + 0 = 0 \text{ or}$$

$$-0.5T_A + T_B \cos 45^\circ = 0 \dots \dots (i)$$

And;

$$\Sigma F_y = T_A \sin 120^\circ + T_B \sin 45^\circ - 50 = 0 \text{ or}$$

$$T_A \sin 120^\circ + T_B \sin 45^\circ = 50 \dots \dots (ii)$$

Solving (i) and (ii) simultaneously gives: $T_A = 36.6\text{N}$; $T_B = 25.9\text{N}$

Tension in string A is 36.6N.

Tension in string B is 25.9N.

Making Sense of the Answer: The tensions are unequal because the strings are inclined at different angles. The string at 60° (string A) provides a larger vertical component, so it carries more of the weight. Their vertical components balance the 50N load, while the horizontal components cancel maintaining equilibrium.

Thinking Like a Physicist: Instead of guessing signs, **measure angles consistently from the positive x-axis**. For example, using 120° rather than 60° automatically gives the correct horizontal and vertical components. This systematic approach reduces sign errors and keeps equilibrium analysis clear.

Method Insight: In this first example, the weight was resolved explicitly using the general vector method to illustrate the procedure clearly. In most later problems, since weight acts vertically downward, we will usually write its components directly without repeating the full resolution each time.

BINDER Example 10

A 5kg block rests on a horizontal table. A horizontal force of 12N is applied to the block. The coefficient of static friction between block and table is $\mu = 0.4$.

- Determine whether the block remains in equilibrium.
- Find the magnitude of the frictional force acting on the block.

$$\text{Take } g = 9.8\text{m/s}^2$$

Solution

Normal reaction: $R = mg = 5\text{kg} \times 9.8\text{N/kg} = 49\text{N}$

Limiting friction: $f = \mu R = 0.4 \times 49\text{N} = 19.6\text{N}$

But the applied force, $F = 12\text{N}$ which is smaller than the limiting friction.

- Since the applied force (12N) is smaller than the limiting friction (19.6N), the static friction is sufficient to prevent motion and hence the block remains in equilibrium.
- When the block is at equilibrium, frictional force equals the applied force (12N) and hence the frictional force acting on the block is 12N.

Making Sense of the Answer: Static friction adjusts itself to oppose the applied force up to its maximum value. Since the applied force (12N) is less than the maximum possible static friction (19.6N), the block does not move, and friction simply matches the applied force to maintain equilibrium.

Thinking Like a Physicist: Do not assume friction is always μR . That expression gives only the maximum static friction. Always compare the applied force with this maximum first; if it is smaller, the actual friction equals the applied force, not μR .

REAL Example 11

Kipanga is pulling his overloaded school bag along the ground using a strap. He can pull horizontally or at an angle upward. Explain why pulling at an angle upward might make it easier to move the bag, even though some of the force is "wasted" going upward instead of forward.

Solution

Pulling the bag at an upward angle reduces the **normal reaction (R)** between the bag and the ground. When Kipanga pulls partly upward, that upward component of the pulling force slightly lifts the bag, so the ground presses on it less strongly.

Since friction depends on the normal reaction ($f = \mu R$), a smaller normal reaction means **less frictional force resisting motion**. Although part of the pulling force is directed upward rather than forward, the reduction in friction usually outweighs this "loss," making the bag easier to move.

Making Sense of the Answer: Physics often involves trade-offs. Sacrificing some forward force to reduce friction can make pulling easier overall. Efficiency is not always about pushing hardest in one direction.

Thinking Like a Physicist: Real-world situations often involve balancing competing effects. The best pulling angle can be analysed mathematically at advanced levels, but the key idea is simple: reducing resistance can be more effective than simply increasing effort.

REAL Example 12

During a practical lesson, **Kipute** places her Physics textbook on a desk that is tilted at a small angle. The book stays in place.

Kipanga: "The book must be glued to the desk! Otherwise gravity would pull it down."

Kipute: "Don't be silly! There's no glue. But Kipanga has a point, Mr. Akilikubwa; why doesn't the book slide?"

Mr. Akilikubwa: (smiling) "Ah! You have discovered friction, the invisible guardian. No glue needed. The desk surface and the book touch, and friction acts like tiny invisible hands holding the book in place, opposing the tendency to slide."

Kipanga: "So friction is always there?"

Mr. Akilikubwa: "Not quite! Friction only appears when there is a tendency to move. Place the same book on a perfectly horizontal desk....., no tendency to slide, so friction takes a break and remains zero. Tilt the desk slightly, and friction wakes up just enough to prevent motion. Tilt it more, and friction increases up to its maximum strength. Tilt it too much, and friction loses the battle; the book slides. Friction is a responsive force, not a permanent one."

Kipute: "So it's like a lazy security guard who only works when needed?"

Mr. Akilikubwa: (laughing) "Exactly! And just like a lazy guard, once the thief (motion) escapes and the book starts sliding, the guard (static friction) stops trying and switches to a different mode called kinetic friction, which is usually weaker."

Question: Based on this dialogue, explain:

- Why friction does not act on a book resting on a horizontal table.
- Why friction increases as the desk tilt increases (up to a limit)
- What happens when the book finally starts sliding.

In (b) and (c), justify your explanation with relevant mathematical equations.

Solution

- When the book rests on a horizontal table, weight acts vertically downward and is balanced by the normal reaction acting vertically upward. There is no force component parallel to the surface that would tend to cause motion. Since friction opposes attempted motion, and there is no such attempt, friction is zero.
- As the desk tilts, weight develops a component parallel to the surface (down the slope). This component tries to pull the book downward. Static friction responds by acting up the slope with exactly the right magnitude to prevent motion. The steeper the tilt, the larger the parallel component of weight, so friction must increase to maintain equilibrium up to its maximum possible value (limiting friction, $f = \mu R$).

Mathematical justification:

As the desk tilts by angle θ :

- Weight develops a component parallel to surface: $W \sin \theta$ (down the slope)
- This parallel component tries to pull the book downward.
- Static friction responds by acting up the slope: $f = W \sin \theta$ (upward)
- The steeper the tilt (larger value of $\sin \theta$), the larger $W \sin \theta$ becomes, so friction must increase to maintain equilibrium.

However, friction has a maximum value: $f_{\max} = \mu R = \mu W \cos \theta$

Friction can increase up to this limit. Beyond this, equilibrium breaks.

- When tilt becomes too steep, the parallel component of weight exceeds the maximum static friction. At this point, static friction can no longer prevent motion, and the book begins to slide. Once motion starts,

kinetic (sliding) friction takes over, which is typically smaller than the maximum static friction and thus the book continues to slide easily.

Mathematical justification:

When tilt becomes too steep: $W\sin\theta > \mu W\cos\theta$ or $\tan\theta > \mu$ (by dividing $W\cos\theta$ both sides)

At this point:

- Static friction reaches its maximum but can no longer prevent motion. So the book begins to slide.
- Static friction no longer operates (surfaces are moving relative to each other).
- Kinetic (sliding) friction takes over: $f_k = \mu_k R$

As $\mu_k < \mu$ (kinetic friction is weaker than maximum static friction), once the book starts sliding, it continues more easily.

Making Sense of the Answer: Friction is not a fixed force; it adjusts itself as needed (up to a limit). Understanding this adaptive behavior prevents the common mistake of assuming friction always equals μR . The equation $f = \mu R$ applies only at limiting equilibrium, just before motion begins.

Thinking Like a Physicist: Mr. Akilikubwa's "lazy security guard" analogy captures an important truth: friction is a passive, responsive force. It doesn't initiate anything; it merely reacts to prevent motion. This is fundamentally different from active forces like applied pushes or gravity, which act regardless of the situation.

BINDER Example 13

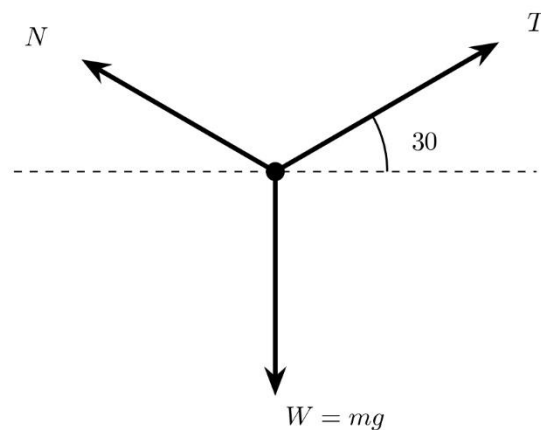
A 10kg block rests on a smooth (frictionless) inclined plane that makes an angle of 30° with the horizontal. The block is held in place by a string parallel to the plane. Take $g = 9.8\text{m/s}^2$.

Calculate:

(a) the normal reaction from the plane and (b) the tension in the string

Solution

Forces acting on the block are shown in the following free body diagram.



Resolving weight into components:

Component parallel to plane (down):

$$W_{\text{parallel}} = mg\sin\theta = 10 \times 9.8\sin 30^\circ = 49\text{N}$$

Component perpendicular to plane (into surface):

$$W_{\text{perp}} = mg\cos\theta = 10 \times 9.8\cos 30^\circ = 98\text{N}\cos 30^\circ$$

Forces acting perpendicular to plane:

- Normal reaction (**away** from surface), R .
- Perpendicular component of weight (**into** surface), $98\text{N}\cos 30^\circ$.

Since the two forces act in opposite directions and the block is at equilibrium (held in place):

$$R = 98\text{N}\cos 30^\circ = 84.9\text{N}$$

Forces acting parallel to plane:

- Tension (upward), T .
- Parallel component of weight (downward), $W_{\text{parallel}} = 49\text{N}$

Again, the two forces act in opposite directions and the block is still at equilibrium. So:

$$T = 49\text{N}$$

(a) The normal reaction is 84.9N .

(b) The tension in string is 49N .

Making Sense of the Answer: The normal reaction (84.9N) is less than the weight (98N) because the plane is tilted. Only the component of weight perpendicular to the plane is balanced by R . The weight component parallel to the plane (49N) tries to pull the block down, so the string must provide equal tension upward along the plane to maintain equilibrium.

Thinking Like a Physicist: On an incline, always resolve weight into components parallel and perpendicular to the surface. This transforms a two-dimensional problem into two separate one-dimensional problems. The steeper the incline, the larger the parallel component (harder to hold) and the smaller the perpendicular component (smaller normal reaction).

BINDER Example 14

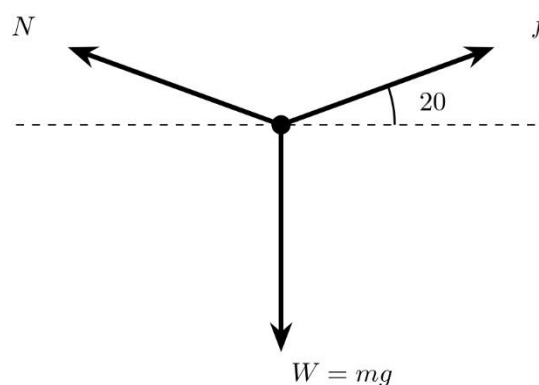
A block of mass 20kg rests on a rough inclined plane inclined at 20° to the horizontal. The block is on the verge of sliding down the plane. The coefficient of static friction between the block and the plane is $\mu = 0.364$. Take $g = 9.8\text{m/s}^2$.

Calculate:

- The normal reaction between the block and the plane.
- The frictional force acting on the block.
- Verify that the block is in limiting equilibrium.

Solution

Forces acting on the block are shown in the following free body diagram.



Resolving weight into components:

Component parallel to plane (downward):

$$W_{\text{parallel}} = mg\sin\theta = 20 \times 9.8\sin 20^\circ = 196\text{N}\sin 20^\circ$$

Component perpendicular to plane (into surface):

$$W_{\text{perp}} = mg\cos\theta = 20 \times 9.8\cos 20^\circ = 196N\cos 20^\circ$$

Forces acting perpendicular to plane:

- Normal reaction (**away** from surface), R .
- Perpendicular component of weight (**into** surface), $196N\cos 20^\circ$.

Since the two forces act in opposite directions and the block is at equilibrium (it is just on verge of sliding, not yet moving):

$$R = 196N\cos 20^\circ = 184.15N$$

(a) The normal reaction is 184.15N.

Forces acting parallel to plane:

- Friction force (upward), f .
- Parallel component of weight (downward), $W_{\text{parallel}} = 196N\sin 20^\circ$.

Again, the two forces act in opposite directions and the block is still at equilibrium. So:

$$f = 196N\sin 20^\circ = 67.04N$$

(b) The frictional force is 67.04N.

Limiting equilibrium is found when: $f_{\text{max}} = \mu R$

Substituting $f_{\text{max}} = 0.364 \times 184.15N = 67.03N \approx 67.04N = f$ (found in (b))

(c) Since the actual frictional force is equal to the limiting friction, the block is in limiting equilibrium.

Alternative verification

Using the condition $\mu = \tan\theta$ at limiting equilibrium.

$$\tan 20^\circ = 0.364$$

$$\text{Given } \mu = 0.364$$

Since $\mu \approx \tan\theta$, limiting equilibrium is confirmed.

Making Sense of the Answer: At limiting equilibrium, friction reaches its maximum value, which exactly balances the down-slope component of weight. Any slight increase in angle or decrease in μ would cause sliding. This is why $\mu = \tan\theta$ is a useful relationship.

Thinking Like a Physicist: Limiting equilibrium represents the boundary between static equilibrium and motion. Understanding this threshold is crucial in engineering: roads must have sufficient friction (high μ) or low slope (small θ) to prevent vehicles from sliding, especially in rain when μ decreases.

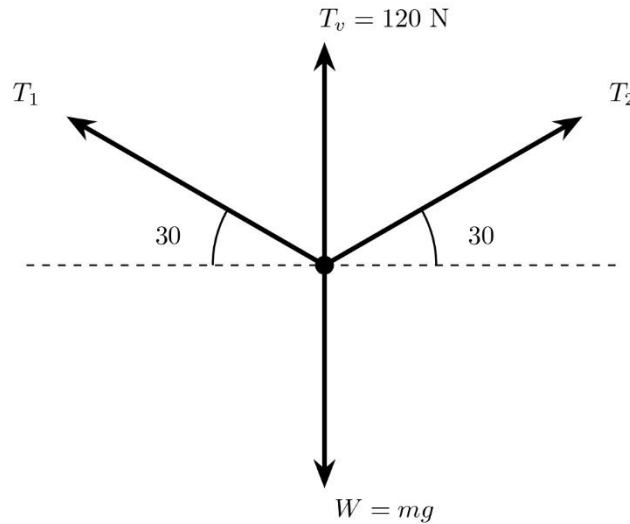
HOT Example 15

A 20kg signboard hangs from a vertical cable attached to its center and two side cables. Each side cable is attached to the edge of the signboard and makes 30° with the horizontal, pulling outward to wall supports. If the vertical cable has tension 120N, calculate the tension in each side cable.

$$\text{Take } g = 9.8\text{m/s}^2.$$

Solution

All forces acting on the signboard are shown in the following free body diagram.



Resolving each force to horizontal and vertical component:

For T_1 :

$$\theta = \text{angle measured from positive x - axis} = 180^\circ - 30^\circ = 150^\circ$$

$$\text{Horizontal component} = T_1 \cos 150^\circ = -T_1 \cos 30^\circ$$

$$\text{Vertical component} = T_1 \sin 150^\circ = T_1 \sin 30^\circ$$

For T_v :

$$\text{Horizontal component} = 0$$

$$\text{Vertical component} = 120\text{N}$$

For T_2 :

$$\theta = \text{angle measured from positive x - axis} = 30^\circ$$

$$\text{Horizontal component} = T_2 \cos 30^\circ$$

$$\text{Vertical component} = T_2 \sin 30^\circ$$

For weight:

$$\text{Horizontal component} = 0$$

$$\text{Vertical component} = -mg = -20 \times 9.8\text{N} = -196\text{N}$$

$$\text{From equilibrium equations: } \Sigma F_x = 0 \text{ and } \Sigma F_y = 0$$

Then;

$$\Sigma F_x = -T_1 \cos 30^\circ + T_2 \cos 30^\circ = 0 \text{ or}$$

$$T_2 \cos 30^\circ = T_1 \cos 30^\circ; \mathbf{T_2 = T_1}$$

And;

$$\Sigma F_y = T_1 \sin 30^\circ + 120 + T_2 \sin 30^\circ - 196 = 0 \text{ or}$$

$$T_1 \sin 30^\circ + T_2 \sin 30^\circ = 76$$

$$\text{But } T_2 = T_1$$

$$T_1 \sin 30^\circ + T_1 \sin 30^\circ = 76; 2T_1 \sin 30^\circ = 76 \text{ or } T_1 = 76\text{N}$$

Tension in each side cable is 76N.

Making Sense of the Answer: The signboard weighs 196N downward, with 120N already supported by the vertical cable. The remaining 76N is shared by the two side cables through their vertical components. The side cables also provide horizontal stabilization, preventing the signboard from swinging.

Thinking Like a Physicist: Real engineering often involves over-constraining systems for safety. A signboard could hang from a single cable, but using three distributes the load and provides backup support. If one side cable fails, the board may tilt, but it does not fall immediately.

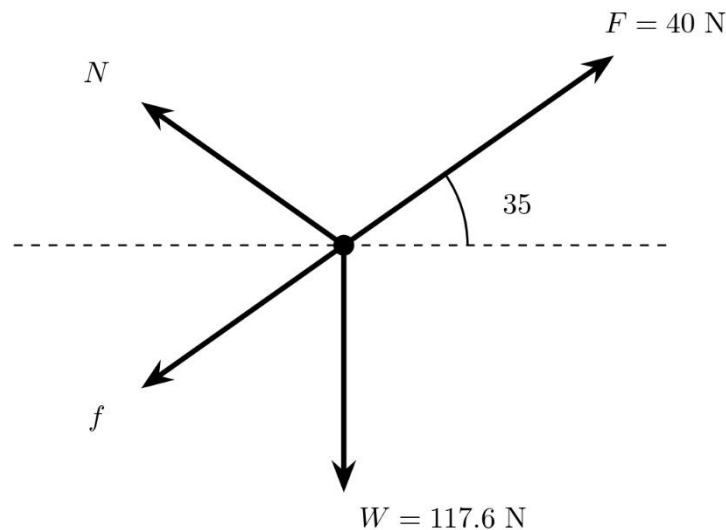
HOT Example 16

A 12kg block rests on a rough plane inclined at 35° to the horizontal. A force of 40N acts on the block parallel to and up the plane. The block is in equilibrium. The coefficient of static friction between block and plane is $\mu = 0.6$. Take $g = 9.8\text{m/s}^2$.

- Calculate the normal reaction.
- Determine the magnitude and direction of the frictional force.
- Verify that the block is not at limiting equilibrium.

Solution

All forces acting on the block are shown in the following free body diagram.



- Forces acting perpendicular to plane are only normal reaction, **R** and the perpendicular component of weight.

Since the two forces act in opposite directions;

$$R = mg\cos\theta = 12 \times 9.8 \times \cos 35^\circ = 96.3\text{N}$$

The normal reaction is 96.3N.

- Forces acting parallel to the plane.

- Parallel weight component (downward): $W_p = -mg\sin\theta = -117.6\sin 35^\circ = -67.4\text{N}$
- Applied force (upward): $F = 40\text{N}$
- Frictional force (unknown direction): f

Applying equilibrium equation parallel to the plane:

$$-67.4\text{N} + 40\text{N} + f = 0; f = 27.4\text{N} \text{ (upward as it is positive)}$$

Direction determination: since weight component down the plane (67.4N) is larger than the applied force (40N) up the plane, the tendency is to slide **down** the plane, so friction must act **up the plane** to prevent the downward motion.

The frictional force is 27.4N up the plane.

(c) Limiting friction: $f_{\max} = \mu R = 0.6 \times 96.3 = 57.8\text{N}$

But actual friction required for equilibrium is 27.4N , which is less than 57.8N .

Since the actual frictional force is less than the limiting friction, the block is not at limiting equilibrium.

Making Sense of the Answer: *The applied force (40N) helps support the block but isn't strong enough to overcome the down-slope pull (67.4N). Friction makes up the difference (27.4N). Together, the applied force and friction provide 67.4N up the slope, exactly balancing the down-slope weight component. The large safety margin (57.8N maximum vs 27.4N actual) means the block is very stable.*

Thinking Like a Physicist: *This problem illustrates an important principle: **friction direction depends on motion tendency, not on which forces are present**. Always first determine which way the block would move if friction were absent, then friction acts opposite to that tendency. In this case, removing friction would cause downward motion ($67.4\text{N} > 40\text{N}$), so friction acts upward.*

With these worked examples now neatly packed away, let us move on and meet the next subtopic; it has been waiting patiently to join the conversation.

MOTION OF CONNECTED BODIES

Introduction

Imagine two students connected by a rope during a tug-of-war revision session. One pulls confidently, the other resists bravely, and suddenly both start moving together, not because either planned it, but because the forces refused to stay balanced. Physics describes such situations as **motion of connected bodies**.

In the previous subtopics, you mastered equilibrium for single objects: books on tables, blocks on inclines, and signboards hanging from cables. But the real world rarely presents isolated objects. Instead, systems work together: elevators carrying people, cranes lifting loads, cars towing trailers, and even your school bag hanging from a desk hook while pulling on the strap.

When two or more objects are physically connected by strings, cables, ropes, chains, or rigid bars, they form what we call a **connected bodies system**. The fascinating aspect of such systems is this: although the objects may have different masses and experience different forces, they often share the same motion. A hanging mass pulls on a string that accelerates a block on a table. The string transmits force between them, and both objects accelerate together (though in different directions).

What makes this different from what we have studied?

Unlike equilibrium situations where forces cancel perfectly, connected bodies often reveal what happens when that balance is slightly disturbed.

Previously, we analysed equilibrium where acceleration was zero. Now we extend our understanding to connected systems where acceleration may occur, but the key principle remains: we apply Newton's laws systematically to each body, paying special attention to how the connection (string or cable) transmits force through tension.

Instead of the equilibrium condition: $\Sigma F = 0$; we now apply Newton's second law: $\Sigma F = ma$.

Thus, this topic does not replace equilibrium principles; it builds directly on them.

Three common configurations:

Connected bodies can move in three main ways:

1. **Vertical motion:** Bodies connected by a string over a pulley, where one hangs and pulls the other vertically, as in an elevator and its counterweight system.
2. **Horizontal motion:** Bodies connected by a string or tow-bar on a horizontal surface, where one pulls the other along the same plane, similar to a car towing a trailer.
3. **Inclined plane motion:** Bodies connected by a string over a pulley at the top of an inclined plane, where one moves up or down the slope while the other hangs vertically, as seen in some mountain rescue lifting systems.

Let us explore each configuration systematically, building your understanding step by step.

Connected Bodies in Vertical Motion

One of the simplest connected-body systems consists of two masses joined by a light string passing over a smooth pulley. Such a system is often called an **Atwood-type machine**.

Consider two masses m_1 and m_2 connected by a light, inextensible string that passes over a smooth (frictionless) pulley fixed to the ceiling. The masses hang freely on either side of the pulley.

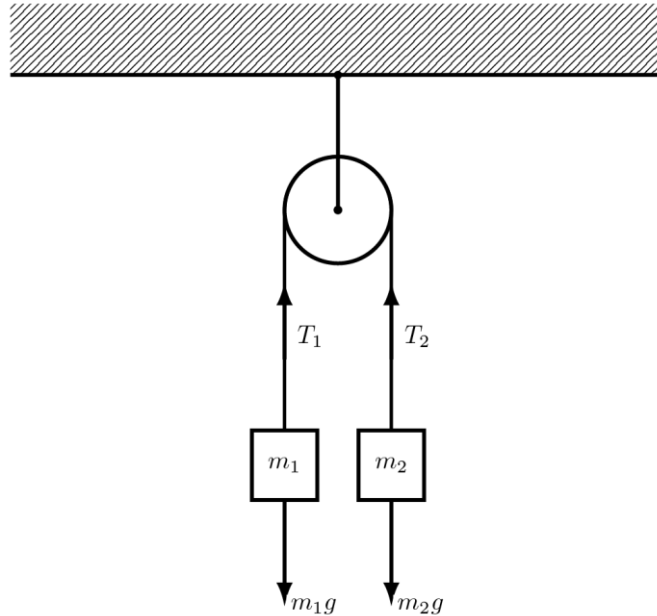


Figure: Two masses m_1 and m_2 connected by a light inextensible string passing over a smooth pulley. Tensions T_1 and T_2 act along the string, while the weights m_1g and m_2g act vertically downward.

Three possible scenarios:

- 1) If $m_1 = m_2$: The system is in equilibrium. There is no acceleration; the masses remain stationary (or move with constant velocity if already in motion).
- 2) If $m_1 < m_2$: The heavier mass m_2 pulls the system downward. Mass m_2 accelerates downward while m_1 accelerates upward. Both have the **same** magnitude of acceleration a .
- 3) If $m_1 > m_2$: The heavier mass m_1 pulls the system downward. Mass m_1 accelerates downward while m_2 accelerates upward, again with the same acceleration magnitude a .

Deriving the acceleration and tension formulas:

Let us analyse the case where m_2 is heavier than m_1 ($m_2 > m_1$)

Acceleration formula:

Forces on mass m_1 (moving upward):

- Weight: $W_1 = m_1g$ (downward)
- Tension: T (upward)

Forces on mass m_2 (moving downward):

- Weight: $W_2 = m_2g$ (downward)
- Tension: T (upward)

Applying Newton's second law to each body

For m_1 (taking upward as positive): $\Sigma F_1 = m_1a$; $T_1 - m_1g = m_1a \dots$ (equation 1)

For m_2 (taking downward as positive): $\Sigma F_2 = m_2a$; $m_2g - T_2 = m_2a \dots$ (equation 2)

Solving the simultaneous equations:

Since the pulley is smooth, $T_1 = T_2 = T$

Adding equation 1 and equation 2:

$$(T - m_1g) + (m_2g - T) = m_1a + m_2a$$

$$m_2g - m_1g = m_1a + m_2a$$

$$(m_2 - m_1)g = (m_1 + m_2)a$$

Hence, acceleration is:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

The final result is our key formula for acceleration in vertical connected bodies.

The formula shows:

- Greater mass difference leads to greater acceleration. If $m_2 \gg m_1$ (much heavier), then a approaches g , meaning the system accelerates nearly in free fall.
- Equal masses ($m_2 - m_1 = 0$) means zero acceleration (equilibrium).
- The acceleration cannot exceed the acceleration due to gravity, g .

$$\left(\text{Since } (m_2 - m_1) < (m_1 + m_2), \left(\frac{m_2 - m_1}{m_1 + m_2} \right) < 1; \text{ hence } a < g \right).$$

Tension formula:

Also from equation 1:

$$T_1 = T = m_1g + m_1a$$

$$T = m_1g + m_1 \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

$$T = m_1g \left(1 + \frac{m_2 - m_1}{m_1 + m_2} \right)$$

$$T = m_1g \left(\frac{m_1 + m_2 + m_2 - m_1}{m_1 + m_2} \right)$$

$$T = m_1g \left(\frac{2m_2}{m_1 + m_2} \right)$$

Hence, tension is:

$$T = \frac{2m_1m_2g}{m_1 + m_2}$$

The same result can be obtained by substituting the expression for a into equation 2.

It is important to note that the tension is always *greater than the lighter weight* and *less than the heavier weight*: $m_1g < T < m_2g$.

Two key insights regarding a vertical connected-body system:

Insight 1: Why same acceleration magnitude?

Because the string is inextensible (does not stretch), any distance m_1 moves up equals the distance m_2 moves down in the same time interval. Therefore, their velocities and accelerations must have the same magnitude, though opposite directions.

Insight 2: The smooth pulley assumption

A smooth pulley means frictionless. This ensures the tension in the string is the same on both sides of the pulley. If friction existed, tension would differ on each side, complicating the analysis significantly. For A-level problems, we almost always assume smooth pulleys unless stated otherwise.

For now, let us pause the theory and bring the ideas down to earth with a few carefully chosen worked examples.

BINDER Example 17

Show that, in a vertical connected-body system, if one body is much heavier than the other, the system accelerates at nearly the acceleration due to gravity, g .

Solution

Consider two masses m_1 and m_2 connected by a light inextensible string over a smooth pulley, with $m_2 > m_1$ so that m_2 moves downward.

The acceleration of the vertical connected-body system is given by the following formula:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

If $m_2 \gg m_1$ (much heavier), then m_2 is very large compared to m_1 . Thus:

$$m_2 - m_1 \approx m_2 \text{ and } m_2 + m_1 \approx m_2$$

Substituting into the acceleration expression:

$$a \approx \left(\frac{m_2}{m_2} \right) g \approx g$$

Hence, when one body is much heavier than the other, the system accelerates at almost g , (almost in free fall).

Making Sense of the Answer: *If the heavier mass is far larger than the lighter one, the lighter mass becomes almost negligible. The heavy mass then behaves nearly like an object falling under gravity with very little “load” to lift, so its acceleration gets very close to g .*

Thinking Like a Physicist: *This is a limiting-case test. If m_1 were zero (an “almost nothing” mass), the heavy mass would essentially be falling freely and the formula would give $a = g$ exactly. Real systems never reach g because the other mass is never truly zero, but as the mass difference becomes extreme, the system approaches free-fall behaviour.*

BINDER Example 18

Show that, in a vertical connected-body system, the tension, T is always less than the weight of the heavier mass and greater than the weight of the lighter mass.

Solution

Assume a light inextensible string over a smooth pulley with $m_2 > m_1$, so the system accelerates with magnitude $a > 0$, with m_2 downward and m_1 upward.

Applying Newton’s second law on each mass:

For m_1 (accelerating upward):

$$T - m_1g = m_1a$$

$$\text{So: } T = m_1g + m_1a$$

$$\text{Since } a > 0, \rightarrow T > m_1g \dots \dots (i)$$

For m_2 (accelerating downward):

$$m_2g - T = m_2a$$

$$\text{So: } T = m_2g - m_2a$$

$$\text{Since } a > 0, \rightarrow T < m_2g \dots \dots (ii)$$

Combining (i) and (ii) gives:

$$m_1g < T < m_2g$$

Hence, the tension is always less than the weight of the heavier mass and greater than the weight of the lighter mass.

Making Sense of the Answer: The heavier mass pulls downward, but part of its weight accelerates the lighter mass upward, so the tension is less than the heavier weight. At the same time, it (tension) must exceed the lighter weight to lift it.

Thinking Like a Physicist: If tension equalled either weight, one mass would not accelerate. Since both move, the tension must lie between the two weights.

BINDER Example 19

Two masses of 4kg and 6kg are connected by a light inextensible string passing over a smooth pulley. The system is released from rest. Take $g = 9.8 \text{ m/s}^2$.

Calculate: (a) the acceleration of each mass (b) the tension in the string

Solution

Using:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

Where:

$$m_2 = 6\text{kg (heavier)}$$

$$m_1 = 4\text{kg (lighter)}$$

Substituting:

$$a = \left(\frac{6\text{kg} - 4\text{kg}}{4\text{kg} + 6\text{kg}} \right) \times 9.8\text{m/s}^2 = 1.96\text{m/s}^2$$

Using:

$$T = \frac{2m_1m_2g}{m_1 + m_2}$$

Substituting:

$$T = \frac{2 \times 4\text{kg} \times 6\text{kg} \times 9.8\text{m/s}^2}{4\text{kg} + 6\text{kg}} = 47.04\text{N}$$

(a) The acceleration is 1.96m/s^2 .

(b) The tension is 47.04N .

Making Sense of the Answer: The heavier mass (6kg) pulls the system downward, so both masses accelerate together at 1.96m/s^2 . The tension (47.04N) lies between the two weights: greater than 4g (39.2N) but less than 6g (58.8N), which confirms the result is physically reasonable.

Thinking Like a Physicist: In the vertical connected-body system, always identify the heavier mass first to determine the direction of motion. Also as a quick check, the tension must satisfy $m_1g < T < m_2g$; if it does, the analysis is likely correct.

BINDER Example 20

Two masses are connected by a string over a smooth pulley. When released from rest, the system accelerates at 1.5 m/s^2 . If one of the masses is 4kg and moves upward, determine the mass of the other body. Take $g = 9.8 \text{ m/s}^2$.

Solution

Since the given mass was moving upward, it is the lighter mass. Thus the given mass is m_1 and the unknown mass is m_2 (heavier mass).

Using:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

Substituting:

$$1.5 \text{ m/s}^2 = \left(\frac{m_2 - 4\text{kg}}{4\text{kg} + m_2} \right) \times 9.8 \text{ m/s}^2$$

Solving for m_2 gives: $m_2 = 5.45\text{kg}$

The mass of the other body is 5.45kg .

Making Sense of the Answer: The unknown mass (5.45kg) is moderately heavier than the known mass (4kg), producing a modest acceleration (1.5m/s^2). If they were equal, acceleration would be zero. If the difference were larger, acceleration would be greater. The result fits the physical expectation perfectly.

Thinking Like a Physicist: Always identify the direction of motion first; the mass moving upward must be the lighter one. Also check whether the result makes physical sense: a slightly heavier opposing mass should produce a moderate acceleration, not zero and not close to free fall (g).

REAL Example 20

Kipanga watches workers installing a new elevator at his uncle's building. He notices a large concrete block hanging on the other side of the pulley.

Kipanga: "Why is there a heavy block hanging there? Isn't the elevator motor strong enough to lift people by itself?"

Kipute: "I think it's a counterweight. It helps balance the elevator so the motor doesn't have to work as hard."

Mr. Akilikubwa: (joining them) "Excellent observation, Kipute! Without that counterweight, the motor would need to lift the full weight of the elevator cabin plus passengers. Imagine lifting a 1000kg load straight up! But with a properly sized counterweight, the motor only needs to overcome the difference. If the cabin with passengers weighs 1000kg and the counterweight is 800kg , the motor effectively only lifts 200kg . Much easier, much more energy efficient."

Kipanga: "So it's like those connected masses we studied in class?"

Mr. Akilikubwa: "Exactly! The elevator and counterweight are connected bodies in vertical motion. When the elevator goes up, the counterweight comes down, just like our pulley system. The tension in the cable does most of the work, and the motor just provides the extra push or pull to accelerate or decelerate the system smoothly. Without the counterweight, your electricity bill would be shocking, and not in a good way!"

Kipute: (smiling) "So Physics saves money too?"

Mr. Akilikubwa: "Always! Good engineering is just applied physics with a practical goal. Every elevator in the world uses this principle. Next time you ride one, thank Newton's laws and the counterweight quietly working behind the scenes."

Question: Based on this conversation:

- (a) Using the given masses, show quantitatively how the counterweight reduces the force required from the elevator motor compared with lifting the cabin alone.
- (b) If an elevator cabin with passengers has a total mass of 900kg and the counterweight has a mass of 700kg , calculate:
 - (i) the acceleration if the motor provides an additional upward force of 2450N to the cabin,
 - (ii) the tension in the cable connecting them.

(Take $g = 9.8 \text{ m/s}^2$)

Solution

(a) Minimum motor force required without counterweight:

Without a counterweight, the motor must provide the entire upward force to lift the cabin and overcome its full weight. This requires at least: $F = W = mg$

$$\text{For a } 900\text{kg cabin: } F_{\text{motor}} = 900\text{kg} \times 9.8 \text{ m/s}^2 = 8820\text{N}$$

Minimum motor force required with counterweight:

With a counterweight of 700kg , the downward pull of the counterweight partially balances the cabin's weight. The net force the motor must provide becomes much smaller because it only needs to overcome the difference (200kg):

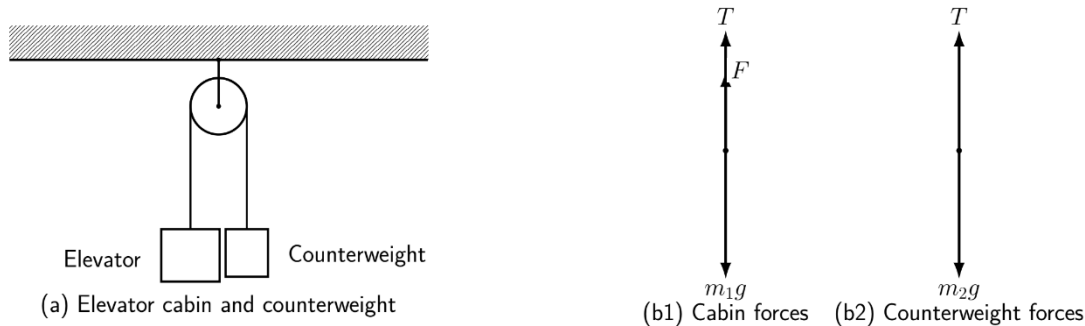
$$F_{\text{motor}} = (m_{\text{cabin}} - m_{\text{counterweight}})g = (900 - 700)\text{kg} \times 9.8\text{m/s}^2 = 1960\text{N}$$

This is only 22% of the original force! The counterweight does most of the "heavy lifting" through gravity, and the motor provides just the extra force needed for acceleration and control.

Additionally, when the cabin descends, the counterweight assists by pulling upward, again reducing motor effort.

This bidirectional efficiency explains why nearly all elevators use counterweights as they dramatically reduce energy consumption and allow smaller, cheaper motors.

(b) To fully understand this, consider the following situation diagram and its corresponding free-body diagram.



Forces acting on the elevator cabin (moving upward):

- Weight of cabin (downward): $W_{\text{cabin}} = m_1g = 900\text{kg} \times 9.8\text{m/s}^2 = 8820\text{N}$
- Tension (upward): T
- Motor force (upward): $F_{\text{motor}} = 2450\text{N}$

$$\text{Total upward forces} = 2450\text{N} + T$$

$$\text{Total downward forces} = 8820\text{N}$$

$$\text{Applying Newton's second law (upward positive): } \Sigma F = m_1a$$

$$(2450 + T) - 8820 = 900a \text{ or}$$

$$T - 900a = 6370 \dots\dots(i)$$

Forces acting on the counterweight (moving downward):

- Weight of counterweight (downward): $W_{\text{counter}} = m_2g = 700\text{kg} \times 9.8\text{m/s}^2 = 6860\text{N}$
- Tension (upward): T

$$\text{Again applying Newton's second law (downward positive for counterweight): } \Sigma F = m_2a$$

$$6860 - T = 700a \text{ or}$$

$$T + 700a = 6860 \dots\dots(ii)$$

$$\text{Solving (i) and (ii) simultaneously gives: } a = 0.31\text{m/s}^2, T = 6646\text{N}$$

(i) The acceleration is 0.306m/s^2 .

(ii) The tension is 6646N .

Making Sense of the Answer: The counterweight principle dramatically reduces the force requirement. Instead of the motor lifting 900kg , it effectively lifts only the difference (200kg), plus any additional force for acceleration. This is why elevators are energy-efficient despite moving heavy loads many times per day. The tension in the cable is substantial (6646N) but shared between supporting the cabin and the counterweight; the motor just provides the marginal difference.

Thinking Like a Physicist: Real engineering applications like elevators demonstrate why understanding connected bodies matters beyond textbooks. Engineers must calculate safe cable tensions, motor power

requirements, and energy efficiency. A poorly designed counterweight (too light or too heavy) wastes energy or creates unsafe accelerations. Physics isn't just formulas, it's the foundation of technologies we use every day without thinking about the Newton's laws quietly working behind the scenes.

HOT Example 21

A container of mass 8kg hangs from one end of a light inextensible string that passes over a smooth pulley. The other end of the string is attached to a 12kg block of ice resting on a platform. The system is initially at rest. After the ice block is released and starts moving, it melts at a constant rate of 0.5kg every 10 seconds due to the pulley mechanism warming the string.

- Calculate the initial acceleration of the system immediately after release.
- Determine the acceleration of the system after 20 seconds.
- Explain clearly what happens to the system's motion as the ice continues to melt.

$$\text{Take } g = 9.8 \text{ m/s}^2$$

Solution

- Initial masses:

$$m_1 = 8\text{kg (container, moving upward)}$$

$$m_2 = 12\text{kg (ice block, moving downward initially)}$$

Using:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

Substituting:

$$a = \left(\frac{12\text{kg} - 8\text{kg}}{8\text{kg} + 12\text{kg}} \right) \times 9.8\text{m/s}^2 = 1.96\text{m/s}^2$$

The initial acceleration is 1.96m/s^2 .

- Calculating mass of ice after melting:

Melting rate = 0.5kg per 10 seconds

Time elapsed = 20 seconds

$$\text{Mass melted} = \frac{0.5 \text{ kg}}{10\text{s}} \times 20\text{s} = 1\text{kg}$$

$$\text{New mass of ice block: } m'_2 = 12 - 1 = 11\text{kg}$$

Again using the acceleration formula:

$$a' = \left(\frac{m'_2 - m_1}{m_1 + m'_2} \right) g$$

Substituting:

$$a' = \left(\frac{11\text{kg} - 8\text{kg}}{8\text{kg} + 11\text{kg}} \right) \times 9.8\text{m/s}^2 = 1.55\text{m/s}^2$$

The acceleration after 20 seconds is 1.55m/s^2 .

- As ice melts, the mass difference decreases, causing acceleration to decrease. At 80 seconds, masses become equal (both 8kg) and acceleration becomes zero, making the system to move at constant velocity. If melting continues beyond this point, the container becomes heavier, causing acceleration to reverse direction. Eventually, if all the ice melts, the container is left unsupported (the system is no longer connected bodies) and will undergo free fall with acceleration 9.8m/s^2 until it hits the ground or another obstacle.

The worked examples have filled the plate nicely; now it is time to enjoy the next subtopic and see what new ideas it brings to the table.

Connected Bodies on Horizontal Plane

In the previous subtopic, we analysed bodies connected by a string over a pulley where both masses hung vertically. Now we shift to a different but equally common configuration: one or more bodies moving

horizontally on a surface, connected by a string or rigid bar. You will recognise this arrangement in everyday situations such as a car towing a trailer, a locomotive pulling carriages, or a person dragging a chain of boxes across a floor. Although the connection between bodies remains similar, the source of motion is no longer the same.

The key difference lies in what drives the motion. In the vertical system, the driving force came from the **difference in weights** of the two hanging masses. On a horizontal plane, however, the situation changes slightly: a block resting on a table cannot use its weight to produce horizontal motion. Instead, movement must be provided by an **external pull**, which then transmits motion through the connecting string or bar.

When analysing connected bodies on a horizontal surface, two distinct situations naturally arise depending on the nature of the surface. In the first case, the **surface is smooth** (frictionless), so the only horizontal forces are the applied pull and the tensions in the connecting string or bar, making the motion relatively straightforward to analyse. In the second case, the **surface is rough**, meaning friction acts alongside the applied pull and tension. This additional resistive force modifies the acceleration and must be carefully included when applying Newton's laws. These two cases form the foundation for understanding horizontal connected-body systems.

The general setup:

Consider two bodies of masses m_1 and m_2 connected by a light inextensible string on a horizontal surface. A horizontal force F is applied to mass m_1 , which then pulls mass m_2 through the connecting string. Since the surface is horizontal, the motion is entirely in the horizontal direction and both masses share the same acceleration, a .

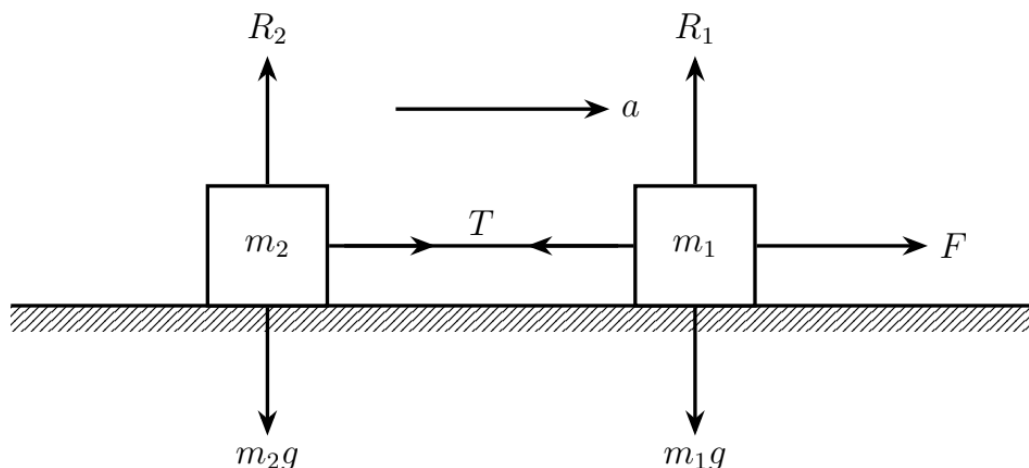


Figure: A diagram showing two masses m_1 and m_2 connected by a light inextensible string on a horizontal surface.

First case: Smooth horizontal surface (no friction)

Forces acting on m_1 (being pulled by F):

- Applied force: F (horizontal, forward)
- Tension: T (horizontal, backward: string pulls m_1 back toward m_2)
- Weight: m_1g (downward)
- Normal reaction: $R_1 = m_1g$ (upward)

Since no vertical motion (vertical forces balance), only horizontal forces (forward and backward forces) will be considered.

Applying Newton's second law:

$$F - T = m_1a \dots (\text{equation 1})$$

Forces acting on m_2 (being pulled by string):

- Tension: T (horizontal, forward: string pulls m_2 forward)
- Weight: m_2g (downward)

- Normal reaction: $R_2 = m_2g$ (upward)

Again, only horizontal forces will be considered. Thus:

$$T = m_2a \dots (\text{equation 2})$$

Adding equations 1 and 2:

$$F - T + T = m_1a + m_2a$$

$$F = (m_1 + m_2)a$$

Therefore, acceleration is:

$$a = \frac{F}{m_1 + m_2}$$

Substituting the acceleration expression into equation (2):

$$T = m_2 \times \frac{F}{m_1 + m_2}$$

Thus, the tension is:

$$T = \frac{m_2 F}{m_1 + m_2}$$

What physical meaning does each formula convey?

- The acceleration formula shows the entire applied force, F , accelerates the combined total mass ($m_1 + m_2$) of the system, just as if it were pushing a single object.
- The tension formula shows the following:
 - ✓ Tension, T in the string is only responsible for accelerating m_2 alone ($T = m_2a$). It does not need to accelerate m_1 because m_1 is already driven by the external force F .
 - ✓ Tension (T) is always less than applied force (F) (since $m_2 < m_1 + m_2$, $\frac{m_2}{m_1 + m_2} < 1$), meaning the string never experiences the full applied force.

Second case: Rough horizontal surface (with friction)

When the surfaces are rough, friction opposes the motion of each body. Friction acts backward (opposing forward motion) on both masses.

Let μ be the coefficient of **kinetic** friction between each mass and surface.

Then, frictional force (f) acting on each mass will be as follows:

$$f_1 = \mu R_1 = \mu m_1g \text{ (on } m_1, \text{ opposing its forward motion)}$$

$$f_2 = \mu R_2 = \mu m_2g \text{ (on } m_2, \text{ opposing its forward motion)}$$

Applying Newton's second law:

For m_1 :

$$F - T - f_1 = m_1a$$

$$F - T - \mu m_1g = m_1a \dots (\text{equation 1})$$

For m_2 :

$$T - f_2 = m_2a$$

$$T - \mu m_2g = m_2a \dots (\text{equation 2})$$

Adding equations 1 and 2:

$$F - \mu m_1g - \mu m_2g = m_1a + m_2a$$

$$F - \mu(m_1 + m_2)g = (m_1 + m_2)a$$

Therefore, acceleration is:

$$a = \frac{F - \mu(m_1 + m_2)g}{m_1 + m_2}$$

Also from equation (2):

$$T = m_2 a + \mu m_2 g$$

$$T = m_2(a + \mu g)$$

But:

$$a = \frac{F - \mu(m_1 + m_2)g}{m_1 + m_2} = \frac{F}{m_1 + m_2} - \mu g$$

It follows that:

$$T = m_2(a + \mu g) = m_2 \left(\frac{F}{m_1 + m_2} - \mu g + \mu g \right) = \frac{m_2 F}{m_1 + m_2}$$

Hence, the tension is:

$$T = \frac{m_2 F}{m_1 + m_2}$$

(The same as the **smooth** case)

What does the acceleration formula reveal?

- For motion to occur at all, the applied force must overcome total friction:
 $F > \mu(m_1 + m_2)g$ (where, $\mu(m_1 + m_2)g$ is the total friction)

If $F \leq \mu(m_1 + m_2)g$ (which would give $a \leq 0$), the system remains stationary and static friction keeps everything at rest. The derived equations apply only when motion occurs.

- In **this particular arrangement**, friction reduces the acceleration by μg . Thus:

$$\text{Acceleration in rough} \left(\frac{F}{m_1 + m_2} - \mu g \right) = \text{Acceleration in smooth} \left(\frac{F}{m_1 + m_2} \right) - \mu g$$

BINDER Example 22

Two blocks of masses 4kg and 6kg are connected by a light inextensible string on a smooth horizontal table. A horizontal force of 20N is applied to the 6kg block, pulling both blocks along the table. Take $g = 9.8 \text{ m/s}^2$.

Calculate: (a) the acceleration of the system (b) the tension in the string connecting the two blocks.

Solution

Identifying the system:

$$m_1 = 6\text{kg (block with applied force)}$$

$$m_2 = 4\text{kg (block being pulled by string)}$$

Surface: smooth (no friction).

(a) Since the smooth was surface, the acceleration is given by:

$$a = \frac{F}{m_1 + m_2} = \frac{20\text{N}}{(6 + 4)\text{kg}} = 2\text{m/s}^2$$

The acceleration is 2m/s^2 .

(b) The tension is given by:

$$T = m_2 a = 4\text{kg} \times 2\text{m/s}^2 = 8\text{N}$$

The tension in the string is 8N.

Making Sense of the Answer: The 20N force accelerates a total mass of 10kg, giving 2m/s^2 which is the same as pushing a single 10kg object with 20N on a smooth surface. The tension (8N) is less than the applied force (20N) because it only needs to accelerate the 4kg block, not the entire system.

Thinking Like a Physicist: The tension in a connecting string is usually less than the applied force because it accelerates only part of the system. This explains why tow ropes in vehicles can snap: When a vehicle brakes suddenly, the rope must transmit a large tension to slow the trailer quickly. Understanding tension in connected systems is therefore not just theoretical, it is directly linked to engineering safety.

BINDER Example 23

Two boxes of masses 3kg and 5kg are connected by a light inextensible string on a rough horizontal floor. A horizontal force of 40N is applied to the 5kg box, pulling both boxes. The coefficient of kinetic friction between each box and the floor is $\mu = 0.3$. Take $g = 9.8 \text{ m/s}^2$.

Calculate: (a) the acceleration of the system (b) the tension in the string.

Solution

Identifying the system:

$m_1 = 5\text{kg}$ (box with applied force)

$m_2 = 3\text{kg}$ (box being pulled by string)

Surface: rough with $\mu = 0.3$.

(a) The acceleration is given by:

$$a_{\text{rough}} = a_{\text{smooth}} - \mu g = \frac{F}{m_1 + m_2} - \mu g = \frac{40\text{N}}{(5 + 3)\text{kg}} - (0.3 \times 9.8 \text{ m/s}^2) = 2.06 \text{ m/s}^2$$

The acceleration is 2.06 m/s^2 .

Alternative solution

Treating the whole system as one body:

Total mass, $m_t = m_1 + m_2 = 5\text{kg} + 3\text{kg} = 8\text{kg}$

Total frictional force, $f_t = \mu m_t g = 0.3 \times 8\text{kg} \times 9.8 \text{ m/s}^2 = 23.52\text{N}$

Resultant force, $F_R = F - f_t = 40\text{N} - 23.52\text{N} = 16.48\text{N}$

Then from $F = ma$ or $a = \frac{F}{m}$;

The acceleration of the system, $a = \frac{F_R}{m_t} = \frac{16.48\text{N}}{8\text{kg}} = 2.06 \text{ m/s}^2$

(b) The tension is given by:

$$T = m_2 a_{\text{smooth}} = \frac{m_2 F}{m_1 + m_2} = \frac{3\text{kg} \times 40\text{N}}{(5 + 3)\text{kg}} = 15\text{N}$$

The tension is 15N.

Making Sense of the Answer: The acceleration is smaller than it would be on a smooth surface because friction opposes motion.

Thinking Like a Physicist: Friction reduces acceleration but does not directly dictate the tension formula, which depends on how the force is shared between the masses.

REAL Example 24

Tow ropes used to pull trailers sometimes snap when the towing vehicle brakes suddenly. Explain why this happens.

Solution

When the vehicle brakes suddenly, it slows down quickly but the trailer tends to keep moving forward due to inertia. The tow rope then becomes tightly stretched and develops a large tension as it tries to stop the trailer in a short time. If this tension becomes too great, the rope may snap.

Making Sense of the Answer: The trailer does not stop immediately when the vehicle brakes because it has inertia. The tow rope must therefore provide a large tension to slow it down quickly. If this tension exceeds

the rope's strength, it snaps. This matches everyday experience where sudden braking produces stronger forces than gradual stopping.

Thinking Like a Physicist: Whenever motion changes rapidly, large forces are involved. In connected-body systems, tension is the force that transmits these changes. Sudden deceleration increases tension significantly, which is why engineers design tow ropes and cables with safety limits.

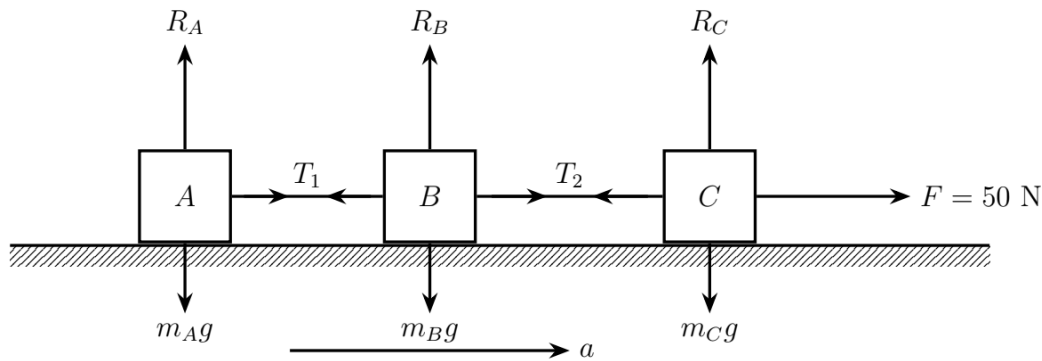
HOT Example 25

Three blocks A, B and C of masses 2kg, 3kg and 5kg respectively are connected in series by two light inextensible strings on a rough horizontal surface. Block A is at the back, connected to block B by string 1, and block B is connected to block C by string 2. A horizontal force $F = 50\text{N}$ is applied to block C at the front, pulling the entire system forward. The coefficient of kinetic friction between each block and surface is $\mu = 0.25$. Take $g = 9.8\text{ m/s}^2$.

- Calculate the acceleration of the system.
- Find the tension T_1 in the string between A and B.
- Find the tension T_2 in the string between B and C.
- Show that $T_2 > T_1$ and explain why this must always be the case.

Solution

- Consider the following diagram.



Treating the entire system as one body:

Total mass, $m_t = m_A + m_B + m_C = 2\text{kg} + 3\text{kg} + 5\text{kg} = 10\text{kg}$

Total frictional force, $f_t = \mu m_t g = 0.25 \times 10\text{kg} \times 9.8\text{ m/s}^2 = 24.5\text{N}$

Resultant force, $F_R = F - f_t = 50\text{N} - 24.5\text{N} = 25.5\text{N}$

Then $a = \frac{F_R}{m_t} = \frac{25.5\text{N}}{10\text{kg}} = 2.55\text{m/s}^2$

The acceleration of the system is 2.55m/s^2 .

- Horizontal forces acting on block A:

- Tension: T_1 (forward)
- Frictional force: $f_A = \mu m_A g = 0.25 \times 2\text{kg} \times 9.8\text{ m/s}^2 = 4.9\text{N}$ (backward)

Applying Newton's second law:

$$F_R = T_1 - f_A = m_A a \text{ or } T_1 = m_A a + f_A$$

Substituting:

$$T_1 = 2\text{kg} \times 2.55\text{m/s}^2 + 4.9\text{N} = 10\text{N}$$

The tension T_1 is 10N.

- Horizontal forces acting on block B:

- Tension: T_1 (backward)

- Tension: T_2 (forward)
- Frictional force: $f_B = \mu m_B g = 0.25 \times 3\text{kg} \times 9.8 \text{ m/s}^2 = 7.35\text{N}$ (backward)

Applying Newton's second law:

$$F_R = T_2 - T_1 - f_B = m_B a \text{ or } T_2 = m_B a + T_1 + f_B$$

Substituting:

$$T_2 = 3\text{kg} \times 2.55\text{m/s}^2 + 10\text{N} + 7.35\text{N} = 25\text{N}$$

The tension T_2 is 25N.

Alternative solution

From the diagram, it is clearly understood that the tension T_2 is responsible for pulling both block A and block B.

So total mass pulled by $T_2 = m_A + m_B = (2 + 3)\text{kg} = 5\text{kg}$

And total friction (on block A and B) = $\mu mg = 0.25 \times 5\text{kg} \times 9.8 \text{ m/s}^2 = 12.25\text{N}$ (backward)

Applying Newton's second law:

$$F_R = T_2 - f = ma \text{ or } T_2 = ma + f$$

Substituting:

$$T_2 = ma + f = 5\text{kg} \times 2.55\text{m/s}^2 + 12.25\text{N} = 25\text{N}$$

(d) From results in (b) and (c): $T_1 = 10\text{N}$, $T_2 = 25\text{N}$; hence: $T_2 > T_1$ (shown).

Explanation

T_2 pulls both A and B (total mass 5kg) against greater friction, while T_1 pulls only A (2kg) against less friction; therefore, T_2 must be larger than T_1 .

Making Sense of the Answer: The three tensions in this problem tell a clear story: $F = 50\text{N}$ (applied to C) $\rightarrow T_2 = 25\text{N}$ (transmitted to B and A) $\rightarrow T_1 = 10\text{N}$ (transmitted to A only). Each string experiences progressively less tension as it has fewer masses to pull. This cascade of decreasing tension from front to back is a universal feature of any train of connected bodies.

Thinking Like a Physicist: Railway engineers exploit this principle when designing train couplings. Each coupling must be rated for the maximum tension it will experience, and the front coupling is always under the greatest tension. When a locomotive accelerates a long train from rest, the front coupling experiences the highest tension; a failure there would split the train. This is also why heavy trucks use reinforced front tow hitches. The front connection always bears the greatest mechanical load in a chain of connected vehicles.

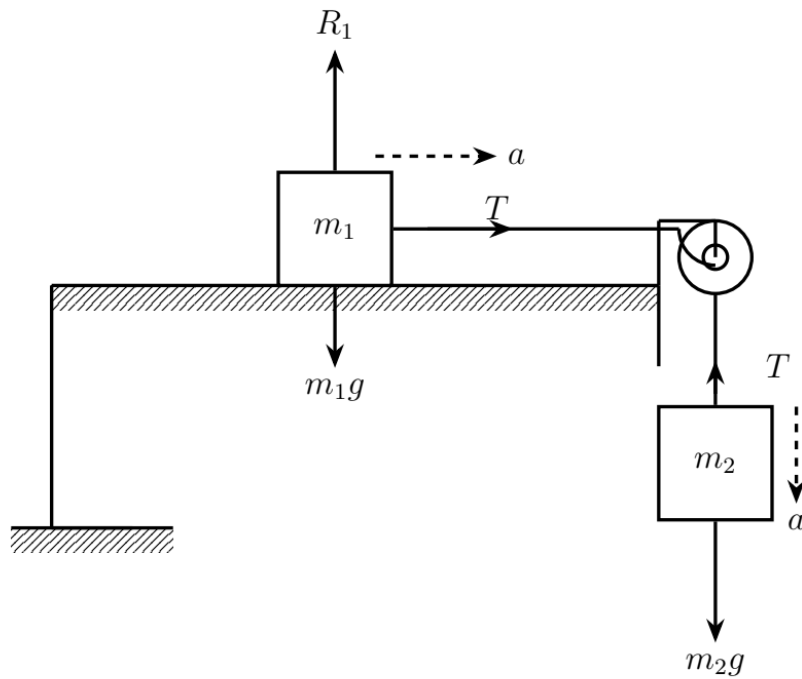
HOT Example 26

A block of mass 4kg rests on a horizontal table. It is connected by a light inextensible string passing over a smooth pulley fixed at the edge of the table to a hanging mass of 3kg. The system is released from rest. Take $g = 9.8 \text{ m/s}^2$.

- Assuming the table surface is smooth (frictionless), calculate:
 - the acceleration of the system,
 - the tension in the string.
- Now assume the table surface is rough with coefficient of kinetic friction $\mu = 0.25$. Calculate:
 - the acceleration of the system,
 - the tension in the string.

Solution

Consider the following diagram.



(a) **For smooth surface (No friction):**

Horizontal forces acting on m_1 (on table, moving horizontally, forward):

- Tension, T (horizontal, forward)

Vertical forces acting on m_2 (hanging, moving vertically, downward):

- Weight, m_2g (downward)
- Tension, T (upward)

Since the string is inextensible, both masses have the same acceleration magnitude, a .

For m_1 :

$$T = m_1a \dots (\text{equation 1})$$

For m_2 :

$$m_2g - T = m_2a \dots (\text{equation 2})$$

Adding equations (1) and (2):

$$\begin{aligned} T + m_2g - T &= m_1a + m_2a \\ m_2g &= (m_1 + m_2)a \end{aligned}$$

Rearrange to make a the subject:

$$a = \frac{m_2g}{m_1 + m_2}$$

Substituting values:

$$a = \frac{3\text{kg} \times 9.8\text{m/s}^2}{4\text{kg} + 3\text{kg}} = 4.2\text{m/s}^2$$

The acceleration on a smooth surface is 4.2m/s^2 .

From equation (1):

$$T = m_1a = 4\text{kg} \times 4.2\text{m/s}^2 = 16.8\text{N}$$

For smooth surface, the tension is 16.8N.

A quicker way to calculate acceleration:

With this method, you only need to identify the source of the external pull and then apply Newton's second law of motion, $F = ma$, directly.

In this example, the pulling force comes from the weight of the hanging mass, m_2g . This weight provides the force that accelerates both m_1 and m_2 .

Thus:

$$F = m_2g, m = m_1 + m_2$$

It follows that:

$$m_2g = (m_1 + m_2)a \text{ or } a = \frac{m_2g}{m_1 + m_2} = \frac{3\text{kg} \times 9.8\text{m/s}^2}{4\text{kg} + 3\text{kg}} = 4.2\text{m/s}^2$$

Making Sense of the Answer (Part a): *The acceleration (4.2m/s^2) is less than g because the 4kg block on the table resists the downward pull of the 3kg hanging mass (the hanging mass also accelerates the block horizontally). The tension (16.8N) is less than the weight of the hanging mass (29.4N) because the hanging mass is accelerating downward; if it were stationary, the tension would equal its full weight.*

(b) For smooth surface (No friction):

In this case, the pulling force (m_2g) has to overcome frictional force (μm_1g) before accelerating both m_1 and m_2 with acceleration a .

So the resultant force, $F_R = F - f = m_2g - \mu m_1g$

Applying Newton's second law of motion:

$$m_2g - \mu m_1g = (m_1 + m_2)a$$

From which:

$$a = \frac{m_2g - \mu m_1g}{m_1 + m_2} = \frac{(3\text{kg} \times 9.8\text{m/s}^2) - (0.25 \times 4\text{kg} \times 9.8\text{m/s}^2)}{(4 + 3)\text{kg}} = 2.8\text{m/s}^2$$

The acceleration on a rough surface is 2.8m/s^2 .

Considering the hanging mass, m_2 .

$$m_2g - T = m_2a \text{ or } T = m_2g - m_2a = m_2(g - a) = 3\text{kg}(9.8 - 2.8)\text{m/s}^2 = 21\text{N}$$

For rough surface, the tension is 21N.

Making Sense of the Answer (Part b): *The friction force (9.8N) significantly reduces the acceleration from 4.2m/s^2 to 2.8m/s^2 . This makes physical sense: friction opposes the forward motion of the block, requiring part of the hanging mass's weight to overcome friction rather than producing acceleration. Interestingly, **the tension increases from 16.8N to 21N. Why?** Because now the string must not only accelerate the block but also overcome friction acting on it. The string works harder in the rough case.*

Thinking Like a Physicist: *This problem illustrates a fundamental principle: adding friction to any part of a connected system reduces overall acceleration but may increase forces in connecting elements. This has practical implications: in elevator systems, increased cable friction means the motor must work harder (higher cable tension) even though the elevator accelerates more slowly. Similarly, towing a vehicle on a muddy road (high friction as it has greater μ value) puts greater tension on the tow cable despite slower acceleration. Engineers must design cables and connections to withstand these higher tensions, not just the smooth-surface ideal case.*

Connected Bodies on an Inclined Plane

You have now mastered two connected-body configurations: masses hanging vertically over a pulley, and masses moving horizontally on surfaces. Both involved motion in a single direction: pure vertical or pure horizontal. Now we combine these ideas: one mass moves along an incline while another hangs vertically.

This is perhaps the most common connected-body scenario in real engineering such as mountain railway systems, mine hoists, cable cars, and even simple construction pulleys lifting materials up slopes.

The general setup:

Consider a mass m_1 rests on an inclined plane making angle θ with the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the incline to a mass m_2 hanging freely on the vertical side.

When released, the system moves in a direction determined by the relative magnitudes of m_1 , m_2 , and angle θ . Because the string is inextensible, if m_1 moves distance d up the slope, then m_2 descends distance d vertically. Therefore, both masses share the same acceleration magnitude a , though they move in different directions (one along the incline, another vertically).

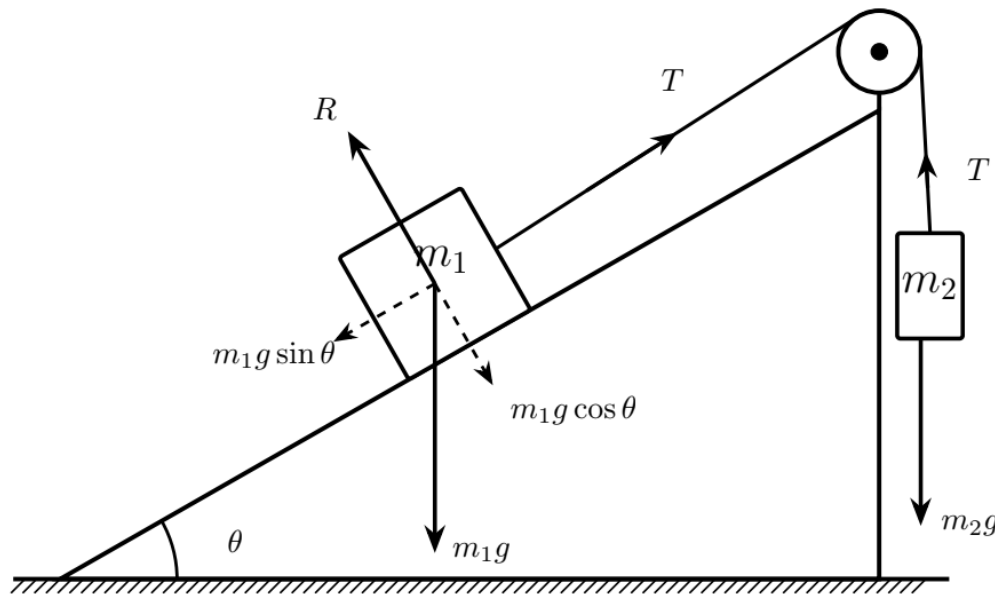


Figure: Connected-body system on an inclined plane.

First case: Smooth incline (no friction)

Forces acting on m_1 :

- Weight: m_1g (vertically downward)
 - ✓ Component parallel to incline: $m_1g\sin\theta$ (down the slope)
 - ✓ Component perpendicular to incline: $m_1g\cos\theta$ (into the incline)
- Tension: T (up the slope, parallel to incline)
- Normal reaction: $R = m_1g\cos\theta$ (perpendicular to slope, outward)

Forces acting on m_2 (being pulled by string):

- Weight: m_2g (downward)
- Tension: T (upward)

Identifying forces trying to move the system:

- Weight of hanging mass, m_2 : $W = m_2g$ (try to move m_1 up the slope while m_2 is moving vertically downward).
- Weight component of m_1 parallel to incline: $m_1g\sin\theta$ (try to move m_2 vertically upward while m_1 is moving down the slope).

So two possibilities exist on direction of motion:

- 1) If $m_2g > m_1g\sin\theta$: Mass m_2 is heavier (or the incline is gentle as θ is small), so m_2 descends and m_1 moves up the slope.
- 2) If $m_2g < m_1g\sin\theta$: Weight component of m_1 dominates, so m_1 slides down and m_2 ascends.

For analysis, assume case (1): m_2 descends, m_1 ascends.

Applying Newton's second law:

For m_1 :

$$T - m_1 g \sin \theta = m_1 a \dots (\text{equation 1})$$

For m_2 :

$$m_2 g - T = m_2 a \dots (\text{equation 2})$$

Finding acceleration:

Adding equations (1) and (2):

$$T - m_1 g \sin \theta + m_2 g - T = m_1 a + m_2 a$$

$$m_2 g - m_1 g \sin \theta = (m_1 + m_2) a$$

Making a the subject:

$$a = \frac{m_2 g - m_1 g \sin \theta}{m_1 + m_2}$$

Hence:

$$a = \frac{(m_2 - m_1 \sin \theta) g}{m_1 + m_2}$$

Alternatively; the pulling force ($m_2 g$) must overcome weight component of m_1 parallel to incline ($m_1 g \sin \theta$) before accelerating both m_1 and m_2 with acceleration a . Thus:

$$m_2 g - m_1 g \sin \theta = (m_1 + m_2) a$$

From which:

$$a = \frac{m_2 g - m_1 g \sin \theta}{m_1 + m_2} = \frac{(m_2 - m_1 \sin \theta) g}{m_1 + m_2}$$

Finding tension:

From equation (1):

$$T = m_1 a + m_1 g \sin \theta = m_1 (a + g \sin \theta)$$

Substituting expression for a :

$$T = m_1 \left(\frac{(m_2 - m_1 \sin \theta) g}{m_1 + m_2} + g \sin \theta \right)$$

$$T = m_1 g \left(\frac{m_2 - m_1 \sin \theta + m_1 \sin \theta + m_2 \sin \theta}{m_1 + m_2} \right)$$

$$T = m_1 g \left(\frac{m_2 (1 + \sin \theta)}{m_1 + m_2} \right)$$

Hence:

$$T = \frac{m_1 m_2 g (1 + \sin \theta)}{m_1 + m_2}$$

Interesting physical interpretation

From the acceleration and tension equations, we can deduce that:

- If $\theta = 0^\circ$ (horizontal surface): $\sin \theta = 0$, so $a = m_2 g / (m_1 + m_2)$, $T = m_1 m_2 g / (m_1 + m_2)$ which matches the horizontal case we studied earlier.
- If $\theta = 90^\circ$ (vertical pulley system): $\sin \theta = 1$, therefore $a = (m_2 - m_1) g / (m_1 + m_2)$, and $T = 2 m_1 m_2 g / (m_1 + m_2)$ which matches the vertical case (Atwood machine).

These limiting cases confirm that our general results are consistent with the familiar horizontal and vertical configurations. Let us now extend the analysis further by introducing friction, leading us to the more realistic case of motion on a rough incline.

Second case: Rough incline (With friction)

When the incline surface is rough with coefficient of kinetic friction μ , friction opposes the motion of m_1 along the slope.

Friction force: $f = \mu R = \mu m_1 g \cos \theta$

The direction of friction depends on the direction of motion of m_1 .

- If m_1 moves up the slope: Friction acts down the slope (opposing upward motion).
- If m_1 moves down the slope: Friction acts up the slope (opposing downward motion).

Now, let us return to our assumed scenario where m_2 descends, m_1 ascends.

For this case, the pulling force ($m_2 g$) must overcome both weight component of m_1 parallel to incline ($m_1 g \sin \theta$) and frictional force ($\mu m_1 g \cos \theta$) before accelerating both m_1 and m_2 with acceleration a . Thus:

$$m_2 g - m_1 g \sin \theta - \mu m_1 g \cos \theta = (m_1 + m_2) a$$

From which:

$$a = \frac{m_2 g - m_1 g \sin \theta - \mu m_1 g \cos \theta}{m_1 + m_2} = \frac{m_2 g - (m_1 g \sin \theta + \mu m_1 g \cos \theta)}{m_1 + m_2}$$

$$a = \frac{m_2 g - m_1 g (\sin \theta + \mu \cos \theta)}{m_1 + m_2}$$

Hence:

$$a = \frac{(m_2 - m_1 (\sin \theta + \mu \cos \theta)) g}{m_1 + m_2}$$

Once the acceleration has been determined, the tension can be found by considering forces acting on the hanging mass.

That is:

$$m_2 g - T = m_2 a$$

From which:

$$T = m_2 g - m_2 a$$

Never overlook this important condition!

For motion to begin from rest, the driving force must first overcome static friction. In this system, the weight of the hanging mass must exceed the combined resisting forces along the incline:

$$m_2 g > m_1 g \sin \theta + \mu_s m_1 g \cos \theta$$

Where μ_s is the coefficient of static friction. If this condition is not met, the system remains stationary.

The ideas have finished cooking; now let us sit down and enjoy them through a few worked examples.

BINDER Example 27

A block of mass 6kg rests on a smooth plane inclined at 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a hanging mass of 4kg. The system is released from rest. Take $g = 9.8 \text{ m/s}^2$.

- Determine the acceleration of the system.
- Determine the tension in the string.

Solution

Identifying the system:

$m_1 = 6 \text{ kg}$ (on incline)

$$m_2 = 4\text{kg (hanging)}$$

$$\theta = 30^\circ$$

Surface: smooth (no friction)

Comparing m_2g and $m_1g\sin\theta$ to determine the direction of motion:

$$m_2g = 4\text{kg} \times 9.8\text{m/s}^2 = 39.2\text{N (pulls } m_1 \text{ up)}$$

$$m_1g\sin\theta = 6\text{kg} \times 9.8\text{m/s}^2 \times \sin 30^\circ = 29.4\text{N (pulls } m_1 \text{ down)}$$

Since m_2g (39.2N) > $m_1g\sin\theta$ (29.4N), the hanging mass moves downward and pulls the block up the plane.

- (a) The weight of hanging mass (m_2g) must overcome weight component of m_1 parallel to incline ($m_1g\sin\theta$) before accelerating both m_1 and m_2 with acceleration a . Thus:

$$m_2g - m_1g\sin\theta = (m_1 + m_2)a$$

From which:

$$a = \frac{m_2g - m_1g\sin\theta}{m_1 + m_2} = \frac{9.8\text{m/s}^2(4\text{kg} - 6\text{kg}\sin 30^\circ)}{6\text{kg} + 4\text{kg}} = 0.98\text{m/s}^2$$

The acceleration is 0.98m/s^2 .

- (b) Considering forces acting on the hanging mass, m_2 .

$$m_2g - T = m_2a \text{ or } T = m_2g - m_2a$$

Substituting:

$$T = (4\text{kg} \times 9.8\text{m/s}^2) - (4\text{kg} \times 0.98\text{m/s}^2) = 33.32\text{N}$$

The tension is 35.28N.

Making Sense of the Answer: The hanging side is stronger (39.2N) than the block's down-slope pull (29.4N), so the system accelerates but only gently because both masses must be moved together. The tension is slightly less than 39.2N (since the hanging mass is accelerating downward) and greater than 29.4N (since the block is being pulled up the plane), which fits $T = 35.28\text{N}$.

Thinking Like a Physicist: On inclined connected-body problems, always compare m_2g with $m_1g\sin\theta$ first to predict the direction. That single comparison prevents wrong friction directions later and saves you from sign confusion. A quick check is that the tension should lie between the two opposing pulls along the string.

BINDER Example 28

A block of mass 9kg rests on a rough plane inclined at 40° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a hanging mass of 4kg. The coefficient of friction between the block and the plane is $\mu = 0.20$. The system is released from rest. Take $g = 9.8\text{m/s}^2$.

Calculate: (a) the acceleration of the system, (b) the tension in the string.

Solution

Identifying the system:

$$m_1 = 9\text{kg (on incline)}$$

$$m_2 = 4\text{kg (hanging)}$$

$$\theta = 40^\circ$$

Surface: rough with $\mu = 0.20$

Determining the direction motion:

In this case, the driving force must overcome both the opposing pull from the other mass and the frictional force. Therefore, the comparison is either between m_2g and $(m_1g\sin\theta + f)$, or between $m_1g\sin\theta$ and $(m_2g + f)$, where:

- $m_2g = 4\text{kg} \times 9.8\text{m/s}^2 = 39.2\text{N}$
- $m_1g\sin\theta = 9\text{kg} \times 9.8\text{m/s}^2 \times \sin 40^\circ = 56.7\text{N}$
- $f = \mu m_1g\cos\theta = 0.2 \times 9\text{kg} \times 9.8\text{m/s}^2 \times \cos 40^\circ = 13.5\text{N}$

So $m_1g\sin\theta$ (56.7N) is greater than $m_1g\sin\theta + f$ (52.7N). Hence, the block moves down the plane and the hanging mass moves upward.

- (a) The driving force ($m_1g\sin\theta$) must overcome all resistive forces so as to accelerate both m_1 and m_2 with acceleration a . Thus:

$$m_1g\sin\theta - m_2g - f = (m_1 + m_2)a \text{ or } a = \frac{m_1g\sin\theta - m_2g - f}{m_1 + m_2}$$

Substituting:

$$a = \frac{(56.7 - 39.2 - 13.5)\text{N}}{(9 + 4)\text{kg}} = 0.31\text{m/s}^2$$

The acceleration is 0.31m/s^2 .

- (b) Considering forces acting on the hanging mass, m_2 .

$$T - m_2g = m_2a \text{ or } T = m_2g + m_2a$$

Substituting:

$$T = 39.2\text{N} + (4\text{kg} \times 0.31\text{m/s}^2) = 40.44\text{N}$$

The tension is 40.44N.

Making Sense of the Answer: The block's down-slope pull (56.7N) is slightly larger than the combined resistance from the hanging weight and friction (52.7N), so motion occurs but slowly. The tension lies between the opposing pulls, which confirms the result is physically reasonable.

Thinking Like a Physicist: Never assume the hanging mass must move downward. Always compare the possible driving force with its corresponding resistive forces. That quick comparison tells you the direction before any algebra and saves many sign mistakes.

REAL Example 29

During road construction, a small machine is lowered down a sloping surface using a rope connected over a pulley to a counterweight. Workers observe that when the slope is made steeper, the machine begins to move downward even though the counterweight remains unchanged. Account for this observation.

Solution

As the slope becomes steeper, the component of the machine's weight acting down the slope increases. This increases the driving pull downward along the rope. If this down-slope pull becomes greater than the upward pull provided by the counterweight (together with friction), the machine will begin to move downward. Therefore, increasing the slope angle changed the balance of forces and altered the direction of motion even though the masses remained the same.

Making Sense of the Answer: A steeper slope (greater value of $\sin\theta$) means a stronger downward pull along the ramp. Even without changing the counterweight, the balance can shift simply because geometry ($\sin\theta$) changes the effective force along the slope.

Thinking Like a Physicist: Always remember: on an inclined plane, the angle controls how much of the weight acts along the slope. Changing the angle changes the driving force even if the mass stays constant.

HOT Example 30

A block of mass 10kg rests on a rough plane inclined at 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a hanging mass of 4kg. The

coefficient of static friction between the block and the plane is $\mu_s = 0.25$, and the coefficient of kinetic friction is $\mu_k = 0.2$. The system is released from rest. Take $g = 9.8 \text{ m/s}^2$.

- Determine whether the system moves. If it moves, state the direction of motion.
- If it moves, calculate the acceleration. If it does not move, determine the frictional force and the tension in the string.

Solution

- Calculating key forces in the system:

For the 10kg block:

Weight component parallel to the plane: $m_1 g \sin 30^\circ = 10 \times 9.8 \times 0.5 = 49 \text{ N}$

Limiting friction: $f_{\max} = \mu_s R = \mu m_1 g \cos 30^\circ = 0.25 \times 10 \text{ kg} \times 9.8 \text{ m/s}^2 \times \sin 30^\circ = 21.22 \text{ N}$

For the hanging mass:

Weight: $m_2 g = 4 \text{ kg} \times 9.8 \text{ m/s}^2 = 39.2 \text{ N}$

Testing possible directions:

First possibility: Suppose the hanging mass goes down (block moves up the plane).

The weight of hanging mass must overcome: $m_1 g \sin 30^\circ + f_{\max} = (49 + 21.22) \text{ N} = 70.22 \text{ N}$

But the weight, $m_2 g = 39.2 \text{ N} < 70.22 \text{ N}$ ($m_1 g \sin 30^\circ + f_{\max}$). So motion in this direction is impossible.

Second possibility: Suppose the block goes down the plane (hanging mass moves upward).

The weight acting along the plane must overcome: $m_2 g + f_{\max} = (39.2 + 21.22) \text{ N} = 60.42 \text{ N}$

But the weight acting along the plane, $m_1 g \sin 30^\circ = 49 \text{ N} < 60.42 \text{ N}$ ($m_2 g + f_{\max}$). So motion in this direction is impossible too.

Conclusion: The system does **not** move from rest. Static friction holds it in equilibrium.

- Finding frictional force and tension (since $a = 0$).

Since $m_1 g \sin 30^\circ (49 \text{ N}) > m_2 g (39.2 \text{ N})$, the block on the plane attempts to move down the plane.

To move in this direction, the weight component has to overcome the weight of hanging mass and static friction. Thus for the block along the plane in **equilibrium**:

$$m_1 g \sin 30^\circ - m_2 g - f = 0 \text{ or } f = m_1 g \sin 30^\circ - m_2 g$$

Substituting:

$$f = (49 - 39.2) \text{ N} = 9.8 \text{ N}$$

The frictional force is 9.8 N.

Considering forces acting on the hanging mass in equilibrium, m_2 .

$$T - m_2 g = 0 \text{ or } T = m_2 g = 39.2 \text{ N}$$

The tension is 39.2 N.

Making Sense of the Answer: Even though the block has a strong down-slope pull (49N), the hanging mass provides a 39.2N counter-pull through the tension. Only 9.8N of friction is needed to stop motion, and since the surface can provide up to 21.22N, the system stays comfortably at rest. Technically, the system is "locked" by friction.

Thinking Like a Physicist: In real engineering, this "locking" effect is both useful and dangerous. Useful: prevents cable cars from sliding backward when motors are off. Dangerous: can trap loads on slopes, requiring extra force to start motion. Engineers must calculate whether static friction will hold loads safely OR whether it will prevent necessary motion. Sometimes "stuck" is good; sometimes it is catastrophic.

As the worked examples quietly leave the table, the next subtopic arrives; not to overwhelm us, but to be understood and enjoyed.

Motion in a Lift (Elevator)

You step into a lift on the ground floor of a tall building. As the lift starts moving upward, you feel slightly heavier as your legs feel the extra pressure. When the lift reaches the top floor and begins to stop, you feel momentarily lighter, almost as if you are floating. Yet your mass has not changed at all.

This sensation is one of the most direct experiences of Newton's laws in daily life. The lift is a connected-body system: you (the passenger) and the lift cabin move together with the same acceleration. The normal reaction from the lift floor (what you feel as your "weight") changes depending on whether the lift accelerates upward, downward, or moves at constant velocity.

Understanding lift motion is not just strengthening your grasp of tension, weight, acceleration, and equilibrium. It also governs elevator design, astronaut training (simulating weightlessness), amusement park rides, and even the sensation pilots experience during takeoff and landing.

Note on classification: While motion in a lift differs from traditional connected body problems (where we analyse forces between two separate masses connected by strings), we include it here because: (1) the person and lift move together with the same acceleration, similar to connected systems, (2) the analysis uses the same Newton's second law approach we have developed in this section, and (3) it demonstrates practical applications of forces in accelerating systems. Strictly speaking, this is an example of motion in an accelerating reference frame, where we analyse forces on a single body (the person) rather than interactions between connected bodies.

The physical setup:

Consider a person of mass **m** standing on a weighing scale inside a lift (elevator).

The weighing scale reads the **normal reaction R** between the person and the lift floor. It is sometimes called **apparent weight**. So:

$$\text{Apparent weight} = \text{Normal reaction (R)}$$

This may differ from the actual weight depending on the lift's motion.

Forces acting on the person:

- Weight: $W = mg$ (downward, always constant)
- Normal reaction, R from the lift floor (upward, not constant)

The **resultant force** on the person determines the acceleration according to Newton's second law.

With this setup in mind, we can now examine how the lift's motion affects the normal reaction and therefore the apparent weight by considering the different possible cases of motion.

Case 1: Lift at rest or moving with constant velocity

When the lift is stationary or moving at constant velocity (either upward or downward), the acceleration, **a = 0**. This is equilibrium in vertical motion.

Applying Newton's second law: $R - mg = 0$ or **R = mg**

Conclusion: *The normal reaction equals the actual weight. The person feels their normal weight.*

Case 2: Lift accelerating upward

When the lift accelerates upward with acceleration **a**, the person must also accelerate upward (they move together with the lift).

In this case, the resultant force on a person must be upward to produce upward acceleration.

Applying Newton's second law:

$$R - mg = ma \text{ or } R = mg + ma$$

Therefore: **R = m(g + a)**

Conclusion: *The normal reaction, **R > mg**. The person feels **heavier** than usual.*

Physical interpretation: *The lift floor must push upward with extra force to accelerate the person upward. This extra push is what makes you feel heavier when a lift starts moving upward.*

Case 3: Lift accelerating downward

When the lift accelerates downward with acceleration **a**, the person also accelerates downward.

In this case, the resultant force on a person must be downward to produce downward acceleration.

Applying Newton's second law:

$$mg - R = ma \text{ or } R = mg - ma$$

Therefore: $R = m(g - a)$

Conclusion: The normal reaction, $R < mg$. The person feels lighter than usual.

Physical interpretation: The lift floor does not need to push as hard because gravity is helping to accelerate the person downward. This reduced push makes you feel lighter when a lift slows down while moving upward, or speeds up while moving downward.

Case 4: Free fall (cable breaks)

If the lift cable breaks and the lift falls freely under gravity alone, the acceleration equals the acceleration due to gravity: $a = g$ (downward).

Applying Newton's second law:

$$mg - R = ma$$

From which (as shown earlier):

$$R = m(g - a)$$

Substituting $a = g$

$$R = m(g - g) = 0$$

Conclusion: The normal reaction, $R = 0$. The person experiences **weightlessness**.

Physical interpretation: Both the person and the lift fall together at the same rate. The lift floor no longer pushes against the person's feet (they do not "press" against each other). This is exactly how astronauts experience weightlessness in orbit. Fortunately, real lifts have strong safety systems (including emergency brakes and shock absorbers) preventing true free fall.

Summary Table: Apparent Weight in Different Lift Conditions

Lift Motion	Acceleration	Normal reaction	Apparent Weight R	Sensation
At rest	$a = 0$	$R = mg$	Normal	Normal weight
Constant velocity (up or down)	$a = 0$	$R = mg$	Normal	Normal weight
Accelerating upward	a (upward)	$R = m(g + a)$	Increased	Feel heavier
Decelerating upward	a (downward)	$R = m(g - a)$	Decreased	Feel lighter
Accelerating downward	a (downward)	$R = m(g - a)$	Decreased	Feel lighter
Decelerating downward	a (upward)	$R = m(g + a)$	Increased	Feel heavier
Free fall	$a = g$ (downward)	$R = 0$	Zero	Weightless

A crucial distinction to remember:

- Accelerating upward means either starting to move upward **or slowing down while moving downward**.
- Accelerating downward means either starting to move downward **or slowing down while moving upward**.

Always focus on the direction of acceleration, not the direction of motion.

With the ideas now simmering nicely, let us serve them properly through a few worked examples and enjoy the flavour of physics in action.

BINDER Example 31

A person of mass 70kg stands on a weighing scale inside a lift. Take $g = 9.8 \text{ m/s}^2$. Determine the scale reading (in kg):

- (a) When the lift is at rest.
- (b) When the lift accelerates upward at 3 m/s^2 ?
- (c) When the lift accelerates downward at 2.5 m/s^2 ?

Solution

- (a) When at rest, acceleration $a = 0$.

$$R = mg = 70\text{kg} \times 9.8\text{m/s}^2 = 686\text{N}$$

Scale reading in kilograms:

$$\text{Reading} = \frac{R}{g} = \frac{686\text{N}}{9.8\text{m/s}^2} = 70\text{kg}$$

The scale reads 70kg.

- (b) When lift is accelerating upward:

$$R = m(g + a) = 70\text{kg}(9.8 + 3)\text{m/s}^2 = 896\text{N}$$

Scale reading in kilograms:

$$\text{Reading} = \frac{896\text{N}}{9.8\text{m/s}^2} = 91.4\text{kg}$$

The scale reads 91.4kg.

- (c) When lift is accelerating downward:

$$R = m(g - a) = 70\text{kg}(9.8 - 2.5)\text{m/s}^2 = 511\text{N}$$

Scale reading in kilograms:

$$\text{Reading} = \frac{511\text{N}}{9.8\text{m/s}^2} = 52.1\text{kg}$$

The scale reads 52.1kg.

Making Sense of the Answer: The actual weight (686N) never changes, that is determined by mass and gravity. What changes is the normal reaction (what you feel), which depends on acceleration. When accelerating upward, the floor must push harder (896N) to accelerate you. When accelerating downward, the floor pushes less (511N) because gravity does some of the work.

Thinking Like a Physicist: The percentage change in apparent weight depends on the ratio a/g . For $a = 3 \text{ m/s}^2$, the change is $\frac{896\text{N} - 686\text{N}}{686\text{N}} = \frac{3\text{m/s}^2}{9.8\text{m/s}^2} \approx 31\%$. Similarly for $a = 2.5 \text{ m/s}^2$, it is 26%. Fast elevators in skyscrapers typically limit acceleration to about 2 m/s^2 for passenger comfort since larger accelerations would be uncomfortable and potentially dangerous.

REAL Example 32

Kipute and Kipanga visit their aunt who lives on the 20th floor of a tall building in Dar es Salaam. As they ride the elevator up, Kipanga places his school bag (mass 8kg) on a weighing scale on the elevator floor.

Kipanga: "Look, Kipute! When the lift started moving, the scale showed 10kg for a moment. But my bag is only 8 kg!"

Kipute: "That's because the lift was accelerating upward. The scale had to push harder against the bag to make it accelerate. What does it show now?"

Kipanga: "Now it shows exactly 8kg. We must be moving at constant velocity."

Kipute: "Correct! And watch what happens when we approach the 20th floor..."

(The lift begins to slow down)

Kipanga: "Wow! Now it shows only 6kg! Did my bag lose weight?"

Kipute: (laughing) "No, silly! The lift is decelerating, it's slowing down while moving upward, which means it's accelerating downward. So the scale pushes less against the bag."

Question:

- (a) Calculate the upward acceleration of the lift when the scale read 10kg.
- (b) Calculate the downward acceleration when the scale read 6kg.
- (c) If the lift cable were to break, what would the scale read?

$$\text{Take } g = 9.8 \text{ m/s}^2.$$

Solution

- (a) The scale reading represents the normal reaction R.

$$R = 10\text{kg} \times 9.8\text{m/s}^2 = 98\text{N}$$

For upward acceleration:

$$R = m(g + a)$$

Substituting:

$$98\text{N} = 8\text{kg}(9.8\text{m/s}^2 + a); a = 2.45 \text{ m/s}^2$$

The lift was accelerating upward at 2.45 m/s^2 .

- (b) When scale reads 6 kg:

$$R = 6\text{kg} \times 9.8\text{m/s}^2 = 58.8\text{N}$$

For downward acceleration (deceleration while moving up):

$$R = m(g - a)$$

$$58.8\text{N} = 8\text{kg}(9.8\text{m/s}^2 - a); a = 2.45 \text{ m/s}^2$$

The lift was accelerating downward at 2.45 m/s^2 .

- (c) In free fall: $a = g$.

$$R = m(g - g) = 0$$

The scale would read 0kg.

Making Sense of the Answer: Interestingly, the magnitude of acceleration is the same (2.45m/s^2) whether speeding up or slowing down. This is typical of well-designed elevator that maintains constant acceleration/deceleration rates for smooth rides. The scale reading changed by $\pm 2\text{kg}$ (from 8 to 10, or 8 to 6), which represents a $\pm 25\%$ change. This corresponds to $a/g = 2.45/9.8 = 0.25$ or 25%.

Thinking Like a Physicist: Elevator designers must balance several competing factors: velocity (to move people quickly), acceleration (affects comfort and safety), and cable strength (must support apparent weight during upward acceleration, which exceeds actual weight). Modern high-speed elevators use sophisticated control systems to minimize jerky motion and thus they gradually increase acceleration at the start and gradually decrease it before stopping.

HOT Example 33

A person of mass 60kg stands in a lift and holds a 5kg bag suspended by a light string.

- (a) Calculate the tension in the string when the lift moves upward with constant velocity of 2m/s .
- (b) Calculate the tension in the string when the lift accelerates upward at 2.5 m/s^2 .
- (c) Calculate the tension in the string when the lift accelerates downward at 3m/s^2 .
- (d) The person feels they can comfortably hold a tension of up to 70N. What is the maximum acceleration the lift can have without the person dropping the bag?
- (e) Calculate the total force that the lift floor exerts on the person for case (b). Explain why this differs from the person's weight.

$$\text{Take } g = 9.8 \text{ m/s}^2.$$

Solution

(a) Forces acting on the bag:

- Tension: T (upward)
- Weight: $W = mg$ (downward)

Applying Newton's second law:

$$T - mg = ma$$

But at constant velocity, acceleration $a = 0$.

$$T - mg = 0 \text{ or } T = mg = 5\text{kg} \times 9.8\text{m/s}^2 = 49\text{N}$$

The tension is 49N.

(b) If the lift accelerating upward at 2.5 m/s^2 , the bag must also accelerate upward at 2.5 m/s^2 .

Again using:

$$T - mg = ma \text{ or } T = mg + ma = m(g + a) = 5\text{kg}(9.8 + 2.5)\text{m/s}^2 = 61.5\text{N}$$

The tension is 61.5N.

(c) If the lift is accelerating downward:

$$mg - T = ma \text{ or } T = mg - ma = m(g - a) = 5\text{kg}(9.8 - 3)\text{m/s}^2 = 34\text{N}$$

The tension is 34N.

(d) When the lift accelerates downward, the tension decreases; when it accelerates upward, the tension increases. Therefore, for the tension to rise to the limiting value of 70N, the lift must be accelerating upward.

For upward acceleration:

$$T - mg = ma \text{ or } T = mg + ma$$

From which:

$$a = \frac{T - mg}{m}$$

It follows that:

$$a_{\max} = \frac{T_{\text{lim}} - mg}{m} = \frac{70\text{N} - 5\text{kg} \times 9.8\text{m/s}^2}{5\text{kg}} = 4.2\text{m/s}^2$$

Hence, the maximum acceleration is 4.2m/s^2 **upward**.

Understand that: Above this acceleration, the required tension would exceed 70N and the person would drop the bag.

(e) Forces acting on a person (holding a bag):

- Normal reaction: R (upward)
- Tension: T (downward)
- Weight of the person: mg

Applying Newton's second law (for upward acceleration):

$$R - T - mg = ma \text{ or } R = ma + T + mg$$

Substituting:

$$R = 60\text{kg} \times 2.5 \text{ m/s}^2 + 61.5\text{N} + 60\text{kg} \times 9.8\text{m/s}^2 = 799.5\text{N}$$

The total force that the lift floor exerts on the person is 799.5N.

Alternative solution

Treating both a person and a bag as one object: $m_t = (60 + 5)\text{kg} = 65\text{kg}$

Applying Newton's second law:

$$R - m_t g = m_t a \text{ or } T = m_t g + m_t a = m_t (g + a) = 65\text{kg}(9.8 + 2.5)\text{m/s}^2 = 799.5\text{N}$$

The total force is 799.5N.

Explanation:

This differs from the person's actual weight (588 N) because the person is accelerating upward, which requires an additional upward force. In addition, the bag being held must also be accelerated upward. Consequently, the floor must provide enough force to accelerate the entire system (person plus bag).

Making Sense of the Answer: *Constant velocity means zero acceleration, so the tension equals the bag's weight (49N). Upward acceleration increases the required tension, while downward acceleration reduces it. The floor force on the person is larger than their weight because it must both accelerate the person upward and counter the downward pull from the bag.*

Thinking Like a Physicist: *Only acceleration changes forces, not velocity. Decide the direction of acceleration first, then apply Newton's second law. Remember: upward acceleration increases apparent weight and tension; downward acceleration decreases them.*

That brings our subtopic-by-subtopic worked examples to a satisfying close. The plates are cleared! Now let us enjoy the full buffet, where all the ideas of this topic come together in miscellaneous worked examples.

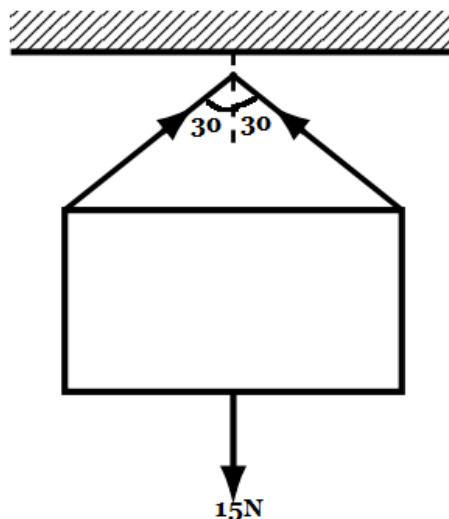
MISCELLANEOUS WORKED EXAMPLES ON EQUILIBRANT FORCES**Example 34**

- (a) Explain why a tow rope becomes very tight when a vehicle suddenly accelerates while pulling a trailer.
 (b) A picture frame of weight 15N hangs from a single nail by a string. The string makes an angle of 30° with the vertical on each side of the nail.
 (i) Calculate the tension in the string.
 (ii) Calculate the horizontal force exerted by the nail on each side of the string.

Solution

- (a) When the vehicle accelerates suddenly, the trailer tends to remain in its original state of motion due to inertia. The tow rope must then provide a large force (tension) to accelerate the trailer in a short time. This sudden increase in required force makes the rope to tighten sharply.
 (b) Each side of the string makes **30° with the vertical**, the system is symmetrical and hence tensions on both sides are equal (**T**).

The frame is in equilibrium as illustrated in the following diagram:



- (i) Resolving vertically:

Vertical component in each string = $T\cos 30^\circ$ (both upward)

$$\sum F_y = T\cos 30^\circ + T\cos 30^\circ - W = 0 \text{ or } 2T\cos 30^\circ = W \text{ or } T = \frac{W}{2\cos 30^\circ}$$

Substituting:

$$T = \frac{15\text{N}}{2\cos 30^\circ} = 8.66\text{N}$$

The tension T is 8.66N.

(ii) Resolving horizontally:

Horizontal component in each string = $T\sin 30^\circ$ (one left side, another right side)

Substituting:

$$8.66N\sin 30^\circ = 4.33N$$

The horizontal force exerted by the nail is 4.33N on each side of the string.

Example 35

- (a) A block rests on a rough inclined plane without sliding. Explain the role of friction in maintaining equilibrium and describe what would happen if the angle of the incline were gradually increased.
- (b) A wooden crate of mass 50kg is kept on a ramp inclined at 30° to the horizontal. The coefficient of static friction (μ_s) between the crate and ramp is 0.45 and that of kinetic friction (μ_k) is 0.3. Take $g = 9.8 \text{ m/s}^2$.
- Calculate the maximum angle at which the crate would remain in equilibrium without sliding.
 - Explain whether the crate is on the verge of sliding or if it could withstand a slight increase in the angle.
 - Calculate the frictional force acting on the crate.
 - Find the acceleration of the crate.

Solution

- (a) When the block rests on the inclined plane, the component of its weight down the plane ($mg\sin\theta$) tends to make it slide. Static friction acts up the plane to oppose this tendency and maintain equilibrium. As the angle of the incline increases, $mg\sin\theta$ increases while the normal reaction ($mg\cos\theta$) decreases, thereby reducing the maximum static friction available. Eventually, the limiting friction is reached where $\tan\theta = \mu_s$, and beyond this angle the block slides down the plane.
- (b) The solution of each part is as follows:
- The maximum angle is found when the system is at limiting equilibrium.

At limiting equilibrium:

$$f_{\max} = \mu_s R = \mu_s mg\cos\theta = \text{driving force} = mg\sin\theta$$

Taking:

$$\mu_s mg\cos\theta = mg\sin\theta;$$

$$\mu_s \cos\theta = \sin\theta;$$

$$\mu_s = \frac{\sin\theta}{\cos\theta} = \tan\theta;$$

$$\theta = \tan^{-1} \mu_s = \tan^{-1} 0.45 = 24.2^\circ$$

The maximum angle is 24.2° .

(ii) Compare:

$$\tan 30^\circ = 0.577 > 0.45(\mu_s)$$

Since $\tan 30^\circ > \mu_s$, the static friction is not enough to hold the crate.

So at 30° , the crate will slide (it is beyond the equilibrium limit). It cannot withstand any increase as it is already slipping.

(iii) Because it is sliding, friction is kinetic:

$$f_k = \mu_k R = \mu_k mg\cos 30^\circ = 0.3 \times 50\text{kg} \times 9.8\text{m/s}^2 \times \cos 30^\circ = 127\text{N}$$

The frictional force is 127N (up the plane).

(iv) Applying Newton's second law:

$$mg\sin 30^\circ - f_k = ma$$

$$mg\sin 30^\circ - \mu_k mg\cos 30^\circ = ma$$

$$a = \frac{mg\sin 30^\circ - \mu_k mg\cos 30^\circ}{m} = g\sin 30^\circ - \mu_k g\cos 30^\circ = g(\sin 30^\circ - \mu_k \cos 30^\circ)$$

Substituting:

$$a = 9.8\text{m/s}^2(\sin 30^\circ - 0.3\cos 30^\circ) = 2.35\text{m/s}^2$$

The acceleration is 2.35m/s^2 (down the plane).

Example 36

- (a) Explain why the tension in a string connecting two blocks being pulled across a horizontal surface is not equal to the applied force, even when the surface is smooth (frictionless).
- (b) Two blocks of masses $m_1 = 4\text{kg}$ and $m_2 = 6\text{kg}$ are connected by a light inextensible string on a smooth horizontal table. A horizontal force $F = 30\text{N}$ is applied to the 4kg block, pulling both blocks to the right. Take $g = 9.8\text{ m/s}^2$.
- Calculate the acceleration of the system.
 - Calculate the tension in the connecting string.
 - If the force F were instead applied to the 6kg block (pulling both blocks), how would the tension change? Calculate the new tension and explain why it differs from part (ii).

Solution

- (a) Even with no friction, the applied force is used to accelerate both blocks. However, the tension only needs to accelerate one of the blocks (whichever block is not directly receiving the applied force). Since part of the applied force is spent in accelerating the pulled block itself, only the remaining part is transmitted through the string. Therefore, tension is generally less than applied force.
- (b) Solution for each part is as follows

Identifying the system:

$m_1 = 4\text{kg}$ (block with applied force)

$m_2 = 6\text{kg}$ (block being pulled by string)

Surface: smooth (no friction).

- (i) Using Newton's second law for the whole system:

$$F = (m_1 + m_2)a \text{ or } a = \frac{F}{m_1 + m_2} = \frac{30\text{N}}{(6+4)\text{kg}} = 3\text{m/s}^2$$

The acceleration is 3m/s^2 .

- (ii) Consider the 6kg block alone (it only experiences tension):

$$T = m_2a = 6\text{kg} \times 3\text{m/s}^2 = 18\text{N}$$

The tension in the string is 18N .

- (iii) If the force (30N) is applied to the 6kg block instead; the acceleration remains the same (same total force, same total mass). But now tension must accelerate the 4kg block (not the 6kg block):

$$T = m_1a = 4\text{kg} \times 3\text{m/s}^2 = 12\text{N}$$

The tension would decrease to 12N .

Explanation of difference:

New tension (12N) is less than the previous one (18N) because tension now accelerates the lighter 4kg (smaller mass) block instead of the heavier 6kg (larger mass) block.

Example 37

- (a) A block of mass m rests on a rough plane inclined at angle θ to the horizontal. A force F is applied up the plane. The coefficient of static friction is μ_s . Prove that the block can remain in equilibrium only if:
- $$mg(\sin\theta - \mu_s\cos\theta) \leq F \leq mg(\sin\theta + \mu_s\cos\theta)$$
- (b) A package of mass 30kg rests on a rough plane inclined at 15° to the horizontal. A force of 220N is applied up the plane. If the coefficients of friction are $\mu_s = 0.40$ and $\mu_k = 0.30$ for static and kinetic friction respectively, determine:
- whether the package remains at rest or moves,

- (ii) its acceleration.

$$\text{Take } g = 9.8\text{m/s}^2.$$

Solution

(a) Proof:

$$\text{Normal reaction: } R = mg\cos\theta$$

$$\text{Maximum static friction: } f_{\max} = \mu_{sR} = \mu_s mg\cos\theta$$

Static friction adjusts in magnitude (0 to $f_{\max}$) and acts opposite to the tendency of motion.

Lower limit of F (tendency of motion: down the plane):

If F is small ($F < mg\sin\theta$), the block tends to slide down the plane, so friction acts up the plane (in the same direction as F but opposite to $mg\sin\theta$).

For equilibrium at the limiting case (just about to slide down):

$$F + f_{\max} = mg\sin\theta$$

So:

$$F = mg\sin\theta - \mu_s mg\cos\theta$$

$$F = mg(\sin\theta - \mu_s \cos\theta) = \text{Minimum } F \text{ to prevent downward sliding}$$

Hence, for equilibrium:

$$F \geq mg(\sin\theta - \mu_s \cos\theta)$$

Upper limit of F (tendency of motion: up the plane):

If F is large ($F > mg\sin\theta$), the block tends to move up the plane, so friction acts down the plane (in the same direction as $mg\sin\theta$ but opposite to F).

For equilibrium at the limiting case (just about to move up):

$$F = mg\sin\theta + f_{\max}$$

So:

$$F = mg\sin\theta + \mu_s mg\cos\theta$$

$$F = mg(\sin\theta + \mu_s \cos\theta) = \text{Maximum } F \text{ just before upward sliding}$$

Hence, for equilibrium:

$$F \leq mg(\sin\theta + \mu_s \cos\theta)$$

Combining the two conditions gives:

$$mg(\sin\theta - \mu_s \cos\theta) \leq F \leq mg(\sin\theta + \mu_s \cos\theta)$$

(b) Solution for each part is as follows:

- (i) Comparing F and $mg\sin\theta$ to determine the tendency of motion:

$$mg\sin\theta = 30\text{kg} \times 9.8\text{m/s}^2 \times \sin 15^\circ = 76.1\text{N} < 220\text{N} \text{ (applied force } F)$$

Since the applied force is greater than the weight component acting along the plane, the tendency of motion is up the plane. So the applied force has upper limit for achieving limiting equilibrium and it is given by (from (a)):

$$F = mg(\sin\theta + \mu_s \cos\theta) = F_{\max}$$

Substituting:

$$F_{\max} = 30\text{kg} \times 9.8\text{m/s}^2 (\sin 15^\circ + 0.4 \cos 15^\circ) = 189.7\text{N} < 220\text{N} \text{ (applied force)}$$

Since the applied upward force is greater than opposing (resistive) forces, the package **moves** up the plane.

- (ii) As the package moves up the plane, the **kinetic** frictional force acts down the plane.

But the kinetic friction is given by:

$$f_k = \mu_k R = \mu_k mg \cos \theta = 0.3 \times 30 \text{ kg} \times 9.8 \text{ m/s}^2 \cos 15^\circ = 85.2 \text{ N}$$

$$\text{Total resistive forces} = mg \sin \theta + f_k = 76.1 \text{ N} + 85.2 \text{ N} = 161.3 \text{ N}$$

Applying Newton's second law:

$$\text{Applied force} - \text{Resistive forces} = ma \text{ or } a = \frac{\text{Applied force} - \text{Resistive forces}}{m}$$

Substituting:

$$a = \frac{(220 - 161.3) \text{ N}}{30 \text{ kg}} = 1.96 \text{ m/s}^2$$

The acceleration is 1.96 m/s^2 .

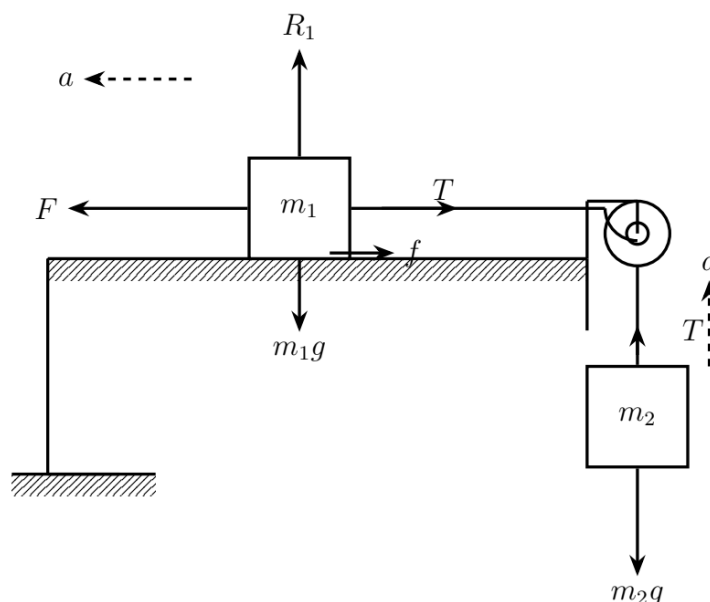
Example 38

- (a) Explain why does a hanging signboard remain at rest even though several forces act on it?
- (b) A block of mass 6 kg rests on a rough horizontal table with coefficient of static friction, $\mu_s = 0.4$ and coefficient of kinetic friction $\mu_k = 0.3$. It is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a hanging mass of 2 kg . A horizontal force $F = 50 \text{ N}$ is applied to the 6 kg block pulling it away from the pulley. The system is released from rest.
- Determine whether the system will move and, if so, in which direction.
 - Calculate the acceleration of the system and the tension in the string.
 - What is the minimum value of the applied force, F , required to prevent the hanging mass from descending?

$$\text{Take } g = 9.8 \text{ m/s}^2.$$

Solution

- (a) The signboard remains at rest because the forces acting on it are balanced. The upward components of the tensions in the supporting strings add up to equal the weight of the signboard, while the horizontal components cancel each other. Therefore, the resultant force is zero and hence there is no acceleration.
- (b) Consider the following diagram.



- (i) Identifying forces trying to move the system:
- Driving force: $F = 50 \text{ N}$ (try to move the system **away** from pulley, leftward).
 - Weight of hanging mass, m_2 : $W = m_2 g = 2 \text{ kg} \times 9.8 \text{ m/s}^2 = 19.6 \text{ N}$ (try to move the system **toward** pulley, rightward).

So: Driving force(50N) > Weight of hanging mass(19.6N). From this alone we can conclude that the system tries to move away from pulley.

For actual movement to occur, the driving force must overcome **all** resisting forces (static friction and weight of hanging mass).

The static friction is maximum when: $f_{\max} = \mu_s R_1 = \mu_s m_1 g = 0.4 \times 6\text{kg} \times 9.8\text{ m/s}^2 = 23.52\text{N}$

So total resistive forces, the driving force has to overcome for actual motion to occur is:

$$(19.6 + 23.52)\text{N} = 43.12\text{N} < 50\text{N}(\text{Driving force})$$

Since the driving force is greater than total resistive forces, the system move.

Conclusion: Yes, the system moves with m_1 going leftward (away from pulley) and m_2 ascending. This is because the applied force $F = 50\text{N}$ exceeds the combined resistance from the weight of hanging mass and static friction.

- (ii) Once the system moves, the static friction is no longer functioning and the kinetic friction takes place. Thus:

$$\text{Friction force, } f_k = \mu_k m_1 g = 0.3 \times 6\text{kg} \times 9.8\text{ m/s}^2 = 17.64\text{N}$$

So total resistive forces during the motion = $W_{m_2} + f_k = (19.6 + 17.64)\text{N} = 37.24$

The net driving force, will accelerate both m_1 and m_2 . Therefore:

$$\text{Driving force} - \text{Resistive force} = (m_1 + m_2)a$$

Substituting:

$$50\text{N} - 37.24\text{N} = (6\text{kg} + 2\text{kg})a; a = 1.595\text{m/s}^2$$

The acceleration is 1.595m/s^2 .

For hanging mass:

$$T - m_2 g = m_2 a \text{ or } T = m_2 a + m_2 g = m_2(a + g) = 2\text{kg}(1.595 + 9.8)\text{m/s}^2 = 22.79\text{N}$$

The tension is 22.79N.

- (iii) For m_2 not to descend, the system must either be at rest or moving leftward. The critical condition is when m_2 is on the verge of descending (*limiting equilibrium with $a = 0$*).

At limiting equilibrium:

- Friction force is equal to the maximum static friction = 23.52N.
- Total resistive forces = $f_{\max} + W_{m_2} = (23.52 + 19.6)\text{N} = 43.12\text{N}$

So:

- If driving force, $F < 43.12\text{N}$, the hanging mass descends (pulls m_1 rightward).
- If $F \geq 43.12\text{N}$, the hanging mass is not descending.

Hence, the minimum value of the applied force required to prevent the hanging mass from descending is 43.12N.

Example 39

- (a) Why do passengers sometimes feel heavier when a lift starts moving upward?
- (b) Two bodies of masses 3kg and 5kg are connected by a light inextensible string passing over a smooth pulley and are released from rest while hanging freely. After the system has been moving for 1.5s, the string suddenly snaps just above the 5kg body. Take $g = 9.8\text{m/s}^2$. Calculate:

- (i) The time taken for the 5kg body to reach the ground after the string snaps, given that at the instant the string snaps it is 2m above the ground.
- (ii) The maximum height reached by the 3kg body above its initial position if the two bodies were initially at the same level.
- (iii) The total time taken by the 3kg body to reach the ground.

Solution

- (a) When the lift accelerates upward, the floor must push up on passengers with a force greater than their weight in order to accelerate them upward. This increased normal reaction is what make them feel heavier.
- (b) Before the string snapped, the system was undergoing motion similar to an Atwood machine.

For the Atwood machine:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

Where:

$$m_2 = 5\text{kg (heavier)}$$

$$m_1 = 3\text{kg (lighter)}$$

Substituting:

$$a = \left(\frac{5\text{kg} - 3\text{kg}}{3\text{kg} + 5\text{kg}} \right) \times 9.8\text{m/s}^2 = 2.45\text{m/s}^2$$

Velocity at the instant of snapping:

$$v = at = 2.45\text{m/s}^2 \times 1.5\text{s} = 3.675\text{m/s}$$

Distance moved in 1.5s (each body):

$$s = ut + 0.5at^2 = 0\text{m/s} \times 1.5\text{s} + 0.5 \times 2.45\text{m/s}^2 \times (1.5\text{s})^2 = 2.756\text{m}$$

So, at the instant the string snaps:

- The 5kg mass is moving downward at 3.675m/s.
- The 3kg mass is moving upward at 3.675m/s.
- Each body moved through a vertical distance of 2.756m.

- (i) After snapping, the 5kg body moves under gravity with initial downward velocity 3.675m/s.

Using:

$$s = ut + 0.5gt^2$$

Substituting:

$$2 = 3.675t + 0.5(9.8)t^2 \text{ or } 4.9t^2 + 3.675t - 2 = 0$$

Solving the quadratic equation, gives the practical value of t which is:

$$t = 0.37\text{s}$$

The time taken for the 5kg body to reach the ground is 0.37s.

- (ii) After snapping, the 3kg mass continues to rise with a deceleration of 9.8m/s^2 until its velocity becomes zero at the highest point.

Using:

$$v^2 = u^2 + 2as; \text{ with } v = 0, u = 3.675\text{m/s}, a = -9.8\text{m/s}^2, s = h$$

Substituting:

$$0^2 = (3.675\text{m/s})^2 - 2(9.8\text{m/s}^2)h; h = 0.689\text{m}$$

Height gained before snapping was $s = 2.756\text{m}$, so maximum height above initial position:

$$H_{\text{max}} = s + h = 2.756\text{m} + 0.689\text{m} = 3.445\text{m}$$

The maximum height is 3.445m.

- (iii) At snapping, the 5kg body is 2m above ground, and it has descended $s = 2.756\text{m}$ since release, so its initial height above ground was:

$$H_0 = 2\text{m} + 2.756\text{m} = 4.756\text{m}$$

The 3kg body started at the same level, so it also started 4.756m above ground.

It rose by $s = 2.756\text{m}$ before snapping, so its height above ground at snapping was:

$$H_{\text{snap}} = 4.756\text{m} + 2.756\text{m} = 7.512\text{m}$$

The time to reach the ground from the snapping time can be found by using:

$$s = ut + 0.5at^2; \text{ with } s = -7.512\text{m}, u = 3.675\text{m/s}, a = -9.8\text{m/s}^2$$

Substituting:

$$-7.512 = 3.675t - 0.5(9.8)t^2 \text{ or } 4.9t^2 - 3.675t - 7.512 = 0$$

Solving the quadratic equation, gives the practical value of t which is:

$$t = 1.67\text{s}$$

Total time from release to ground for the 3kg body $= 1.5 + t = 1.5\text{s} + 1.67\text{s} = 3.17\text{s}$

The total time is 3.17s.

If the ideas now feel familiar and connected, you are ready! Let us understand and enjoy them even more in the Digging Deeper Exercise in the next page.

DIGGING DEEPER EXERCISE 4

EXERCISE 4A: BINDER QUESTIONS

Question 1

Two ropes are tied to a ring and pulled in different directions. A third rope must be attached to keep the ring in equilibrium. Explain why the direction of the third rope is uniquely determined by the first two ropes.

Question 2

In an Atwood machine with two unequal masses, the lighter mass accelerates upward. Explain why the tension in the string is **not** equal to the weight of either mass.

Question 3

A block on an inclined plane experiences a normal reaction perpendicular to the surface. Explain why the normal reaction is less than the block's weight.

Question 4

When you stand in a lift that accelerates upward, you feel heavier. Explain what actually changes and what remains constant.

Question 5

Two blocks are connected by a string on a horizontal surface. When pulled by a force F , they accelerate together. Explain why the string tension is less than the applied force F .

Question 6

A block rests on a rough inclined plane at angle θ below the critical angle. Explain why friction acts up the slope even though no other force is pulling the block upward.

Question 7

In a pulley system, the string is described as "light and inextensible." Explain the physical significance of each of these terms.

Question 8

A car tows a trailer on a horizontal road. The tension in the tow-bar is less when traveling at constant velocity than when accelerating. Explain why.

Question 9

A student writes: "*Tension always equals weight for a hanging object.*" Give a counter-example using a system in equilibrium and explain why tension can differ from weight.

Question 10

Two strings hold a load at a junction. One string is closer to horizontal than the other. Without calculation, explain which string must have the larger tension and why.

EXERCISE 4B: REAL QUESTIONS

Question 11

A market vendor in Kariakoo hangs a 20 kg bag of rice from a ceiling hook using a single rope. The rope makes a V-shape with both sides going up to the same hook. Explain why the tension in each side of the rope is greater than half the weight of the bag.

Question 12

A bus traveling from Dar es Salaam to Mwanza carries luggage on the roof rack. The ropes tying down the luggage must be tightened periodically during the journey. Explain why the ropes become loose even without any luggage shifting position.

Question 13

Kipute and Kipanga are helping load maize sacks onto a pickup truck using a wooden plank as a ramp.

Kipanga: "Let's make the ramp steeper; it's shorter, so we'll move the sacks faster!"

Kipute: "But won't a steeper ramp make it harder to push the sacks up?"

Mr. Akilikubwa: "Kipute has a good point. Let me ask you both: what happens to the force needed to push a sack up the ramp as we increase the angle?"

Explain why a steeper ramp requires more force to push the same sack at constant velocity, even though the distance is shorter.

Question 14

A traditional water well uses a rope over a pulley to raise a bucket. An experienced well user knows to pull the rope smoothly and steadily rather than in quick jerks. Explain why jerking the rope increases the risk of the rope breaking.

Question 15

A daladala climbs a steep road while carrying many passengers. When it stops on the slope, the driver says, "If the handbrake is weak, the bus will slide even if the engine is off." Explain what force must act to prevent sliding and why it is required even when the bus is at rest.

Question 16

Kipanga leans a wooden plank against a rough wall. Sometimes it slips down, but when he pushes the bottom of the plank harder into the floor, the plank stops slipping. Explain why pressing harder can help prevent slipping.

Question 17

A lorry carrying sand is tied down using a rope that slopes downward toward the trailer. The driver claims: "The rope helps in two ways: it stops the load from sliding sideways and also presses it down more firmly." Explain how one rope can do both jobs at the same time.

Question 18

A signboard is fixed to a wall using a hinge and supported by a cable at an angle. When the cable is removed, the signboard falls. Explain why the hinge alone cannot keep the signboard in equilibrium.

EXERCISE 4C: HOT QUESTIONS

Take $g = 9.8 \text{ m/s}^2$

Question 19

A picture frame of mass 2.5kg is suspended from a nail by a string. The string makes an angle of 35° with the vertical on each side of the nail. Calculate:

- (a) The tension in the string
- (b) The horizontal force exerted by the nail on the string.

Question 20

An Atwood machine consists of masses 4kg and 7kg connected by a light inextensible string over a smooth pulley. The system is released from rest. Calculate:

- (a) The acceleration of the system.
- (b) The tension in the string.
- (c) The distance traveled by the 7kg mass in the first 2 seconds.

Question 21

Two blocks of masses 6kg and 4kg are connected by a light string on a smooth horizontal table. A horizontal force of 35N is applied to the 6kg block.

- (a) Calculate the acceleration of the system.
- (b) Calculate the tension in the connecting string.
- (c) If the surface had coefficient of kinetic friction 0.3, what would be the new acceleration?

Question 22

A block of mass 8 kg rests on an inclined plane at 25° to the horizontal. The coefficient of static friction is 0.45 and the coefficient of kinetic friction is 0.35.

- (a) Determine whether the block remains at rest or slides down.
- (b) If a force of 40N is applied parallel to and up the incline, calculate the acceleration of the block.

Question 23

An Atwood machine has masses $m_1 = 2$ kg and m_2 . When released, the system accelerates at 2.5m/s^2 and m_2 moves downward.

- (a) Calculate the mass m_2 .
- (b) The tension in the string.
- (c) If the initial separation between the masses is 3m, how long until they are at the same height?

Question 24

A lift of mass 800 kg carries 4 passengers each of mass 70kg. Calculate:

- (a) The tension in the lift cable when accelerating upward at 1.5m/s^2
- (b) The tension when moving upward at constant velocity of 3m/s
- (c) The tension when decelerating at 2m/s^2 while moving upward.

Question 25

Two blocks **A** (5kg) and **B** (3kg) are connected by a string. Block **A** is on a smooth horizontal table and block **B** hangs over the edge via a pulley. Block **B** is initially held at rest 2m above the ground. When released:

- (a) Calculate the acceleration of the system.
- (b) Calculate the velocity of **B** just before it hits the ground.
- (c) How far does **A** travel before **B** hits the ground?

Question 26

Three blocks of masses 2kg, 3kg, and 4kg are connected by light inextensible strings on a smooth horizontal surface. A force of 45N is applied to the 2kg block. Calculate:

- (a) The acceleration of the system.
- (b) The tension in the string between the 2kg and 3kg blocks.
- (c) The tension in the string between the 3kg and 4kg blocks.

Question 27

Two blocks lie on smooth opposite inclines and are connected by a light string over a smooth pulley. Block A has mass 10kg on a 35° incline. Block B has mass 8kg on a 25° incline. An additional force P is applied to block A parallel to the plane to keep the system in equilibrium. Find:

- (a) the tension in the string,
- (b) the magnitude and direction of P (state whether it acts up or down the plane on A).

Question 28

A 8kg block rests on a smooth plane inclined at 30° and is connected by a light string over a smooth pulley to a hanging 6kg mass. The system is in equilibrium due to friction between the block and the plane. Find:

- (a) the tension in the string,
- (b) the frictional force on the block (magnitude and direction along the plane),
- (c) the minimum coefficient of friction required.

Question 29

A load of 120 N is suspended from a junction by two light strings. The string on the left makes an angle of 30° with the horizontal, while the string on the right makes an angle of 45° with the horizontal.

- (a) Without calculation, state which string has the greater tension, and why this is expected.
- (b) Calculate the tension in each string.

Question 30

A person stands on a weighing scale in a lift. The scale reading is 20% greater than the person's true weight.

- (a) Determine the acceleration of the lift and state its direction.
- (b) If the person's mass is 75kg, calculate the force exerted by the scale on the person.

Chapter 5

MOMENTUM AND COLLISIONS**INTRODUCTION**

Imagine this scene: It is a busy afternoon at Kariakoo market in Dar es Salaam. A heavily loaded *mkokoteni* (handcart), stacked high with sacks of rice, is being pushed steadily by a strong young man. From the opposite direction, a small boy on a bicycle riding with no hands and showing off to his friends comes speeding down the narrow road between the small shops.

Everyone sees what is coming. The boy's mother shouts. Vendors scatter. And then... CRASH!

When the dust settles, the bicycle has stopped completely and is now moving backwards. The *mkokoteni* has barely slowed down and continues forward, now with a bent wheel and a very embarrassed boy stuck in the sacks of rice.

Why did the light bicycle bounce backward while the heavy *mkokoteni* barely noticed the collision? Why could the boy's small mass not stop the massive cart, no matter how fast he was going? And most importantly, why did the market women immediately start laughing and taking photos?

Welcome to the fascinating world of momentum and collisions where size and velocity combine to create effects that can't be explained by force or mass alone. You will discover why a slow-moving truck is harder to stop than a speeding motorcycle, why airbags save lives, how rockets work in the vacuum of space (where there is nothing to "push against"), and why relaxing your body during a fall can actually reduce injury.

By the end of this chapter, you will understand the fundamental principle that governs everything from billiard balls to planetary motion, from car crashes to the recoil of a gun. You will even understand why that boy on the bicycle should have been paying attention to physics class instead of showing off!

MOMENTUM

In **Chapter 2**, we learned about velocity, how fast an object moves and in which direction. In **Chapter 3**, we discovered Newton's laws and learned that force causes changes in motion. We even encountered Newton's second law in its most general form:

$$F = \frac{dp}{dt}$$

Where p represents a quantity called **momentum**. This tells us that force does not simply change velocity; it changes momentum. To understand why this distinction matters, imagine the following two situations:

Situation 1: A 10kg stone falls from a second-floor window onto your toe. Ouch! That hurts.

Situation 2: The same 10kg stone is handed gently to you. You catch it comfortably.

In both cases, the stone has the same mass (10kg) and experiences the same gravitational force. Yet the effects are dramatically different. What is missing from our description? The answer is velocity. The falling stone is moving rapidly when it reaches you, while the handed stone has almost no motion. To fully describe such situations, we need a quantity that combines both mass and velocity. This combined effect of mass and velocity is what we call **momentum**.

Concisely, **momentum** is the product of an object's mass and its velocity.

$$p = mv$$

Where:

p = momentum (measured in kgm/s)

m = mass (kg)

v = velocity (m/s)

Key insights about momentum

Now that we have defined momentum, it is useful to highlight what this definition really tells us. The following points summarise its most important features:

1. **Momentum is a vector quantity:** Since velocity is a vector, momentum is also a vector. This means momentum has both magnitude and direction, and its direction is always the same as that of the velocity.
2. **Momentum depends on both mass and velocity:** Momentum exists only when both mass and velocity are present. An object at rest ($v = 0$) has zero momentum, no matter how large its mass may be.
3. **Momentum represents the “quantity of motion”:** A heavy object moving slowly can have the same momentum as a lighter object moving faster. In both cases, the overall amount of motion being carried is the same.

Impulse and the Impulse-Momentum Theorem

We have learned that Newton's second law in its most general form tells us that **force changes momentum**. But *how exactly does force change momentum? How much momentum change results from a given force? And does it matter whether the force acts for a long time or a short time?*

These questions lead us to one of the most practical and powerful concepts in mechanics: **impulse**.

Before exploring impulse in depth, let us first understand a closely related concept known as **impulsive forces**. To build clear intuition, consider the following everyday situations:

Situation 1: A cricket batsman hits a ball. The bat touches the ball for only about 0.01 seconds, yet the ball's momentum changes dramatically from moving toward the batsman to flying away at high speed.

Situation 2: When you catch a fast-moving ball, your hands experience a large force for a brief moment.

Situation 3: During a car collision, the passengers experience enormous forces, but only for a fraction of a second.

In all these cases, forces act for very short times. Such forces (*short-duration forces*) are difficult to measure directly because they start at zero, shoot up to very large values, then drop back to zero, all within milliseconds. We call these, **impulsive forces**.

How can we analyse motion when forces change so rapidly? The answer is surprisingly elegant: instead of tracking the force at every instant, we focus on the **combined effect** of force and time. This combined effect is called **impulse**.

Understanding impulse

When a force F acts on an object for a time interval Δt , the impulse J delivered to the object is defined as:

$$J = F\Delta t$$

Where:

J = impulse (measured in Ns or kgm/s)

F = force applied (N)

Δt = time interval during which the force acts (s)

So in a simple physical meaning, **impulse is the product of force and the time duration over which that force acts**.

The following insights will help you grasp the concept of impulse more clearly and see how it operates in real physical situations.

1. **Impulse is a vector quantity:** It has the same direction as the force that produces it.
2. **Same impulse, different ways:** The same impulse can be delivered by:
 - A **large force** acting for a **short time**, or
 - A **small force** acting for a **long time**.
3. **Impulse measures the "effort" of the force:** A strong force acting briefly can have the same effect as a weak force acting for a long time.

The Impulse-Momentum Theorem

Now comes the beautiful connection. Starting from Newton's second law:

$$F = \frac{dp}{dt}$$

We can rearrange this to:

$$F dt = dp$$

Integrate over a time interval:

$$\int_{t_1}^{t_2} F dt = \int_{p_1}^{p_2} dp$$

If F is **constant** (or F means **average force**), then the above integral becomes;

$$F(t_2 - t_1) = p_2 - p_1$$

But:

$$t_2 - t_1 = \Delta t \text{ and } p_2 - p_1 = \Delta p$$

It follows that:

$$F \Delta t = \Delta p$$

Remember $F \Delta t$ is exactly what we defined as impulse (J)!

Hence:

$$J = \Delta p$$

This equation is called the **Impulse-Momentum Theorem**. It states that:

Impulse delivered to an object is equal to the change in its momentum.

This theorem is extraordinarily useful because:

1. **It connects force, time, and momentum change:** You don't need to know how the force varies with time, only the average force and total time.
2. **It works for impulsive forces:** Even when forces change rapidly (like during collisions), we can still calculate momentum change without knowing the detailed force variation.
3. **It provides practical insights:** It explains many safety features and sports techniques we encounter daily.

Variable Forces and Average Force

In real situations, forces rarely stay constant. During a collision, the force might start at zero, increase to a maximum, then decrease back to zero. *How do we handle such situations?*

The answer is to use the **average force**, F_{avg} . The impulse-momentum theorem still works:

$$F_{\text{avg}} \times \Delta t = \Delta p$$

The **average force** is the constant force that, if it acted over the same time interval, would produce the same change in momentum.

This idea becomes even clearer when we represent the interaction graphically. If we plot force against time, the area under the force–time curve represents the impulse. This is because impulse is defined as the product of force and time. When the force varies with time, each small time interval contributes a small impulse. Adding all these contributions together gives the total impulse, which is represented graphically by the area under the force–time curve.

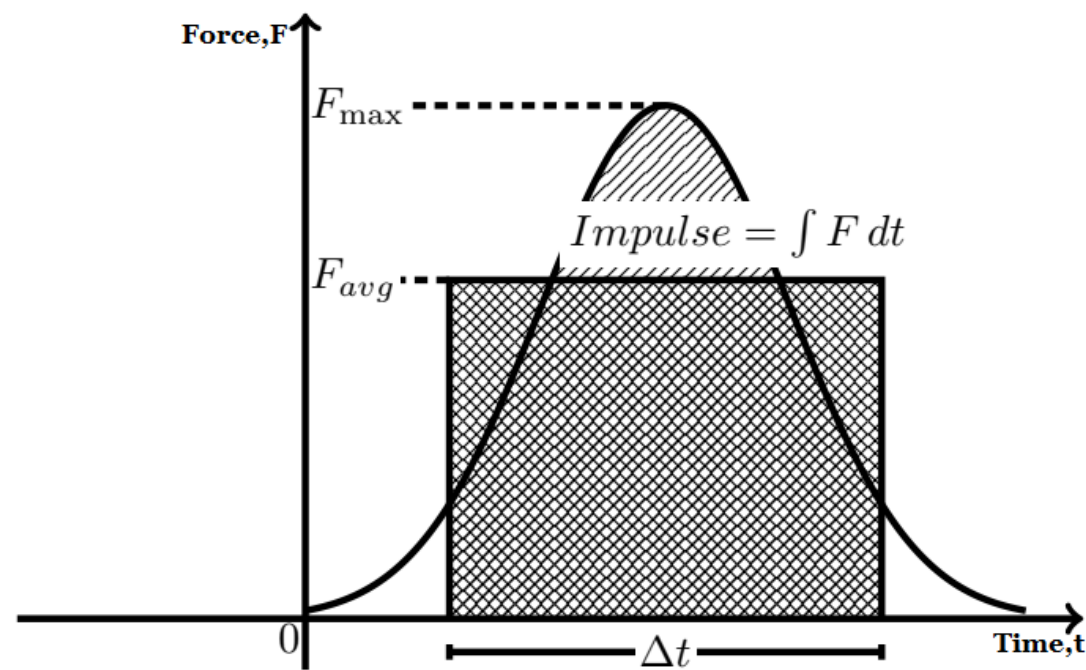


Figure: Force–time graph for a variable force during a collision. The shaded area under the curve represents the impulse, which is equal to the change in momentum. The rectangle shows the constant average force that produces the same impulse over the same time interval (Δt).

Practical Applications of Impulse-Momentum Theorem

The impulse-momentum theorem explains many real-life phenomena:

1. Why airbags save lives

An airbag increases the time Δt over which the momentum changes during a crash. Same Δp , but larger Δt means smaller average force F_{avg} on the body.

2. Why athletes "follow through"

In cricket or baseball, following through increases the time the bat contacts the ball, delivering more impulse and thus more momentum change to the ball.

3. Why you bend your knees when landing

Bending your knees increases the time Δt over which your momentum changes, thereby reducing the force on your legs.

4. Why a karate expert can break boards

By striking very quickly (small Δt), the expert delivers the required impulse with a very large force F .

5. Why it hurts more to catch a fast ball with stiff arms

Stiff arms mean short Δt , leading to large F . Relaxed, "giving" arms increase Δt , therefore reduce the force experienced.

With these foundational ideas firmly in place, let us now enjoy some carefully crafted worked examples that bring momentum to life.

BINDER Example 1

A tennis ball of mass 0.06kg moving at 20m/s is hit by a racket and rebounds in the opposite direction at 30m/s. The ball is in contact with the racket for 0.008s. Calculate:

- The change in momentum of the ball.
- The impulse delivered to the ball.
- The average force exerted by the racket on the ball.

Solution

Interpreting the data:

$$m = 0.06\text{kg}$$

Initial velocity: $u = 20\text{m/s}$ (let us take this as positive direction)

Final velocity: $v = -30\text{m/s}$ (opposite direction, hence negative)

Contact time: $\Delta t = 0.008\text{s}$

(a) Initial momentum:

$$p_1 = mu = 0.06\text{kg} \times 20\text{m/s} = 1.2\text{kgm/s}$$

Final momentum:

$$p_2 = mv = 0.06\text{kg} \times (-30\text{m/s}) = -1.8\text{kgm/s}$$

Change in momentum:

$$\Delta p = p_2 - p_1 = (-1.8 - 1.2)\text{kgm/s} = -3\text{kgm/s}$$

The negative sign indicates the change is in the opposite direction to the initial motion.

The change in momentum is -3kgm/s (or 3kgm/s in the opposite direction)

(b) Using the impulse-momentum theorem:

$$J = \Delta p = -3\text{Ns}$$

The impulse is 3Ns .

(c) Using:

$$J = F_{\text{avg}} \times \Delta t$$

From which:

$$F_{\text{avg}} = \frac{J}{\Delta t} = \frac{-3\text{Ns}}{0.008\text{s}} = -375\text{N}$$

The average force is -375N (or 375N in the direction opposite to initial motion).

Making Sense of the Answer: The force is quite large (375N , about 38kg -weight!) even though the ball is light. This happens because the momentum change is large (ball reversed direction) and the contact time is very short (0.008s). The negative sign means the force acts opposite to the ball's initial direction, which makes sense, since the racket pushed the ball backward.

Think Like a Physicist: When an object reverses direction, the momentum change is especially large because you must first stop the object (eliminate its initial momentum) then accelerate it in the opposite direction (add momentum in new direction). This is why hitting a ball back requires more force than simply catching it.

BINDER Example 2

A force of 50N acts on a body of mass 5kg initially at rest for 4 seconds. Calculate:

- (a) The impulse delivered to the body.
- (b) The final velocity of the body.

Solution

(a) Impulse is given by:

$$J = F\Delta t = 50\text{N} \times 4\text{s} = 200\text{Ns}$$

The impulse is 200Ns .

(b) Using impulse-momentum theorem:

$$J = \Delta p = p_2 - p_1$$

Since the body starts from rest ($p_1 = 0$):

$$J = p_2 = mv$$

It follows that:

$$v = \frac{J}{m} = \frac{200\text{kgm/s}}{5\text{kg}} = 40\text{m/s}$$

The final velocity is 40m/s.

Making Sense of the Answer: A 50N force acting for 4 seconds delivers substantial impulse (200Ns). Starting from rest, this impulse accelerates the 5kg body to 40m/s. We can verify this using $F = ma$: acceleration $a = F/m = 50/5 = 10\text{m/s}^2$, and $v = u + at = 0 + 10 \times 4 = 40\text{m/s}$ ✓.

Think Like a Physicist: The impulse-momentum approach ($J = \Delta p$) gave us the answer directly without needing to find acceleration first. This is the beauty of impulse as it connects force, time, and velocity change in one step.

REAL Example 3

One afternoon, **Kipanga** and **Kipute** are playing catch with a cricket ball. Kipanga throws the ball hard to Kipute. She catches it two different ways and notices something interesting.

First catch: Kipute keeps her hands stiff and catches the ball quickly. "Ouch! That hurt!"

Second catch: Kipute relaxes her arms and lets them move backward as she catches the ball. "Ah, much better, that didn't hurt at all."

Mr. Akilikubwa, watching nearby, asks: "Why did the same ball, thrown with the same speed, feel so different in the two catches?"

Explain the physics behind Kipute's observation.

Solution

Explanation:

The same momentum change occurred in both catches, requiring the same impulse. Stiff hands mean short contact time (small Δt), resulting in large force that causes pain. Relaxed, "giving" hands increase the contact time (large Δt), resulting in much smaller force that feels comfortable.

Demonstration:

This demonstrates that impulse can be delivered by large force over short time or small force over long time.

Making Sense of the Answer: Experienced cricket players and goalkeepers instinctively understand this principle. They always "give" with the ball rather than resisting it rigidly. This is not weakness, it is physics!

Think Like a Physicist: This same principle explains why car crumple zones save lives (increase Δt , decrease F), why parachutes work (increase Δt for landing, decrease F on the body), and why landing on soft sand hurts less than landing on concrete (sand increases Δt).

REAL Example 4

At a physics demonstration, Mr. Akilikubwa places a raw egg on a table and challenges his students:

Mr. Akilikubwa: "I will drop this hammer onto the egg, and the egg will not break. Who wants to bet against me?"

Kipanga (confident): "That's impossible, sir! A hammer will definitely break an egg!"

Mr. Akilikubwa places a thick foam pad between the hammer and the egg. He drops the hammer from a height. The hammer hits the foam, compresses it slowly, and the egg remains intact.

Kipute: "Oh! I see what happened. But Kipanga, what if sir had dropped the hammer directly onto the egg without the foam?"

Kipanga: "The egg would break immediately!"

Mr. Akilikubwa: "Exactly! Now, explain why the foam saved the egg using impulse ideas."

Explain the physics principle Mr. Akilikubwa is demonstrating.

Solution**Explanation:**

The foam increases the time interval Δt over which the hammer's momentum changes to zero. Since impulse $J = \Delta p$ must be the same (same hammer, same fall), increasing Δt dramatically decreases the average force F_{avg} . The reduced force is small enough that the eggshell can withstand it.

Demonstration:

This demonstrates that the same impulse can be delivered with large force over short time (egg breaks) or small force over long time (egg safe).

Making Sense of the Answer: *This is exactly how packaging materials work. They increase the stopping time during impacts, reducing forces on fragile items. It is also why modern cars have crumple zones that deform during crashes, protecting passengers by extending the collision time.*

Think Like a Physicist: *Safety devices like airbags, seatbelts, helmets with foam padding, and gymnastics mats all work on this principle: increase Δt to decrease F for the same momentum change. Nature uses this too; for example, cats spread their bodies when landing to increase contact time and reduce impact forces!*

HOT Example 5

A ball of mass 0.2kg moving horizontally at 10m/s strikes a vertical wall and rebounds horizontally at 8m/s. The ball is in contact with the wall for 0.05s. Calculate:

- The impulse received by the ball from the wall.
- The average force exerted by the wall on the ball.
- The average force exerted by the ball on the wall.
- If the wall were softer and the contact time increased to 0.15s with the same rebound velocity, calculate the new average force.

Solution

Interpreting the data:

$$m = 0.2\text{kg}$$

Initial velocity: $u = 10\text{m/s}$ (toward wall, take as positive direction)

Final velocity: $v = -8\text{m/s}$ (away from wall, negative)

Contact time: $\Delta t = 0.05\text{s}$

- (a) Initial momentum:

$$p_1 = mu = 0.2\text{kg} \times 10\text{m/s} = 2\text{kgm/s}$$

Final momentum:

$$p_2 = mv = 0.2\text{kg} \times (-8\text{m/s}) = -1.6\text{kgm/s}$$

Impulse (change in momentum):

$$\Delta p = p_2 - p_1 = (-1.6 - 2)\text{kgm/s} = -3.6\text{kgm/s}$$

The negative sign indicates the change is in the opposite direction to the initial motion.

The impulse is -3.6Ns (or 3.6Ns away from the wall)

- (b) Using:

$$J = F_{\text{avg}} \times \Delta t$$

From which:

$$F_{\text{avg}} = \frac{J}{\Delta t} = \frac{-3.6\text{Ns}}{0.05\text{s}} = -72\text{N}$$

The average force is -72N (or 72N directed away from the wall).

- (c) By Newton's third law, the ball exerts an equal and opposite force on the wall:

$$F_{\text{on wall}} = -F_{\text{on ball}} = -(-72) = +72\text{N}$$

The average force is +72N (or 72N directed into the wall).

(d) The impulse remains the same ($J = -3.6\text{Ns}$) because the momentum change is the same (same initial and final velocities).

With new contact time $\Delta t = 0.15\text{s}$:

$$F_{\text{avg}} = \frac{J}{\Delta t} = \frac{-3.6\text{Ns}}{0.15\text{s}} = -24\text{N}$$

The new average force is -24N (or 24N directed away from the wall).

(The force decreased from 72N to 24N which is equivalent to a reduction to one-third of the original value because the contact time tripled).

Making Sense of the Answer: The negative signs indicate opposite directions of force: the wall exerts a force on the ball away from the wall, while the ball exerts an equal and opposite force on the wall toward it (Newton's third law). If the wall is softer, the contact time is longer, so the same change in momentum occurs over a greater time, reducing the force. This is why soft-surfaced walls in sports facilities help reduce the risk of injury.

Think Like a Physicist: Notice that the rebound velocity (8m/s) is less than the approach velocity (10m/s). This means some kinetic energy was lost, converted to heat and sound during collision. We will explore this more when we study collisions later in this chapter. Also, observe that Newton's third law applies throughout the collision: at every instant, the force on the ball equals the force on the wall but in opposite direction.

The worked examples have beautifully illustrated the power of impulse and the impulse-momentum theorem. Our understanding is now deep and practical. We are ready to welcome the next subtopic with confidence and curiosity.

Conservation of Linear Momentum

We have seen that momentum describes the *quantity of motion* an object possesses, while impulse represents the change in that momentum (Δp). From this emerges a powerful and elegant conclusion: **forces change momentum**. Whenever a force acts on an object, it can increase its momentum, decrease it, or simply redirect it without changing its magnitude.

But now comes a profound question: **What happens to the total momentum when two objects interact with each other?**

For example, when two billiard balls collide, each ball experiences forces and its momentum changes. But *what about the **total momentum** of both balls combined? Does it increase, decrease, or stay the same?*

The answer to this question is one of the most beautiful and powerful principles in all of physics: **the law of conservation of linear momentum**.

Why do we study Conservation of Momentum?

Consider these situations:

Situation 1: A gun fires a bullet. The bullet shoots forward while the gun recoils backward. *How are these motions related?*

Situation 2: Two ice skaters push against each other on a frozen lake. Both move in opposite directions. *Why?*

Situation 3: A rocket in space ejects exhaust gases backward and accelerates forward even though there is nothing to "push against." *How is this possible?*

In all these cases, analysing individual forces would be extremely complicated. But conservation of momentum provides an elegant shortcut that makes solutions almost trivial.

To develop a clear and confident understanding of the law of conservation of momentum, let us examine what happens when two objects collide or interact each other. During their interaction:

- Object A exerts force F_{AB} on object B
- Object B exerts force F_{BA} on object A

Now applying the impulse-momentum theorem to each object over the interaction time Δt :

For object A:

$$F_{BA} \times \Delta t = \Delta p_A \dots (\text{equation 1})$$

For object B:

$$F_{AB} \times \Delta t = \Delta p_B \dots (\text{equation 2})$$

Adding equations (1) and (2):

$$F_{BA} \times \Delta t + F_{AB} \times \Delta t = \Delta p_A + \Delta p_B$$

But from Newton's third law (*Chapter 3*), F_{AB} and F_{BA} are equal in magnitude but opposite in direction:

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}$$

It follows that:

$$\begin{aligned} F_{BA} \times \Delta t - F_{BA} \times \Delta t &= \Delta p_A + \Delta p_B \\ 0 &= \Delta p_A + \Delta p_B \end{aligned}$$

Therefore:

$$\Delta \mathbf{p}_A + \Delta \mathbf{p}_B = \mathbf{0}$$

But:

$$\Delta p_A = (p_A)_{\text{final}} - (p_A)_{\text{initial}}$$

And:

$$\Delta p_B = (p_B)_{\text{final}} - (p_B)_{\text{initial}}$$

So the equation, $\Delta p_A + \Delta p_B = 0$ becomes:

$$(p_A)_{\text{final}} - (p_A)_{\text{initial}} + (p_B)_{\text{final}} - (p_B)_{\text{initial}} = 0$$

Rearranging:

$$(\mathbf{p}_A)_{\text{final}} + (\mathbf{p}_B)_{\text{final}} = (\mathbf{p}_A)_{\text{initial}} + (\mathbf{p}_B)_{\text{initial}}$$

The final result is the **law of conservation of linear momentum** which can be stated as:

If no external force acts on a system, the total momentum of the system remains constant.

Or equivalently:

In an isolated system, the total momentum before any interaction equals the total momentum after the interaction.

Mathematically the law can be re-written as follows:

$$\sum p_{\text{initial}} = \sum p_{\text{final}}$$

If only two objects interact with their respective initial and final velocities: u_1, u_2 and v_1, v_2 .

Then:

$$\sum p_{\text{initial}} = m_1 u_1 + m_2 u_2$$

And:

$$\sum p_{\text{final}} = m_1 v_1 + m_2 v_2$$

Hence:

$$\mathbf{m}_1 \mathbf{u}_1 + \mathbf{m}_2 \mathbf{u}_2 = \mathbf{m}_1 \mathbf{v}_1 + \mathbf{m}_2 \mathbf{v}_2$$

When Is Momentum Really Conserved?

The law of conservation of momentum applies only when there is **no net external force** acting on a system. This immediately raises an important question: *What exactly do we include in the "system"?* The answer determines whether momentum will be conserved or not.

Within a chosen system, we may have **internal forces** which are *forces that objects inside the system exert on each other*. These forces *always occur in equal and opposite pairs* (Newton's third law). Although they can change the momentum of individual objects, they cannot change the **total** momentum of the system. For example, during a collision, the forces between the colliding objects are internal; momentum may transfer from one object to another, but the total remains unchanged.

In contrast, **external forces** come from outside the system such as friction with the ground, gravity, or air resistance. Unlike internal forces, *external forces can change the total momentum* of the system because they are not balanced by equal and opposite partners within the system.

The key distinction, therefore, is this:

- **Momentum is conserved** when all interacting objects are included in the system and there is no net external force.
- **Momentum is not conserved** when an unbalanced external force acts on the system.

In practice, even when external forces like gravity exist, if they are balanced or act for a very short time during a collision, their effect may be negligible. In such cases, we can often treat momentum as approximately conserved during the interaction.

Examples of isolated systems:

To make these ideas concrete, let us examine a few real-life situations where systems can be treated as isolated and momentum is conserved.

1. Two ice skaters pushing apart:

On frictionless ice, no external horizontal forces exist. Momentum is conserved.

2. Gun and bullet:

Before firing, total momentum is zero. After firing, bullet goes forward with momentum $+p$, gun recoils backward with momentum $-p$, so total momentum remains zero.

3. Rocket in space:

The rocket and ejected gases form an isolated system. Total momentum is conserved.

4. Colliding billiard balls:

During the brief collision (milliseconds), external forces like friction and air resistance have negligible effect. Momentum is approximately conserved during the collision.

Momentum Conservation in Different Directions: Signs, Application, and Common Misconceptions

Since momentum is a **vector quantity**, its conservation must be considered separately along each direction. This makes analysis clearer, especially in collisions and interactions involving motion in more than one direction.

In one dimension (along x-axis):

$$\sum (p_x)_{\text{initial}} = \sum (p_x)_{\text{final}}$$

In two dimensions:

$$\sum (p_x)_{\text{initial}} = \sum (p_x)_{\text{final}}$$

$$\sum (p_y)_{\text{initial}} = \sum (p_y)_{\text{final}}$$

This means you can apply the law of conservation of momentum separately in the x-direction and y-direction. To do this effectively:

1. Choose a positive direction (usually **right** or **upward**).
2. Velocities in that direction are positive.
3. Velocities opposite are negative.
4. Apply the conservation: $\Sigma p_{\text{initial}} = \Sigma p_{\text{final}}$ with proper signs.

However, some common misconceptions can interfere with correct application. This include the following:

Misconception 1: "Momentum is always conserved."

Truth: *Momentum is conserved only when no net external force acts on the system.*

Misconception 2: "If two objects collide, each keeps its original momentum."

Truth: *Individual momenta change; only the total momentum is conserved.*

Misconception 3: "Conservation of momentum means objects do not change velocity."

Truth: *Objects can and do change velocity. What stays constant is the total momentum.*

With this solid theoretical foundation established, let us now bring these ideas to life through carefully crafted worked examples.

BINDER Example 6

A boy of mass 40kg stands on a stationary skateboard of mass 2kg. He jumps off the skateboard horizontally with a velocity of 3m/s. Calculate the velocity of the skateboard immediately after he jumps.

Solution

Interpreting the data:

Mass of boy: $m_1 = 40\text{kg}$

Mass of skateboard: $m_2 = 2\text{kg}$

Initial velocity of both (at rest): $u_1 = u_2 = 0\text{m/s}$

Final velocity of boy: $v_1 = 3\text{m/s}$ (take as positive direction)

Final velocity of skateboard: v_2 (to find)

Applying conservation of momentum:

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

Since both start at rest:

$$0 + 0 = m_1 v_1 + m_2 v_2$$

Rearranging to make v_2 the subject:

$$v_2 = \frac{-m_1 v_1}{m_2}$$

Substituting:

$$v_2 = -\frac{40\text{kg} \times 3\text{m/s}}{2\text{kg}} = -60\text{m/s}$$

The skateboard moves at 60m/s in the opposite direction to the boy's jump.

Making Sense of the Answer: *The negative sign indicates the skateboard moves backward (opposite to the boy's jump direction). The skateboard moves much faster (60m/s) than the boy (3m/s) even though the boy is much heavier. This happens because momentum depends on both mass and velocity: the light skateboard needs high velocity to balance the momentum of the heavy boy moving at low velocity.*

Think Like a Physicist: *Before the jump, total momentum was zero. After the jump, the boy's momentum is +120 kg·m/s and the skateboard's momentum is -120 kg·m/s. They cancel perfectly, keeping total momentum at zero. This demonstrates that **when a system starts at rest, objects must move in opposite directions with equal and opposite momenta.***

BINDER Example 7

Two ice skaters stand facing each other on a horizontal ice surface (assume frictionless). Skater A has mass 50kg and skater B has mass 60kg. They push against each other and move apart. If skater A moves backward at 2.4m/s, find the velocity of skater B.

Solution

Interpreting the data:

Mass of A: $m_A = 50\text{kg}$

Mass of B: $m_B = 60\text{kg}$

Initial velocities: $u_A = u_B = 0\text{m/s}$ (both at rest)

Final velocity of A: $v_A = -2.4\text{ m/s}$ (backward, take as negative)

Final velocity of B: v_B (to find)

Again applying conservation of momentum:

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

$$0 = m_A v_A + m_B v_B$$

Making v_B the subject and substituting values:

$$v_B = -\frac{m_A v_A}{m_B} = -\frac{50\text{kg} \times (-2.4\text{m/s})}{60\text{kg}} = +2\text{m/s}$$

The velocity of skater B is 2m/s forward.

Making Sense of the Answer: The positive sign indicates skater B moves forward (opposite to skater A). Skater B is heavier (60kg vs 50kg) so moves more slowly (2m/s vs 2.4 m/s) to achieve equal and opposite momentum. Check: momentum of A = $50 \times (-2.4) = -120\text{ kg}\cdot\text{m/s}$; momentum of B = $60 \times 2.0 = +120\text{ kg}\cdot\text{m/s}$. They cancel! ✓

Think Like a Physicist: The skaters pushed against each other with equal and opposite forces (Newton's third law). These are internal forces that changed individual momenta but not the total. This is why momentum conservation is so powerful; with it we did not need to know how hard they pushed or for how long!

REAL Example 8

A fisherman of mass 80kg stands at one end of a stationary boat of mass 120kg floating on still water. He walks 4 metres toward the other end of the boat. Assuming the water offers negligible resistance, describe what happens to the boat.

Solution

Since the fisherman and the boat form an isolated system (water resistance is negligible) and both start from rest, the **total momentum remains zero** throughout the motion.

As the fisherman walks forward, he gains momentum in the forward direction. To keep the total momentum zero, the boat must gain an equal momentum in the opposite direction. Hence, **the boat moves backward**.

Applying conservation of momentum:

$$0 = m_f v_f + m_b v_b$$

Where: m_f , m_b are the masses of fisherman and boat, and v_f , v_b are their velocities relative to ground (water).

Making v_b the subject and substituting values:

$$v_b = -\frac{m_f v_f}{m_b} = -\frac{80\text{kg} v_f}{120\text{kg}} = -\frac{2}{3} v_f \text{ or } \frac{2}{3} = -\frac{v_b}{v_f}$$

But for constant time (both fisherman and boat moved at the same time): $v \propto \text{displacement, } s$. Thus:

$$\frac{v_b}{v_f} = \frac{s_b}{s_f}$$

It follows that:

$$\frac{2}{3} = -\frac{s_b}{s_f}$$

But the fisherman moves **4m relative to the boat**, not relative to the water (ground). Therefore:

$$\text{Relative displacement} = s_f - s_b = 4; s_f = 4 + s_b$$

Substituting:

$$\frac{2}{3} = -\frac{s_b}{4 + s_b} \text{ or } 8 + 2s_b = -3s_b$$

Solving the equation gives:

$$s_b = -1.6\text{m (and } s_f = 4 + s_b = 2.4\text{m)}$$

Hence, the boat moves 1.6m backward, opposite to the fisherman's motion.

Making Sense of the Answer: *The fisherman is lighter than the boat, so he moves a greater distance (2.4m) while the boat moves less (1.6m). But because the boat is heavier, their momenta can still cancel. The fisherman thinks he walked 4m along the boat, but relative to the shore, he only moved 2.4m (4m–1.6m) forward!*

Think Like a Physicist: *This example shows that momentum conservation applies even when objects do not collide or push apart suddenly. Any internal forces (fisherman's feet pushing on boat) cannot change the total momentum of the system. This is why you cannot make a boat move forward by standing inside it and pushing on its walls; you are part of the system!*

REAL Example 9

Kipute and Kipanga are playing with a toy cannon at school during physics club. They load it with a small ball and place it on a smooth table.

Kipanga: "Watch! When I fire the cannon, the ball will shoot forward very fast!"

He fires. The ball shoots forward at high speed, but the cannon rolls backward off the table and crashes to the floor.

Kipanga: "Oh no! I didn't expect that!"

Kipute (laughing): "That's momentum conservation, Kipanga! You forgot about the cannon recoiling!"

Mr. Akilikubwa (walking over): "Excellent demonstration, though unintentional! Kipanga, can you calculate how fast the cannon was moving when it fell off the table?"

Information:

- Mass of ball: 0.05kg
- Mass of cannon: 2kg
- Velocity of ball after firing: 40m/s forward
- Both cannon and ball were initially at rest.

Questions:

- Calculate the recoil velocity of the cannon.
- If the cannon were bolted firmly to the table so it could not recoil, explain what would happen and state whether momentum would still be conserved.

Solution

- Initial total momentum = 0 (both at rest)

By conservation of momentum:

$$m_{\text{ball}}v_{\text{ball}} + m_{\text{cannon}}v_{\text{cannon}} = 0$$

From which:

$$v_{\text{cannon}} = -\frac{m_{\text{ball}}v_{\text{ball}}}{m_{\text{cannon}}} = -\frac{0.05\text{kg} \times 40\text{m/s}}{2\text{kg}} = -1\text{m/s}$$

The recoil velocity is 1m/s backward.

- (b) If the cannon were bolted firmly to the table, the cannon, table, and Earth would recoil together. However, because the Earth's mass is extremely large, the resulting recoil velocity would be extremely small and practically undetectable. Momentum would still be conserved, but the Earth's motion would be negligible and impossible to detect.

Making Sense of the Answer: The cannon moved much slower (1m/s) than the ball (40m/s) because it is much heavier (2kg vs 0.05kg; that is 40 times heavier!). Despite the huge difference in speeds, their momenta are equal and opposite: ball's momentum = +2kgm/s, cannon's momentum = -2kgm/s. Total = zero, as required.

Think Like a Physicist: This is exactly how real cannons and guns work. The recoil is unavoidable; it is a direct consequence of momentum conservation. Artillery soldiers must brace cannons to prevent them from rolling backward. This also explains rocket propulsion: rockets do not need anything to "push against"; they eject gases backward (like the cannon ball) and recoil forward (like the cannon), all by momentum conservation!

HOT Example 10

A railway wagon of mass 2000kg moving at 5m/s collides and couples with a stationary wagon of mass 3000kg. Calculate:

- The common velocity of the two wagons after coupling.
- The loss in kinetic energy during the collision.
- Explain clearly what happened to the "lost" kinetic energy.

Solution

Interpreting the data:

Mass of moving wagon: $m_1 = 2000\text{kg}$

Mass of stationary wagon: $m_2 = 3000\text{kg}$

Initial velocity of moving wagon: $u_1 = 5\text{m/s}$

Initial velocity of stationary wagon: $u_2 = 0\text{m/s}$

After coupling, both move together: $v_1 = v_2 = v$ (common velocity)

- (a) Applying conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v + m_2v$$

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

Making v the subject:

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

Substituting:

$$v = \frac{2000\text{kg} \times 5\text{m/s} + 3000\text{kg} \times 0\text{m/s}}{2000\text{kg} + 3000\text{kg}} = 2\text{m/s}$$

The common velocity is 2m/s.

- (b) The loss in kinetic energy is given by:

Loss in K.E = Initial kinetic energy – Final kinetic energy

$$\begin{aligned} \text{Initial kinetic energy} &= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}(2000\text{kg})(5\text{m/s})^2 + \frac{1}{2}(3000\text{kg})(0\text{m/s})^2 \\ &= 25000\text{J} \end{aligned}$$

$$\text{Final kinetic energy} = \frac{1}{2}(m_1 + m_2)v^2 = \frac{1}{2}(5000\text{kg})(2\text{m/s})^2 = 10000\text{J}$$

$$\text{Loss in K.E} = 25000\text{J} - 10000\text{J} = 15000\text{J}$$

The loss in kinetic energy is 15000J.

- (c) The lost kinetic energy (15,000 J) was converted into other forms of energy such as:

1. **Heat (thermal) energy:** Friction between the coupling mechanisms and deformation of metal generated heat
2. **Sound energy:** The loud "clang" when wagons collided carried away energy as sound waves.
3. **Deformation energy:** Slight permanent deformation of the coupling devices and wagon structures.
4. **Vibrational energy:** The wagons vibrated after impact, which eventually dissipated as heat.

So the energy was not destroyed (energy is always conserved), but it changed from organised kinetic energy into disorganised forms that cannot be recovered as useful motion.

Making Sense of the Answer: The final velocity (2m/s) is less than the initial velocity (5m/s) of the moving wagon, which makes sense because the moving wagon had to accelerate the stationary wagon, sharing its momentum. Notice that 60% of the kinetic energy was lost (15000 out of 25000J), yet momentum was perfectly conserved. This demonstrates that **momentum conservation and kinetic energy conservation are independent principles!**

Think Like a Physicist: Momentum is **always** conserved in isolated systems (it is a fundamental law), but kinetic energy is only conserved in elastic collisions. Most real collisions are inelastic as some kinetic energy always converts to heat and sound. We will explore elastic vs inelastic collisions in depth later in this chapter.

HOT Example 11

A projectile of mass 10kg moving horizontally at 50m/s explodes into two fragments (**A** and **B**). Fragment **A** has mass 4kg and moves at 80m/s in the original direction. Calculate:

- (a) The velocity of fragment **B**.
- (b) The energy released by the explosion.

Solution

Interpreting the data:

Initial mass: $m = 10\text{kg}$

Initial velocity: $u = 50\text{m/s}$ (horizontal, take as positive)

Mass of fragment A: $m_A = 4\text{kg}$

Velocity of fragment A: $v_A = 80\text{m/s}$ (positive direction)

Mass of fragment B: $m_B = 10\text{kg} - 4\text{kg} = 6\text{kg}$

Velocity of fragment B: v_B (to find)

- (a) Applying conservation of momentum:

$$mu = m_A v_A + m_B v_B$$

Making v_B the subject:

$$v_B = \frac{mu - m_A v_A}{m_B}$$

Substituting:

$$v_B = \frac{10\text{kg} \times 50\text{m/s} - 4\text{kg} \times 80\text{m/s}}{6\text{kg}} = 30\text{m/s}$$

The velocity of fragment B is 30m/s in the original direction.

- (b) The energy released is given by:

Energy released = Gain in K.E = Final kinetic energy – Initial kinetic energy

Where:

Initial kinetic energy = Kinetic energy before explosion

$$= \frac{1}{2}mu^2 = \frac{1}{2}(10\text{kg})(50\text{m/s})^2 = 12500\text{J}$$

Final kinetic energy = Kinetic energy after explosion

$$= \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{2}(4\text{kg})(80\text{m/s})^2 + \frac{1}{2}(6\text{kg})(30\text{m/s})^2 = 15500\text{J}$$

$$\text{Energy released} = 15500\text{J} - 12500\text{J} = 3000\text{J}$$

The energy released by the explosion is 3000J.

Making Sense of the Answer: *Fragment A (the lighter piece) was blown forward faster than the original projectile (80m/s vs 50m/s), while fragment B (the heavier piece) slowed down to 30m/s. Both moved forward because the explosion happened while the projectile was already moving. The explosion added 3000J of energy (from chemical energy of the explosive) to the system, increasing the total kinetic energy from 12500 J to 15500J.*

Think Like a Physicist: *This example shows that:*

1. Momentum is always conserved (even during explosions).
2. Energy is conserved if we account for **all** forms (the 3000J came from chemical energy).
3. Kinetic energy alone is **not** conserved during explosions; it increases! (This is opposite to inelastic collisions where kinetic energy decreases).

The worked examples have beautifully demonstrated the power and universality of momentum conservation. Whether objects collide, push apart, explode, or move within a system, the total momentum remains constant as long as no external forces act. This principle will serve as the foundation for our study of collisions in the next subtopic.

COLLISIONS

We have now developed a complete understanding of momentum and its conservation. We have learned that when objects interact in an isolated system, the total momentum before equals the total momentum after, regardless of how violent or gentle the interaction.

But everyday experience shows us that collisions come in many varieties:

- When two billiard balls collide, they bounce apart crisply with sharp clicking sounds.
- When two lumps of clay collide, they stick together and move as one.
- When a tennis ball bounces off a wall, it returns at nearly the same speed.
- When a ball of dough hits a wall, it sticks and does not bounce at all.

All these collisions obey conservation of momentum but they look and feel completely different. *So what distinguishes one type of collision from another?*

The answer lies in what happens to **kinetic energy** during the collision. But before going to that, we have to understand first what is collision. But before we explore how kinetic energy behaves, we must first ask a more fundamental question:

What exactly is a collision?

A **collision** is an event in which two or more objects come into contact with each other for a short time, exerting relatively large forces on each other.

A collision is typically identified by the following characteristics:

1. Brief duration

Collisions typically last from microseconds (atomic collisions) to milliseconds (e.g. vehicle collisions). This short time is crucial.

2. Large forces

During collision, the forces between objects are much larger than any external forces (like friction or gravity). This is why we can often treat momentum as conserved even when small external forces exist.

3. Localised interaction

Objects are in direct contact (or very close, in the case of atomic/molecular collisions).

Now, **why do we study collisions?**

Collisions are fundamental to understanding:

- Vehicle safety design (crumple zones, airbags).

- Sports (hitting balls, tackling in rugby).
- Particle physics (understanding matter at atomic scale).
- Astronomy (formation of planets, asteroid impacts).
- Engineering (impact testing, materials science).

Classification of Collisions

Collisions are classified according to what happens to **kinetic energy** during the interaction. In an isolated system, momentum is **always** conserved. However, kinetic energy may or may not be conserved. This difference in the behaviour of kinetic energy is what distinguishes one type of collision from another.

On this basis, collisions are commonly discussed in three forms:

- 1) Elastic collisions
- 2) Inelastic collisions
- 3) Perfectly inelastic collisions (*a special case of **inelastic collisions** in which the bodies stick together*)

Elastic Collisions

An **elastic collision** is one in which both momentum and kinetic energy are conserved. It is characterised by the following features:

- Objects bounce apart after collision.
- No energy is converted to heat, sound, or permanent deformation.
- Total kinetic energy before = Total kinetic energy after.
- Objects may exchange kinetic energy, but the total remains constant.

Examples of approximately elastic collision include:

1. **Steel ball bearings colliding:** Very little energy lost to deformation.
2. **Billiard balls:** Slight energy loss to sound and deformation, but close to elastic.
3. **Atomic and molecular collisions in gases:** Perfectly elastic at atomic scale.
4. **Super balls bouncing:** High elasticity materials.

However, the reader should note that perfectly elastic collisions are only idealisation. In reality, no collision is perfect elastic; all collisions lose some energy to heat, sound, or deformation. Nevertheless, many collisions are "nearly elastic" where energy loss is small enough to ignore.

Mathematical conditions for elastic collisions:

Conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \dots \text{(equation 1)}$$

Conservation of kinetic energy:

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \dots \text{(equation 2)}$$

These two equations can be solved simultaneously to find the final velocities v_1 and v_2 .

Also rearranging equation 1:

$$m_1u_1 - m_1v_1 = m_2v_2 - m_2u_2$$

From which:

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \dots \text{(equation 3)}$$

From equation 2 (multiplying both sides by 2):

$$m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2$$

Rearranging:

$$m_1u_1^2 - m_1v_1^2 = m_2v_2^2 - m_2u_2^2$$

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$$

Factorising:

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2) \dots \text{(equation 4)}$$

Dividing equation (4) by equation (3) gives:

$$u_1 + v_1 = v_2 + u_2$$

Rearranging:

$$u_1 - u_2 = -(v_1 - v_2) \text{ or } u_1 - u_2 = v_2 - v_1$$

In words: *The relative velocity of approach before collision equals the relative velocity of separation after collision (but in opposite direction).*

Inelastic Collisions

An **inelastic collision** is one in which momentum is conserved but kinetic energy is **not** conserved. It is characterised by the following features:

- Some kinetic energy is converted to other forms (heat, sound, deformation, vibrations).
- Total kinetic energy after collision < Total kinetic energy before collision.
- Momentum is conserved (as always).
- Objects may or may not stick together.

Examples of inelastic collisions include:

1. **Car crashes:** Crumpling metal, breaking glass, heat, sound (a lot of kinetic energy converted to other forms).
2. **Dropping a ball of dough:** Sticks to floor, does not bounce.
3. **Bullet embedding in wood:** Kinetic energy is converted to heat and deformation.
4. **Tackling in rugby:** Players move together after collision.

Mathematical conditions for inelastic collisions:

Conservation of momentum (still applies):

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Kinetic energy is not conserved:

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 > \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

So in an inelastic collision, the total kinetic energy decreases. This happens because part of the initial kinetic energy is converted into:

1. **Heat:** Friction and deformation generate thermal energy.
2. **Sound:** The "thud" or "crash" of collision carries energy as sound waves.
3. **Permanent deformation:** Bending, crumpling, breaking of objects.
4. **Internal vibrations:** Objects vibrate after impact.

Perfectly Inelastic Collisions

A **perfectly inelastic collision** is a special type of inelastic collision in which **objects stick together after collision** and move with a common final velocity after impact. This situation results in the maximum possible loss of kinetic energy while still conserving momentum.

It is therefore important to distinguish between a general inelastic collision and a perfectly inelastic one:

- In an **inelastic collision**, the objects may either separate after impact or stick together, but some kinetic energy is always lost.
- In a **perfectly inelastic collision**, the objects must stick together and move as a single body. This represents the greatest possible loss of kinetic energy for the given initial conditions.

Examples of perfectly inelastic collisions include:

1. A **bullet embedding in a block of wood**, where the bullet remains lodged in the block after impact.
2. **Railway wagons coupling together**, moving as a single unit after they collide.
3. **Clay balls colliding and sticking**, forming one combined mass.
4. A **meteorite striking and merging with a celestial body**, becoming part of it.
5. **Catching a moving ball and continuing to move with it**, where the ball and catcher momentarily share a common velocity.

Mathematical conditions for perfectly inelastic collisions:

After collision: $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}$ (common velocity)

Conservation of momentum:

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$$

Kinetic energy is not conserved (maximum loss in K.E):

$$\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 > \frac{1}{2}(m_1 + m_2)v^2$$

Coefficient of Restitution

We have classified collisions as either elastic or inelastic. However, in real life, most collisions are not perfectly elastic and not perfectly inelastic either. They fall somewhere in between.

Some collisions bounce strongly. Others barely rebound at all. So instead of asking only “*Is this collision elastic or inelastic?*”, we should ask a better question:

How elastic is the collision?

To answer this, we introduce a numerical measure called the **coefficient of restitution**, represented by the symbol e .

The coefficient of restitution tells us how much “bounciness” a collision has. It allows us to measure elasticity on a scale, rather than simply labeling a collision as one type or the other.

To understand the coefficient of restitution more precisely, we focus on what happens during the collision itself.

Before the collision, the two objects move toward each other with a certain **relative velocity of approach**. After the collision, if they separate, they move away from each other with a certain **relative velocity of separation**.

The coefficient of restitution (e), is defined as:

The ratio of the relative velocity of separation to the relative velocity of approach.

In mathematical form:

$$e = \frac{\text{relative velocity of separation}}{\text{relative velocity of approach}}$$

The coefficient of restitution, e , tells us what fraction of the relative speed of approach is recovered as relative speed of separation after the collision. In other words, it measures how much of the “closing speed” before impact reappears as “rebounding speed” after impact.

The value of e therefore reveals the nature of the collision. Specifically, if:

- If $e = 1$, relative velocity of separation = relative velocity of approach. The collision is **perfectly elastic**.
- If $e = 0$, relative velocity of separation = 0 (there is no separation after collision as the objects move together). The collision is **perfectly inelastic**.
- If $0 < e < 1$, relative velocity of separation < relative velocity of approach. The collision is **inelastic (partially elastic)**.

Understand how to apply e in calculations

When calculating the value of e , we use the magnitudes of the relative velocities. This ensures that the coefficient of restitution is always a positive quantity.

For **one-dimensional** collisions:

Magnitude of relative velocity of separation = $v_2 - v_1$

Magnitude of relative velocity of approach = $u_1 - u_2$

Therefore:

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

With the restitution equation now introduced, we have three fundamental equations available for analysing collisions:

1) Conservation of momentum (always):

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

2) Coefficient of restitution:

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

3) Conservation of energy (only if $e = 1$):

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2.$$

For elastic collisions ($e = 1$), we can use equations 1 and 3, or equations 1 and 2.

For inelastic collisions ($e < 1$), we must use equations 1 and 2.

Warning!

The relation $e = (v_2 - v_1) / (u_1 - u_2)$ is valid **only for one-dimensional collisions**, where all velocities lie along the same straight line (the line of impact). It must **not** be applied directly in two-dimensional collisions. In such cases, the coefficient of restitution applies only to the components of velocity **along the line of impact**, not to the total velocities.

Summary of Collision Types

Type	Momentum Conserved?	KE Conserved?	Coefficient e	After Collision
Elastic	Yes	Yes	$e = 1$	Objects separate
Inelastic	Yes	No (Partial loss)	$0 < e < 1$	Objects may separate
Perfectly Inelastic	Yes	No (Maximum loss)	$e = 0$	Objects stick together

With this comprehensive theoretical foundation firmly established, we are now ready to bring these concepts to life through carefully crafted worked examples that demonstrate how to apply collision principles to solve real problems.

BINDER Example 12

Two balls A and B collide head-on. Ball A has mass 2kg and moves at 6m/s. Ball B has mass 3kg and moves at 4m/s in the opposite direction. After collision, ball A rebounds with a velocity of 2m/s.

- Calculate the velocity of ball B after collision.
- Determine whether the collision is elastic or inelastic.

Solution

Interpreting the data:

Mass of A: $m_A = 2\text{kg}$

Mass of B: $m_B = 3\text{kg}$

Initial velocity of A: $u_A = 6\text{m/s}$ (take as positive direction)

Initial velocity of B: $u_B = -4\text{m/s}$ (opposite direction, negative)

Final velocity of A: $v_A = -2\text{m/s}$ (rebounds, negative)

Final velocity of B: v_B (to find)

- Applying conservation of momentum:

$$m_Au_A + m_Bu_B = m_Av_A + m_Bv_B$$

Substituting values:

$$2\text{kg} \times 6\text{m/s} + 3\text{kg} \times -4\text{m/s} = 2\text{kg} \times -2\text{m/s} + 3\text{kg} \times v_B$$

On solving gives; $v_B = 1.33\text{m/s}$

(b) Comparing initial and final kinetic energy:

Initial kinetic energy:

$$(K.E)_{\text{before collision}} = \frac{1}{2}m_A u_A^2 + \frac{1}{2}m_B u_B^2 = \frac{1}{2}(2\text{kg})(6\text{m/s})^2 + \frac{1}{2}(3\text{kg})(-4\text{m/s})^2 = 60\text{J}$$

Final kinetic energy:

$$(K.E)_{\text{after collision}} = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{2}(2\text{kg})(-2\text{m/s})^2 + \frac{1}{2}(3\text{kg})(1.33\text{m/s})^2 = 6.7\text{J}$$

Since Initial kinetic energy (60J) $\neq$ Final kinetic energy (6.7J), kinetic energy is not conserved and hence, the collision is inelastic.

Making Sense of the Answer: Ball A reversed direction (from +6 m/s to -2m/s) while ball B also reversed direction (from -4 m/s to +1.33m/s). Both balls bounced backward from their original directions, which is typical in head-on collisions. The large energy loss (89% of initial KE) indicates that the collision is very inelastic (like collision of two soft rubber balls or balls covered in felt).

Think Like a Physicist: Never assume a collision is elastic just because objects bounce apart. Always verify by comparing kinetic energies. Momentum is conserved in all collisions (it must be), but kinetic energy is only conserved in elastic collisions.

BINDER Example 13

A ball of mass 0.5kg moving at 10m/s collides elastically with a stationary ball of mass 1.5kg. Calculate the velocities of both balls after collision.

Solution

Interpreting the data:

Mass of ball 1: $m_1 = 0.5\text{kg}$

Mass of ball 2: $m_2 = 1.5\text{kg}$

Initial velocity of ball 1: $u_1 = 10\text{m/s}$

Initial velocity of ball 2: $u_2 = 0\text{ m/s}$ (at rest)

$e = 1$ (balls collided **elastically**)

Final velocities: v_1, v_2 (to find)

Applying conservation of momentum:

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$0.5 \times 10 + 1.5 \times 0 = 0.5v_1 + 1.5v_2$$

$$0.5v_1 + 1.5v_2 = 5 \dots (1)$$

Using coefficient restitution equation:

$$e = 1 = \frac{v_2 - v_1}{u_1 - u_2}$$

Thus:

$$v_2 - v_1 = u_1 - u_2$$

$$v_2 - v_1 = 10 - 0$$

$$v_2 - v_1 = 10 \dots (2)$$

Solving equation (1) and (2) simultaneously gives: $v_1 = -5\text{m/s}$, $v_2 = 5\text{m/s}$

After collision, ball 1 moves at 5m/s backward (opposite to original direction), and ball 2 moves at 5m/s forward (in ball 1's original direction).

Making Sense of the Answer: Ball 1 (lighter, moving) hit ball 2 (heavier, stationary) and bounced backward at half its original velocity. Ball 2 moved forward at half of ball 1's original velocity. The heavier

ball "won" the collision, sending the lighter ball backward. Despite this, kinetic energy was perfectly conserved (the energy was transferred from ball 1 to ball 2).

Think Like a Physicist: When a light object collides elastically with a heavier stationary object, the lighter object typically rebounds. As the mass of the second object becomes much larger, the rebound becomes more pronounced. In the limiting case where the second mass is effectively infinite, such as a ball striking a rigid wall, the lighter object bounces back with nearly the same velocity but in the opposite direction.

REAL Example 14

A 60kg student running at 5m/s jumps onto a 20kg stationary skateboard and continues moving together with it after landing.

- Calculate the velocity of the student–skateboard system immediately after landing.
- Calculate the kinetic energy lost during the collision.
- Where did the "lost" kinetic energy go? Explain.

Solution

- Conservation of momentum:

$$m_s u_s + m_b u_b = (m_s + m_b)v$$

Where: *s* stands for student and *b* stands for skateboard.

Substituting given values:

$$60\text{kg} \times 5\text{m/s} + 20\text{kg} \times 0\text{m/s} = (60\text{kg} + 20\text{kg})v; v = 3.75\text{m/s}$$

The velocity is 3.75m/s.

- Kinetic energy lost is given by the following equation:

$$(K.E)_{\text{lost}} = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)v^2$$

Substituting:

$$(K.E)_{\text{lost}} = \frac{1}{2}(60\text{kg})(5\text{m/s})^2 + 0 - \frac{1}{2}(60 + 20)\text{kg} \times (3.75\text{m/s})^2 = 187.5\text{J}$$

The kinetic energy lost is 187.5J.

- The 187.5J of "lost" kinetic energy was converted into:

- Heat:** Friction between the student's feet and skateboard surface generated thermal energy.
- Sound:** The "thud" of landing produced sound waves carrying energy away.
- Deformation:** Small compression of skateboard deck and student's shoe soles.
- Vibrations:** The skateboard vibrated after impact, eventually dissipating as heat through internal friction.

Making Sense of the Answer: The student slowed from 5m/s to 3.75m/s (25% reduction) by sharing momentum with the skateboard. This 25% speed reduction resulted in a 25% energy loss (187.5J out of 750J). This is typical for perfectly inelastic collisions; significant kinetic energy loss even though momentum is conserved.

Think Like a Physicist: Perfectly inelastic collisions always have the maximum possible energy loss while still conserving momentum. If the student had jumped onto a moving skateboard going in the opposite direction, the energy loss would be even greater. This is why crashes between vehicles moving in opposite directions are more dangerous as they dissipate more energy.

REAL Example 15

During a physics demonstration, Mr. Akilikubwa sets up a **Newton's cradle**: five identical steel balls suspended by strings so they can swing and collide.

He pulls back one ball and releases it. It swings down and hits the row of stationary balls. Surprisingly, only the ball at the far end swings up, while the other balls remain almost stationary.

Kipanga: "I don't understand! Why didn't all the balls move a little bit? Wouldn't that share the momentum?"

Kipute: "But if all balls moved together, that would be perfectly inelastic. These are **steel balls**, they bounce elastically!"

Mr. Akilikubwa: "Excellent thinking, Kipute! Since the balls collide elastically, let's analyse this with physics."

Setup:

- 5 identical steel balls, each mass m
- Ball 1 swings in at velocity u
- Balls 2, 3, 4, 5 initially at rest
- After collision, ball 5 swings out at velocity v
- Balls 1, 2, 3, 4 remain nearly at rest

Mr. Akilikubwa asks: "What must be the velocity of ball 5?"

Now, assist Kipute and Mr. Akilikubwa in answering the question.

Solution

Conservation of momentum:

$$\text{Total initial momentum} = \text{Total final momentum}$$

$$m \times u + m \times 0 + m \times 0 + m \times 0 + m \times 0 = m \times 0 + m \times 0 + m \times 0 + m \times 0 + m \times v$$

$$\text{From which: } mu = mv$$

$$\text{Therefore: } v = u$$

Conservation of kinetic energy:

Since the collision is elastic:

$$\text{Initial kinetic energy} = \text{Final kinetic energy}$$

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2$$

$$\text{This also gives: } v = u$$

Both conservation laws give the same answer: ball 5 must leave with exactly the same velocity that ball 1 arrived with.

Hence, Ball 5 swings out at velocity $v = u$ (same velocity as ball 1 came in).

Making Sense of the Answer: Newton's cradle demonstrates that elastic collisions between equal masses produce a unique result: complete energy and momentum transfer. The incoming ball stops; the outgoing ball moves at the incoming velocity. The middle balls act as perfect transmitters, passing the energy along without keeping any. This only works because the balls are identical and the collision is elastic.

Think Like a Physicist: Newton's cradle demonstrates a powerful principle: for elastic collisions between identical masses, applying **both** conservation laws (momentum and energy) simultaneously forces one unique outcome whereby the moving ball stops completely and the stationary ball moves at the original velocity. If momentum alone were considered, we could imagine all five balls moving together slowly (momentum conserved), but this would violate energy conservation. If only energy mattered, we could imagine the first ball continuing while others moved differently (energy conserved), but this would violate momentum conservation. Only the one-in-one-out pattern satisfies both laws, and this is why Newton's cradle is such a popular physics demonstration; it makes conservation laws visible!

HOT Example 16

A ball is dropped from a height of 2m onto a hard floor. It rebounds to a height of 1.2m. Calculate:

- The velocity just before hitting the floor.
- The velocity just after leaving the floor.
- The coefficient of restitution between the ball and floor.
- The percentage of kinetic energy lost in the collision

$$\text{Take } g = 9.8\text{m/s}^2$$

Solution

- Using $v^2 = u^2 + 2as$ for free fall from rest with h_1 as initial drop height:

$$v^2 = 0^2 + 2gh_1 \text{ or } v = \sqrt{2gh_1} = \sqrt{2 \times 9.8\text{m/s}^2 \times 2\text{m}} = 6.26\text{m/s (downward)}$$

The velocity is 6.26m/s (downward)

(ii) Using $v^2 = u^2 + 2as$ for upward motion to reach height h_2 , with $a = -g$, $v = 0\text{m/s}$.

$$0^2 = u^2 - 2gh_2 \text{ or } u = \sqrt{2gh_2} = \sqrt{2 \times 9.8\text{m/s}^2 \times 1.2\text{m}} = 4.85\text{m/s (upward)}$$

The velocity is 4.85m/s (upward).

(iii) Taking upward as positive:

Before collision: $u_{\text{ball}} = -6.26\text{m/s}$ (downward)

Before collision: $u_{\text{floor}} = 0\text{m/s}$ (floor is stationary)

After collision: $v_{\text{ball}} = +4.85\text{m/s}$ (upward)

After collision: $v_{\text{floor}} = 0\text{m/s}$ (floor still stationary)

$$e = \frac{v_{\text{ball}} - v_{\text{floor}}}{u_{\text{floor}} - u_{\text{ball}}} = \frac{(4.85 - 0)\text{m/s}}{(0 - (-6.26))\text{m/s}} = 0.77$$

The coefficient of restitution is 0.77.

(iv) Kinetic energy before collision = $\frac{1}{2}mv^2 = \frac{1}{2}m(6.26^2) = 19.6\text{m}$

Kinetic energy after collision = $\frac{1}{2}mu^2 = \frac{1}{2}m(4.85^2) = 11.76\text{m}$

$$\begin{aligned} \text{Percentage of kinetic energy lost} &= \frac{\text{Kinetic energy lost}}{\text{Kinetic energy before collision}} \times 100\% \\ &= \frac{(19.6\text{m} - 11.76\text{m})}{19.6\text{m}} \times 100\% = 40\% \end{aligned}$$

The percentage is 40%.

Making Sense of the Answer:

The ball rebounded to 60% of its original height ($1.2\text{m}/2\text{m} = 0.6$). Since height is proportional to mechanical energy, the rebound to 60% of the original height means the ball retains 60% of its energy and loses **40%**. However, velocity is proportional to the square root of height, which is why the coefficient of restitution is $e = \sqrt{0.6} = 0.77$, not 0.6.

Think Like a Physicist: The "lost" 40% of kinetic energy went into heat (ball and floor warmed slightly), sound (the bounce made noise), and small permanent deformations. If we dropped the ball repeatedly, it would bounce lower each time until eventually stopping when all its initial gravitational potential energy would have been converted to heat and other non-mechanical forms. That is why bouncing balls do not bounce forever!

HOT Example 17

Two smooth spheres can move freely on a horizontal surface. A sphere of mass 2kg moves with a velocity of 6m/s in a straight line and collides with a stationary sphere of mass 3kg. After the collision, the 2kg sphere is observed to move off with a velocity of 4m/s at an angle of 30° above its original direction of motion. A student claims that the collision was perfectly elastic.

- Determine magnitude and direction of the velocity of the 3kg sphere immediately after the collision.
- Hence determine whether the student's claim is physically valid. Justify your answer clearly.

Solution

(a) *Conserving momentum horizontally:*

$$\sum (p_x)_{\text{initial}} = \sum (p_x)_{\text{final}}$$

$$m_1(u_1)_x + m_2(u_2)_x = m_1(v_1)_x + m_2(v_2)_x$$

Where:

$$(u_1)_x = 6\text{m/s (Initially 2kg sphere was moving horizontally)}$$

$$(u_2)_x = 0 \text{ (At rest)}$$

$$(v_1)_x = 4\cos 30^\circ$$

$$m_1 = 2\text{kg}, m_2 = 3\text{kg}$$

Substituting:

$$2 \times 6 + 3 \times 0 = 2 \times 4\cos 30^\circ + 3(v_2)_x$$

$$\text{Solving gives: } (v_2)_x = 1.69\text{m/s}$$

Conserving momentum vertically:

$$\sum (p_y)_{\text{initial}} = \sum (p_y)_{\text{final}}$$

$$m_1(u_1)_y + m_2(u_2)_y = m_1(v_1)_y + m_2(v_2)_y$$

Where:

$$(u_1)_y = 0 \text{ (No initial vertical component of velocity because 2kg sphere was moving horizontally)}$$

$$(u_2)_y = 0 \text{ (At rest)}$$

$$(v_1)_y = 4\sin 30^\circ$$

Substituting:

$$0 = 2 \times 4\sin 30^\circ + 3(v_2)_y$$

$$\text{Solving gives: } (v_2)_y = -1.33\text{m/s}$$

The negative sign indicates that the velocity component is directed opposite to the chosen positive vertical direction; that is, along the negative y-direction (below the original line of motion).

$$\text{Magnitude: } v_2 = \sqrt{((v_2)_x)^2 + ((v_2)_y)^2} = \sqrt{(1.69\text{m/s})^2 + (-1.33\text{m/s})^2} = 2.15\text{m/s}$$

$$\text{Direction: } \tan \theta = \frac{(v_2)_y}{(v_2)_x} = \frac{-1.33\text{m/s}}{1.69\text{m/s}} = -0.787$$

$$\theta = \tan^{-1} -0.787 = -38.3^\circ$$

The negative angle means that the angle was measured below the horizontal (clockwise from positive x-axis).

Hence, the velocity of the 3kg sphere is 2.15m/s at an angle of 38.3° below the original direction.

(b) Comparing initial and final kinetic energy:

Initial kinetic energy:

$$(K.E)_{\text{before collision}} = \frac{1}{2}m_1u_1^2 + 0 = \frac{1}{2}(2\text{kg})(6\text{m/s})^2 = 36\text{J}$$

Final kinetic energy:

$$(K.E)_{\text{after collision}} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(2\text{kg})(4\text{m/s})^2 + \frac{1}{2}(3\text{kg})(2.15\text{m/s})^2 = 22.94\text{J}$$

Since Initial kinetic energy (36J) $\neq$ Final kinetic energy (22.94J), kinetic energy is not conserved and hence, the collision is inelastic.

Conclusion: student's claim was **invalid**.

Making Sense of the Answer: In *oblique collisions*, the vector nature of momentum becomes crucial. 2kg sphere deflected upward at 30°, so ball 3kg sphere must deflect downward to conserve y-momentum.

Think Like a Physicist: In 2D collisions, always:

1. Resolve velocities into components

2. Apply momentum conservation separately in x and y directions

3. Remember that momentum is a vector; direction matters.

The negative y-component of velocity of 3kg sphere makes physical sense: since 2kg sphere went upward, 3kg sphere must go downward to keep total y-momentum at zero.

HOT Example 18

Two identical pucks collide on a smooth horizontal air-hockey table. Before the collision, puck A moves with a velocity of 8m/s in a straight line, while puck B is at rest. The collision is perfectly elastic. After impact, puck A moves off at an angle of 60° to its original direction of motion.

Determine:

- The velocity of puck A immediately after the collision.
- The velocity of puck B immediately after the collision, stating its magnitude and direction relative to the original line of motion.

Solution

(a) Conserving momentum horizontally:

$$\sum (p_x)_{\text{initial}} = \sum (p_x)_{\text{final}}$$

$$m_A(u_A)_x + m_B(u_B)_x = m_A(v_A)_x + m_B(v_B)_x$$

Where:

$$(u_A)_x = 8\text{m/s}, (u_B)_x = 0, (v_A)_x = v_A \cos 60^\circ = 0.5v_A, (v_B)_x = v_B \cos \theta, m_1 = m_2 = m$$

Substituting:

$$8m + m \times 0 = 0.5mv_A + mv_B \cos \theta$$

Dividing by m throughout gives:

$$0.5v_A + v_B \cos \theta = 8$$

Making $\cos \theta$ the subject:

$$\cos \theta = \frac{8 - 0.5v_A}{v_B} \dots (i)$$

Conserving momentum vertically:

$$\sum (p_y)_{\text{initial}} = \sum (p_y)_{\text{final}}$$

$$m_A(u_A)_y + m_B(u_B)_y = m_A(v_A)_y + m_B(v_B)_y$$

Where:

$$(u_A)_y = 0, (u_B)_y = 0, (v_A)_y = v_A \sin 60^\circ = \frac{\sqrt{3}}{2}v_A, (v_B)_y = v_B \sin \theta$$

Substituting:

$$0 = mv_A \sin 60^\circ + mv_B \sin \theta$$

Again dividing by m throughout gives:

$$\frac{\sqrt{3}}{2}v_A + v_B \sin \theta = 0$$

Making $\sin \theta$ the subject:

$$\sin \theta = \frac{-\frac{\sqrt{3}}{2}v_A}{v_B} = -\frac{(\sqrt{3})v_A}{2v_B} \dots (ii)$$

Conserving kinetic energy:

Since the collision was perfect elastic, Initial K.E = Final K.E:

$$\frac{1}{2}mu_A^2 + 0 = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2$$

From which:

$$u_A^2 = v_A^2 + v_B^2$$

Where: $u_A = 8\text{m/s}$

Then:

$$v_A^2 + v_B^2 = 64 \dots (\text{iii})$$

Using identity $\sin^2\theta + \cos^2\theta = 1$ to eliminate θ :

From (i):

$$\cos^2\theta = \left(\frac{8 - 0.5v_A}{v_B} \right)^2 = \frac{64 - 8v_A + 0.25v_A^2}{v_B^2}$$

From (ii):

$$\sin^2\theta = \left(-\frac{(\sqrt{3})v_A}{2v_B} \right)^2 = \frac{3v_A^2}{4v_B^2}$$

Thus:

$$\sin^2\theta + \cos^2\theta = \frac{3v_A^2}{4v_B^2} + \frac{64 - 8v_A + 0.25v_A^2}{v_B^2} = \frac{v_A^2 - 8v_A + 64}{v_B^2} = 1$$

From which:

$$v_A^2 - 8v_A - v_B^2 = -64 \dots (\text{iv})$$

But from (iii):

$$v_B^2 = 64 - v_A^2$$

Substituting to (iv):

$$\begin{aligned} v_A^2 - 8v_A - (64 - v_A^2) &= -64 \\ 2v_A^2 &= 8v_A; v_A = 4\text{m/s} \end{aligned}$$

Then:

$$v_B^2 = 64 - 4^2 = 48; v_B = 6.93\text{m/s}$$

Also from (i):

$$\cos\theta = \frac{8 - 0.5v_A}{v_B}$$

Substituting:

$$\begin{aligned} \cos\theta &= \frac{8 - 0.5 \times 4}{6.93} = 0.866 \\ \theta &= \cos^{-1} 0.866 = 30^\circ \end{aligned}$$

Hence:

- (a) The velocity of puck A is 4m/s.
- (b) The velocity of puck B is 6.93m/s at an angle of 30° below the original direction of motion.

Alternative solution

For **elastic collision** between **equal masses** where one is initially at **rest**, there is a special property: *the objects move at right angles to each other after collision.*

This means if puck A deflects at 60° above the original direction of motion, puck B must deflect at $(60^\circ - 90^\circ) = -30^\circ$ (or at 30° below x-axis).

(a) *Conserving momentum horizontally:*

$$\begin{aligned} m_A(u_A)_x + m_B(u_B)_x &= m_A(v_A)_x + m_B(v_B)_x \\ 8m + m \times 0 &= 0.5mv_A + mv_B\cos 30^\circ \end{aligned}$$

Dividing by m throughout gives:

$$0.5v_A + v_B\cos 30^\circ = 8 \dots (\text{i})$$

Conserving momentum vertically:

$$m_A(u_A)_y + m_B(u_B)_y = m_A(v_A)_y + m_B(v_B)_y$$

$$0 = mv_A \sin 60^\circ + mv_B \sin(-30^\circ)$$

Again dividing by m throughout gives:

$$v_A \sin 60^\circ - v_B \sin 30^\circ = 0 \dots (ii)$$

Solving (i) and (ii) simultaneously (by using calculator) gives: $v_A = 4\text{m/s}$, $v_B = 6.93\text{m/s}$

So as before, answers are: $v_A = 4\text{m/s}$, $v_B = 6.93\text{m/s}$, $\theta = 30^\circ$ below the original direction.

Making Sense of the Answer: For elastic collisions between equal masses (one initially at rest), the two objects always separate at right angles. Puck A deflected 60° upward, so puck B deflected 30° downward ($60^\circ + 30^\circ = 90^\circ$ between their paths). The slower puck (A at 4m/s) went in the more deflected direction, while the faster puck (B at 6.93m/s) went in the less deflected direction.

Think Like a Physicist: The right-angle separation for equal-mass elastic collisions is a beautiful consequence of conservation laws. This is why billiard balls (nearly equal mass, nearly elastic) often separate at angles close to 90° after collisions. Professional players use this property intuitively when planning shots!

HOT Example 19

A bullet of mass 0.02kg is fired horizontally into a wooden block of mass 2kg suspended by a string (forming a ballistic pendulum). The bullet embeds in the block, and the block swings to a maximum height of 0.3m above its initial position. Taking $g = 9.8\text{ m/s}^2$, calculate:

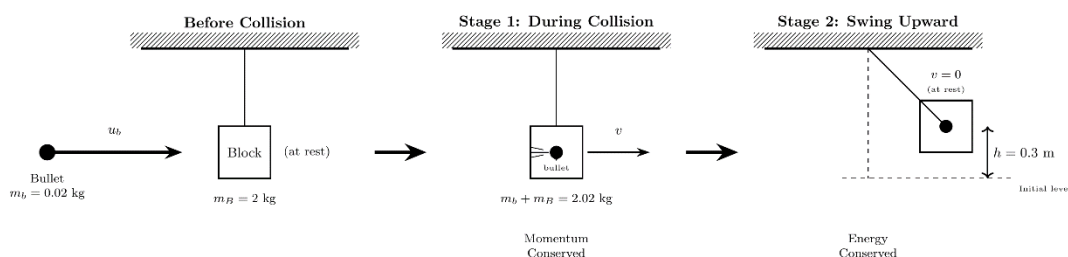
- The velocity of the block immediately after the bullet embeds.
- The initial velocity of the bullet.
- The kinetic energy lost during the collision.
- The percentage of the bullet's initial kinetic energy that was lost.

Solution

The motion occurs in two distinct stages:

- Collision stage:** Bullet embeds in block in perfectly inelastic collision (conservation of linear momentum).
- Swing stage:** Block embedded with bullet swing upward (conservation of mechanical energy).

The following diagram, clarifies these stages.



- Considering stage 2: Swing upward (mechanical energy conservation):

Let m and M denote the masses of the bullet and the block respectively.

At the bottom (just after collision):

$$\text{Kinetic energy} = \frac{1}{2}(m + M)v^2$$

$$\text{Potential energy} = 0 \text{ (reference level)}$$

At maximum height:

$$\text{Kinetic energy} = 0 \text{ (momentarily at rest, } v = 0)$$

$$\text{Potential energy} = (m + M)gh$$

By conservation of mechanical energy:

$$\text{Total mechanical energy at the bottom} = \text{Total mechanical energy at maximum height}$$

$$\frac{1}{2}(m + M)v^2 = (m + M)gh$$

Cancelling $m + M$:

$$v^2 = 2gh \text{ or } v = \sqrt{2gh}$$

Substituting values:

$$v = \sqrt{2 \times 9.8 \text{ m/s}^2 \times 0.3 \text{ m}} = 2.42 \text{ m/s}$$

The velocity of the block is 2.42 m/s.

(b) Considering stage 1: Swing upward (momentum conservation):

Before collision:

- Bullet momentum: mu_{bullet}
- Block momentum: 0 (at rest)

After collision (bullet and block move together):

Combined momentum: $(m + M)v$

By conservation of momentum:

Momentum before collision = Momentum after collision

$$mu_{\text{bullet}} = (m + M)v$$

Making u_{bullet} the subject and substituting values:

$$u_{\text{bullet}} = \frac{(m + M)v}{m} = \frac{(0.02 + 2) \text{ kg} \times 2.42 \text{ m/s}}{0.02 \text{ kg}} = 244.4 \text{ m/s}$$

The initial velocity of bullet is 244.4 m/s.

(c) Kinetic energy lost is given by the following equation:

$$(K.E)_{\text{lost}} = \frac{1}{2}mu_{\text{bullet}}^2 - \frac{1}{2}(m + M)v^2$$

Substituting:

$$(K.E)_{\text{lost}} = \frac{1}{2}(0.02 \text{ kg})(244.4 \text{ m/s})^2 - \frac{1}{2}(0.02 + 2) \text{ kg} \times (2.42 \text{ m/s})^2 = 591.4 \text{ J}$$

The kinetic energy lost is 591.4 J.

(d) The percentage of the kinetic energy lost

$$= \frac{\text{Kinetic energy lost}}{\text{Initial kinetic energy}} \times 100\% = \frac{591.4 \text{ J}}{\frac{1}{2}(0.02 \text{ kg})(244.4 \text{ m/s})^2} \times 100\% = 99\%$$

The percentage is 99%.

Making Sense of the Answer: The bullet was traveling at an extremely high velocity (244 m/s), typical for firearms. When it embedded in the much heavier block (100 times heavier!), almost all its kinetic energy converted to heat, sound, and deformation of wood. Only 1% remained as kinetic energy of the swinging block. This is why bullets cause so much damage, as they deposit almost all their energy into the target.

Think Like a Physicist: The ballistic pendulum is a classic device historically used to measure bullet speeds before modern chronographs existed. By measuring how high the pendulum swings (easily observable), you can calculate the bullet's initial velocity (hard to measure directly). This is a beautiful application of conservation principles working together: momentum during collision, then energy during the swing.

HOT Example 20

A stationary bomb of mass 12 kg explodes into three fragments. Fragment A (mass 3 kg) flies north at 20 m/s. Fragment B (mass 4 kg) flies east at 15 m/s. Find:

- The velocity of fragment C (magnitude and direction).
- The kinetic energy released by the explosion.

Take $g = 9.8 \text{ m/s}^2$

Solution

Interpreting the data:

Total mass: $M = 12 \text{ kg}$

Fragment A: $m_A = 3 \text{ kg}$, $v_A = 20 \text{ m/s}$ north

Fragment B: $m_B = 4 \text{ kg}$, $v_B = 15 \text{ m/s}$ east

Fragment C: $m_C = 12\text{kg} - (3 + 4)\text{kg} = 5\text{kg}$, $v_C = ?$

(a) Conservation of linear momentum (initially at rest, total momentum = 0):

Taking north as +y direction and east as +x direction.

x-direction:

$$0 = m_A(v_A)_x + m_B(v_B)_x + m_C(v_C)_x$$

$$0 = 3 \times 0 + 4 \times 15 + 5(v_C)_x$$

$$(v_C)_x = -12\text{m/s} \text{ (the negative sign along x-axis means west direction).}$$

y-direction:

$$0 = m_A(v_A)_y + m_B(v_B)_y + m_C(v_C)_y$$

$$0 = 3 \times 20 + 4 \times 0 + 5(v_C)_y$$

$$(v_C)_y = -12\text{m/s} \text{ (the negative sign along y-axis means south direction).}$$

$$\text{Magnitude: } v_C = \sqrt{((v_C)_x)^2 + ((v_C)_y)^2} = \sqrt{((-12\text{m/s})^2 + (-12\text{m/s})^2)} = 16.97\text{m/s}$$

Direction: $\tan \theta = \frac{(v_C)_y}{(v_C)_x} = \frac{-12\text{m/s}}{-12\text{m/s}} = 1$; $\theta = \tan^{-1} 1 = 45^\circ$ $\theta = 45^\circ$ (but both components are negative, so it is in the third quadrant).

The velocity of fragment C is 16.97m/s at 45° south of west (or southwest direction).

(b) Initial KE = 0 (bomb at rest)

Final KE (sum of all fragments)

$$= \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 + \frac{1}{2}m_C v_C^2$$

$$= \frac{1}{2}(3\text{kg})(20\text{m/s})^2 + \frac{1}{2}(4\text{kg})(15\text{m/s})^2 + \frac{1}{2}(5\text{kg})(16.97\text{m/s})^2 = 1770\text{J}$$

$$\text{Energy released} = \text{Final KE} - \text{Initial KE} = 1770\text{J} - 0 = 1770\text{J}$$

The energy released is 1770J.

Making Sense of the Answer: The bomb started with zero momentum and zero kinetic energy. After explosion, the vector sum of momenta is still zero (each component sums to zero), but kinetic energy increased dramatically. This energy came from chemical energy stored in the explosive. Fragment C had to move southwest to balance the northward motion of A and eastward motion of B.

Think Like a Physicist: Explosions are "reverse collisions." In collisions, kinetic energy usually decreases (converted to heat/sound/deformation). In explosions, kinetic energy increases (chemical energy converts to kinetic energy). But momentum is conserved in both cases.

The worked examples have comprehensively demonstrated all types of collisions and how to analyse them using momentum conservation, energy considerations, and the coefficient of restitution. We can now confidently solve any collision problem by identifying the type of collision and applying the appropriate conservation laws.

VARIABLE MASS SYSTEMS

Throughout this chapter, we have analysed collisions and momentum conservation for systems where mass remains constant. We used Newton's second law in the form $F = ma$, and later learned the more general form $F = dp/dt$. But what happens when the mass of a moving object changes continuously?

Consider these fascinating situations:

Situation 1: A rocket in space burns fuel and ejects exhaust gases backward. Its mass decreases continuously while it accelerates forward, even though nothing external pushes it!

Situation 2: Rain falls vertically onto a toy car rolling horizontally on a smooth surface. The car's mass increases as it collects rainwater. What happens to its velocity?

Situation 3: A conveyor belt moves at constant velocity while sand falls onto it from a stationary hopper above. What force must the motor provide to maintain constant velocity?

These are **variable mass systems** (objects whose mass changes with time). They cannot be analyzed using $F = ma$ because that equation assumes constant mass. We must return to the fundamental form: $\mathbf{F} = \frac{d\mathbf{p}}{dt}$.

Why $F = ma$ Fails for Variable Mass

Newton's second law in the form $F=ma$ is derived from $F = dp/dt$ by assuming mass is constant:

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m \left(\frac{dv}{dt} \right) + v \left(\frac{dm}{dt} \right)$$

If mass is constant ($\frac{dm}{dt} = 0$), this simplifies to:

$$F = m \left(\frac{dv}{dt} \right) = ma$$

But when mass changes ($\frac{dm}{dt} \neq 0$), we cannot ignore the second term. The complete equation is:

$$F = m \left(\frac{dv}{dt} \right) + v \left(\frac{dm}{dt} \right)$$

Or equivalently:

$$F = ma + v \left(\frac{dm}{dt} \right)$$

Where:

F = external force on the system,

m = instantaneous mass at time t ,

$a = \frac{dv}{dt}$ = acceleration,

v = velocity of the system,

$\frac{dm}{dt}$ = rate of mass change.

It is important to understand that the term $v \left(\frac{dm}{dt} \right)$ represents the force associated with the mass flow; either mass leaving the system (e.g. a rocket) or mass joining the system (e.g. rain on cart). Therefore, we have two categories of variable mass systems:

Category 1: Mass decreasing systems

- Examples: Rockets, jet engines, leaking containers.
- $\frac{dm}{dt} < 0$ (mass decreases).
- Typically: external force $F \approx 0$ (rocket in space).
- Result: Object accelerates as it loses mass.

Category 2: Mass increasing systems

- Examples: Rain falling on cart, sand on conveyor, snow on train.
- $\frac{dm}{dt} > 0$ (mass increases).
- Often: maintain constant velocity ($v = \text{constant}$, $a = 0$).
- Result: Force needed to prevent slowing down (to maintain constant velocity).

Let us explore each category in depth.

Mass Decreasing Systems: Rockets and Jets**Rocket Propulsion:**

A common question students ask is: *How can a rocket move in empty space if there is nothing to push against?*

The answer is that a rocket does not rely on air, the ground, or any external surface. Its motion is entirely explained by the **conservation of momentum**.

A rocket moves by expelling exhaust gases at very high velocity in the backward direction. When the gases are pushed backward, they carry momentum with them. To conserve total momentum, the rocket gains an equal amount of forward momentum. This interaction occurs entirely within the rocket–exhaust system, which is why a rocket functions perfectly even in a vacuum.

As the fuel burns:

- Rocket's mass decreases with time: $\frac{dm}{dt} < 0$.

- Exhaust gases are expelled with velocity u relative to rocket.
- Rocket accelerates forward (by conservation of momentum).
- The backward momentum of the expelled gases produces a forward thrust on the rocket.

Thus, the rocket accelerates forward as a direct consequence of momentum conservation in a variable-mass system.

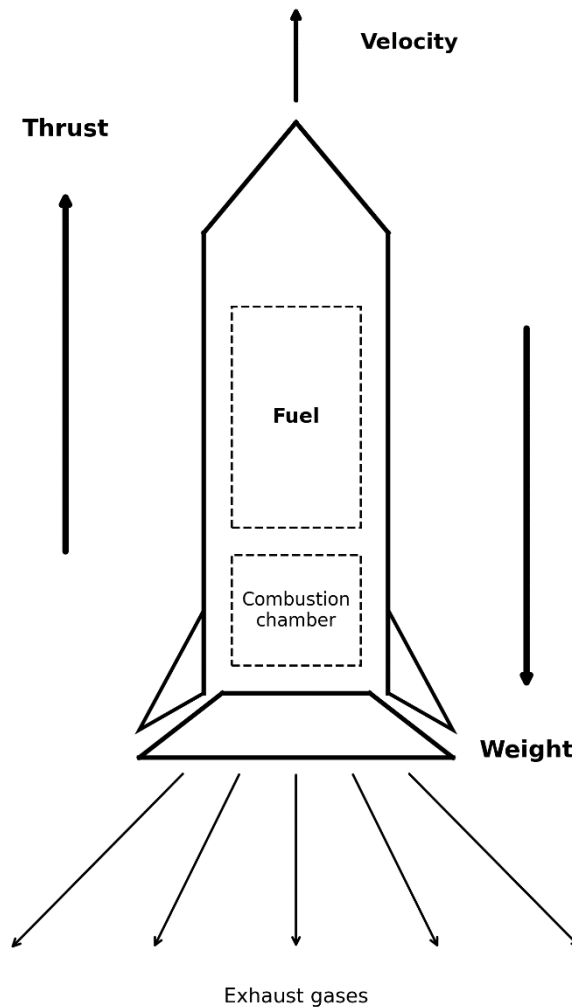


Figure: Rocket propulsion mechanism. Expelled gases carry momentum downward, generating upward thrust on the rocket by conservation of momentum.

Deriving Rocket Equations:

Consider a rocket of instantaneous mass m moving with velocity v in free space (no external forces).

In a small time interval dt :

- Rocket ejects a small mass dm of exhaust.
- The exhaust leaves with velocity u_{rel} relative to the rocket (directed backward).
- Rocket mass decreases to $(m - dm)$.
- Rocket velocity increases to $(v + dv)$.

Taking the rocket as our reference frame initially:

Before ejection (in rocket's initial rest frame, where $v = 0$):

Total momentum = 0

After ejection (in the same initial rest frame):

- Rocket momentum = $(m - dm)(v + dv) = (m - dm)(0 + dv) = (m - dm)dv$.

- Exhaust momentum = $dm(-u_{\text{rel}})$ (negative because backward).

Conservation of momentum:

$$0 = (m - dm)(dv) + dm(-u_{\text{rel}})$$

$$0 = m(dv) - dm(dv) - u_{\text{rel}}(dm)$$

Since dm and dv are very small quantities, $dm(dv) \approx 0$ (negligible). So:

$$0 = m(dv) - u_{\text{rel}}(dm)$$

$$m(dv) = u_{\text{rel}}(dm)$$

Rearranging:

$$dv = u_{\text{rel}} \left(\frac{dm}{m} \right)$$

Important note on sign convention: In this derivation, u_{rel} is taken as a **positive** exhaust velocity relative to the rocket, and dm represents the **positive** mass expelled, so the rocket's mass decreases by this amount; that is $m_{\text{rocket}} = m - dm$.

Tsiolkovsky Rocket Equation:

Integrating the differential equation obtained from momentum conservation leads to the Tsiolkovsky rocket equation, which relates the rocket's change in velocity to the exhaust velocity and the ratio of its initial mass to its final mass.

From the differential equation:

$$dv = u_{\text{rel}} \left(\frac{dm}{m} \right)$$

Integrating from initial mass m_i to final mass m_f , with their respective change in velocity, v_i to v_f where total mass expelled = m_i (upper limit) $- m_f$ (lower limit) and total increase in rocket's velocity = v_f (upper limit) $- v_i$ (lower limit).

$$\begin{aligned} \int_{v_i}^{v_f} dv &= u_{\text{rel}} \int_{m_f}^{m_i} \frac{dm}{m} \\ v_f - v_i &= u_{\text{rel}} [\ln m]_{m_f}^{m_i} = u_{\text{rel}} (\ln m_i - \ln m_f) \\ v_f &= v_i + u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right) \end{aligned}$$

The final equation is known as **Tsiolkovsky Rocket Equation**. In this form, the equation applies if and only if **no external forces act on the rocket**.

Alternatively, the equation can be written as:

$$v_f - v_i = \Delta v = u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right)$$

The Tsiolkovsky Rocket Equation reveals several important practical insights:

1. Exhaust velocity and mass ratio:

A higher exhaust velocity (u_{rel}) or a larger mass ratio m_i/m_f produces a greater final velocity. The exhaust velocity depends on engine design and fuel quality, while the mass ratio depends on how much fuel the rocket can carry relative to its structure and payload.

2. Logarithmic limitation:

The velocity depends logarithmically on the mass ratio. This means that achieving large increases in velocity requires disproportionately large increases in fuel mass. For example, to double the velocity increase, the mass ratio must be squared. This logarithmic behaviour is one reason space travel is extremely expensive.

3. Staged rockets:

Instead of building a single massive rocket with an extremely high mass ratio, engineers use multiple stages. Each stage burns its fuel and is then discarded, reducing the remaining mass. This greatly improves efficiency and makes high velocities achievable in practice.

The Thrust Force:

The thrust produced by a rocket arises from the rate of change of momentum of the expelled gases.

$$\mathbf{F}_{\text{thrust}} = \mathbf{u}_{\text{rel}} \left(\frac{d\mathbf{m}}{dt} \right)$$

Where $\frac{dm}{dt}$ is the rate at which mass is ejected (positive value).

Thus, thrust is directly proportional to both the exhaust velocity and the rate of fuel consumption.

The relationship between thrust and resultant force of rocket in absence of external forces can be derived as follows:

From Rocket Differential Equation:

$$dv = u_{\text{rel}} \left(\frac{dm}{m} \right)$$

Multiplying by m both sides:

$$m(dv) = u_{\text{rel}}(dm)$$

Dividing by dt throughout:

$$m \left(\frac{dv}{dt} \right) = u_{\text{rel}} \left(\frac{dm}{dt} \right)$$

Where $m \left(\frac{dv}{dt} \right)$ is the rocket's resultant (net) force = ma

If external forces (e.g. gravity, drag) exist:

Resultant force = Thrust + External forces

$$m \left(\frac{dv}{dt} \right) = u_{\text{rel}} \left(\frac{dm}{dt} \right) + F_{\text{external}}$$

For vertical launch against gravity where the rocket must first overcome gravity (mg) before accelerating upward, $F_{\text{external}} = -mg$; thus:

$$m \left(\frac{dv}{dt} \right) = u_{\text{rel}} \left(\frac{dm}{dt} \right) - mg$$

From this equation we can derive modified Tsiolkovsky Rocket Equation under presence of force of gravity as follows:

Multiplying by dt and dividing by m throughout the above equation:

$$dv = u_{\text{rel}} \left(\frac{dm}{m} \right) - gdt$$

Integrating:

$$\begin{aligned} \int_{v_i}^{v_f} dv &= u_{\text{rel}} \int_{m_i}^{m_f} \frac{dm}{m} - g \int_0^t dt \\ v_f - v_i &= u_{\text{rel}} [\ln m]_{m_i}^{m_f} - g(t - 0) = u_{\text{rel}} (\ln m_i - \ln m_f) - gt \\ v_f &= v_i + u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right) - gt \end{aligned}$$

The final equation is the modified Tsiolkovsky rocket equation for vertical flight under gravity (neglecting drag).

Jet Propulsion:

Jet engines operate on the same fundamental principle as rockets: thrust is produced by the rate of change of momentum of expelled gases. However, there is one crucial difference in how combustion is sustained.

Rockets: Carry both fuel and oxidiser, allowing them to operate independently of the surrounding environment. As a result, rockets function in space as well as in the atmosphere.

Jet engines: Draw in atmospheric air to supply oxygen for combustion. Therefore, they can operate only within the atmosphere.

This difference leads to important practical consequences.

Advantage: Since jet engines do not carry oxidiser, they are much more efficient in the atmosphere. The absence of heavy oxidiser significantly reduces the aircraft's mass.

Disadvantage: Because they depend on atmospheric oxygen, jet engines cannot operate in space, where no air is available.

Operation of a Jet Engine

The basic working process of a jet engine can be summarised as follows:

1. Air intake:

Air enters the front of the engine with velocity v_{in} (relative to the ground).

2. Compression and combustion:

The incoming air is compressed, mixed with fuel, and ignited in the combustion chamber.

3. Exhaust:

The high-temperature, high-pressure gases expand and exit the rear of the engine with velocity, v_{out} (relative to the ground).

4. Thrust generation:

The difference between the incoming and outgoing momentum of the air–gas mixture produces a net forward thrust.

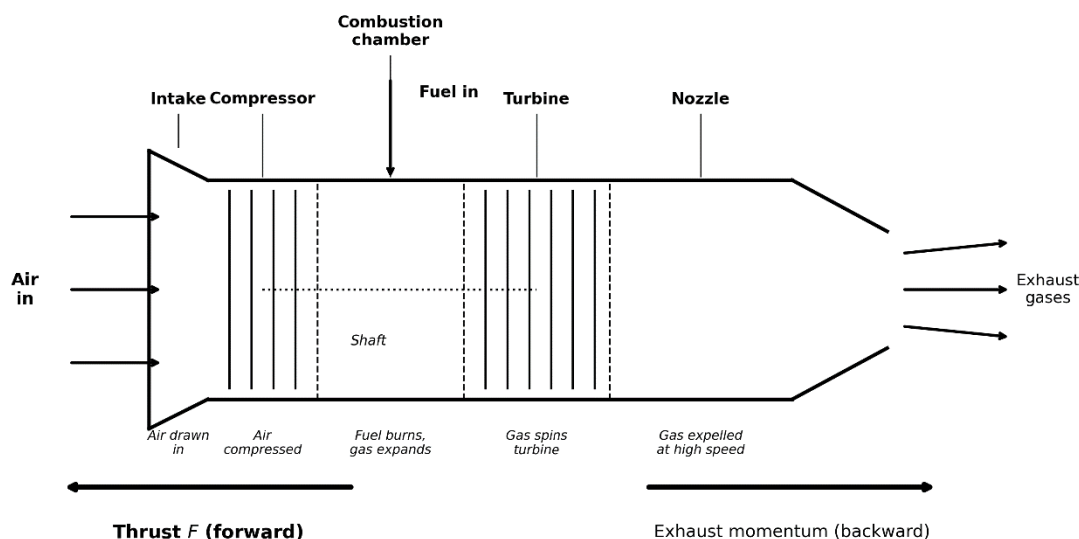


Figure: Jet propulsion mechanism. Air is drawn in at the front, compressed, mixed with fuel and burned in the combustion chamber, then expelled at high velocity through the nozzle. The backward momentum of the exhaust produces a forward thrust by Newton's third law. Unlike a rocket, a jet engine takes in atmospheric air as the working mass.

Thrust Equation for a Jet Engine

A jet engine produces thrust by changing the momentum of the air–fuel mixture passing through it. Let:

$\dot{m}_{air}$ = mass flow rate of incoming air,

$\dot{m}_{\text{fuel}}$ = mass flow rate of injected fuel,

v_{in} = velocity of incoming air (relative to ground),

v_{out} = velocity of exhaust gases (relative to ground).

Applying momentum principle:

Thrust is equal to the rate of change of momentum:

$$F_{\text{thrust}} = \frac{d(\text{momentum})}{dt}$$

For steady flow:

$$F_{\text{thrust}} = (\text{momentum out per second}) - (\text{momentum in per second})$$

Momentum in:

Only air enters the engine from outside. So:

$$\text{momentum out per second} = \dot{m}_{\text{air}} v_{\text{in}}$$

You should understand that fuel is injected inside the engine, so it does not contribute to incoming external momentum.

Momentum out:

Both air and fuel leave as exhaust gases.

So total mass leaving per second = $\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}}$

And momentum out per second = $(\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})v_{\text{out}}$

Net thrust:

$$F_{\text{thrust}} = (\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})v_{\text{out}} - \dot{m}_{\text{air}}v_{\text{in}}$$

Rearranging:

$$F_{\text{thrust}} = \dot{m}_{\text{air}}(v_{\text{out}} - v_{\text{in}}) + \dot{m}_{\text{fuel}}v_{\text{out}}$$

Hence, the final thrust equation for a jet engine is:

$$F_{\text{thrust}} = \dot{m}_{\text{air}}(v_{\text{out}} - v_{\text{in}}) + \dot{m}_{\text{fuel}}v_{\text{out}}$$

Where $\dot{m} = \frac{dm}{dt}$ (mass flow rate)

Simplified form:

If the aircraft velocity equals the incoming air velocity:

$$v_{\text{in}} = v_{\text{jet}}$$

And if fuel mass flow is small compared to air flow, then the thrust equation for jet engine can be simplified as follows:

From one the form of the thrust equation above:

$$F_{\text{thrust}} = (\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})v_{\text{out}} - \dot{m}_{\text{air}}v_{\text{in}}$$

Subtracting and adding $\dot{m}_{\text{fuel}}v_{\text{in}}$ (this is just adding zero in a useful form):

$$F_{\text{thrust}} = (\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})v_{\text{out}} - \dot{m}_{\text{air}}v_{\text{in}} - \dot{m}_{\text{fuel}}v_{\text{in}} + \dot{m}_{\text{fuel}}v_{\text{in}}$$

Grouping the first three terms:

$$F_{\text{thrust}} = (\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})v_{\text{out}} - v_{\text{in}}(\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}}) + \dot{m}_{\text{fuel}}v_{\text{in}}$$

$$F_{\text{thrust}} = (\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})(v_{\text{out}} - v_{\text{in}}) + \dot{m}_{\text{fuel}}v_{\text{in}}$$

Applying the common jet assumption $v_{\text{in}} = v_{\text{jet}}$

$$F_{\text{thrust}} = (\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})(v_{\text{out}} - v_{\text{jet}}) + \dot{m}_{\text{fuel}}v_{\text{jet}}$$

In most aircraft engines, $\dot{m}_{\text{fuel}} \ll \dot{m}_{\text{air}}$, and thus the extra term $\dot{m}_{\text{fuel}}v_{\text{jet}}$ is relatively small compared with the main momentum-change term and hence it can be neglected. Therefore, the simplified thrust equation for jet engine becomes:

$$F_{\text{thrust}} = (\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})(v_{\text{out}} - v_{\text{jet}})$$

This form emphasizes that thrust depends on:

- The total mass flow rate,
- The difference between exhaust velocity and aircraft velocity.

So we can conclude that:

- Thrust increases if exhaust velocity increases.
- Thrust increases if mass flow rate increases.
- If $v_{\text{out}} = v_{\text{in}} \approx v_{\text{jet}}$, thrust is zero.
- If exhaust velocity is only slightly greater than aircraft velocity, thrust is small.

Warning!

The simplified form

$$F_{\text{thrust}} = (\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})(v_{\text{out}} - v_{\text{jet}})$$

is an **approximation**. It neglects the additional momentum term associated with the fuel entering the engine at aircraft velocity. **In solving problems, always use the full momentum expression unless explicitly told otherwise.**

Mass Increasing Systems: Constant Velocity

So far, we have considered systems that lose mass, such as rockets. Let us now examine the opposite situation: systems whose mass increases with time.

Examples include:

- Rain falling onto a moving cart
- Sand pouring onto a moving conveyor belt
- Water flowing from a pipe onto a moving platform
- Grain being loaded into a moving truck

In this case, the key question is this: *if mass is being added to a moving system, what force is required to maintain constant velocity?*

When mass is added to a moving system, the added mass must be accelerated from rest (or its initial velocity) to match the system's velocity. This requires a force, even if the system's velocity does not change! If no force is applied, the system slows down.

Deriving the required force:

Consider a system of mass m moving at constant velocity v . In time dt , additional mass dm is added (initially at rest in the ground frame).

Conservation of momentum approach:

Before addition:

- System momentum: $p_1 = mv$
- Added mass momentum: $p_2 = 0$ (at rest)
- Total: $p_{\text{initial}} = mv$

After addition (system maintains velocity v):

- Combined momentum: $p_{\text{final}} = (m + dm)v$

Change in momentum:

$$\Delta p = p_{\text{final}} - p_{\text{initial}}$$

$$\Delta p = (m + dm)v - mv = v(dm)$$

This momentum change must be provided by an external force, F .

$$\Delta p = F \times dt = v(dm)$$

Hence:

$$\mathbf{F} = v \left(\frac{dm}{dt} \right)$$

The equation tells that to maintain constant velocity while mass increases, you must apply a force equal to the velocity times the rate of mass increase.

Alternative derivation using $\mathbf{F} = \frac{dp}{dt}$:

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m \left(\frac{dv}{dt} \right) + v \left(\frac{dm}{dt} \right)$$

If velocity is constant $\left(\frac{dv}{dt} = 0 \right)$:

$$F = v \left(\frac{dm}{dt} \right)$$

The equation reveals several important practical insights:

- 1. Force is needed even though velocity is constant** ($a = 0$ but $F \neq 0$!)
- 2. F is proportional to v :** Faster systems need more force to maintain velocity while collecting mass.
- 3. F is proportional to dm/dt :** Heavier mass flow requires more force.
- 4. Direction:** Force must be in the direction of motion.

With this comprehensive theoretical foundation established, we are now ready to bring these concepts to life through carefully crafted worked examples.

BINDER Example 21

A toy rocket has a total mass of 500g (including 300g of fuel). The rocket engine ejects exhaust gases at a speed of 800m/s relative to the rocket. Calculate:

- The final velocity of the rocket after all fuel is burned (assume it starts from rest in space).
- The thrust force if fuel burns at a rate of 20g/s.

Solution

Interpreting the data:

Initial mass: $m_0 = 500\text{g} = 0.5\text{kg}$

Fuel mass: $300\text{g} = 0.3\text{kg}$

Final mass (after fuel burned): $m_f = (500 - 300)\text{g} = 200\text{g} = 0.2\text{kg}$

Exhaust velocity: $u_{\text{rel}} = 800\text{m/s}$

Initial velocity: $v_i = 0$ (starts from rest)

Fuel burn rate: $\frac{dm}{dt} = 20\text{g/s} = 0.02\text{kg/s}$

- Using the integral rocket equation:

$$v_f = v_i + u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right)$$

With $v_i = 0$, the above equation becomes:

$$v_f = u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right)$$

Substituting values:

$$v_f = 800\text{m/s} \times \ln\left(\frac{0.5\text{kg}}{0.2\text{kg}}\right) = 733\text{m/s}$$

The velocity is 733m/s.

(b) The thrust force is given by the following equation:

$$F_{\text{thrust}} = u_{\text{rel}} \left(\frac{dm}{dt} \right)$$

Substituting values:

$$F_{\text{thrust}} = 800\text{m/s} \times 0.02\text{kg/s} = 16\text{N}$$

The thrust force is 16N.

Making Sense of the Answer: The rocket reached 733m/s (about 2600km/h) by burning 60% of its initial mass as fuel. The thrust of 16N is modest but sufficient for a 500g rocket as it provides initial acceleration of $a = F/m = 16\text{N}/0.5\text{kg} = 32\text{m/s}^2$, which is impressive! Notice that the thrust remains constant as long as u_{rel} and dm/dt stay constant, but the acceleration increases as the mass of rocket decreases.

Think Like a Physicist: The rocket equation $v_f = u_{\text{rel}} \times \ln(m_i/m_f)$ shows a logarithmic relationship. To double the final velocity, you would need to square the mass ratio, $m_i:m_f$ (from 2.5 to 6.25), meaning even more fuel. This is why **multi-stage rockets are used for space missions because they drop empty fuel tanks to improve the mass ratio**. Also notice that thrust is independent of the rocket's mass - it depends only on exhaust velocity and mass flow rate.

BINDER Example 22

A small jet aircraft is flying horizontally at 200m/s. The jet engine takes in 50kg of air per second and ejects exhaust gases at 500m/s relative to the ground. The fuel consumption rate is 2kg/s. Calculate:

- (b) The thrust produced by the engine.
(c) The net force on the aircraft if air resistance is 8000N

Solution

Interpreting the data:

Aircraft velocity: $v_{\text{jet}} = 200\text{m/s}$

Air intake rate: $\dot{m}_{\text{air}} = 50\text{kg/s}$

Exhaust velocity (relative to ground): $v_{\text{out}} = 500\text{m/s}$

Fuel burn rate: $\dot{m}_{\text{fuel}} = 2\text{kg/s}$

Air resistance: $F_{\text{drag}} = 8000\text{N}$

- (a) Air enters the engine at the aircraft's velocity: $v_{\text{in}} = v_{\text{jet}} = 200\text{m/s}$ (relative to ground).

Using the jet thrust equation:

$$F_{\text{thrust}} = \dot{m}_{\text{air}}(v_{\text{out}} - v_{\text{in}}) + \dot{m}_{\text{fuel}}v_{\text{out}}$$

Substituting values:

$$F_{\text{thrust}} = 50\text{kg/s}(500 - 200)\text{m/s} + 2\text{kg/s} \times 500\text{m/s} = 16000\text{N}$$

The thrust is 16000N (or 16kN).

- (b) Net force = Thrust – Air resistance

$$F_{\text{net}} = F_{\text{thrust}} - F_{\text{drag}} = (16000 - 8000)\text{N} = 8000\text{N}$$

The net force is 8000N (or 8kN).

Making Sense of the Answer: The engine produces 16kN of thrust by increasing the momentum of the flow through it. Each second, a 52kg of mass (air + fuel) leaves as exhaust at 500m/s, while 50kg of incoming air enters at 200m/s, giving a momentum increase equivalent to a velocity change of 300m/s for the main airflow. The net forward force of 8kN means the aircraft is accelerating. The large air intake (50kg/s) is typical for

jet engines; they use atmospheric oxygen rather than carrying it, making them much more efficient than rockets in atmosphere.

Think Like a Physicist: Jets are fundamentally different from rockets: jets take in air (free mass + oxygen), while rockets carry everything. This is why jets are efficient in atmosphere but useless in space, while rockets work everywhere but are less efficient in atmosphere (must carry oxidiser). The thrust depends on the velocity difference ($v_{out} - v_{in}$), which is why jets are less efficient at higher speeds because as the aircraft speeds up, the relative exhaust velocity decreases.

BINDER Example 23

A jet aircraft is flying horizontally at 250m/s. The engines take in 80kg of air per second and eject it at 600m/s relative to the ground. The fuel burn rate is 3kg/s. Calculate:

- (a) The thrust produced by the engines.
- (b) The mass of fuel consumed to travel 100km at this velocity.

Solution

Interpreting the data:

Aircraft velocity: $v_{jet} = v_{in} = 250\text{m/s}$

Air intake rate: $\dot{m}_{air} = 80\text{kg/s}$

Exhaust velocity (relative to ground): $v_{out} = 600\text{m/s}$

Fuel burn rate: $\dot{m}_{fuel} = 3\text{kg/s}$

Distance: $d = 100\text{ km} = 100000\text{m}$

- (a) Using the jet thrust equation:

$$F_{thrust} = \dot{m}_{air}(v_{out} - v_{in}) + \dot{m}_{fuel}v_{out}$$

Substituting values:

$$F_{thrust} = 80\text{kg/s}(600 - 250)\text{m/s} + 3\text{kg/s} \times 600\text{m/s} = 29800\text{N}$$

The thrust is 29800N.

- (b) Time to travel 100km:

$$t = \frac{d}{v} = \frac{100000\text{m}}{250\text{m/s}} = 400\text{s}$$

Mass of fuel consumed = $\dot{m}_{fuel} \times t = 3\text{kg/s} \times 400 = 1200\text{kg}$

The mass consumed is 1200kg.

Making Sense of the Answer: The thrust (29800N) is substantial, typical for a small jet engine. The exhaust exits at only 350m/s faster than it entered ($600 - 250 = 350\text{m/s}$), but the large mass flow (83kg/s) produces significant thrust. In 400 seconds (6.7 minutes), the aircraft burned 1,200 kg (1.2 tonnes) of fuel! This illustrates how rapidly fuel is used in jet propulsion and why fuel efficiency is a critical factor in aircraft operation and design.

Think Like a Physicist: Jet engines are more efficient than rockets in atmosphere because they do not carry oxidiser as they get oxygen "free" from the air. The 80: 3 ratio (air:fuel) is typical, showing that most mass flow is air, not fuel. This is why jets are practical for aircraft but rockets are needed for space.

BINDER Example 24

A flatbed railway wagon moves at constant velocity of 10m/s along a horizontal track. Coal is being dropped vertically onto the wagon from a stationary hopper at a rate of 40kg/s. Calculate the horizontal force required to maintain the wagon's constant velocity.

Solution

Interpreting the data:

Wagon velocity: $v = 10\text{m/s}$ (constant)

Coal loading rate: $\frac{dm}{dt} = 40\text{kg/s}$

Coal initially has zero horizontal velocity (drops vertically)

For mass increasing system at constant velocity:

$$F = v \left(\frac{dm}{dt} \right)$$

Substituting values:

$$F = 10\text{m/s} \times 40\text{kg/s} = 400\text{N}$$

The force is 400N acting in the direction of motion.

Making Sense of the Answer: The force requirement (400N) seems modest, but it must be applied continuously as long as coal is being loaded.

Think Like a Physicist: This problem demonstrates an important principle: maintaining constant velocity while mass increases requires force, even though acceleration is zero ($a = 0$ but $F \neq 0$). This seems to contradict $F = ma$, but remember the complete equation is $F = m(dv/dt) + v(dm/dt)$. When $dv/dt = 0$, we still have $F = v(dm/dt)$. The falling coal must be given horizontal momentum, and that requires a horizontal force on the system. This is why conveyor belt motors must work harder when material is being loaded, even if belt speed stays constant.

BINDER Example 25

Water flows horizontally from a stationary fire hose at a rate of 20kg/s with velocity 25m/s. The water strikes a vertical wall and falls vertically downward (assume no bouncing). Calculate:

- The force exerted by water on the wall.
- If the wall is mounted on wheels and can move freely, at what rate will it accelerate if the wall's mass is 500kg?

Solution

Interpreting the data:

Water flow rate: $\frac{dm}{dt} = 20\text{kg/s}$

Water velocity: $v = 25\text{m/s}$ (horizontal)

Wall mass: $m_{\text{wall}} = 500\text{kg}$

Water falls vertically after hitting (final horizontal velocity = 0)

- The horizontal velocity of the water changes from **25m/s to 0**.

Rate of change of momentum = $F = \dot{m}(v_{\text{final}} - v_{\text{initial}}) = 20\text{kg/s} (0 - 25)\text{m/s} = -500\text{N}$

The negative indicates that momentum of water decreases (water slows horizontally).

The force exerted is 500N (pushing the wall horizontally).

- If the wall is free to move, this force (500N) acts on it.

Using Newton's second law:

$$a = \frac{F}{m} = \frac{500\text{N}}{500\text{kg}} = 1\text{m/s}^2$$

The acceleration would be 1m/s^2 .

Making Sense of the Answer: The force of 500N arises because every second, 20kg of water moving at 25m/s has its horizontal velocity reduced to zero. Stopping this steady stream of horizontal momentum requires a continuous force. Even though the water simply falls downward after impact, the key effect is that its horizontal momentum is destroyed, and the wall must provide the force to make that happen. If the wall is free to move, this 500N force produces an acceleration of 1m/s^2 , which is quite noticeable for a 500 kg mass.

Thinking Like a Physicist: Whenever a moving fluid strikes a surface, do not focus on velocity but on **momentum change per second**. The force comes from how much momentum is being removed (or redirected) each second. Even if the velocity of the fluid remains constant before impact, a change in direction or a reduction to zero velocity still represents a change in momentum. Always ask:

- What is the initial velocity?
- What is the final velocity?
- What is the rate at which mass is flowing?

Force in fluid problems is almost always a **momentum-rate problem**, not an energy problem.

REAL Example 26

A space agency is designing a launch vehicle. One engineer suggests building one very large rocket with enough fuel to reach orbit. Another proposes a multi-stage rocket where empty fuel tanks are discarded during flight.

By using the rocket equation, explain which design is more suitable.

Solution

Suitable choice: Multi-stage rocket.

Explanation

A rocket gains velocity by ejecting mass backward at high speed. The total gain in velocity (Δv) is given by the integral (Tsiolkovsky) rocket equation:

$$\Delta v = u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right)$$

This shows that velocity gain increases as the mass ratio, m_i/m_f increases. For a single massive rocket, the initial mass (m_i) includes: fuel, structure and payload. After fuel is burned, the final mass (m_f) still includes the entire empty structure. That empty structure contributes no further thrust but must still be accelerated. This reduces the effective mass ratio.

In a multi-stage rocket, once fuel in one stage is exhausted, that empty structure is discarded. The next stage then accelerates a smaller mass. This increases the effective mass ratio for each stage and allows a greater total velocity increase. Therefore, multi-stage rockets are preferred because they increase achievable velocity without requiring unrealistically large fuel masses.

Making Sense of the Answer: The rocket equation contains a logarithm. This means that to double the final velocity, the mass ratio must increase exponentially. Carrying empty tanks is inefficient because they not only increase final mass (which makes the mass ratio smaller) but also do not contribute to thrust. So discarding them improves efficiency dramatically.

Thinking Like a Physicist: Focus on the mass ratio, not just the fuel amount. The rocket equation shows that useless mass in the final mass reduces the achievable velocity. Multi-staging works because it removes dead weight, improving the mass ratio at each stage. Better mass ratio means greater velocity gain.

REAL Example 27

During a safety drill, Kipute uses a CO_2 fire extinguisher. When she presses the handle, the extinguisher jerks backward in her hands. Explain why this recoil occurs, even though the extinguisher is not “pushing against the air.”

Solution

When the extinguisher is activated, carbon dioxide is expelled at high velocity in one direction. The escaping gas carries momentum. According to conservation of momentum, if the gas gains momentum in one direction, the extinguisher must gain equal momentum in the opposite direction.

This produces a backward thrust on the extinguisher. The force experienced equals the rate at which momentum is expelled:

$$F_{\text{thrust}} = u_{\text{rel}} \left(\frac{dm}{dt} \right)$$

Thus, the recoil is not due to pushing against air but due to the continuous ejection of mass. The faster and greater the mass flow rate, the stronger the recoil.

Making Sense of the Answer: *The extinguisher behaves exactly like a small rocket. It does not need air to push against; it moves because it throws mass forward. The same principle explains rocket propulsion in space.*

Thinking Like a Physicist Ask: *what momentum leaves the system each second? If mass is expelled forward, momentum is carried away. By conservation of momentum, the source must recoil backward. Thrust is simply the rate at which momentum is expelled.*

HOT Example 28

A rocket in deep space has an initial mass of 3000kg (including 2400kg of fuel). The rocket engines eject exhaust gases at a speed of 2500m/s relative to the rocket. Calculate:

- The final velocity of the rocket after all fuel is burned (starting from rest).
- The thrust force when the fuel is being burned at a rate of 15kg/s.
- The time taken to burn all the fuel at this rate.

Solution

Interpreting the data:

Initial mass: $m_i = 3000\text{kg}$

Fuel mass: 2400kg

Final mass (after fuel burned): $m_f = (3000 - 2400)\text{kg} = 600\text{kg}$

Exhaust velocity: $u_{\text{rel}} = 2500\text{m/s}$

Initial velocity: $v_i = 0$ (starting from rest)

Burn rate: $\frac{dm}{dt} = 15\text{kg/s}$

- Using the integral rocket equation with $v_i = 0$:

$$v_f = u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right)$$

Substituting values:

$$v_f = 2500\text{m/s} \times \ln \left(\frac{3000\text{kg}}{600\text{kg}} \right) = 4023 \text{ m/s}$$

The velocity is 4023m/s.

- The thrust force is given by the following equation:

$$F_{\text{thrust}} = u_{\text{rel}} \left(\frac{dm}{dt} \right)$$

Substituting values:

$$F_{\text{thrust}} = 2500\text{m/s} \times 15\text{kg/s} = 37500\text{N}$$

The thrust force is 37500N.

- Using basic rate equation:

$$\text{Burn rate} = \frac{\text{Total fuel burned}}{\text{Time taken}}$$

From which:

$$\text{Time taken} = \frac{\text{Total fuel burned}}{\text{Burn rate}} = \frac{2400\text{kg}}{15\text{kg/s}} = 160\text{s}$$

The time is 160s.

Making Sense of the Answer: *The rocket's final velocity (4km/s) is impressive but still less than what is needed for Earth orbit (~8 km/s). The mass ratio of 5:1 ($m_i/m_f = 5$) is typical for single-stage rockets. To*

achieve orbital velocity, rockets need multiple stages. The thrust (37500N) would give initial acceleration $a = F/m = 37500/3000 = 12.5 \text{ m/s}^2$, which increases as mass decreases.

Think Like a Physicist: Notice the logarithmic relationship: to double the velocity, you would need to square the mass ratio (from 5 to 25). This is why 90% of a rocket's initial mass is fuel! Also notice that thrust is constant (if dm/dt and u_{rel} are constant), but acceleration increases during the burn because mass decreases.

HOT Example 29

A rocket of mass 800kg (including 600kg fuel) is launched vertically from Earth's surface. The engines produce a thrust of 12000N. Taking $g = 9.8 \text{ m/s}^2$, calculate:

- The initial acceleration of the rocket.
- The acceleration when half the fuel has been burned.
- Compare the performance of a rocket operating in space with that of a rocket launching from Earth. Hence, explain why rockets are generally more efficient in space than during launch from the Earth's surface.

Solution

- Forces acting on rocket at launch:
 - Thrust: $F_{\text{thrust}} = 12000\text{N}$ (upward)
 - Weight: $W = m_i g = 800\text{kg} \times 9.8\text{m/s}^2 = 7840\text{N}$ (downward)

By Newton's second law:

$$\text{Resultant force} = F_{\text{thrust}} - W = m_i a$$

From which:

$$a = \frac{F_{\text{thrust}} - W}{m_i} = \frac{(12000 - 7840)\text{N}}{800\text{kg}} = 5.2 \text{ m/s}^2$$

The initial acceleration is 5.2 m/s^2 upward.

- When half fuel ($\frac{1}{2} \times 600\text{kg} = 300\text{kg}$) burned:
 - Remaining mass: $m_f = (800 - 300)\text{kg} = 500\text{kg}$
 - Thrust: still 12000N (unchanged)
 - Weight: $W = m_f g = 500\text{kg} \times 9.8\text{m/s}^2 = 4900\text{N}$

Again:

$$a = \frac{F_{\text{thrust}} - W}{m_f} = \frac{(12000 - 4900)\text{N}}{500\text{kg}} = 14.2 \text{ m/s}^2$$

The acceleration is 14.2 m/s^2 upward.

(c) Comparison:

On Earth (launching):

- The rocket must overcome Earth's gravitational force; a significant portion of the thrust is used to overcome its weight (mg).
- It experiences air resistance: drag force opposes the motion and waste energy.
- At lower altitudes, the air is denser, resulting in greater drag forces.
- Each kilogram of fuel must not only propel the rocket forward but also lift its own mass against gravity.

In space:

- Gravitational effects are negligible compared to those at Earth's surface.
- There is no air resistance, since space is essentially a vacuum.
- The entire thrust contributes to changing the rocket's momentum (no energy is lost to drag).
- For the same thrust and mass, the rocket achieves greater acceleration because opposing forces are minimal.

Quantitative comparison:

In space:

$$a = \frac{F_{\text{thrust}}}{m}$$

On Earth:

$$a = \frac{F_{\text{thrust}} - mg}{m} = \frac{F_{\text{thrust}}}{m} - g$$

Hence, even when drag is ignored, the rocket on Earth loses exactly g (9.8 m/s^2) of acceleration simply to counteract its weight.

Conclusive explanation:

In space, rockets do not waste thrust in overcoming force of gravity or air resistance. Every newton of thrust goes directly into accelerating the rocket, making them far more efficient in space than on Earth.

Making Sense of the Answer: The acceleration nearly tripled (from 5.2 to 14.2 m/s^2) when half the fuel burned. This happens for two reasons: (1) mass decreased, and (2) weight decreased, so more of the thrust goes into acceleration rather than overcoming gravity. This is why rockets seem to accelerate faster and faster as they rise; they are getting lighter and escaping Earth's gravity!

Think Like a Physicist: The rocket equation $v = u_{\text{rel}} \times \ln(m_i/m_f)$ applies perfectly in space but must be modified for Earth launch to account for gravity losses. A rocket launching from Earth loses velocity $\Delta v \approx g \times t_{\text{burn}}$ due to gravity, where t_{burn} is burn time. This is called "gravity loss" and is why Earth launches need higher mass ratios than theoretical predictions suggest.

HOT Example 30

A toy cart of mass 2 kg rolls freely at 4 m/s on a smooth horizontal surface. Rain begins falling vertically at a rate such that the cart collects water at 0.5 kg/s . Calculate:

- The velocity of the cart after 3 seconds if no horizontal force is applied.
- The horizontal force needed to maintain the cart's velocity at 4 m/s .
- The distance travelled in the first 3 seconds with no applied force.

Solution

- Mass after 3 seconds: m_3

$$m_3 = m_0 + \left(\frac{dm}{dt}\right) \times t = 2 \text{ kg} + 0.5 \text{ kg/s} \times 3 \text{ s} = 3.5 \text{ kg}$$

Conservation of horizontal momentum:

Since the rain was falling vertically, it had zero horizontal velocity initially (and therefore zero horizontal momentum). So:

$$m_0 v_0 = m_3 v_3$$

From which:

$$v_3 = \frac{m_0 v_0}{m_3} = \frac{2 \text{ kg} \times 4 \text{ m/s}}{3.5 \text{ kg}} = 2.29 \text{ m/s}$$

The velocity is 2.29 m/s .

- For mass increasing system at constant velocity:

$$F = v \left(\frac{dm}{dt}\right)$$

Substituting values:

$$F = 4 \text{ m/s} \times 0.5 \text{ kg/s} = 2 \text{ N}$$

The required force is 2 N in the direction of motion.

- This requires integration because velocity decreases as mass increases.

From momentum conservation at any time t :

$$m_0 v_0 = \left(m_0 + \left(\frac{dm}{dt} \right) \times t \right) \times v_t$$

$$v_t = \frac{m_0 v_0}{\left(m_0 + \left(\frac{dm}{dt} \right) \times t \right)} = \frac{2 \times 4}{2 + 0.5 \times t} = \frac{8}{2 + 0.5t}$$

Distance is given by:

$$s = v_t dt = \left(\frac{8}{2 + 0.5t} \right) dt$$

After 3 seconds:

$$s_3 = \int_0^3 \left(\frac{8}{2 + 0.5t} \right) dt = 8 \times 2 [\ln(2 + 0.5t)]_0^3 = 16(\ln 3.5 - \ln 2) = 16 \ln \left(\frac{3.5}{2} \right) = 8.96 \text{ m}$$

The distance travelled is 8.96m.

Making Sense of the Answer: The cart slowed from 4m/s to 2.29 m/s as it collected rain, losing 43% of its velocity. This happened because horizontal momentum was conserved whereby the cart shared its original momentum with the collected rainwater (which had zero horizontal momentum initially). To prevent slowing, a modest 2N force is needed to continuously accelerate the incoming rain from zero to 4m/s horizontally.

Think Like a Physicist: This demonstrates an important principle: when a moving object collects stationary mass, it slows down (if no force applied) because momentum is conserved but mass increases. Industrial conveyor systems must account for this when calculating motor requirements as the motor must provide extra force to accelerate incoming material to the belt's speed.

HOT Example 31

A flatbed truck of mass 2000kg moves at constant velocity 15m/s along a horizontal road. Sand falls vertically onto the truck bed from a stationary hopper at a rate of 50kg/s. The truck maintains its velocity by increasing engine power. Taking coefficient of friction between truck and road as $\mu = 0.4$ and $g = 9.8 \text{ m/s}^2$, calculate:

- The force needed to accelerate the falling sand horizontally.
- The friction force between truck and road after 10 seconds.
- The total forward force needed from the engine after 10 seconds.

Solution

- For mass increasing system at constant velocity:

$$F = v \left(\frac{dm}{dt} \right)$$

Substituting values:

$$F = 15 \text{ m/s} \times 50 \text{ kg/s} = 750 \text{ N}$$

The force needed is 750N in the direction of motion.

- After 10 seconds:

$$m_{\text{total}} = m_{\text{truck}} + m_{\text{sand}} = m_{\text{truck}} + \left(\frac{dm}{dt} \right) t = 2000 \text{ kg} + 50 \text{ kg/s} \times 10 \text{ s} = 2500 \text{ kg}$$

$$\text{Friction force: } f = \mu R = \mu mg = 0.4 \times 2500 \text{ kg} \times 9.8 \text{ m/s}^2 = 9800 \text{ N}$$

The friction force is 9800N.

- The engine must overcome:

- Friction force: $f = 9800 \text{ N}$ (opposes motion)
- Force to accelerate incoming sand: $F_{\text{sand}} = 750 \text{ N}$ (continuous)

Thus, the total force needed will be:

$$F_{\text{total}} = f + F_{\text{sand}} = 9800 \text{ N} + 750 \text{ N} = 10550 \text{ N}$$

The total forward force needed is 10550N.

Making Sense of the Answer: The engine must continuously provide 750N just to accelerate the incoming sand (50kg/s from 0 to 15m/s). Additionally, as sand accumulates, the truck gets heavier, increasing friction. After 10 seconds, friction dominates (9800N) compared sand acceleration force (750N).

Think Like a Physicist: Notice that the sand acceleration force (750N) is constant because dm/dt and v are constant. But friction increases linearly with time as mass accumulates. Eventually, friction becomes the dominant force requirement. This is why trucks can start moving easily when empty but need much more power when fully loaded; friction increases with weight.

Important note: We assumed the truck maintains constant velocity throughout. In reality, the engine would need to increase power continuously as friction increases. Maximum engine power limits how much sand can be loaded while maintaining velocity.

That brings our subtopic-by-subtopic worked examples to a satisfying close. The plates are cleared! Now let us enjoy the full buffet, where all the ideas of this topic come together in miscellaneous worked examples.

MISCELLANEOUS WORKED EXAMPLES ON MOMENTUM AND COLLISIONS

Example 32

- (a) Kipanga says: "When a moving car brakes to a stop, its momentum is destroyed." Is Kipanga correct? Explain briefly.
- (b) A cricket ball of mass 160g is bowled horizontally at 25m/s toward a batsman. The batsman hits it straight back toward the bowler at 30m/s. The ball is in contact with the bat for 0.01s. Calculate:
- the change in momentum of the ball,
 - the average force exerted by the bat on the ball,
 - the change in kinetic energy of the ball.
 - Using your answers in (i), (ii), and (iii), explain whether the collision between bat and ball is elastic or inelastic, giving clear reasoning.

Solution

- (a) Kipanga is not correct.

Explanation:

Momentum is not destroyed. When a car brakes, friction between the tyres and the road exerts a force that reduces the car's momentum. At the same time, the Earth gains an equal amount of momentum in the opposite direction. Thus, the total momentum of the car–Earth system remains constant.

- (b) The solution of each part is as follows:

- (i) Taking initial direction as positive:

$$\Delta p = m(v - u) = 0.16\text{kg}(-30 - 25)\text{m/s} = -8.8\text{kgm/s}$$

The change in momentum is -8.8kgm/s (negative indicates change is opposite to initial direction).

- (ii) The average force is given by:

$$F_{\text{avg}} = \frac{\Delta p}{\Delta t} = -\frac{8.8\text{kgm/s}}{0.01\text{s}} = -880\text{N}$$

The average force is 880N (directed opposite to the ball's initial motion).

- (iii) Calculating initial and final kinetic energy:

$$\text{Initial KE} = \frac{1}{2}mu^2 = \frac{1}{2}(0.16\text{kg})(25\text{m/s})^2 = 50\text{J}$$

$$\text{Final KE} = \frac{1}{2}mv^2 = \frac{1}{2}(0.16\text{kg})(30\text{m/s})^2 = 72\text{J}$$

$$\text{Change in KE} = \text{Final KE} - \text{Initial KE} = (72 - 50)\text{J} = 22\text{J}$$

The kinetic energy increased by 22J.

- (iv) Collision type: Inelastic.

Reasoning:

Although the kinetic energy increased (rather than decreased), this does not make the collision elastic. In an elastic collision, kinetic energy is conserved ($\Delta\text{KE} = 0$). Here, $\Delta\text{KE} = +22\text{J}$, meaning energy was added to

the system. The batsman's muscles did work on the ball through the bat, adding 22J of energy. This external energy input means the collision is inelastic. The collision would only be elastic if the ball rebounded with kinetic energy equal to its initial kinetic energy with no external energy added.

Example 33

- (a) Explain why a rocket can accelerate in empty space even though there is nothing to "push against."
- (b) A rocket of mass 1500kg (including 1000kg fuel) is in deep space, initially at rest. The engines eject exhaust at 2500m/s relative to the rocket.
 - (i) Calculate the thrust force if fuel burns at 8kg/s.
 - (ii) Calculate the rocket's final velocity after all fuel is burned.
 - (iii) If instead the same rocket were launched vertically from Earth's surface with the same thrust, explain whether its final velocity would be greater, smaller, or the same as in part (ii), and why. Take $g = 9.8\text{m/s}^2$.

Solution

- (a) The rocket works by conservation of momentum, not by "pushing." When exhaust gases are ejected backward, they carry backward momentum. Since total momentum must remain zero (no external forces in space), the rocket gains equal forward momentum. The rocket and exhaust push on each other (Newton's third law), not on external objects.
- (b) The solution of each part is as follows:
 - (i) The thrust is given by:

$$F_{\text{thrust}} = u_{\text{rel}} \left(\frac{dm}{dt} \right) = 2500\text{m/s} \times 8\text{kg/s} = 20000\text{N}$$

The thrust is 20000N.

- (ii) Using the integral rocket equation with $v_i = 0$:

$$v_f = u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right)$$

Substituting values:

$$v_f = 2500\text{m/s} \times \ln \left(\frac{1500\text{kg}}{500\text{kg}} \right) = 2747.5\text{m/s}$$

The final velocity is 2747.5m/s.

- (iii) Final velocity: Smaller

Reasoning:

On Earth, the rocket must overcome gravity. Part of the thrust is used just to support the rocket's weight ($F = mg$), leaving less thrust available for acceleration. Additionally, during the burn time ($t = 1000/8 = 125\text{s}$), gravity continuously acts downward, causing "gravity losses." The rocket loses velocity $\Delta v \approx g \times t_{\text{burn}} \approx 9.8 \times 125 \approx 1225\text{m/s}$ due to gravity. Therefore, the final velocity would be significantly smaller than the 2747.5 m/s achieved in space.

Example 34

- (a) Two objects collide and stick together. Explain whether the momentum of the system is conserved, and whether the kinetic energy is conserved.
- (b) A rocket of mass 2000kg (including 1400kg fuel) is moving at 100m/s in deep space. The engines are fired, ejecting exhaust at 2800m/s relative to the rocket. Take $g = 9.8\text{ m/s}^2$.
 - (i) Calculate the thrust force if fuel burns at 12kg/s.
 - (ii) Calculate the rocket's velocity after all fuel is burned.
 - (iii) If instead the same rocket started from rest (with the same thrust and exhaust velocity), calculate its final velocity.
 - (iv) Explain why the velocity gain in part (ii) is the same as in part (iii), even though the rocket started with different initial velocities.

Solution

- (a) Momentum: Conserved.
Kinetic energy: Not conserved.

Explanation:

When objects stick together (perfectly inelastic collision), momentum is conserved because momentum conservation depends only on absence of external forces, not on the collision type. However, kinetic energy is not conserved; some converts to heat, sound, and deformation during the collision.

- (b) The solution of each part is as follows:

- (i) The thrust is given by:

$$F_{\text{thrust}} = u_{\text{rel}} \left(\frac{dm}{dt} \right) = 2800 \text{ m/s} \times 12 \text{ kg/s} = 33600 \text{ N}$$

The thrust is 33600 N.

- (ii) Using the integral rocket equation:

$$v_f = v_i + u_{\text{rel}} \ln \left(\frac{m_i}{m_f} \right)$$

Substituting values:

$$v_f = 100 \text{ m/s} + 2800 \text{ m/s} \times \ln \left(\frac{2000 \text{ kg}}{600 \text{ kg}} \right) = 3471 \text{ m/s}$$

The final velocity is 3471 m/s.

- (iii) Starting from rest ($v_i = 0$):

$$v_f = 0 \text{ m/s} + 2800 \text{ m/s} \times \ln \left(\frac{2000 \text{ kg}}{600 \text{ kg}} \right) = 3371 \text{ m/s}$$

The final velocity is 3371 m/s.

- (iv) The velocity gain (Δv) in both cases is the same: 3371 m/s. In part (ii), the rocket went from 100 m/s to 3471 m/s (gain = 3371 m/s). In part (iii), it went from 0 to 3371 m/s (gain = 3371 m/s). The velocity change depends only on the exhaust velocity and mass ratio ($\Delta v = u_{\text{rel}} \times \ln(m_i/m_f)$), not on the initial velocity. This is because the rocket equation describes the change in velocity, which is independent of the reference frame.

Example 35

- (a) A ball bounces on the floor and rises to 80% of its original height. Explain what happened to the "missing" 20% of the ball's energy.
- (b) A rubber ball is dropped from rest from a height $h_1 = 2.5 \text{ m}$ onto a hard floor. After bouncing, it rises to a maximum height $h_2 = 1.6 \text{ m}$. Take $g = 9.8 \text{ m/s}^2$.
- Calculate the coefficient of restitution between ball and floor.
 - Calculate the percentage of mechanical energy lost during the bounce.
 - The ball is now dropped from a new height H such that after bouncing it rises to exactly 2.5 m. Calculate H .
 - Explain whether the percentage of energy lost in part (iii) is greater than, less than, or equal to the percentage lost in part (ii).

Solution

- (a) The "missing" 20% of energy was converted to heat (warming the ball and floor slightly), sound (the bounce noise), and permanent deformation of the ball material. Energy was not destroyed but was transformed from organized mechanical energy (gravitational potential energy and kinetic energy) into disorganized thermal energy and other non-recoverable forms.

- (b) The solution of each part is as follows:

- (i) Using $v^2 = u^2 + 2as$ for free fall from rest with h_1 as initial drop height:

$$v^2 = 0^2 + 2gh_1 \text{ or } v = \sqrt{2gh_1} = u_{\text{ball}} \text{ (downward)}$$

Again, using $v^2 = u^2 + 2as$ for upward motion to reach height h_2 , with $a = -g$, $v = 0 \text{ m/s}$.

$$0^2 = u^2 - 2gh_2 \text{ or } u = \sqrt{2gh_2} = v_{\text{ball}} \text{ (upward)}$$

$$e = \frac{v_{\text{ball}} - v_{\text{floor}}}{u_{\text{floor}} - u_{\text{ball}}} = \frac{\sqrt{2gh_2} - 0}{(0 - (-\sqrt{2gh_1}))} = \frac{\sqrt{2gh_2}}{\sqrt{2gh_1}} = \sqrt{\frac{h_2}{h_1}} = \sqrt{\frac{1.6\text{m}}{2.5\text{m}}} = 0.8$$

The coefficient of restitution is 0.8.

$$(ii) \text{ Kinetic energy before collision} = \frac{1}{2}mv^2 = \frac{1}{2}m(6.26^2) = 19.6\text{m}$$

$$\text{Kinetic energy after collision} = \frac{1}{2}mu^2 = \frac{1}{2}m(4.85^2) = 11.76\text{m}$$

Percentage of mechanical energy lost

$$\begin{aligned} &= \frac{\text{Initial PE} - \text{Final PE}}{\text{Initial PE}} \times 100\% \\ &= \frac{mgh_1 - mgh_2}{mgh_1} \times 100\% = \frac{mg(h_1 - h_2)}{mgh_1} \times 100\% \\ &= \frac{h_1 - h_2}{h_1} \times 100\% = \frac{(2.5 - 1.6)\text{m}}{2.5\text{m}} \times 100\% = 36\% \end{aligned}$$

The percentage of mechanical energy lost is 36%.

(iii) Using:

$$e = \sqrt{\frac{h_2}{h_1}}$$

Substituting:

$$0.8 = \sqrt{\frac{2.5\text{m}}{H}}$$

Solving gives: $H = 3.91\text{m}$

The required drop height is 3.91m.

(iv) Percentage lost: **Equal to part (ii)** (also 36%).

Reasoning:

The coefficient of restitution e is a property of the materials (ball and floor) and remains constant regardless of drop height. Since $e = 0.8$ is constant, the ratio $h_2/h_1 = e^2 = 0.64$ is always constant. This means the ball always rises to 64% of its drop height, losing 36% of its energy in every bounce, regardless of the initial height. Therefore, the percentage energy loss is the same in both cases.

Example 36

- (a) Kipanga says: "In a perfectly elastic collision between two objects, they must bounce apart with high velocity." Is this statement always true? Explain.
- (b) A 0.04kg bullet is fired horizontally at 500m/s into a 4kg wooden block suspended by a string of length 2m. The bullet embeds in the block and the block swings upward. Take $g = 9.8\text{m/s}^2$.
- Calculate the velocity of the block immediately after the bullet embeds.
 - Calculate the maximum height the block rises above its initial position.
 - Calculate the maximum angle the string makes with the vertical.
 - If the string length were doubled to 4m but everything else remained the same, explain whether the maximum angle would be greater, smaller, or the same.

Solution

- (a) Statement: Not always true.

Explanation:

In a perfectly elastic collision where one object has much greater mass than the other (e.g., tennis ball hitting a wall), the lighter object may bounce back at high velocity while the heavier object experiences negligible motion.

Also, if two objects of equal mass collide elastically and one is initially at rest, the moving object stops completely while the stationary object moves at the original velocity and thus the first object has zero final velocity, not "high velocity."

In conclusion, an elastic collision means that kinetic energy is conserved; it does not necessarily imply that the final velocity must be large.

(b) The solution of each part is as follows:

(i) Let m and M denote the masses of the bullet and the block respectively.

By conservation of linear momentum:

Momentum before collision = Momentum after collision

$$mu_{\text{bullet}} = (m + M)v$$

From which:

$$v = \frac{mu_{\text{bullet}}}{m + M} = \frac{0.04\text{kg} \times 500\text{m/s}}{(0.04 + 4)\text{kg}} = 4.95\text{m/s}$$

The velocity immediately after embedding is 4.95m/s.

(ii) By conservation of mechanical energy:

Total mechanical energy at the bottom = Total mechanical energy at maximum height

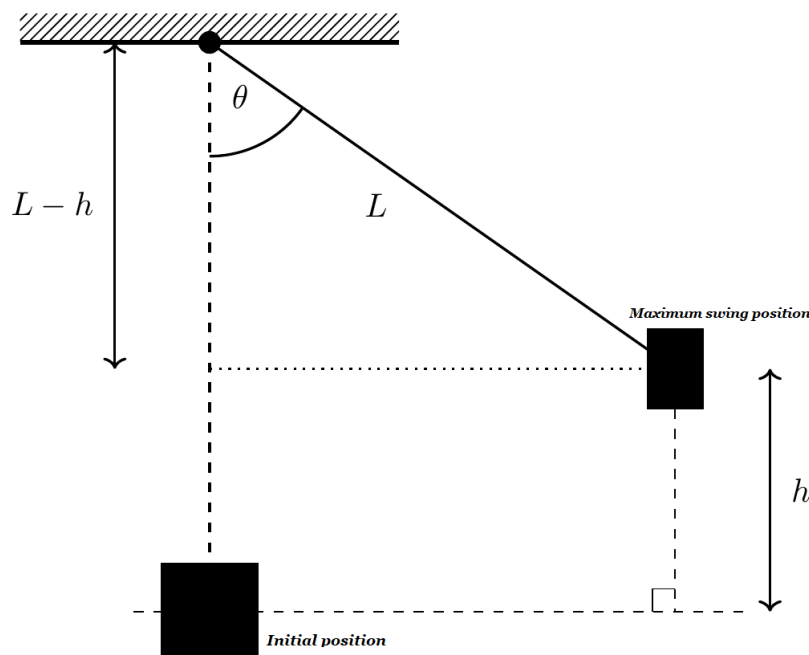
$$\frac{1}{2}(m + M)v^2 = (m + M)gh$$

Cancelling $m + M$:

$$v^2 = 2gh \text{ or } h = \frac{v^2}{2g} = \frac{(4.95\text{m/s})^2}{2 \times 9.8 \text{ m/s}^2} = 1.25\text{m}$$

The maximum height is 1.25m.

(iii) Consider the following diagram:



From geometry of the right-angled triangle:

$$\cos\theta = \frac{L - h}{L} = \frac{(2 - 1.25)m}{2m} = 0.375$$

$$\theta = \cos^{-1}(0.375) = 68^\circ$$

The maximum angle is approximately 68° from vertical.

(iv) Maximum angle: Smaller.

Reasoning:

The maximum height h depends only on the velocity after embedding, which is determined by momentum conservation and remains constant at $h = 1.25m$ regardless of string length.

With doubled string length $L = 4m$, $\cos\theta = \frac{4-1.25}{4} = 0.6875$; $\theta = 47^\circ$

Therefore, the maximum angle decreases from 68° to 47° because the same vertical height corresponds to a smaller angle when the string is longer.

Example 37

- (a) Kipute says: "When two objects collide, the lighter object always experiences greater acceleration than the heavier object." Is Kipute's statement correct? Explain.
- (b) A $0.05kg$ bullet is fired horizontally into a $3kg$ wooden block hanging by a string of length $1m$. The bullet embeds in the block, and the block swings to a maximum angle of 60° from the vertical. Take $g = 9.8 m/s^2$.

Calculate:

- (i) The height the block rises.
- (ii) The velocity of the block immediately after the bullet embeds.
- (iii) The initial velocity of the bullet.

Solution

(a) Statement: Correct.

Explanation:

During collision, both objects experience equal magnitude forces (Newton's third law). Since $F=ma$, for equal forces, acceleration is inversely proportional to mass ($a = F/m$). The lighter object has smaller mass; therefore, it experiences greater acceleration. This is always true during any collision.

(b) The solution of each part is as follows:

(i) Again from:

$$\cos\theta = \frac{L - h}{L};$$

Making h the subject:

$$h = L - L\cos\theta = L(1 - \cos\theta)$$

Substituting values:

$$h = 1m(1 - \cos 60^\circ) = 0.5m$$

The height is $0.5m$.

(ii) By conservation of mechanical energy:

Total mechanical energy at the bottom = Total mechanical energy at maximum height

$$\frac{1}{2}(m + M)v^2 = (m + M)gh$$

Cancelling $m + M$:

$$v^2 = 2gh \text{ or } v = \sqrt{2gh} = \sqrt{2 \times 9.8 m/s^2 \times 0.5m} = 3.13m/s$$

The velocity is 3.13m/s.

(iii) By conservation of linear momentum:

Momentum before collision = Momentum after collision

$$mu_{\text{bullet}} = (m + M)v$$

From which:

$$u_{\text{bullet}} = \frac{(m + M)v}{m} = \frac{(0.05 + 3) \text{kg} \times 3.13 \text{m/s}}{0.05 \text{kg}} = 191 \text{m/s}$$

The initial velocity of bullet is 191m/s.

Example 38

- (a) When a gun fires a bullet, both the bullet and the gun experience the same magnitude of force. Explain why the bullet accelerates much more than the gun.
- (b) A 5kg rifle fires a 0.025kg bullet horizontally. The bullet leaves the barrel at 400m/s.
- Calculate the recoil velocity of the rifle.
 - Calculate the ratio of kinetic energies (bullet : rifle).
 - If the rifle is brought to rest in 0.08s by the shooter's shoulder, calculate the average force exerted by the shoulder.
 - Explain why most of the chemical energy from the gunpowder goes into the bullet's kinetic energy rather than the rifle's kinetic energy, even though they have equal momenta.

Solution

- (a) Both experience equal forces (Newton's third law), but the bullet has much smaller mass than the gun. Since $F = ma$, for the same force F , acceleration a is inversely proportional to mass, m . The bullet's much smaller mass results in much greater acceleration ($a = F/m$), while the gun's large mass results in small acceleration.

(b) The solution of each part is as follows:

(i) By conservation of momentum:

$$0 = m_{\text{bullet}}v_{\text{bullet}} + m_{\text{rifle}}v_{\text{recoil}}$$

From which:

$$v_{\text{recoil}} = \frac{-m_{\text{bullet}}v_{\text{bullet}}}{m_{\text{rifle}}} = \frac{-0.025 \text{kg} \times 400 \text{m/s}}{5 \text{kg}} = -2 \text{m/s}$$

The recoil velocity is 2m/s backward.

(ii) The ratio is given by:

$$\text{Ratio} = \frac{\text{KE of bullet}}{\text{KE of rifle}} = \frac{\frac{1}{2}m_b v_b^2}{\frac{1}{2}m_r v_r^2} = \frac{\frac{1}{2}(0.025 \text{kg})(400 \text{m/s})^2}{\frac{1}{2}(5 \text{kg})(2 \text{m/s})^2} = 200$$

The ratio of kinetic energies is 200: 1 (bullet has 200 times more kinetic energy).

Alternative solution:

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}mv^2 \times \left(\frac{m}{m}\right) = \frac{\frac{1}{2}m^2v^2}{m} = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$$

So kinetic energy and momentum are related by the following equation:

$$\text{KE} = \frac{p^2}{2m}$$

Since both bullet and rifle have the same momentum:

$$\frac{(\text{KE})_b}{(\text{KE})_r} = \frac{p^2}{2m_b} \div \frac{p^2}{2m_r} = \frac{p^2}{2m_b} \times \frac{2m_r}{p^2} = \frac{m_r}{m_b} = \frac{5 \text{kg}}{0.025 \text{kg}} = 200$$

(iii) Using the impulse-momentum theorem:

Impulse = change in momentum

$$F \times \Delta t = 0 - m_{\text{rifle}} v_{\text{recoil}}; F = \frac{-m_{\text{rifle}} v_{\text{recoil}}}{\Delta t} = \frac{-5\text{kg} \times (-2\text{m/s})}{0.08\text{s}} = 125\text{N}$$

The average force exerted by the shoulder is 125N (opposing the recoil).

- (iv) Although bullet and rifle have equal magnitude momenta ($p = mv$), kinetic energy depends on both mass and velocity squared ($KE = \frac{1}{2}mv^2 = \frac{p^2}{2m}$). For equal momentum p , kinetic energy is inversely proportional to mass. Since the bullet has much smaller mass (200 times smaller), it receives much more kinetic energy (200 times more) for the same momentum. The rifle's large mass means most of its momentum comes from small velocity, resulting in small kinetic energy. Therefore, the bullet carries most of the chemical energy despite equal momenta.

Example 39

- (a) Rain falls vertically onto a moving car. Explain why the rain exerts a horizontal force on the car even though the rain falls vertically.
- (b) Rain falls vertically onto a plane roof, 2m square, which is inclined to the horizontal at an angle of 40° . The raindrops strike the roof with a vertical velocity of 4m/s, and a volume of $3 \times 10^{-2}\text{m}^3$ of water is collected from the roof in one minute. Assuming that the rain conditions are steady and that the velocity of the raindrops after impact is zero, calculate:
- the mass of water striking the roof per second,
 - the vertical force exerted on the roof by the impact of the falling rain,
 - the component of this force normal to the roof surface,
 - the pressure exerted normal to the roof due to the impact of the rain.
- (The density of water is 1000kg/m^3)

Solution

- (a) The rain gains horizontal momentum equal to the car's velocity when it lands, requiring the car to exert a horizontal force on the rain, and by Newton's third law, the rain exerts an equal horizontal force on the car.
- (b) Interpreting the data:

Roof area: $A = 2\text{m} \times 2\text{m} = 4\text{m}^2$

Angle of inclination: $\theta = 40^\circ$

Rain velocity: $v = -4\text{m/s}$ (vertical, downward)

Volume collected: $V = 3.0 \times 10^{-2}\text{m}^3$ in $t = 60\text{s}$

Final velocity of rain: 0 (rain sticks to roof)

Density of water: $\rho = 1000\text{kg/m}^3$

- (i) Mass per second

$$= \frac{\text{Total mass collected}}{\text{Time taken in s}} = \frac{\rho V}{t} = \frac{1000 \times 1000\text{kg/m}^3 \times 3 \times 10^{-2}\text{m}^3}{60\text{s}} = 0.5\text{kg/s}$$

The mass of water striking the roof per second is 0.5kg/s.

- (ii) The rain moves downward with velocity $v = 4\text{m/s}$.

$$F_{\text{vertical}} = \Delta v \left(\frac{dm}{dt} \right) = (0 - (-4\text{m/s})) \times 0.5\text{kg/s} = +2\text{N}$$

It is **positive** because the roof exerts an **upward force on the rain** to stop its downward motion.

By Newton's third law, the rain exerts an equal magnitude of downward force on the roof.

Hence, the vertical force on the roof by rain is 2N (downward).

- (iii) The vertical force can be resolved into components parallel and perpendicular to the roof.

$$F_{\text{normal}} = F_{\text{vertical}} \times \cos\theta = 2\text{N} \times \cos 40^\circ = 1.53\text{N}$$

The force component normal to the roof is 1.53N.

(iv) Pressure normal to roof is given by:

$$P = \frac{F_{\text{normal}}}{A} = \frac{1.53\text{N}}{4\text{m}^2} = 0.38\text{N/m}^2$$

The pressure exerted normal to the roof is 0.38N/m^2 .

Example 40

- (a) A stone is thrown horizontally from a cliff. Explain why the stone's horizontal momentum is conserved during its flight, but its vertical momentum is not conserved.
- (b) A ball of mass 0.5kg is thrown vertically upward from ground level with initial velocity 20m/s. Take $g = 9.8\text{m/s}^2$ and upward as positive direction.
- Calculate the momentum of the ball at the instant it is thrown.
 - Calculate the momentum of the ball when it reaches its maximum height.
 - Calculate the change in momentum from launch to maximum height.
 - The ball returns to ground level and is caught by the thrower. Calculate the total change in momentum from the moment of throw to the moment of catch.

Solution

(a) Momentum is conserved only when no net external force acts. Horizontally, no forces act (neglecting air resistance), so horizontal momentum remains constant throughout flight. Vertically, gravity acts continuously downward, exerting an external force that changes the vertical momentum.

(b) The solution of each part is as follows:

(i) $p_{\text{initial}} = mu = 0.5\text{kg} \times 20\text{m/s} = 10\text{kgm/s}$

The initial momentum is $+10\text{kgm/s}$.

(ii) At maximum height, $v = 0$:

$$p_{\text{max}} = mv = 0.5\text{kg} \times 0\text{m/s} = 0\text{kgm/s}$$

The momentum at maximum height is zero.

(iii) $\Delta p = p_{\text{max}} - p_{\text{initial}} = (0 - 10)\text{kgm/s} = -10\text{kgm/s}$

The change in momentum is -10kgm/s .

(iv) When the ball returns to ground level, by symmetry of vertical motion, it has velocity of $v = -20\text{m/s}$ (downward):

$$p_{\text{final}} = mv = 0.5\text{kg} \times (-20\text{m/s}) = -10\text{kgm/s}$$

Total change in momentum:

$$\Delta p_{\text{total}} = p_{\text{final}} - p_{\text{initial}} = -10\text{kgm/s} - (+10\text{kgm/s}) = -20\text{kgm/s}$$

The total change in momentum from throw to catch is -20kgm/s .

Having enjoyed the full combination of ideas, it is time to sharpen our thinking; the Digging Deeper Exercise is ready in the next page.

DIGGING DEEPER EXERCISE 5

EXERCISE 5A: BINDER QUESTIONS

Question 1

A heavy truck and a small car are moving at the same velocity on a highway. They both apply their brakes at the same time with the same braking force. The truck takes much longer to stop than the car. Explain why the truck takes longer to stop, using the concept of momentum.

Question 2

A ball is dropped from a height and bounces back to 75% of its original height.

- (a) Explain what the coefficient of restitution tells us about this collision.
- (b) Explain what happened to the "missing" 25% of the ball's energy.

Question 3

A rocket in deep space fires its engines continuously, ejecting exhaust gases backward.

- (a) Explain why the rocket accelerates forward even though there is nothing in space for the exhaust to "push against."
- (b) Explain why the rocket's acceleration increases during the flight even though the thrust force remains constant.

Question 4

A person standing on a stationary boat throws a heavy object forward. The boat moves backward.

- (a) Explain why the boat moves backward when the object is thrown forward.
- (b) Explain whether the person could make the boat move forward by throwing the object backward.

Question 5

Two identical cars collide head-on with equal velocities. In the first case, both cars stick together after collision. In the second case, both cars bounce apart. Explain which collision (sticking or bouncing) would cause more severe injuries to the passengers, and why.

Question 6

A conveyor belt moves at constant velocity while sand continuously falls onto it from a stationary hopper above. Explain why the motor must provide force to maintain the belt's constant velocity, even though the belt is not accelerating.

Question 7

A ballistic pendulum is used to measure bullet speed. A bullet embeds in a wooden block, causing it to swing upward. Explain why we cannot use conservation of energy for the entire process (from bullet firing to maximum swing height), but we can use it for the swing portion only.

Question 8

When a gun fires a bullet, both the bullet and the gun experience the same magnitude of force, but the bullet travels much faster than the gun recoils.

- (a) Explain why the bullet and gun experience equal forces during firing.
- (b) Explain why the bullet moves much faster than the gun, even though they experience equal forces.
- (c) Explain which has more kinetic energy after firing: the bullet or the gun.

Question 9

A ball is thrown vertically upward. On its way up, gravity acts downward. On its way down, gravity still acts downward.

- (a) Explain how momentum changes during the upward journey.
- (b) Explain how momentum changes during the downward journey.
- (c) Explain the total change in momentum from the moment of throw to the moment it returns to the same height.

EXERCISE 5B: REAL QUESTIONS**Question 10**

At a bus stop in Dar es Salaam, a daladala is moving slowly at constant velocity. Passengers jump onto it one by one while it is moving.

- (a) State what happens to the daladala's velocity as each passenger boards.
- (b) Explain why this happens, using the principle of conservation of momentum.
- (c) Explain why it is easier for the driver to maintain velocity if passengers run alongside first before boarding, rather than jumping from rest.

Question 11

A fisherman stands in the middle of a small boat floating on calm water. He walks toward the front of the boat.

- (a) What happens to the boat as the fisherman walks forward?
- (b) Explain why this happens, even though no external horizontal force acts on the system.
- (c) If the fisherman walks back to his original position, explain where the boat ends up relative to its starting position.

Question 12

During a physics practical at Azania Secondary School, Kipanga and Kipute investigate collisions using two trolleys on a straight track. Trolley A moves toward trolley B, which is initially at rest. In one trial, trolley A stops after collision and trolley B moves forward. In another trial, both trolleys move forward after collision. Kipanga also notices that one collision produces a sharp "click" sound while the other produces a dull "thud".

- (a) Explain under what condition trolley A would stop completely after collision, transferring all its motion to trolley B.
- (b) Explain why the collision sounds different in the two cases, relating this to energy conservation.
- (c) Explain which type of collision (the 'click' or the 'thud') is more likely to be closer to elastic, and why.

Question 13

A maize farmer in Iringa uses a traditional method to test grain quality. He drops grains from the same height onto a wooden board. Good, dense grains bounce to about 40% of the drop height, while lighter, damaged grains bounce to about 70%. The farmer says: *"Good grains are heavier, so they should bounce higher because they have more energy."* Explain whether the farmer's reasoning is correct or incorrect.

Question 14

A water tanker truck drives at constant velocity while water leaks out continuously through a hole at the back.

- (a) Explain what happens to the truck's momentum as water leaks out.
- (b) Explain whether the driver needs to apply more force, less force, or the same force to maintain constant velocity compared to when the tank was full.
- (c) If instead the water sprays backward horizontally (like a jet) rather than just dripping downward, explain how this affects the force needed to maintain constant velocity.

Question 15

Car manufacturers design crumple zones that deform during collisions. Explain why this protects passengers.

Question 16

At Serengeti National Park, a wildlife biologist uses a tranquilizer dart gun to sedate animals. A dart is fired at a running zebra.

- (a) After the dart embeds in the zebra, the zebra's velocity barely changes (decreases by less than 1%). Explain why the effect on the zebra's motion is so small despite the dart traveling at high velocity.
- (b) Explain what happens to the dart's large kinetic energy after it embeds in the zebra.
- (c) The biologist notices that shooting a small antelope with the same dart causes a more noticeable change in the antelope's motion. Explain why.

EXERCISE 5C: HOT QUESTIONS**Question 17**

A fisherman of mass 80kg stands at the front of a stationary boat of mass 120kg floating on calm water. He walks to the back of the boat, covering a distance of 4m relative to the boat.

- (a) Explain why the boat moves when the fisherman walks, even though there is no external horizontal force from the water.
- (b) Calculate:
 - (i) The distance the boat moves relative to the water.
 - (ii) The distance the fisherman moves relative to the water.

Question 18

A maize farmer in Morogoro uses a traditional method to separate grain from chaff by dropping grain onto a slanted wooden board. The grain bounces differently depending on quality.

- (a) Good grain (denser and harder) bounces to 40% of drop height, while chaff (lighter) bounces to 70% of drop height when both are dropped from the same height. Calculate the coefficient of restitution for:
 - (i) Good grain with the wooden board.
 - (ii) Chaff with the wooden board.
- (b) Explain why the coefficient of restitution is different for grain and chaff, even though they hit the same wooden surface.

Question 19

In a mkokoteni workshop in Mwanza, a worker tests the brakes by pushing a loaded mkokoteni into a wall.

- (a) A 150kg mkokoteni (including load) moving at 3m/s hits a rigid wall and rebounds at 1.2m/s. The collision lasts 0.15s. Calculate:
 - (i) The change in momentum of the mkokoteni.
 - (ii) The average force exerted by the wall on the mkokoteni.
- (b) Explain why the wall exerts such a large force even though the mkokoteni was moving relatively slowly.

Question 20

A boda-boda of mass 180kg (including rider) traveling at 12m/s picks up a passenger of mass 70kg standing by the roadside. Assume the passenger jumps onto the moving boda-boda.

- (a) Calculate the new velocity after the passenger boards.
- (b) Calculate the distance the boda-boda travels during the 2.5s it takes for the rider to accelerate back to 12m/s, assuming constant acceleration.
- (c) Explain why it is safer for passengers to run alongside and board gradually rather than jumping onto a moving boda-boda from rest.

Question 21

A coconut falls from a palm tree and lands on soft sandy beach versus hard concrete ground. The coconut has the same velocity (8m/s) just before hitting either surface.

- (a) Explain why the coconut is more likely to crack on concrete than on sand.
- (b) If the coconut (mass 1.5kg) comes to rest in 0.05s on concrete but in 0.25s on sand, calculate:
 - (i) The average force on concrete.
 - (ii) The average force on sand.
 - (iii) The ratio of forces (concrete : sand).

Question 22

A ball is projected horizontally from the top of a cliff with velocity 15m/s. At the same instant, an identical ball is dropped vertically from the same point. Take $g = 9.8 \text{ m/s}^2$.

- (a) Explain whether the two balls have the same momentum at any instant during their flight.
- (b) After 2 seconds:
 - (i) Calculate the horizontal momentum of the projected ball.
 - (ii) Calculate the vertical momentum of both balls.
 - (iii) Calculate the magnitude of total momentum of the projected ball.

- (c) Explain why the horizontal momentum of the projected ball remains constant while its vertical momentum changes.

Question 23

A ball of mass 0.8kg moving at 10m/s collides obliquely with a ball of mass 1.2kg moving at 6m/s. After collision, the 0.8kg ball moves at 5m/s at 60° from its original direction.

- (a) Determine the magnitude and direction of the 1.2kg ball's velocity after collision.
(b) Check whether this collision is elastic or inelastic.
(c) Suggest at least two practical situations where oblique collisions commonly occur.

Question 24

A rocket of mass 8000kg (including 6000kg fuel) is launched vertically from Earth's surface. The engines produce thrust by ejecting exhaust at 3200 m/s relative to the rocket. Fuel burns at 50kg/s. Take $g = 9.8 \text{ m/s}^2$.

- (a) Calculate:
(i) The thrust force produced by the engines.
(ii) The initial acceleration of the rocket at the moment of launch.
(iii) The acceleration when half the fuel has been burned.
(b) Explain why the rocket's acceleration increases during the flight even though thrust remains constant.
(c) After fuel is exhausted, the rocket continues upward. Explain what happens to:
(i) The rocket's momentum.
(ii) The rocket's kinetic energy.
(d) Suggest one advantage and one disadvantage of increasing the fuel burn rate to 100 kg/s while keeping exhaust velocity constant.

Question 25

Rain falls vertically at 8m/s onto a flat roof of area 50m^2 inclined at 30° to the horizontal. The rainfall rate is such that $2 \times 10^{-2} \text{ m}^3$ of water is collected from the roof per minute. The raindrops come to rest upon impact. Density of water = 1000 kg/m^3 , $g = 9.8 \text{ m/s}^2$.

- (a) Calculate:
(i) The mass of water striking the roof per second.
(ii) The vertical force exerted on the roof by the rain.
(iii) The component of this force perpendicular to the roof surface.
(b) During a heavy storm, the rainfall rate doubles. Explain how this affects:
(i) The force on the roof.
(ii) The coefficient of restitution.
(d) If the roof were horizontal instead of inclined (same rainfall), would the perpendicular force be greater, smaller, or the same? Explain.

Question 26

Two ice skaters **A** and **B** (masses 60kg and 80kg) stand facing each other on frictionless ice, holding opposite ends of a rope. They pull on the rope until they meet.

- (a) Explain why they meet at a point closer to skater B than to skater A.
(b) If they were initially 12m apart:
(i) Calculate how far each skater moves.
(ii) Calculate their velocity just before they meet if they pull the rope in steadily over 4 seconds.
(c) After meeting, they push away from each other. Skater A moves at 3m/s. Calculate:
(i) The velocity of skater B.
(ii) The total kinetic energy of the system after pushing.
(d) Where did the kinetic energy in part (c)(ii) come from? Explain in detail.

Chapter 6

PROJECTILE MOTION

INTRODUCTION

It was inter-house sports day at Miono Secondary School, and Mr. Akilikubwa stood at the edge of the football field, clipboard in hand, recording student performances for the shot put competition. Kipanga stepped into the throwing circle, gripping a 4 kg shot put with determination. "Sir," he called out confidently, "I've been practicing! Watch me break the school record!"

Mr. Akilikubwa smiled. "Remember, Kipanga! It's not just about how hard you throw, but also the angle."

"Angle?" **Kipanga** snorted. "Sir, I'm strong! I'll just throw it as hard as I can straight forward!" With a mighty grunt, he launched the shot put at nearly zero degrees above horizontal. The metal ball rocketed forward... and crashed into the ground barely three metres away, creating a small crater in the sandy field.

Kipute, waiting her turn, could not help but giggle. "Kipanga, you threw it into the ground! That's not how shot put works!"

"Fine!" **Kipanga** huffed, retrieving the ball. "If throwing straight doesn't work, I'll throw it straight up!" On his second attempt, he launched the shot put almost vertically. It soared magnificently into the sky, climbing higher and higher and higher... then came straight back down, landing half a metre from where he had thrown it. The students scattered as the heavy ball thudded back to earth.

Mr. Akilikubwa shook his head, trying not to laugh. "Kipanga, you've just demonstrated the two extreme failures of projectile motion! Too flat, and gravity defeats you immediately. Too steep, and you waste all your energy fighting gravity instead of moving forward." He turned to Kipute. "Your turn. Show him how physics works."

Kipute stepped up, closed her eyes briefly to estimate, and released the shot put at approximately 45 degrees. The ball traced a beautiful arc through the air, landing smoothly 8 metres away, a respectable throw!

"But how did you know the right angle?" **Kipanga** demanded, genuinely puzzled.

Mr. Akilikubwa grinned. "Because unlike you, Kipute paid attention in Chapter 2 when we studied vertical motion under gravity. Now she's applying it to motion in two dimensions. Welcome to Chapter 6, everyone, where we learn that every thrown ball, launched javelin, or kicked football becomes a slave to the same beautiful mathematics. Master projectile motion, and you'll understand why goalkeepers kick at certain angles, why water fountains arc gracefully, and why Kipanga just lost five metres of his throwing distance by defying physics!"

"Sir," **Kipanga** protested weakly, "I was just... testing different approaches..."

"Testing?" **Kipute** teased. "Is that what we're calling it now?"

By the end of this chapter, even Kipanga will understand exactly how to calculate the perfect angle for maximum distance. The mathematics that governs his embarrassing shot put attempts is the same mathematics that describes bullets, water spraying from hoses, and rockets **once their engines are no longer firing**. Let us explore how motion in two dimensions emerges from the simpler concepts we mastered in Chapter 2.

FUNDAMENTAL EQUATIONS OF PROJECTILE MOTION

A projectile is any object that, once launched with an initial velocity, moves under the influence of gravity **alone**. Whether it is a shot put released by Kipanga, a stone thrown by hand, a javelin launched by an athlete, or water spraying from a hose, the motion follows the same fundamental principles. Once the object leaves contact with whatever launched it, the only force acting upon it (when air resistance is negligible) is the downward pull of gravity. The path traced by a projectile through space is called its **trajectory**.

Connecting to chapter 2: Do you remember when we studied vertical motion under gravity in Chapter 2? We learned that any object either thrown upward or dropped from rest experiences constant downward acceleration, $g = 9.8\text{m/s}^2$. We also learned three equations of motion ($v = u + at$, $s = ut + \frac{1}{2}at^2$, and $v^2 = u^2 + 2as$) that perfectly describe this one-dimensional vertical motion. The beautiful insight of projectile motion is that these same principles still apply; we simply extend them to two dimensions.

The key to understanding projectile motion lies in recognising that it represents motion in two dimensions simultaneously, yet these two components of motion are completely independent of one another. A projectile exhibits uniform motion in the horizontal direction while simultaneously experiencing uniformly accelerated motion in the vertical direction due to gravity, exactly the same accelerated motion we studied in Chapter 2. This independence means that *what happens horizontally has no effect on what happens vertically, and vice versa*. This fundamental principle, known as **the independence of perpendicular components of motion**, allows us to analyse each direction separately using the equations we developed in Chapter 2 for one-dimensional motion.

Consider Kipanga's disastrous shot put throws from our introduction. From the moment the shot left his hand, it possessed both horizontal and vertical components of velocity. Horizontally, it moved forward at constant velocity as there is no horizontal forces to slow it or speed it up (ignoring air resistance). Vertically, however, gravity continuously acted downward, first slowing any upward motion, bringing it momentarily to rest at its highest point, then accelerating it downward (exactly as we learned in Chapter 2 for vertical motion). These two motions occurred simultaneously and independently, creating the parabolic arc.

To analyse projectile motion systematically, we establish a coordinate system with the x-axis along the horizontal direction and the y-axis pointing vertically upward. The origin is typically placed at the **point of projection** which is the location from which the projectile is launched. By **convention**, we take **upward** as the **positive y-direction**, which means that the acceleration due to gravity, acting downward, has a negative value: $\mathbf{a}_y = -\mathbf{g} = -9.8 \text{ m/s}^2$. In the horizontal direction, where no forces act (again neglecting air resistance), the acceleration is zero: $\mathbf{a}_x = 0$.

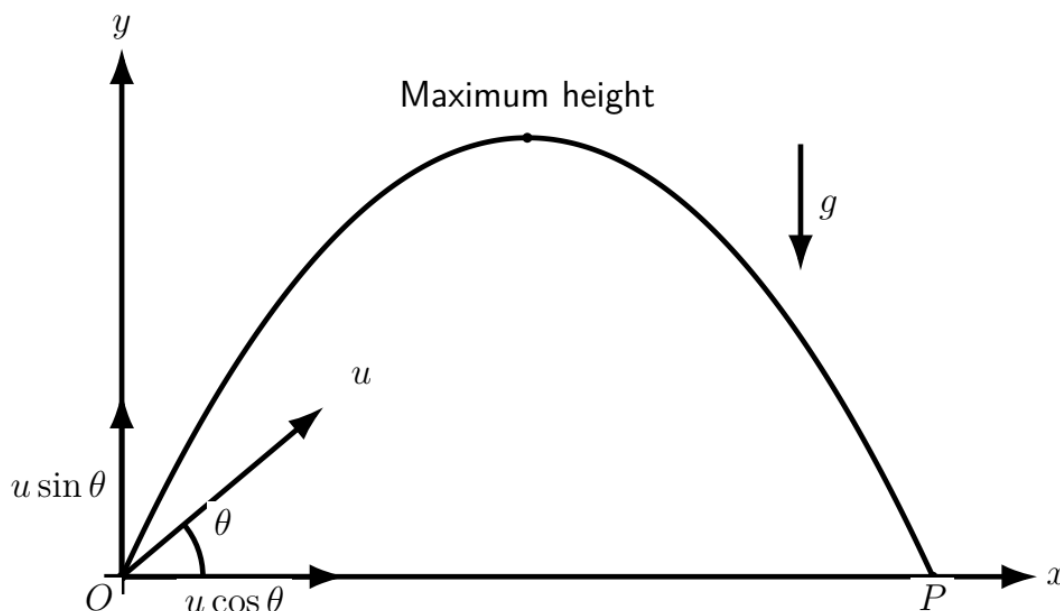


Figure: Projectile launched from the origin O with initial velocity, u at an angle θ above the horizontal. The trajectory is parabolic. The initial velocity is resolved into horizontal component $u \cos \theta$ and vertical component $u \sin \theta$. The highest point represents the maximum height, and the projectile lands at point P on the x-axis while the acceleration due to gravity g acts vertically downward.

Resolving initial velocity into components

When a projectile is launched from point O with initial velocity u at an angle θ above the horizontal, this velocity vector can be resolved into two perpendicular components using basic trigonometry. The horizontal component, u_x , represents the portion of the initial velocity directed along the x-axis, while the vertical component, u_y , represents the portion directed along the y-axis. From the geometry of the velocity vector triangle, these components are given by:

$$u_x = u \cos \theta$$

$$u_y = u \sin \theta$$

The initial velocity u forms the hypotenuse of a right triangle, with u_x and u_y as the two perpendicular sides. The horizontal component uses the cosine function because it lies adjacent to the angle θ , while the vertical

component uses the sine function because it lies opposite to the angle. These trigonometric relationships ensure that the vector sum of the components equals the original velocity vector in both magnitude and direction.

Velocity as a function of time

Since the horizontal acceleration is zero, the horizontal component of velocity remains constant throughout the motion. Thus, at **any time t** after launch, the horizontal velocity is simply:

$$v_x = u \cos \theta = u_x$$

This constancy reflects the fact that, in the absence of horizontal forces, there is nothing to change the projectile's horizontal velocity. *The projectile continues to move forward at the same horizontal rate from launch until landing.*

The vertical component of velocity, however, changes continuously under the influence of gravity. Applying the first equation of motion ($v = u + at$) to the vertical direction, with initial velocity $u_y = u \sin \theta$ and acceleration $a_y = -g$, we obtain:

$$v_y = u \sin \theta - gt$$

The negative sign before g reflects gravity's downward action. Initially, when $t = 0$, the vertical velocity equals $u \sin \theta$, directed upward. As time progresses, gravity steadily reduces this upward velocity. At some point, v_y becomes zero; this marks the instant when the projectile reaches its maximum height and momentarily stops rising. Beyond this point, v_y becomes negative, indicating downward motion as the projectile begins to fall.

Displacement as a function of time

The **horizontal displacement** (s_x) at any time t follows from the second equation of motion, which is: $s = ut + \frac{1}{2}at^2$. With zero horizontal acceleration, the $\frac{1}{2}at^2$ term vanishes, leaving:

$$s_x = (u \cos \theta)t$$

This linear relationship between horizontal displacement and time shows that the projectile covers equal horizontal distances in equal time intervals, which is a characteristic of uniform motion. Thus, the horizontal distance increases steadily for as long as the projectile remains in the air.

For **vertical displacement**, we again apply $s = ut + \frac{1}{2}at^2$ to the vertical direction, substituting $u_y = u \sin \theta$ and $a_y = -g$:

$$s_y = (u \sin \theta)t - \frac{1}{2}gt^2$$

This equation contains two competing terms. *The first term, $(u \sin \theta)t$, represents how high the projectile would rise if gravity did not act (it increases linearly with time). The second term, $-\frac{1}{2}gt^2$, represents how far gravity pulls the projectile downward (it increases quadratically with time).* The net vertical displacement at any instant results from these two opposing effects. Initially, the linear term dominates and the projectile rises. Eventually, the quadratic term overtakes the linear term, and the projectile begins to descend.

The trajectory equation

The equations for horizontal and vertical displacement both contain time as a parameter. By eliminating time between these two equations, we can obtain a direct relationship between s_x and s_y that describes the shape of the trajectory itself.

From $s_x = (u \cos \theta)t$, we can express time as:

$$t = \frac{s_x}{u \cos \theta}$$

Substituting this expression for t into the equation for vertical displacement:

$$s_y = (u \sin \theta) \left(\frac{s_x}{u \cos \theta} \right) - \frac{1}{2}g \left(\frac{s_x}{u \cos \theta} \right)^2$$

Simplifying the first term:

$$s_y = s_x \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) s_x^2$$

This is the trajectory equation.

We may take s_y as y , and s_x as x ; then the trajectory equation becomes:

$$y = x \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) x^2$$

With this form, we can clearly see that the equation is in the standard form of a parabola equation:

$$y = bx + ax^2$$

Where:

$$b \text{ (coefficient of } s_x) = \tan \theta$$

$$a \text{ (coefficient of } s_x^2) = - \left(\frac{g}{2u^2 \cos^2 \theta} \right) \text{ (negative)}$$

So the trajectory equation has the mathematical form of a parabola that opens downward.

Also it is important to note that both coefficients are constants for a given projectile launched with particular initial conditions. This reveals that *regardless of the initial velocity or launch angle, the path of a projectile is always parabolic* when air resistance is negligible.

Magnitude and direction of velocity at any instant

Although we analyse the horizontal and vertical components of velocity separately, the actual velocity of the projectile at any instant is a single vector pointing along the instantaneous direction of motion. The magnitude of this velocity vector can be found by combining its perpendicular components using the Pythagorean theorem:

$$v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(u \cos \theta)^2 + (u \sin \theta g - t)^2}$$

The direction of the velocity vector at time t , measured as an angle α **from the horizontal**, is given by:

$$\tan \alpha = \frac{v_y}{v_x}$$

$$\alpha = \tan^{-1} \left(\frac{v_y}{v_x} \right)$$

As the projectile moves along its trajectory, both the magnitude and direction of its velocity change continuously. The horizontal component remains constant, but the changing vertical component causes both the velocity and direction to vary from moment to moment.

The Principle of Independence

Students often find it puzzling that we can treat horizontal and vertical motions as if they were completely separate, when clearly the projectile moves as a single object through space. This separation is justified by a fundamental principle of mechanics: **perpendicular components of motion are independent**. *What happens in one direction has no influence on what happens in a perpendicular direction, provided no forces act to couple the two directions.*

To understand this more clearly, imagine an observer on a train moving at constant velocity who drops a ball. Relative to the observer, the ball falls straight downward, taking time t to reach the floor. The fact that the train is moving horizontally does not alter the time required for the ball to fall (gravity acts only vertically and is unrelated to the train's horizontal motion). An observer standing beside the track sees the ball follow a parabolic path, but even from this external perspective, the time for the ball to fall remains t . Thus, the horizontal motion is superimposed on the vertical motion without affecting it.

This independence extends to all projectile motion. The time a football remains in the air depends only on its initial vertical velocity and the action of gravity; it does not matter whether the ball was kicked gently or powerfully in the horizontal direction. Similarly, the horizontal distance traveled depends on how long the projectile stays in the air (determined by vertical motion) and how fast it moves horizontally (determined by the horizontal component of initial velocity). The two components of motion proceed simultaneously and independently, yet together they produce the observed parabolic trajectory.

Addressing Common Misconceptions

Several persistent misconceptions about projectile motion deserve explicit attention. Let us address them directly:

Misconception 1: "Heavier objects follow different trajectories than lighter objects."

The truth: *Mass does not appear anywhere in our projectile equations!* A heavy stone and a light pebble, if thrown with the same initial velocity and angle, will follow identical paths. Examination of all the equations we have derived reveals no dependence on mass whatsoever.

Why students believe this: In everyday experience, air resistance affects lighter objects more severely than heavier ones, making them appear to follow different paths. But the fundamental projectile motion equations (which assume negligible air resistance) are completely independent of mass.

Misconception 2: "At the highest point, the velocity is zero."

The truth: At the maximum height, **only** the vertical component of velocity (v_y) becomes zero. The horizontal component remains unchanged at $v_x = u \cos \theta$. *The projectile never stops moving completely, even at the peak of its trajectory, it continues to travel forward.*

Why this matters: If the velocity were truly zero at the highest point, the projectile would simply drop straight downward from there. Instead, it continues its parabolic arc because horizontal motion persists throughout the flight.

Misconception 3: "Gravity acts differently during ascent than during descent."

The truth: Throughout the entire motion; rising, at the peak, and falling, *the acceleration due to gravity remains constant and always acts downward* ($a = -g = -9.8 \text{ m/s}^2$ always), exactly as we learned in Chapter 2. This never changes.

The apparent difference: On the way up, this downward acceleration opposes and reduces the upward velocity. On the way down, the same downward acceleration increases the downward velocity. The acceleration itself remains identical; only its effect on the velocity appears different depending on the direction of motion.

Misconception 4: "Throwing harder changes the shape of the trajectory."

The truth: *The trajectory is always parabolic, regardless of how hard you throw.* Throwing harder increases the horizontal distance and maximum height, creating a larger parabola, but the shape remains fundamentally the same (it is still a parabola). Throwing gently produces a smaller parabola, but still a parabola.

Relating to Kipanga's two attempts: In Kipanga's throwing attempts: one nearly horizontal, another nearly vertical, both failed. In the first case, he created a very flat parabola that hit the ground almost immediately. In the second, he created a very tall, narrow parabola that barely traveled forward. Kipute's 45-degree throw created a parabola with optimal proportions for maximum horizontal distance.

Application to Kipanga's shot put disaster

Returning to the opening scenario, we can now analyse what went wrong with Kipanga's two attempts and why Kipute succeeded.

Kipanga's first throw (nearly horizontal, say $\theta \approx 5^\circ$):

If he threw with velocity $u = 10 \text{ m/s}$ at angle $\theta = 5^\circ$:

- $u_x = 10 \cos 5^\circ \approx 9.96 \text{ m/s}$ (almost all horizontal)
- $u_y = 10 \sin 5^\circ \approx 0.87 \text{ m/s}$ (very little vertical)

*The small vertical component meant the shot barely rose above his release height before gravity pulled it down. With such **minimal time in the air**, even the large horizontal component could not carry it far.*

Kipanga's second throw (nearly vertical, say $\theta \approx 86^\circ$):

If he threw with the same velocity at $\theta = 86^\circ$:

- $u_x = 10 \cos 86^\circ \approx 0.70 \text{ m/s}$ (barely any horizontal)
- $u_y = 10 \sin 86^\circ \approx 9.98 \text{ m/s}$ (almost all vertical)

The large vertical component sent the shot high into the air, giving it plenty of time before landing. But with such minimal horizontal velocity, it barely moved forward during all that airtime!

Kipute's optimal throw ($\theta = 45^\circ$):

With $u = 10\text{m/s}$ at $\theta = 45^\circ$:

- $u_x = 10\cos 45^\circ \approx 7.07\text{m/s}$ (balanced horizontal)
- $u_y = 10\sin 45^\circ \approx 7.07\text{m/s}$ (balanced vertical)

This balance between horizontal and vertical components gives maximum horizontal distance. It allowed the shot to stay in the air long enough while also traveling forward effectively. The result: 8 metres compared to Kipanga's embarrassing 3 metres and 0.5 metres!

The mathematics of projectile motion transforms shot put from pure strength into applied science. Understanding these principles applies equally to javelin throws, long jumps, basketball shots, and yes, even Mr. Akilikubwa's favorite example, the perfect arc of water from a garden hose!

With these fundamental concepts now simmering nicely, let us serve them properly through a few worked examples and enjoy the flavour of physics in action!

BINDER Example 1

A stone is thrown with an initial velocity of 20m/s at an angle of 30° above the horizontal. Take $g = 9.8\text{ m/s}^2$. Calculate:

- the horizontal and vertical components of the initial velocity,
- the horizontal and vertical components of velocity after 1.5s .

Solution

(a) Horizontal component:

$$u_x = u\cos\theta = 20\text{m/s} \times \cos 30^\circ = \mathbf{17.3\text{m/s}}$$

Vertical component:

$$u_y = u\sin\theta = 20\text{m/s} \times \sin 30^\circ = \mathbf{10\text{m/s}}$$

(b) Horizontal velocity (remains constant):

$$v_x = u_x = \mathbf{17.3\text{m/s}}$$

Vertical velocity (changes due to gravity):

$$v_y = u_y - gt = 10\text{m/s} - (9.8\text{m/s}^2)(1.5\text{s}) = \mathbf{-4.7\text{ m/s}}$$

The negative sign indicates the stone is now moving downward.

So the vertical velocity is -4.7 m/s (or 4.7 m/s downward)

Making Sense of the Answer: After 1.5 seconds , the horizontal velocity remains unchanged at 17.3m/s because there's no horizontal force. The vertical velocity changed from $+10\text{m/s}$ (upward) to -4.7 m/s (downward), meaning the stone has passed its maximum height and is now falling. The stone initially rose with vertical velocity 10m/s , slowed to zero at the peak, then began falling, reaching 4.7m/s downward after total time of 1.5s .

Think Like a Physicist: Notice that we could immediately tell the stone had passed its highest point because v_y became negative. At the exact moment of maximum height, $v_y = 0$. We can calculate when this occurred: $0 = 10 - 9.8t$, giving $t = 1.02\text{s}$. So the stone reached maximum height at $t = 1.02\text{s}$, and by $t = 1.5\text{s}$ it was already descending. This demonstrates how the vertical component alone determines when the projectile reaches its peak.

BINDER Example 2

A ball is kicked horizontally from the top of a cliff with initial velocity 15m/s . Calculate the position and velocity of the ball after 2 seconds . Take $g = 9.8\text{ m/s}^2$.

Solution

In this case, $u_x = 15\text{m/s}$, $u_y = 0$ (horizontal launch),

Horizontal displacement:

$$s_x = u_x t = 15\text{m/s} \times 2\text{s} = 30\text{m}$$

Vertical displacement:

$$s_y = u_y t - \frac{1}{2}gt^2 = 0 - \frac{1}{2}(9.8\text{m/s}^2)(2\text{s})^2 = -19.6\text{m}$$

The negative sign indicates displacement downward from the launch point.

Hence, **the position of the ball is 30m horizontally from the cliff and 19.6m below the launch point.**

Horizontal velocity:

$$v_x = u_x = 15\text{m/s}$$

Vertical velocity:

$$v_y = u_y - gt = 0 - (9.8\text{m/s}^2)(2\text{s}) = -19.6\text{m/s}$$

Magnitude of velocity:

$$v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(15\text{m/s})^2 + (-19.6\text{m/s})^2} = 24.7\text{m/s}$$

Direction (below horizontal since the motion is downward):

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-19.6\text{m/s}}{15\text{m/s}}\right) = -52.6^\circ$$

The negative sign means the angle was measured clockwise from the horizontal, and hence the direction of motion is below the horizontal.

Hence, **the velocity of the ball is 24.7m/s directed at 52.6° below horizontal.**

Making Sense of the Answer: Even though the ball was kicked horizontally (no initial vertical velocity), after 2 seconds it has fallen 19.6m and gained significant downward velocity (19.6m/s). The horizontal velocity remained constant at 15m/s throughout. The final velocity (24.7m/s) is greater than the initial velocity (15m/s) because gravity has added a vertical component. The steep angle (52.6° below horizontal) shows the ball is now falling quite rapidly.

Think Like a Physicist: This example illustrates a special case of projectile motion: horizontal launch ($\theta = 0^\circ$). Even though there was no initial upward motion, the ball still follows a parabolic path as it falls. This is exactly what happens when you roll a ball off a table; it continues forward at constant horizontal velocity while simultaneously falling under gravity. Both motions combine to create the characteristic parabolic trajectory, even without an initial upward component.

REAL Example 3

At Kariakoo market, vendors selling oranges often demonstrate the quality of their fruit by tossing an orange to potential customers standing several metres away. An experienced vendor can consistently land the orange gently in a customer's hands, while inexperienced vendors either throw too hard (orange sails over the customer) or too soft (orange falls short).

- Explain why the vendor must adjust both the velocity and angle of throw depending on how far away the customer stands.
- Explain what happens if the vendor uses the correct velocity but wrong angle.

Solution

- The horizontal distance travelled by the orange (the range) depends on both the initial velocity and the angle of projection. For a given distance to the customer, there are actually multiple combinations of velocity and angle that could work: a fast throw at a low angle, or a slower throw at a higher angle, can both reach the same spot. However, experienced vendors prefer a moderate velocity with optimal angle because:
 - High velocity at low angle:** The orange arrives with high horizontal velocity, making it difficult to catch and potentially bruising the fruit on impact
 - Low velocity at high angle:** The orange spends too long in the air, during which wind could deflect it.

The vendor instinctively calculates (through experience) the combination of velocity and angle that makes the orange arrive gently with low velocity, making it easy to catch.

(b) If the vendor uses correct velocity but wrong angle, two problems arise:

Case 1 - Angle too large (too steep):

Too much initial velocity goes into the vertical component, less into horizontal. The orange rises high, spending long time in the air, but does not travel far enough horizontally. It falls short of the customer, landing on the ground in front of them. (This is similar to Kipanga's second shot put attempt.)

Case 2 - Angle too small (too flat):

Too much initial velocity goes into horizontal component, too little into vertical. The orange does not rise high enough or stay in the air long enough. Despite moving forward quickly, it drops to the ground before reaching the customer. (This is similar to Kipanga's first attempt.)

Making Sense of the Answer: This example demonstrates that projectile motion is not just abstract physics; it is embedded in everyday activities. Market vendors, though they have never studied our equations, understand projectile motion intuitively through experience. They know that "how hard" (initial velocity) and "which direction" (angle) are both crucial and must work together. This intuitive understanding is precisely what our mathematical framework makes explicit and quantifiable.

Think Like a Physicist: Notice how real-world projectile motion involves optimization: not just reaching a target, but reaching it in a way that minimizes certain undesirable effects (impact force, time in air, trajectory height, etc.). In sports, warfare, firefighting, and countless other applications, understanding the mathematics allows us to optimize these factors systematically rather than relying purely on trial and error. The vendor's experience represents thousands of unconscious calculations of which our equations let us perform those calculations consciously and precisely.

HOT Example 4

A projectile is launched with initial velocity 25m/s at an angle of 53° above the horizontal from ground level. Taking $g = 9.8 \text{ m/s}^2$ and $\sin 53^\circ = 0.8$, $\cos 53^\circ = 0.6$:

- Derive the equation of the trajectory for this projectile.
- Calculate the height of the projectile when it is at horizontal distance 15m from the launch point.
- Calculate the horizontal distance at which the projectile returns to ground level.

Solution

(a) The general trajectory equation is:

$$s_y = s_x \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) s_x^2$$

Substituting given values:

$$\tan 53^\circ = \frac{\sin 53^\circ}{\cos 53^\circ} = \frac{0.8}{0.6} = 1.333$$

$$\frac{g}{2u^2 \cos^2 \theta} = \frac{9.8}{(2(25)^2(0.6)^2)} = 0.0218$$

Therefore:

$$s_y = 1.333s_x - 0.0218s_x^2$$

Thus the trajectory equation for this particular projectile is:

$$s_y = 1.333s_x - 0.0218s_x^2$$

(b) At $s_x = 15\text{m}$:

$$s_y = 1.333(15) - 0.0218(15)^2 = 15.1 \text{ m}$$

The projectile is at height of **15.1m** when it is 15m horizontally from launch.

(c) The projectile returns to ground level when $s_y = 0$:

$$0 = 1.333s_x - 0.0218s_x^2$$

$$0 = s_x(1.333 - 0.0218s_x)$$

This gives two solutions: $s_x = 0$ (the launch point); or

$$1.333 - 0.0218s_x = 0; s_x = \frac{1.333}{0.0218} = 61.1\text{m}$$

Hence, the projectile lands at horizontal distance 61.1m.

Making Sense of the Answer: The trajectory equation, $s_y = 1.333s_x - 0.0218s_x^2$ is indeed a parabola (it has the form $y = ax + bx^2$ where b is negative). At $x = 15\text{m}$, the height is 15.1m which is still quite high, indicating the projectile is probably near its peak. The horizontal distance of 61.1m means the projectile travels this far before returning to launch height. Notice that the coefficient 1.333 ($= \tan 53^\circ$) determines the initial slope of the trajectory, while 0.0218 determines how quickly gravity curves it downward.

Think Like a Physicist: The trajectory equation is powerful because it relates position to position directly without time as an intermediate variable. This is useful when we care about "where" rather than "when." For instance, if there's an obstacle at a certain location, we can immediately check if the projectile will clear it by substituting that horizontal position and checking if s_y exceeds the obstacle height. The fact that we found two solutions when $s_y = 0$ (namely $s_x = 0$ and $s_x = 61.1$) makes physical sense: the projectile is at ground level at both launch ($x = 0$) and landing ($x = 61.1\text{m}$). The parabola crosses the x -axis at exactly two points, as expected for a symmetric trajectory launched from and returning to the same level.

Summary of Fundamental Equations

For convenient reference, we collect here the essential equations governing projectile motion, valid when air resistance is negligible and the acceleration due to gravity is constant:

Initial velocity components:

$$u_x = u \cos \theta, u_y = u \sin \theta$$

Velocity at time any time t:

$$v_x = u \cos \theta, v_y = u \sin \theta - gt$$

Displacement at any time t:

$$s_x = (u \cos \theta)t, s_y = (u \sin \theta)t - \frac{1}{2}gt^2$$

Trajectory equation:

$$s_y = s_x \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) s_x^2$$

Magnitude of velocity at any instant:

$$v = \sqrt{(v_x)^2 + (v_y)^2}$$

Direction of velocity at any instant:

$$\alpha = \tan^{-1} \left(\frac{v_y}{v_x} \right)$$

These equations form the mathematical foundation for analysing any projectile motion problem. In the following subtopic, we shall use them to derive expressions for important physical quantities such as time of flight, maximum height, and horizontal range.

KEY PARAMETERS OF THE TRAJECTORY

Having developed the fundamental equations governing projectile motion in the previous subtopic, we now turn our attention to deriving expressions for specific quantities that characterize the trajectory. When Kipanga asks "How far will my shot put travel?" or Kipute wonders "How high will the orange I toss reach at its peak?", they are asking about what we call the parameters of projectile motion. These parameters: time of flight, maximum height, and horizontal range, provide complete practical information about any projectile's trajectory. In this subtopic, we shall derive mathematical expressions for each parameter and discover some surprising relationships between them.

Time of flight, T

The **time of flight** is the total time interval from the moment a projectile is launched until the moment it returns to the same vertical level from which it was projected. This parameter is crucial because it determines how long the projectile remains in the air, which in turn affects how far it can travel horizontally.

Consider a projectile launched from ground level with initial velocity u at angle θ above the horizontal. We established in the previous subtopic that the vertical displacement at time t is given by:

$$s_y = (u \sin \theta)t - \frac{1}{2}gt^2$$

The projectile returns to ground level when $s_y = 0$. This occurs at two instants: $t = 0$ (the moment of launch) and at some later time T (the time of flight). Setting $s_y = 0$:

$$0 = (u \sin \theta)t - \frac{1}{2}gt^2$$

Factorising the expression:

$$0 = t(u \sin \theta - \frac{1}{2}gt)$$

This equation has two solutions:

$$t = 0 \text{ and } u \sin \theta - \frac{1}{2}gt = 0.$$

For the second solution, $t = T$, thus:

$$u \sin \theta = \frac{1}{2}gT$$

Solving for T :

$$T = \frac{2u \sin \theta}{g}$$

This is the time of flight for a projectile launched and landing at the same level.

Notice the physical insight embedded in this formula: time of flight depends only on the vertical component of initial velocity ($u \sin \theta$) and gravity. The horizontal component plays no role whatsoever in determining how long the projectile stays in the air. This makes perfect sense from our principle of independence: *the time spent in the air is determined entirely by vertical motion, exactly as if we had thrown the object straight upward with initial velocity $u \sin \theta$, as we studied in Chapter 2.*

Connecting to Chapter 2: Do you remember when we derived the time for an object thrown vertically upward to return to its starting point? We found $t = \frac{2u}{g}$, where u was the upward initial velocity. Here, the vertical component $u \sin \theta$ plays exactly that role, giving us $T = \frac{2(u \sin \theta)}{g}$. The physics is identical; we have simply replaced u with $u \sin \theta$ to account for the angled launch.

Alternative derivation:

We can verify this result using another approach. The projectile reaches maximum height when $v_y = 0$. From the previous subtopic, we know that $v_y = u \sin \theta - gt$. Setting this to zero:

$$0 = u \sin \theta - gt_{\max}$$

$$t_{\max} = \frac{u \sin \theta}{g}$$

This is the time to reach maximum height (the upward journey). By symmetry, the downward journey takes equal time. Therefore, total time of flight is:

$$T = 2t_{\max} = \frac{2(u\sin\theta)}{g}$$

This confirms our result and demonstrates the symmetric nature of projectile motion when launch and landing occur at the same level.

Maximum height, H

The **maximum height** is the greatest vertical displacement achieved by the projectile above its launch point. This occurs at the instant when the vertical component of velocity becomes zero, when the projectile stops rising and is about to begin falling.

We can derive the maximum height using the third equation of motion from Chapter 2:

$$v^2 = u^2 + 2as.$$

Applying this to vertical motion:

$$v_y^2 = u_y^2 + 2a_y s_y$$

At maximum height:

$$v_y = 0, u_y = u\sin\theta, a_y = -g, \text{ and } s_y = H.$$

Substituting:

$$0 = (u\sin\theta)^2 - 2gH$$

Solving for H:

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

This elegant formula reveals that maximum height depends on the square of the vertical component of initial velocity. Doubling the vertical component quadruples the maximum height.

Alternative derivation:

We can also derive this using the time to maximum height and the displacement equation.

$$\text{We found } t_{\max} = \frac{u\sin\theta}{g}.$$

Substituting into $s_y = (u\sin\theta)t - \frac{1}{2}gt^2$:

$$H = u\sin\theta \left(\frac{u\sin\theta}{g} \right) - \frac{1}{2}g \left(\frac{u\sin\theta}{g} \right)^2$$

$$H = \frac{u^2 \sin^2 \theta}{g} - \frac{1}{2} \frac{g(u^2 \sin^2 \theta)}{g^2}$$

$$H = \frac{u^2 \sin^2 \theta}{g} - \frac{u^2 \sin^2 \theta}{2g}$$

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

Both methods yield the same result, providing mathematical confirmation of our formula.

An important observation: If we launch the projectile vertically upward ($\theta = 90^\circ$), then $\sin^2 90^\circ = 1$ (at this angle, the maximum height becomes greatest for a given initial velocity), and we get $H_{\max} = \frac{u^2}{2g}$, which is exactly the formula we derived in Chapter 2 for maximum height of vertical projection. This serves as another verification that our projectile motion analysis correctly reduces to simpler cases.

Horizontal range, R

The **horizontal range** is the total horizontal distance travelled by the projectile from launch point to landing point (when both are at the same level). This is perhaps the most practically important parameter, as it determines how far the projectile will travel.

From the previous subtopic, horizontal displacement is given by:

$$s_x = (u \cos \theta)t$$

At landing ($t = T$), the horizontal displacement is equal to the range R:

$$R = (u \cos \theta)T$$

We already derived $T = \frac{2u \sin \theta}{g}$.

Substituting:

$$R = u \cos \theta \left(\frac{2u \sin \theta}{g} \right)$$

From which:

$$R = \frac{2u^2 \cos \theta \sin \theta}{g}$$

This can be simplified by using the trigonometric identity: $2 \sin \theta \cos \theta = \sin 2\theta$:

$$R = \frac{u^2 \sin 2\theta}{g}$$

This remarkably simple formula captures everything about horizontal range. Notice that range depends on the initial velocity squared, inversely on gravity, and on the sine of twice the angle of projection.

The factor $\sin 2\theta$ deserves special attention. Since the maximum value of sine function is 1 (occurring when the angle is 90°), the range is maximized when:

$$2\theta = 90^\circ$$

Which gives:

$$\theta = 45^\circ$$

And the range formula becomes:

$$R_{\max} = \frac{u^2}{g}$$

This is the famous result: *for a given initial velocity, maximum range occurs at 45° launch angle*. This explains why Kipute's 45° throw in our introduction travelled farther than both of Kipanga's attempts; she intuitively chose the optimal angle!

Physical interpretation: *Why does 45° give maximum range?* At angles below 45° , the projectile has greater horizontal velocity but insufficient vertical velocity, so its time of flight is small. At angles above 45° , the projectile has larger time of flight but moves too slow horizontally. The 45° angle perfectly balances these competing requirements: enough vertical component to provide adequate flight time, and enough horizontal velocity to cover large horizontal distance during that time.

Moreover, the factor $\sin 2\theta$ has an important property. The sine function satisfies the identity:

$$\sin x = \sin(180 - x)$$

If we let $x = 2\theta$. Then:

$$\sin 2\theta = \sin(180 - 2\theta)$$

Since the horizontal range depends on $\sin 2\theta$, the same range will be obtained when:

$$2\theta' = 180 - 2\theta$$

Solving for θ' :

$$\theta' = 90 - \theta$$

Hence, a projectile launched at angle θ will have the same horizontal range as one launched at an angle $(90 - \theta)$. In other words: two projectiles launched at complementary angles (angles that sum to 90°), have the same horizontal range. For example, launches at 30° and 60° produce identical horizontal range, and the same is true for 20° and 70° .

Relationship between parameters

The three parameters we have derived: time of flight, maximum height, and range are not independent. They are intimately connected through the initial conditions u (velocity of projection) and θ (angle of projection). Let us explore some of these relationships.

From our formulas:

- $T = \frac{2u \sin \theta}{g}$
- $H = \frac{u^2 \sin^2 \theta}{2g}$
- $R = \frac{u^2 \sin 2\theta}{g}$

We can express **maximum height in terms of time of flight**:

$$H = \frac{u^2 \sin^2 \theta}{2g} = \frac{(u \sin \theta)^2}{2g}$$

But from $T = \frac{2u \sin \theta}{g}$:

$$u \sin \theta = \frac{gT}{2}$$

So:

$$H = \frac{\left(\frac{gT}{2}\right)^2}{2g} = \frac{g^2 T^2}{4 \times 2g} = \frac{gT^2}{8}$$

Therefore:

$$H = \frac{gT^2}{8}$$

This shows that for any projectile, maximum height is proportional to the square of time of flight. Thus, a projectile that stays in the air four times as long reaches sixteen times the height!

We can also relate **range to maximum height**.

Starting from:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin \theta \cos \theta}{g}$$

And:

$$H = \frac{u^2 \sin^2 \theta}{2g} = \frac{u^2 \sin \theta \sin \theta}{2g}$$

Taking $\frac{R}{H}$:

$$\frac{R}{H} = \frac{2u^2 \sin \theta \cos \theta}{g} \times \frac{2g}{u^2 \sin \theta \sin \theta} = \frac{4 \cos \theta}{\sin \theta} = 4 \cot \theta$$

Hence:

$$R = 4H \cot \theta$$

This relationship tells us that for a given maximum height, the range depends on the launch angle. Shallow angles (small θ) give large $\cot\theta$ and thus large range for the same height. Steep angles give smaller range for the same height. It also shows that for a given angle of projection, if the maximum height increases, the range increases proportionally.

For the special case of 45° launch:

$$\cot 45^\circ = 1$$

So:

$$R = 4H$$

Thus, at 45° launch angle, the range is exactly four times the maximum height. This is a useful rule of thumb: **if a projectile launched at 45° reaches height H , it will land at distance $4H$ from its starting point.**

After all these derivations, the mathematics has revealed the structure of projectile motion. But equations alone can feel abstract until we see them at work. Before they start arguing among themselves, let us invite a few worked examples to settle the matter.

BINDER Example 5

A cricket ball is struck at 30m/s at an angle of 60° above horizontal. Calculate:

(a) the time of flight, (b) the maximum height reached, (c) the horizontal range. Take $g = 9.8 \text{ m/s}^2$.

Solution

(a) Time of flight is given by:

$$T = \frac{2u \sin \theta}{g}$$

Substituting values:

$$T = \frac{2 \times 30\text{m/s} \times \sin 60^\circ}{9.8\text{m/s}^2} = 5.3\text{s}$$

The time of flight is 5.3s.

(b) Maximum height is given by:

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

Substituting values:

$$H = \frac{(30\text{m/s})^2 \times \sin^2 60^\circ}{2 \times 9.8\text{m/s}^2} = 34.44\text{m}$$

The maximum height is 34.44m.

(c) Horizontal range is given by:

$$R = \frac{u^2 \sin 2\theta}{g}$$

Substituting values:

$$R = \frac{(30\text{m/s})^2 \times \sin 120^\circ}{9.8\text{m/s}^2} = 79.55\text{m}$$

The horizontal range is 79.55m.

Making Sense of the Answer: The ball stays in the air for over 5 seconds, reaches a height of 34 metres, and travels nearly 80 metres horizontally. These are realistic values for a powerfully struck cricket ball. Notice that 60° is quite steep, so it gives impressive height but not maximum range. If the same ball were struck at 45° instead, it would travel farther ($R_{\max} = \frac{900}{9.8} = 91.8\text{m}$) but not as high ($H = \frac{900}{39.2} = 23\text{m}$).

Think Like a Physicist: Observe the complementary angle relationship: if we launched at 30° (the complement of 60°), we would get the same range (79.5m) but different flight time and height. We can verify: $\sin 2(30^\circ) = \sin 60^\circ = 0.866$, same as $\sin 2(60^\circ) = \sin 120^\circ = 0.866$. This symmetry is not coincidental; it is built into the trigonometry of projectile motion. In cricket, fielders must position themselves based on the likely range, which depends on both the batsman's power (determining u) and shot selection (determining θ).

BINDER Example 6

A stone is thrown from ground level and reaches a maximum height of 20m. If the stone was thrown at 45° , calculate:

(a) the initial velocity of the stone, (b) the range of the stone, (c) the time of flight.

Take $g = 9.8 \text{ m/s}^2$.

Solution

(a) From the maximum height formula:

$$H = \frac{u^2 \sin^2 \theta}{2g}$$

Making u the subject:

$$u = \sqrt{\frac{2gH}{\sin^2 \theta}}$$

Substituting values:

$$u = \sqrt{\frac{2 \times 9.8 \text{ m/s}^2 \times 20\text{m}}{\sin^2 45}} = 28\text{m/s}$$

The initial velocity is 28m/s.

(b) If the angle of projection is 45° :

$$R = 4H = 4 \times 20\text{m} = 80\text{m}$$

The range is 80m.

(c) Using:

$$T = \frac{2u \sin \theta}{g} = \frac{2 \times 28\text{m/s} \times \sin 45^\circ}{9.8 \text{ m/s}^2} = 4.04\text{s}$$

The time of flight is 4.04s.

Making Sense of the Answer: The special angle of 45° creates the elegant relationship $R = 4H$, making part (b) trivial once we know the height. The initial velocity of 28m/s is enough fast to reach 20 metres height. The 4-second flight time is substantial, giving plenty of time for the stone to travel the 80-metre range. This example demonstrates how knowing one parameter (height) and the launch angle allows us to determine all other parameters.

Think Like a Physicist: This example reverses the usual problem: instead of starting with initial conditions (u and θ) and finding the trajectory parameters, we start with a parameter (H) and work backward to find initial velocity. This "inverse problem" approach is common in real applications. For instance, if you observe a projectile reaching a certain height and range, you can deduce its initial velocity and launch angle. This is how military analysts estimate the capabilities of observed missile launches; they measure the trajectory and work backward to the launch conditions.

BINDER Example 7

A javelin thrower achieves a throw of 60m when the javelin leaves her hand at 25m/s. Calculate:

- (a) the two possible angles at which the javelin could have been launched,
 (b) explain which angle is more realistic for javelin throwing. Take $g = 9.8 \text{ m/s}^2$.

Solution

(a) Javelin is thrown for maximum horizontal distance, so we are going to apply the range formula:

$$R = \frac{u^2 \sin 2\theta}{g}$$

Substituting values:

$$60\text{m} = \frac{(25\text{m/s})^2 \times \sin 2\theta}{9.8\text{m/s}^2}$$

From which:

$$\sin 2\theta = 0.941 \text{ or } 2\theta = \sin^{-1}(0.941)$$

Thus:

$$2\theta = 70.2^\circ \text{ or } 2\theta = 180^\circ - 70.2^\circ = 109.8^\circ$$

And:

$$\theta = \frac{70.2^\circ}{2} = 35.1^\circ \text{ or } \frac{109.8^\circ}{2} = 54.9^\circ$$

two possible angles are 35.1° and 54.9° .

(b) For javelin throwing, the smaller angle (35.1°) is more realistic.

Explanation:

In real situations, air resistance reduces the horizontal range, and this effect becomes more significant for steeper launch angles because the javelin spends more time in the air. Consequently, the optimum angle for maximum range becomes less than 45° (typically around 35° – 40°).

Making Sense of the Answer: The two angles (35.1° and 54.9°) are complementary (they sum to 90°), which is exactly what we expect from the theory: complementary angles give the same range. Both are physically possible, but aerodynamics favour the shallower trajectory. Notice that neither angle is exactly 45° because the range (60m) is less than the maximum possible range: $R_{\max} = \frac{u^2}{g} = \frac{625}{9.8} = 63.8\text{m}$. The maximum range would require 45° .

Think Like a Physicist: This inverse problem requires finding angle from range has two solutions because of the symmetry in the sine function: $\sin 2\theta_1 + \sin 2\theta_2$ when $2\theta_1 + 2\theta_2 = 180^\circ$. In sports, military applications, and firefighting, this symmetry creates interesting choices. In practice, other factors (obstacles, wind, equipment limitations) often make one choice clearly superior.

BINDER Example 8

Two identical stones are thrown from ground level with the same initial velocity of 20m/s, but at different angles: stone A at 30° and stone B at 60° . Taking $g = 9.8 \text{ m/s}^2$. Calculate:

- (a) the horizontal range for each stone and verify they are equal,
- (b) the maximum height reached by each stone,
- (c) the time of flight for each stone.

Solution

(a) Range for stone A:

$$R_A = \frac{u^2 \sin 2\theta_A}{g} = \frac{(20\text{m/s})^2 \times \sin 60^\circ}{9.8\text{m/s}^2} = 35.3\text{m}$$

Range for stone B:

$$R_B = \frac{u^2 \sin 2\theta_B}{g} = \frac{(20\text{m/s})^2 \times \sin 120^\circ}{9.8\text{m/s}^2} = 35.3\text{m}$$

The horizontal range for each stone is 35.3m.

(b) Maximum height for stone A:

$$H_A = \frac{u^2 \sin^2 \theta_A}{2g} = \frac{(20\text{m/s})^2 \times \sin^2 30^\circ}{2 \times 9.8\text{m/s}^2} = 5.1\text{m}$$

Maximum height for stone B:

$$H_B = \frac{u^2 \sin^2 \theta_B}{2g} = \frac{(20\text{m/s})^2 \times \sin^2 60^\circ}{2 \times 9.8\text{m/s}^2} = 15.3\text{m}$$

The maximum height of stone A is 5.1m.

The maximum height of stone B is 15.3m. (Stone B goes 3 times higher!)

(c) Time of flight for stone A:

$$T_A = \frac{2u \sin \theta_A}{g} = \frac{2 \times 20\text{m/s} \times \sin 30^\circ}{9.8\text{m/s}^2} = 2.04\text{s}$$

Time of flight for stone B:

$$T_B = \frac{2u \sin \theta_B}{g} = \frac{2 \times 20\text{m/s} \times \sin 60^\circ}{9.8} = 3.54\text{s}$$

The time of flight for stone A is 2.04s.

The time of flight for stone B is 3.54s.

Making Sense of the Answer: Although both stones travel the same horizontal distance, stone B accomplishes this by staying in the air longer while moving more slowly horizontally, while stone A covers the same distance by moving faster horizontally in less time. So the complementary angle relationship creates fascinating trade-offs: same range, but completely different flight characteristics. Stone A (30°): low, fast trajectory which is good for minimizing flight time. Stone B (60°): high, slow trajectory which is good for clearing obstacles. If there were a 10-metre wall in the path, only stone B would clear it (15.3m > 10m), even though both eventually land at the same spot. This explains why artillery can hit the same target with two different firing solutions: one "flat" trajectory for quick strikes, one "high" trajectory for clearing obstacles.

Think Like a Physicist: This example reveals a profound principle: there are often multiple ways to achieve the same outcome in physics, each with different trade-offs. Both angles reach 35.3m, but the "how" differs dramatically. In engineering and warfare, these choices matter: flat trajectories are faster but cannot clear obstacles; high trajectories clear obstacles but take longer and are more affected by wind. Similarly, in life, different paths can lead to the same destination: some fast and risky, some slow and safe! Physics teaches us to quantify these trade-offs rather than just acknowledge them qualitatively.

REAL Example 9

During the annual school sports day, the long jump competition takes place on a sand pit. After the competition ends, Kipanga sits dejectedly on the bench; he managed only 4.8 metres while his classmate Musa won with 6.5 metres.

Kipanga: (frustrated) "It's not fair, sir! I ran down that track just as fast as Musa, maybe even faster! But he jumped way farther. He must have longer legs or something."

Mr. Akilikubwa: (sitting down beside him) "Kipanga, I watched both your jumps carefully. Your running speed looked similar to Musa's, that's true. But tell me, what happened at the moment you left the takeoff board?"

Kipanga: "I jumped as hard as I could, sir! I remember going quite high. I could see the whole sand pit below me for a moment. It felt like flying!"

Kipute: (joining them) "I noticed that too! Kipanga jumped very high. Musa's jump looked different; he barely went upward. His jump was much flatter."

Mr. Akilikubwa: (smiling) "Interesting observations. But neither of you used the best possible angle for your running speed. Long jump actually hides an important secret of **projectile motion**."

Kipanga: (puzzled) "Projectile motion? Like the shot put we studied?"

Mr. Akilikubwa: "Exactly. The moment you leave the board; your body behaves like a projectile moving through the air. Two things determine how far you travel horizontally: how long you stay in the air, and how fast you move forward while airborne."

Kipute: "But sir, those seem to contradict each other! To stay in the air longer, you need to jump upward more. But to move fast horizontally, you need to jump forward more. You can't do both!"

Mr. Akilikubwa: (enthusiastically) "Exactly, Kipute! That's the heart of it! Your running speed gives you a certain amount of initial velocity. At takeoff, you must decide how to share this velocity between going up and going forward. Jump too steep like Kipanga did, and yes, you'll be airborne for a long time, but you're crawling forward so slowly that you waste all that airtime. Jump too flat like Musa, and yes, you're rocketing forward, but you hit the sand so quickly that you don't have time to travel far."

Kipanga: "So... there's a perfect angle in the middle somewhere?"

Mr. Akilikubwa: "Now you're thinking like a physicist! In ideal projectile motion, when the launch point and landing point are at the same height and air resistance is neglected, the mathematics shows that the maximum horizontal range occurs when the launch angle is 45° . At that angle, the initial velocity is shared equally between the horizontal and vertical directions. The vertical component keeps the projectile in the air long enough, while the horizontal component carries it forward efficiently. This balance produces the greatest possible range"

Kipute: (excited) "It's like that time we learned about the see-saw! The balance point is in the middle!"

Mr. Akilikubwa: "Beautiful analogy, Kipute! And here's the magical part: if you could somehow jump at exactly 45° , the distance you travel horizontally will always be exactly four times the maximum height you reach. Always! Whether you're a child jumping 2 metres, or an Olympic athlete jumping 8 metres; 45° makes the range exactly four times the height."

Kipanga: (thoughtful) "But sir, I've watched Olympic long jumpers on TV. They don't seem to go that high..."

Mr. Akilikubwa: "Ah! Sharp observation, Kipanga. You're right! Do you know why?"

Kipanga: "Um... they're making a mistake?"

Mr. Akilikubwa: (laughing) "Not at all! They are actually being clever. Real long-jump motion is more complicated than the simple projectile model. The athlete does not land at exactly the same height as the take-off point, and air resistance also affects the motion. Because of these factors, the best take-off angle in real long jumping is usually slightly less than 45° ."

Kipute: "So Kipanga wasn't entirely wrong to jump upward, and Musa wasn't entirely wrong to jump flat; they were both just too extreme?"

Mr. Akilikubwa: "Precisely! And here's your homework: both of you, practice finding that middle ground. Kipanga, try to drive your leading knee forward more, not just up. Musa.... well, he needs the opposite advice: lift that knee higher at takeoff. You're both capable of jumping over 6 metres if you find your 45° sweet spot."

Kipanga: (grinning) "Next year, sir, I'm going to beat Musa! Now that I know the secret physics formula!"

Mr. Akilikubwa: "That's the spirit! But remember, knowing the physics is one thing, training your body to execute it is another. Even knowing the optimal angle, it takes years of practice to actually achieve it consistently. Physics tells you the target; dedication gets you there!"

Based on this dialogue, explain:

- Explain clearly why both Kipanga's high jump (steep angle) and Musa's flat jump (shallow angle) failed to achieve maximum distance, even though they had similar running speeds.
- Explain why the optimal take-off angle of 45° gives the maximum range, and explain the meaning of Mr. Akilikubwa's statement that "range equals four times the height" for 45° -degree launches.
- Explain why Olympic long jumpers actually use angles slightly below 45° rather than exactly 45° , as Mr. Akilikubwa explained.

Solution

- Both jumpers failed because they did not balance the vertical and horizontal components of their initial velocity optimally.

Kipanga (too steep, $\theta > 45^\circ$): By jumping at a steep angle, too much of his initial velocity went into the vertical component ($u \sin \theta$) and too little into the horizontal component ($u \cos \theta$).

Result: He rose very high and stayed in the air for a long time (large $T = 2u \sin \theta / g$), but his horizontal velocity was so small that he barely traveled forward during all that airtime. High time of flight but low horizontal velocity leads to mediocre range.

Musa (too flat, $\theta < 45^\circ$): By jumping at a shallow angle, too much velocity went into horizontal component and too little into vertical.

Result: He moved forward very quickly (large $u \cos \theta$), but he did not stay in the air for long enough (small T because small $u \sin \theta$) to take full advantage of that velocity. High horizontal velocity but low time of flight leads to mediocre range again.

(b) Explanation for maximum range at 45° :

Mathematically, $R = \frac{u^2 \sin 2\theta}{g}$. Since the maximum value of sine function is 1, range is maximized when $\sin 2\theta = 1$, which means $2\theta = 90^\circ$, giving $\theta = 45^\circ$.

At this angle, the initial velocity is distributed equally between the horizontal and vertical directions, giving the best balance between: sufficient **time of flight**, and sufficient **horizontal velocity** and the range becomes maximum.

Explanation on the range-height relationship:

For 45° launch:

- Maximum height: $H = \frac{u^2 \sin^2 45^\circ}{2g} = \frac{u^2 \times 0.5}{2g} = \frac{u^2}{4g}$
- Range: $R = \frac{u^2 \sin 90^\circ}{g} = \frac{u^2}{g}$ (since $\sin 2 \times 45^\circ = \sin 90^\circ = 1$)

$$\text{Therefore: } \frac{R}{H} = \frac{\frac{u^2}{g}}{\frac{u^2}{4g}} = 4$$

So: $R = 4H$

This means that for a projectile launched at 45° , the horizontal range is always **four times the maximum height reached**. For example, if the projectile rises to a maximum height of **2m**, then its range is **8m**.

(c) Real long-jump athletes like Olympic long jumpers usually use take-off angles slightly less than 45° because actual long-jump motion is not exactly the same as ideal projectile motion.

In the ideal model, the projectile is launched and lands at the same height and air resistance is neglected. Under those conditions, 45° gives maximum range. In real long jump, however:

- the athlete's landing position is effectively **lower than the take-off position**, and
- **air resistance** also affects the motion.

When the landing point is lower than the launch point, the projectile can stay in the air long enough even with a slightly smaller launch angle. Therefore, the angle for practical maximum range becomes slightly less than 45° .

Making Sense of the Answer: *The dialogue beautifully captures the central insight of projectile optimization: extreme strategies fail, balanced approaches succeed. Kipanga and Musa represent the two failure modes: too much vertical emphasis versus too much horizontal emphasis. Mr. Akilikubwa guides them to the middle ground where physics achieves the optimal result. The conversation also reveals that real-world applications (Olympic long jump) must account for factors beyond the idealized physics (like landing height differences), showing how theory guides practice even when perfect application is not possible.*

Think Like a Physicist: *This example demonstrates how physics provides both explanation and optimization strategy. Without physics, an athlete might spend years trying random angles, hoping to improve. With physics, we know immediately that 45° (or slightly below for long jump) is the target, and training can focus on achieving that specific goal. The dialogue format makes the abstract mathematics tangible: students see Kipanga's frustration, understand his confusion, and follow his journey from "it's not*

fair" to "now I know the secret formula." This is how physics transforms from formulas on paper into actionable knowledge that changes outcomes.

HOT Example 10

A projectile is fired from ground level with initial velocity u at angle θ . At the highest point of its trajectory, the projectile explodes into two equal fragments. One fragment falls straight down from the explosion point and lands directly below.

- (a) Show that the second fragment lands at a distance $\frac{3R}{2}$ from the launch point, where R is the range the projectile would have achieved without exploding.
- (b) If the projectile was fired at 40m/s at 45° , calculate:
- the range the projectile would have travelled if no explosion occurred,
 - the position where each fragment lands,
 - the horizontal velocity of the second fragment immediately after explosion.

Take $g = 9.8 \text{ m/s}^2$.

Solution**(a) Before explosion:**

At the highest point:

- Horizontal position = $\frac{R}{2}$ (halfway through the trajectory)
- Vertical velocity = 0
- Horizontal velocity = $u \cos \theta$ (unchanged throughout flight)

Total horizontal momentum before explosion:

$$p_{\text{before}} = m(u \cos \theta) \text{ (where } m \text{ is the projectile mass)}$$

After explosion:

Fragment 1 (mass $m/2$) falls straight down; thus, horizontal velocity = 0.

Fragment 2 (mass $m/2$) has horizontal velocity v_2 .

Conservation of horizontal momentum:

$$m(u \cos \theta) = \left(\frac{m}{2}\right)(0) + \left(\frac{m}{2}\right)v_2$$

Solving for v_2 :

$$v_2 = 2u \cos \theta \text{ (Fragment 2 has twice the original horizontal velocity!)}$$

Landing position:

Fragment 2 travels from horizontal position $R/2$ with velocity $2u \cos \theta$ for time t (time to fall from maximum height H).

But the time to reach the maximum height is equal to the falling time. Thus:

$$t = t_{\text{max}} = \frac{u \sin \theta}{g}$$

Additional horizontal distance traveled by fragment 2:

$$\Delta x = v_2 \times t = (2u \cos \theta) \times \left(\frac{u \sin \theta}{g}\right) = \frac{2u^2 \sin \theta \cos \theta}{g} = \frac{u^2 \sin 2\theta}{g} = \text{original range } R$$

Therefore: $\Delta x = R$

Fragment 2 starts at horizontal position, $R/2$ and travels additional distance R :

$$\text{Landing position of fragment 2} = \frac{R}{2} + R = \frac{3R}{2}$$

Hence, the second fragment lands at a distance $\frac{3R}{2}$ from the launch point.

(b) The solution of each part is as follows:

(i) Using:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(40 \text{ m/s})^2 \times \sin 90^\circ}{9.8 \text{ m/s}^2} = 163.3 \text{ m}$$

The range the projectile would be 163.3m.

(ii) Fragment 1 falls vertically from height H at horizontal position R/2. So it lands at:

$$\frac{R}{2} = \frac{163.3 \text{ m}}{2} = 81.7 \text{ m}$$

Fragment 2 lands at:

$$\frac{3R}{2} = 3 \times \frac{163.3 \text{ m}}{2} = 245 \text{ m}$$

Fragment 1 lands at 81.7m, while fragment 2 lands at 245m.

(iii) Horizontal velocity of fragment 2:

$$v_2 = 2u \cos \theta = 2 \times 40 \text{ m/s} \times \cos 45^\circ = 56.6 \text{ m/s}$$

Making Sense of the Answer: Fragment 2 lands at 245m which is equivalent to a 50% farther than the original projectile would have traveled (163m)! This seems counterintuitive until we realize that the explosion doubles its horizontal velocity from 28.3 m/s to 56.6 m/s. Even though it only has half the time left to travel, the doubled horizontal velocity allows the fragment to travel a further horizontal distance of 163.3m (R) before reaching the ground. Fragment 1 lands exactly halfway along the original trajectory, which makes sense since it falls straight down from the peak.

Think Like a Physicist: This problem beautifully demonstrates momentum conservation in two dimensions. The explosion provides internal forces between fragments, but no external horizontal force acts on the system. Therefore, total horizontal momentum must be conserved. Since one fragment stops horizontally (falls straight down), the other must carry the entire horizontal momentum, and having half the mass, it must travel at twice the original horizontal velocity.

HOT Example 11

A ball is thrown from ground level at 20m/s at an angle of 40° toward a wall. The wall is 25m away and 3.5m high. Taking $g = 9.8 \text{ m/s}^2$ and using $\sin 40^\circ = 0.643$, $\cos 40^\circ = 0.766$, $\sin 80^\circ = 0.985$:

- Show that the ball clears the wall.
- Calculate by how much the ball clears the top of the wall.
- Calculate where the ball lands beyond the wall.
- If the wall height were increased, determine the maximum wall height that the ball could just clear.

Solution

(a) We need to find the height of the ball when it is at horizontal position of $s_x = 25 \text{ m}$.

Using the trajectory equation:

$$s_y = s_x \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) s_x^2$$

Where: $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$$s_y = s_x \left(\frac{\sin 40^\circ}{\cos 40^\circ} \right) - \left(\frac{g}{2u^2 \cos^2 40^\circ} \right) s_x^2$$

Substituting values:

$$s_y = 25 \text{ m} \left(\frac{0.643}{0.766} \right) - \left(\frac{9.8 \text{ m/s}^2}{2(20 \text{ m/s})^2 (0.766)^2} \right) (25 \text{ m})^2 = 7.91 \text{ m}$$

Since $s_y = 7.91 \text{ m} > 3.5 \text{ m}$ (wall height), **the ball clears the wall.**

(b) Clearance = height of ball – height of wall

$$\text{Clearance} = (7.91 - 3.5) \text{ m} = 4.41 \text{ m}$$

The ball clears the wall by 4.41m.

(c) Total horizontal distance travelled by ball:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin\theta \cos\theta}{g} = \frac{2(20\text{m/s})^2(0.643)(0.766)}{9.8 \text{ m/s}^2} = 40.2\text{m}$$

Distance beyond wall = $40.2\text{m} - 25\text{m} = 15.2\text{m}$

The ball lands 15.2m beyond the wall.

(d) For the ball to just clear the wall: $s_y = h$ at $s_x = 25\text{m}$.

But from (a):

When $s_x = 25\text{m}$, $s_y = 7.91\text{m} = h$

The maximum wall height is 7.91m (as long as the wall stays at 25m distance).

Making Sense of the Answer: The ball reaches 7.91m height at the wall location, clearing the 3.5m wall by a comfortable margin of 4.41m which is more than the wall height itself! This is not surprising since the ball reaches maximum height of 8.43m (by applying maximum height formula) at its peak (which occurs before the wall because $R/2$ is smaller than 25m). If the wall were higher than 7.91m, the ball would hit it. The ball's trajectory is already descending when it reaches the wall position, which is why it lands only 15.1 m beyond the wall rather than much farther.

Think Like a Physicist: In real applications (sports, military, construction), clearance calculations are critical: will the basketball clear the defender's outstretched hand? Will the artillery shell clear the hill? Will the water from the hose reach over the fence? The mathematics gives definitive answers. Notice how the maximum clearable wall height depends on where the wall is positioned; obstacles at different distances pose different challenges, even for the same projectile. A wall placed at the peak position ($R/2$ which is about 20m for our example) is the easiest to clear because the projectile is at its maximum height there.

Summary of key formulas

For a projectile launched at angle θ with initial velocity u from and returning to the same level:

Time of flight: $T = \frac{2u \sin \theta}{g}$

Maximum height: $H = \frac{u^2 \sin^2 \theta}{2g}$

Horizontal range: $R = \frac{u^2 \sin 2\theta}{g}$

Special relationships:

Maximum range occurs at $\theta = 45^\circ$: $R_{\max} = \frac{u^2}{g}$

At 45° , the range is four times the height: $R = 4H$

Complementary angles (θ and $90^\circ - \theta$) give the same range

Maximum height relates to time of flight: $H = \frac{gT^2}{8}$

These formulas, combined with those from previous subtopic, provide complete tools for analysing any projectile motion problem where launch and landing occur at the same level. In the next subtopic, we shall extend our analysis to more complex cases where these conditions do not hold.

PROJECTILES FROM HEIGHT AND ON INCLINED PLANES

In the previous two subtopics, we developed a complete mathematical framework for projectile motion under the assumption that the projectile is launched and lands at the same horizontal level. While this idealized case is fundamental for understanding the basic principles, it represents only a limited range of real-world situations. In practice, projectiles are often launched and land at different heights. For example, a basketball player releases the ball about 2m above the ground toward a hoop 3m high. A stone may be thrown from the edge of a cliff toward the ocean below. A skier may launch off a slope and land farther down the same

incline. In each of these situations, the projectile does not return to its original launch height, and therefore the formulas developed earlier must be modified to account for the difference in vertical levels.

This subtopic extends our analysis to two important special cases that frequently arise in practice: **projectiles launched from a height above the landing level**, and **projectiles moving on inclined planes**. In both situations, the projectile does not return to the same vertical level from which it was launched.

We shall see that the fundamental principles remain unchanged. The horizontal and vertical components of velocity still act independently, and the trajectory of the projectile remains parabolic. However, the specific parameters such as the time of flight, horizontal range, and landing velocity must be recalculated to account for the asymmetry between the launch and landing conditions.

Horizontal Projection from a Height

Consider the simplest extension of our previous analysis: a projectile launched horizontally from height **h** above the ground. This scenario occurs when you roll a ball off a table, kick a football from a cliff edge, or drop supplies from a moving aircraft, and other many familiar circumstances. Since the initial velocity is horizontal, there is no upward component. The projectile therefore begins falling immediately while simultaneously moving forward.

To analyse the motion clearly, let us first define a suitable coordinate system and specify the initial conditions.

Let the origin be at the point of projection, with the x-axis horizontal (direction of initial velocity) and the y-axis vertical (upward positive). The initial conditions are:

- $u_x = u$ (horizontal initial velocity).
- $u_y = 0$ (no vertical component).
- **Initial position:** (0, 0).
- **Landing position:** (R, -h) where R is the range and h is the launch height.

Time of flight (time to hit the ground)

The vertical displacement equation gives:

$$s_y = u_y t - \frac{1}{2}gt^2$$

At landing: $s_y = -h$ (negative because downward from origin) and $u_y = 0$:

$$-h = 0 - \frac{1}{2}gt^2$$

Making **t** the subject:

$$t = \sqrt{\frac{2h}{g}}$$

This is exactly the same result we obtained in **Chapter 2**: *fall time depends only on height, not on horizontal velocity*. Whether the ball rolls off the table slowly or is strongly kicked horizontally, **the time to hit the ground** remains identical, determined solely by **h** and **g**.

Horizontal range

During time **t**, the projectile travels horizontally at constant velocity $u_x = u$:

$$R = ut = u \times \sqrt{\frac{2h}{g}}$$

This elegant formula reveals that range is proportional to both initial velocity and the square root of height. Doubling the launch height increases range by factor $\sqrt{2} \approx 1.41$. Doubling the initial velocity doubles the range directly.

Landing velocity

At time $t = \sqrt{\frac{2h}{g}}$, the velocity components are:

$v_x = u$ (unchanged horizontal velocity)

$$v_y = u_y - gt$$

$$v_y = 0 - g\sqrt{\frac{2h}{g}} = -\sqrt{2gh} \text{ (The negative sign means the velocity is **downward**.)}$$

The magnitude of landing velocity:

$$v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{u^2 + 2gh}$$

The landing angle α satisfies:

$$\tan \alpha = \frac{v_y}{v_x}$$

However, since v_y is negative while v_x is positive, the value of $\tan \alpha$ becomes negative. This indicates that the angle α lies **below the horizontal direction**. To determine the magnitude of this angle directly, we take the absolute value of v_y . Thus:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right)$$

Where α represents the **angle below the horizontal**.

Notice that landing velocity exceeds initial velocity ($v > u$) because gravity has added a vertical component of velocity during the fall.

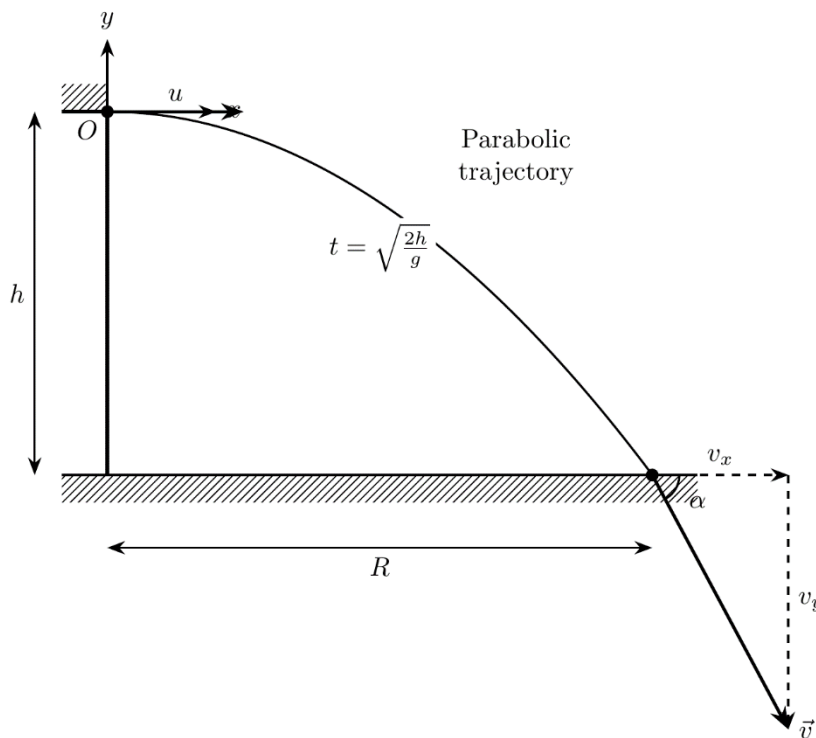


Figure: Horizontal projection from a height h . The projectile is launched with horizontal velocity u and follows a parabolic trajectory, reaching the ground after time $t = \sqrt{\frac{2h}{g}}$ at horizontal range R . The landing velocity has horizontal and vertical components v_x and v_y .

Projection at an Angle from a Height

The more general case occurs when a projectile is launched at angle θ above horizontal from height h . This describes situations like a basketball shot, a water fountain on elevated ground, or an archer shooting from a castle wall. The analysis combines our previous work on angled projection with the modifications introduced by the height difference.

Now, let us specify the initial conditions for this case:

- $u_x = u \cos \theta$ (horizontal component)
- $u_y = u \sin \theta$ (vertical component, upward)
- **Initial position:** (0, 0)
- **Landing position:** (R, -h)

Time of flight

Again, this requires solving the vertical displacement equation:

$$s_y = (u \sin \theta)t - \frac{1}{2}gt^2$$

At landing, $s_y = -h$:

$$-h = (u \sin \theta)t - \frac{1}{2}gt^2$$

Rearranging to standard quadratic equation in t:

$$\frac{1}{2}gt^2 - (u \sin \theta)t - h = 0$$

Using the quadratic formula with: $a = \frac{1}{2}g$, $b = -(u \sin \theta)$, $c = -h$:

$$t = \frac{(u \sin \theta \pm \sqrt{(u \sin \theta)^2 + 2gh})}{g}$$

Since $\sqrt{(u \sin \theta)^2 + 2gh} > u \sin \theta$, the negative root make the numerator negative and therefore give a **negative value of t**. Because negative time has no physical meaning, that root must be rejected. Hence, only the **positive root** is physically meaningful, and therefore:

$$t = \frac{u \sin \theta + \sqrt{u^2 \sin^2 \theta + 2gh}}{g}$$

Comparing with level-ground formula: When $h = 0$, this reduces to:

$$t = \frac{(u \sin \theta + \sqrt{(u \sin \theta)^2})}{g} = \frac{(u \sin \theta + u \sin \theta)}{g} = \frac{2u \sin \theta}{g}$$

This matches our result we derived earlier, confirming that our new formula generalizes the previous one.

Notice the following important physical insight: The term $\sqrt{(u \sin \theta)^2 + 2gh}$ is always greater than $u \sin \theta$ when $h > 0$. So time of flight from a height is always longer than for the same launch on level ground. The projectile gets "bonus time" in the air from the additional height.

Range

Substituting the time of flight into $R = (u \cos \theta)t$:

$$R = u \cos \theta \left(\frac{u \sin \theta + \sqrt{u^2 \sin^2 \theta + 2gh}}{g} \right)$$

This formula can be simplified but is often left in this form for calculation purposes.

Maximum height above launch point

The projectile reaches maximum height when $v_y = 0$:

$$v_y = u \sin \theta - gt_{\max} = 0$$

$$t_{\max} = \frac{u \sin \theta}{g}$$

The height above launch point is:

$$H = (u \sin \theta)t_{\max} - \frac{1}{2}gt_{\max}^2 = \frac{u^2 \sin^2 \theta}{2g}$$

This is the same as the formula we derived earlier. *The maximum height above the launch point depends only on the vertical component of initial velocity, regardless of the absolute elevation of the launch point.*

Keep in mind that the maximum height above the ground is $H + h$ (the height above launch plus the launch elevation).

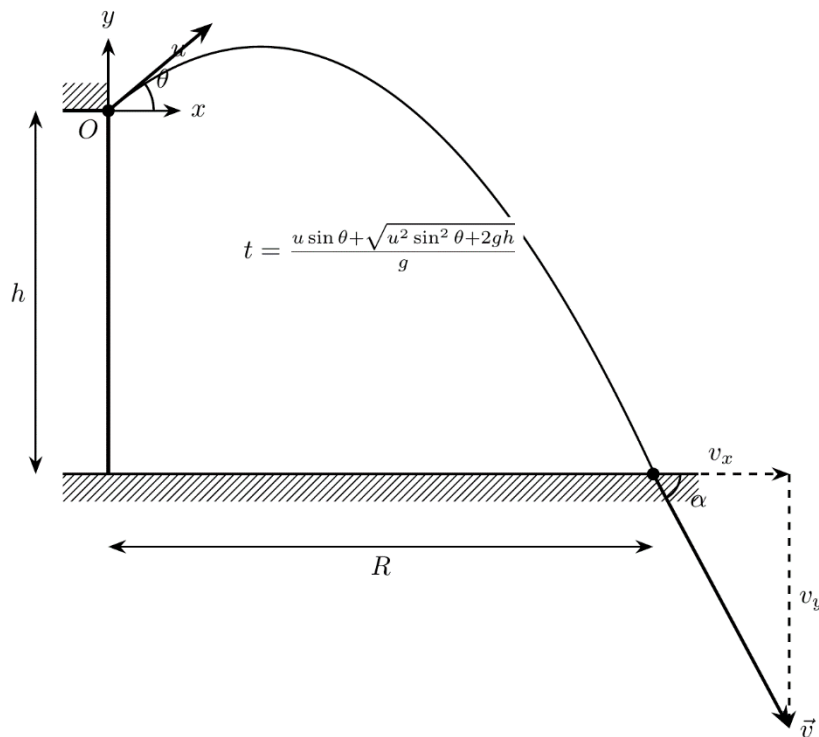


Figure: Projectile launched from a height h above the ground with initial velocity u at angle θ . The projectile follows a parabolic trajectory and lands at horizontal range R . The landing velocity has horizontal and vertical components v_x and v_y , making an angle α below the horizontal.

For now, give theory a short holiday and let us sharpen our understanding with some useful worked examples.

BINDER Example 12

A ball is rolled off a table 1.2m high with horizontal velocity 3m/s. Taking $g = 9.8 \text{ m/s}^2$; Calculate:

- the time taken for the ball to reach the floor,
- the horizontal distance from the table edge where the ball lands
- the velocity of the ball just before it hits the floor.

Solution

- (a) Time for the ball to reach the floor is given by:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1.2\text{m}}{9.8\text{m/s}^2}} = 0.495\text{s}$$

The time is 0.495s.

- (b) Horizontal distance is given by:

$$R = ut = 3\text{m/s} \times 0.495\text{s} = 1.49\text{m}$$

The horizontal distance is 1.49m.

- (c) Velocity components at landing:

$$v_x = u = 3\text{m/s} \text{ (unchanged)}$$

$$v_y = \sqrt{2gh} = \sqrt{2 \times 9.8\text{m/s}^2 \times 1.2\text{m}} = 4.85\text{m/s} \text{ (downward)}$$

$$\text{Magnitude: } v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(3\text{m/s})^2 + (4.85\text{m/s})^2} = 5.7\text{m/s}$$

Angle below horizontal:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{4.85}{3}\right) = 58.3^\circ$$

The velocity is 5.7m/s at 58.3° below horizontal.

Making Sense of the Answer: The ball takes half a second to fall 1.2 m, reasonable for free fall. The landing velocity (5.7m/s) exceeds initial velocity (3m/s) because gravity added vertical component of velocity.

Think Like a Physicist: In projectile motion, the horizontal and vertical components of motion are independent. Gravity affects only the vertical motion, which is why the fall time depends solely on the height (and of course the vertical component of velocity).

BINDER Example 13

A stone is thrown horizontally from the top of a cliff 45 m high with velocity 15m/s. Taking the value of $g = 9.8 \text{ m/s}^2$; calculate:

- how long the stone is in the air,
- how far from the base of the cliff the stone lands,
- the angle at which the stone strikes the ground.

Solution

- (a) Time of flight:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 45\text{m}}{9.8\text{m/s}^2}} = 3.03\text{s}$$

The stone stays in the air for 3.03s.

- (b) Range:

$$R = ut = 15\text{m/s} \times 3.03\text{s} = 45.45\text{m}$$

The stone lands 45.5m from the base of the cliff.

- (c) Landing velocity components:

$$v_x = 15\text{m/s}$$

$$v_y = \sqrt{2gh} = \sqrt{2 \times 9.8\text{m/s}^2 \times 45.45} = \sqrt{882} = 29.85\text{m/s}$$

Landing angle:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{29.85}{15}\right) = 63.4^\circ$$

The stone strikes at 63.4° below horizontal.

Making Sense of the Answer: The 3s fall time and 45.45m range are both substantial, reflecting the considerable cliff height. The steep landing angle (63°) shows the stone is falling quite rapidly by the time it reaches the ground (the vertical velocity (29.7m/s) is nearly double the horizontal velocity (15m/s)).

Think Like a Physicist: Compare this to dropping the stone straight down ($u = 0$). The fall time would be identical (3.03s), but the landing would occur at the cliff base ($R = 0$) instead of 45.45m away. The horizontal motion is "free"; it adds range without costing any flight time. This principle explains why supply drops from aircraft are so effective: the aircraft's forward velocity gives range without reducing the time available for parachutes to deploy.

BINDER Example 14

A projectile is launched at 25m/s at 37° above horizontal from a platform 5m above ground level. Taking $g = 9.8 \text{ m/s}^2$, calculate:

- (a) the time of flight,
- (b) the range,
- (c) the maximum height above the ground.

Solution

(a) Time of flight from height is given by:

$$t = \frac{u \sin \theta + \sqrt{u^2 \sin^2 \theta + 2gh}}{g} = \frac{25 \sin 37^\circ + \sqrt{(25 \sin 37^\circ)^2 + 2 \times 9.8 \times 5}}{9.8} = 3.37 \text{ s}$$

The time of flight is 3.37 s.

(b) Range is given by:

$$R = (u \cos \theta)t = 25 \cos 37^\circ \times 3.37 = 67.3 \text{ m}$$

The range is 67.3 m.

(c) Maximum height above the ground

$$= H + h = \frac{u^2 \sin^2 \theta}{2g} + h = \frac{(25 \sin 37^\circ)^2}{2 \times 9.8} + 5 = 16.55 \text{ m}$$

The height is 16.55 m.

Making Sense of the Answer: The time of flight (3.37 s) is longer than it would be for level ground launch ($T = 3.06 \text{ s}$) because the 5 m platform height provides extra airtime. The range (67.3 m) reflects this longer flight time combined with the horizontal velocity of about 20 m/s.

Think Like a Physicist: The maximum height above the launch point (11.55 m) depends only on the vertical velocity component, exactly as in level-ground projection. The 5 m platform height simply adds to this, giving 16.55 m total above ground. This clear separation of effects shows that the vertical motion is independent of the initial elevation.

REAL Example 15

A construction worker accidentally kicks a hammer off the roof of a building. At the same moment, another worker standing on the roof simply drops a similar hammer without giving it any horizontal push. Explain why both hammers reach the ground at the same time, even though one travels forward while the other falls straight down.

Solution

Both hammers reach the ground at the same time because the time taken to fall depends only on the vertical motion. This is due to the fact that in both cases, the vertical motion begins with zero vertical velocity and the only force acting vertically is gravity. Therefore, each hammer falls with the same acceleration and from the same height leading to equal time of fall. The horizontal velocity only changes the landing position, not the landing time.

Making Sense of the Answer: The kicked hammer moves forward while falling, but gravity acts only vertically. Therefore, both hammers fall for the same duration, even though one lands farther from the building.

Think Like a Physicist: Projectile motion can be separated into independent horizontal and vertical motions. Horizontal motion determines **where** the object lands, while vertical motion determines **when** it reaches the ground.

REAL Example 16

During a rescue operation, supplies are dropped from a moving helicopter flying horizontally over a disaster area. Explain why the supplies land some distance ahead of the point directly below the helicopter, even though they are simply released and not thrown forward.

Solution

When the supplies are released from the helicopter, they already possess the same horizontal velocity as the helicopter at that instant. After release, there is no horizontal force acting on the supplies (neglecting air resistance), so they continue to move forward with this horizontal velocity while falling under gravity.

Consequently, they land some distance ahead of the point directly below the helicopter at the moment of release.

Making Sense of the Answer: *Although the supplies are simply dropped, they already have the same forward velocity as the helicopter. During the fall they keep moving forward while gravity pulls them downward.*

Think Like a Physicist: *An object released from a moving body retains the horizontal velocity it had at the moment of release. Without a horizontal force to change it, that velocity remains constant while gravity controls the vertical motion.*

HOT Example 17

A basketball player shoots from 2m above the ground, releasing the ball at 7.5m/s at 50° above horizontal. The basket is 3.05 m above the ground and 5m away horizontally. Taking $g = 9.8 \text{ m/s}^2$, $\sin 50^\circ = 0.766$, $\cos 50^\circ = 0.643$:

- Calculate the time for the ball to reach the basket's horizontal position.
- Determine whether the ball passes through the basket or misses (too high or too low).
- Calculate the velocity of the ball when it reaches the basket location.
- If the shot misses the basket, determine by how much it misses and, without performing further calculations, suggest possible corrections.

Solution

- (a) The horizontal displacement after any time, t is given by: $x = (u \cos \theta)t$

Thus, the time to reach horizontal distance of 5m:

$$t = \frac{x}{u \cos \theta} = \frac{5\text{m}}{7.5\text{m/s} \times 0.643} = 1.04\text{s}$$

The time is 1.04s.

- (b) Height of ball at $t = 1.04\text{s}$ (measured from release point at 2m):

$$s_y = (u \sin \theta)t - \frac{1}{2}gt^2$$

$$s_y = (7.5\text{m/s} \times 0.766)(1.04\text{s}) - \frac{1}{2}(9.8\text{m/s}^2)(1.04\text{s})^2 = 0.67\text{m}$$

The vertical displacement is 0.67m above release point.

$$\text{Height above ground} = s_y + h_0 = 0.67\text{m} + 2\text{m} = 2.67\text{m}$$

The ball vertical position above the ground = 2.67m < 3.05m (basket position above the ground)

Hence, the ball **misses too low** (passes under the basket).

- (c) Velocity at $t = 1.04\text{s}$:

$$v_x = u \cos \theta = 7.5\text{m/s} \times 0.643 = 4.82\text{m/s}$$

$$v_y = u \sin \theta - gt = 7.5\text{m/s} \times 0.766 - 9.8\text{m/s}^2 \times 1.04 = -4.45 \text{ m/s (downward)}$$

$$\text{Magnitude: } v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(4.82\text{m/s})^2 + (-4.45\text{m/s})^2} = 6.56\text{m/s}$$

$$\text{Direction: } \alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{4.45}{4.82}\right) = 42.7^\circ \text{ below horizontal.}$$

The velocity is 6.56m/s at 42.7° below horizontal.

- (d) But basket height is 3.05m.

$$\text{Difference} = 3.05\text{m} - 2.67\text{m} = 0.38\text{m}$$

The shot misses by **0.38m low**.

Possible corrections:

- Increasing angle of projection.
- Increasing velocity of projection (initial velocity).

3. Launching the ball from higher position (increasing arm extension or jumping higher)

Making Sense of the Answer: The ball reaches correct horizontal position but is 38cm too low; a clear miss! By the time it arrives (1.04 s), it is already descending (v_y negative), having peaked earlier.

Think Like a Physicist: Basketball shooting is applied projectile motion. The 2-metre release height helps by reducing vertical distance to the 3.05m basket (only 1.05m to climb), making the trajectory less extreme. Professional players develop intuition through thousands of shots, but physics is unforgiving: wrong velocity or angle means a miss, regardless of skill.

HOT Example 18

A stone is thrown from the top of a cliff at 15m/s at 30° below the horizontal. The cliff is 40m high. Taking $g = 9.8 \text{ m/s}^2$, calculate:

- Calculate the time taken for the stone to reach the ground.
- Calculate the horizontal distance from the cliff base where the stone lands.
- Calculate the magnitude and direction of stone's velocity when it hits the ground.

Solution

Since the angle of projection is **below the horizontal**, the ball is thrown **downward**. Therefore, in the calculations the angle is taken as **negative**, according to the usual sign convention.

Components of initial velocity:

$$u_x = u \cos(-30^\circ) = 15 \text{ m/s} \times \cos(-30^\circ) = 13 \text{ m/s}$$

$$u_y = u \sin(-30^\circ) = 15 \text{ m/s} \times \sin(-30^\circ) = -7.5 \text{ m/s (downward)}$$

(a) Using vertical displacement equation:

$$s_y = u_y t - \frac{1}{2} g t^2$$

At $s_y = -40 \text{ m}$ (downward displacement is taken as negative)

$$-40 = -7.5t - \frac{1}{2}(9.8)t^2$$

$$4.9t^2 + 7.5t - 40 = 0$$

Solving the quadratic equation by using calculator gives the practical (positive) value of $t = 2.19 \text{ s}$

The time taken is 2.19s.

Alternative solution

By using formula (where $u \sin \theta = -7.5 \text{ m/s}$):

$$t = \frac{u \sin \theta + \sqrt{u^2 \sin^2 \theta + 2gh}}{g} = \frac{-7.5 + \sqrt{(-7.5)^2 + 2 \times 9.8 \times 40}}{9.8} = 2.19 \text{ s}$$

The time is 2.19s.

(b) The horizontal distance from the cliff base:

$$R = u_x t = 13 \text{ m/s} \times 2.19 \text{ s} = 28.47 \text{ m}$$

The horizontal distance is 28.47m.

(c) Final velocity components:

$$v_x = u_x = 13 \text{ m/s}$$

$$v_y = u_y - gt = -7.5 \text{ m/s} - 9.8 \text{ m/s}^2(2.19 \text{ s}) = -29 \text{ m/s}$$

$$\text{Magnitude: } v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(13 \text{ m/s})^2 + (-29 \text{ m/s})^2} = 31.8 \text{ m/s}$$

$$\text{Direction: } \alpha = \tan^{-1} \left(\frac{|v_y|}{v_x} \right) = \tan^{-1} \left(\frac{29}{13} \right) = 65.9^\circ \text{ below horizontal.}$$

Making Sense of the Answer: Starting with downward velocity (-7.5m/s) and falling for 2.19s adds significant vertical velocity, reaching -29m/s vertically. The steep landing angle (66°) shows the stone is falling rapidly, with vertical velocity more than double the horizontal velocity.

Think Like a Physicist: Throwing downward at 30° gives the stone an initial downward "head start" of 7.5m/s , reducing fall time compared to throwing horizontally (which would be about 2.86s). This trade-off between faster impact versus more horizontal distance, appears in many applications, from cliff diving to aerial delivery of supplies.

Projectiles on Inclined Planes

An entirely different special case arises when a projectile is launched from and lands on an inclined plane. This situation occurs in many practical contexts, for example: a skier launching from a jump on a slope, a ball thrown uphill along a mountain path, or supplies released onto an inclined conveyor belt. In such cases, the landing point does not lie on a horizontal surface. Instead, the projectile returns to the **sloping surface**, which introduces a natural asymmetry. Depending on the launch direction, the projectile may land "below" its launch point (**downhill projection**) or "above" it (**uphill projection**), measured along the incline.

To analyse this situation, consider an inclined plane that makes an angle β with the horizontal. A projectile is launched from a point on the plane with initial velocity u at an angle θ measured from the horizontal (not from the incline).

Two coordinate systems may be used for this analysis:

- 1) The horizontal and vertical axes (the standard x - y system).
- 2) Axes parallel and perpendicular to the incline.

Although both approaches are valid, we use standard x - y , which connects directly to our previous work and makes the mathematics more straightforward.

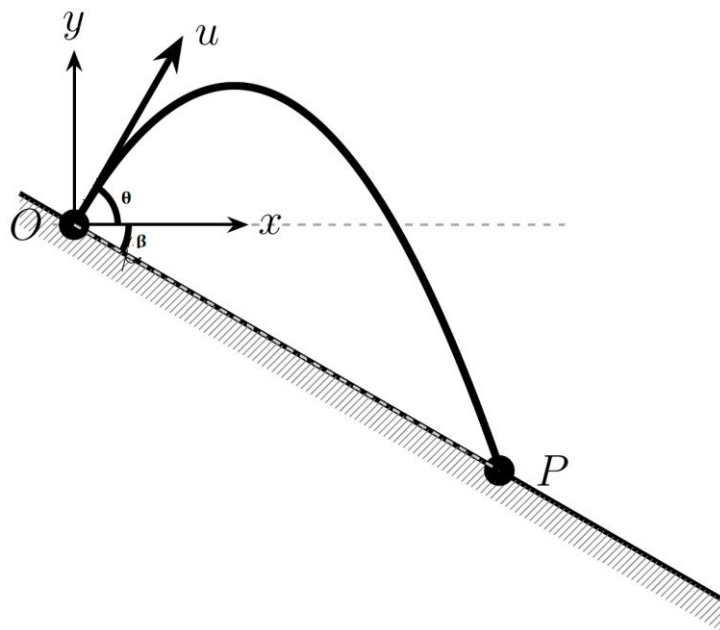


Figure: Downhill projection on an inclined plane. A projectile is launched with initial velocity u at an angle θ above the horizontal from a point on an incline that makes an angle β below the horizontal. The projectile follows a parabolic path and lands at point P on the slope.

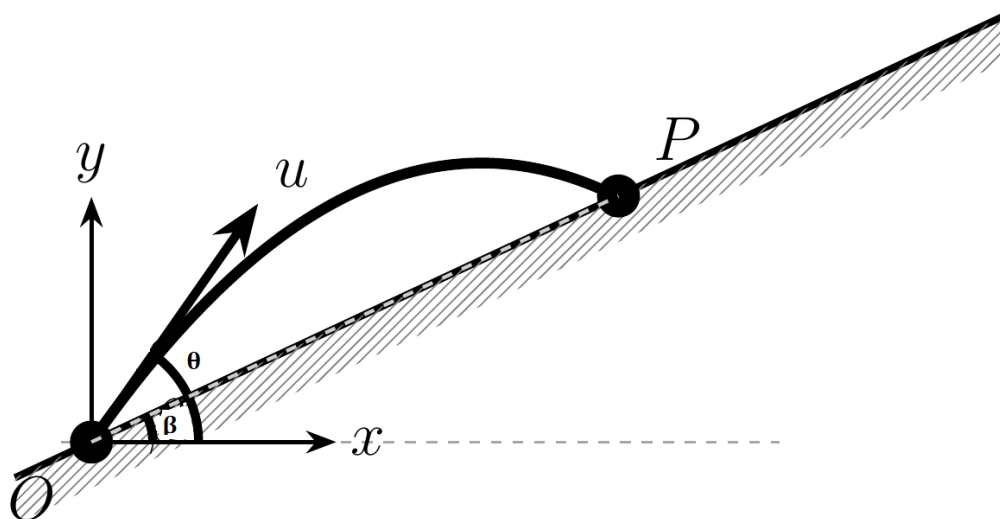


Figure: Uphill projection on an inclined plane. A projectile is launched with initial velocity u at an angle θ above the horizontal from a point on an incline that makes an angle β above the horizontal. The projectile follows a parabolic path and lands at point P on the slope.

From diagrams:

- θ is the **angle of projection** (between u and the horizontal axis).
- β is the **angle of the incline** relative to the horizontal.
- OP is the **range along the incline** (the distance measured along the sloping surface from the point of projection O to the landing point P). This distance will be denoted by L .

Important: Even though the projectile is launched from an inclined plane, the angle of projection is still defined relative to the horizontal, not relative to the slope. If the angle θ were measured from the incline itself, the corresponding angle of projection with respect to the horizontal would be $\theta - \beta$ for downhill projection and $\theta + \beta$ for uphill projection.

Range along the incline, L

The derivation and resulting formula are the same for both **uphill** and **downhill** projections. The only difference arises when applying the formula: for **uphill projection**, the angle β is taken as **positive** (above the horizontal), whereas for **downhill projection**, β is taken as **negative** (below the horizontal). Therefore, for the purpose of this derivation, we shall consider the **uphill case**.

Writing the incline equation

From the diagram:

$$\tan \beta = \frac{y}{x}$$

From which, the equation of the inclined plane is:

$$y = x \tan \beta \dots (i)$$

(It is equivalent to general equation of a straight line through the origin, $y = mx$, where m is the slope. Here the slope of the incline is $\tan \beta$).

Comparing with trajectory equation:

$$y = x \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) x^2 \dots (ii)$$

The projectile lands where it intersects the incline. At this point, the coordinates (x, y) must satisfy **both** equations simultaneously.

Thus, by equating (i) and (ii):

$$x \tan \beta = x \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) x^2$$

Dividing by x throughout (if $x \neq 0$)

$$\tan \beta = \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) x$$

Making x the subject:

$$x = \left(\frac{2u^2 \cos^2 \theta}{g} \right) (\tan \theta - \tan \beta)$$

Simplifying $(\tan \theta - \tan \beta)$

$$\tan \theta - \tan \beta = \frac{\sin \theta}{\cos \theta} - \frac{\sin \beta}{\cos \beta} = \frac{\sin \theta \cos \beta - \cos \theta \sin \beta}{\cos \theta \cos \beta}$$

Using the sine difference formula: $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$\sin \theta \cos \beta - \sin \beta \cos \theta = \sin(\theta - \beta)$$

Therefore:

$$\tan \theta - \tan \beta = \frac{\sin(\theta - \beta)}{\cos \theta \cos \beta}$$

Substituting back into expression for x

$$x = \left(\frac{2u^2 \cos^2 \theta}{g} \right) \left(\frac{\sin(\theta - \beta)}{\cos \theta \cos \beta} \right) = \frac{2u^2 \cos \theta \sin(\theta - \beta)}{g \cos \beta}$$

This is the **horizontal distance to the landing point**.

Finding the range L along the incline

Also from the diagram:

$$\cos \beta = \frac{x}{OP} = \frac{x}{L} \text{ or } x = L \cos \beta$$

Equating:

$$L \cos \beta = \frac{2u^2 \cos \theta \sin(\theta - \beta)}{g \cos \beta}$$

$$L = \frac{2u^2 \cos \theta \sin(\theta - \beta)}{g \cos^2 \beta}$$

This is the **range along the inclined plane**.

Simplifying the formula

$2 \cos \theta \sin(\theta - \beta)$ can be simplified by using the product-to-sum formula:

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

With $A = \theta$ and $B = \theta - \beta$:

$$2 \cos \theta \sin(\theta - \beta) = \sin(\theta + (\theta - \beta)) - \sin(\theta - (\theta - \beta)) = \sin(2\theta - \beta) - \sin \beta$$

Substituting into our expression for L:

$$L = \frac{u^2 (\sin(2\theta - \beta) - \sin \beta)}{g \cos^2 \beta}$$

Hence, the simplified formula for the **range along the inclined plane** is:

$$L = \frac{u^2}{g \cos^2 \beta} (\sin(2\theta - \beta) - \sin \beta)$$

This compact result is remarkably general. It applies equally well to both **uphill** and **downhill** projections; the only difference lies in the sign of β . For an uphill incline β is positive, while for a downhill incline β is taken as negative as mentioned earlier.

A useful check on the correctness of this formula is obtained by considering the special case where the plane becomes horizontal. If $\beta = 0$, the incline disappears and the situation reduces to ordinary ground-level projection. Substituting $\beta = 0$ gives:

$$L = \frac{u^2}{g \cos^2 0} (\sin(2\theta - 0) - \sin 0) = \frac{u^2 \sin 2\theta}{g},$$

which is exactly the familiar **range formula for level ground**. This confirmation shows that the inclined-plane result is simply a natural extension of the standard projectile theory developed earlier.

Having obtained the expression for the range, we can now ask an important question: **for what angle of projection will the range along the incline be greatest?** Since the θ containing term $\sin(2\theta - \beta)$ determines the variable part of the expression (β is constant for given incline, so other factors with β are constant, they do not affect the maximum), the range is maximized when:

$$\sin(2\theta - \beta) = 1$$

This occurs when:

$$2\theta - \beta = 90^\circ$$

Solving for θ gives:

$$\theta = \frac{90^\circ + \beta}{2} = 45^\circ + \frac{\beta}{2}$$

Hence, the **optimal angle** for maximum range along the incline is given by:

$$\theta_{\text{optimal}} = 45^\circ + \frac{\beta}{2}$$

This result also connects beautifully with the familiar case of projection on level ground. When the incline disappears ($\beta = 0$), the expression reduces immediately to $\theta = 45^\circ$, the well-known angle that produces **maximum horizontal range** on a flat surface.

Time of flight

We can derive this by using the vertical displacement equation.

The vertical position at any time t is given by: $y = (u \sin \theta)t - \frac{1}{2}gt^2$

At landing (time $t = T$), the projectile is on the incline at horizontal position x : $y = x \tan \beta$

And also: $x = (u \cos \theta)T$

So: $y = (u \cos \theta)T \times \tan \beta$

The vertical displacement equation after time $t = T$ (at landing) is:

$$y = (u \sin \theta)T - \frac{1}{2}gT^2$$

Where: $y = (u \cos \theta)T \times \tan \beta$

Substituting:

$$(u \cos \theta)T \times \tan \beta = (u \sin \theta)T - \frac{1}{2}gT^2$$

Divide by T (if $T \neq 0$):

$$(u \cos \theta) \tan \beta = u \sin \theta - \frac{1}{2}gT$$

$$\frac{1}{2}gT = u \sin \theta - (u \cos \theta) \tan \beta = u \sin \theta - (u \cos \theta) \frac{\sin \beta}{\cos \beta} = u \sin \theta - \frac{u \cos \theta \sin \beta}{\cos \beta}$$

$$\frac{1}{2}gT = \frac{u(\sin \theta \cos \beta - \cos \theta \sin \beta)}{\cos \beta} = \frac{u(\sin(\theta - \beta))}{\cos \beta}$$

Hence, by making T the subject:

$$T = \frac{2u(\sin(\theta - \beta))}{g\cos\beta}$$

After a rather lengthy chain of algebra and trigonometric manipulation, it is easy for the main physical ideas to become lost in the mathematics. Before those ideas begin to drift away, let us pause for a moment and anchor them with a few carefully chosen worked examples. These examples will translate the formulas we have just derived into concrete situations, showing how the theory predicts the motion of projectiles on inclined surfaces and revealing the practical meaning behind the equations.

BINDER Example 19

A skier launches off a jump on a slope inclined at 30° below horizontal. The skier leaves the jump with velocity 20m/s at 25° above horizontal. Use $g = 9.8\text{ m/s}^2$ to calculate:

- how far down the slope (measured along the slope) the skier lands,
- the time the skier spends in the air before landing.

Solution

Interpreting the data:

$$\theta = 25^\circ \text{ (above horizontal).}$$

$$\beta = -30^\circ \text{ (negative for downhill).}$$

- Range along the incline is given by:

$$L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)$$

Substituting values:

$$L = \frac{(20\text{m/s})^2}{9.8\text{ m/s}^2 \times \cos^2(-30^\circ)}(\sin(2 \times 25^\circ + 30^\circ) - \sin(-30^\circ)) = 80.8\text{m}$$

The skier lands at distance of 80.8m down the slope.

- From $x = (u\cos\theta)t$

Just before landing:

- $t = T$
- $x = L\cos\beta$

$$\text{Thus: } (u\cos\theta)T = L\cos\beta$$

From which:

$$T = \frac{L\cos\beta}{u\cos\theta} = \frac{80.8\text{m} \times \cos(-30^\circ)}{20\text{m/s} \times \cos 25^\circ} = 3.86\text{s}$$

The time the skier spends in the air is 3.86s .

Alternative solution

By using the time of flight formula for an incline projection:

$$T = \frac{2u(\sin(\theta - \beta))}{g\cos\beta}$$

Substituting values:

$$T = \frac{2 \times 20\text{m/s} \times \sin(25^\circ + 30^\circ)}{9.8\text{ m/s}^2 \times \cos(-30^\circ)} = 3.86\text{s}$$

Making Sense of the Answer: The skier travels 80.8m along the slope in just 3.86s . The 25° launch angle is close to the optimal 30° for maximum range in 30° downslope, explaining the large range achieved.

Think Like a Physicist: The optimal angle for maximum range on a -30° inclined plane is: $45^\circ + \frac{-30^\circ}{2} = 30^\circ$. The skier's 25° is close, that is why good range was achieved. Interestingly, experienced ski jumpers naturally learn to use angles below 45° when jumping downhill. The mathematics simply confirms what skilled athletes discover through practice.

BINDER Example 20

A ball is thrown with velocity 18m/s up a hill inclined at 25° to the horizontal. Take $g = 9.8 \text{ m/s}^2$:

- Calculate the optimal angle of projection for maximum range up the slope,
- Calculate the maximum range achieved at this optimal angle,
- Compare this with the maximum range the ball would achieve on horizontal ground.

Solution

- The optimal angle is given by:

$$\theta_{\text{optimal}} = 45^\circ + \frac{\beta}{2}$$

Where, $\beta = 25^\circ$ (uphill, positive)

$$\theta_{\text{optimal}} = 45^\circ + \frac{25^\circ}{2} = 57.5^\circ$$

The optimal angle is 57.5° above the horizontal.

- Range along the incline is given by:

$$L = \frac{u^2}{g \cos^2 \beta} (\sin(2\theta - \beta) - \sin \beta)$$

For optimal angle, ($\theta = 57.5^\circ$), L becomes maximum.

Substituting values:

$$L_{\text{max}} = \frac{(18 \text{ m/s})^2}{9.8 \text{ m/s}^2 \times \cos^2(25^\circ)} (\sin(2 \times 57.5^\circ - 25^\circ) - \sin 25^\circ) = 23.2 \text{ m}$$

The maximum range is 23.2m.

- For horizontal ground, range is given by:

$$R = \frac{u^2 \sin 2\theta}{g}$$

The optimal angle of projection for maximum range is 45° .

$$R_{\text{max}} = \frac{(18 \text{ m/s})^2 \sin 90^\circ}{9.8 \text{ m/s}^2} = 33.1 \text{ m} > L_{\text{max}}(23.2 \text{ m})$$

The maximum range on horizontal ground is greater than that on an uphill inclined plane because, when moving uphill, the ball must overcome both **gravity** and the **rising slope**.

Making Sense of the Answer: The optimal 57.5° angle is steeper than the familiar 45° because the ball must climb the 25° slope. The uphill range (23.2m) is therefore smaller than the range on horizontal ground (33.1m), as expected; with the same initial velocity, a projectile cannot travel farther uphill than it can on level ground.

Think Like a Physicist: This illustrates how changing the reference frame (level ground versus inclined plane) changes the optimal strategy. On level ground, 45° maximizes horizontal distance. On a 25° upslope, the optimal angle shifts to 57.5° to maximize distance along that slope but the actual range achieved is still less than what is possible on flat ground. Physics respects the fundamental constraint: going uphill is harder than going on level ground.

REAL Example 21

At a ski training ground, Kipute notices that ski jumpers launching from a downward slope rarely jump at the familiar 45° angle taught in physics for maximum range on level ground. Instead, their launch angles appear noticeably smaller.

- Explain why the optimal launch angle on a downhill slope is less than 45° .
- Explain why ski jumpers can travel much farther (measured along the slope) than they could on flat ground with the same launch velocity.

Solution

- On a downhill slope, the ground falls away beneath the jumper, giving extra flight time. A flatter angle (less than 45°) maximizes horizontal distance while the falling ground provides the vertical component naturally. The optimal angle is $45^\circ + \frac{\beta}{2}$; for downhill β is negative, so optimal angle $< 45^\circ$.
- There two reasons that combine to give much greater range:
 - Falling time:** The falling ground gives the jumper extra time in the air compared to flat ground, allowing more horizontal distance to accumulate.
 - Measurement along slope:** Distance is measured along the slope, not horizontally. Since jumper lands at a much lower elevation, the sloped path from launch to landing is geometrically longer than just the horizontal distance.

Making Sense of the Answer: *The launch angle for downslope may appear quite shallow, but this is reasonable because the ground drops away beneath the projectile as it moves forward. This increases the time before it meets the surface again, allowing the projectile to travel a much greater distance along the slope.*

Think Like a Physicist: *This illustrates optimization in physics: what is "optimal" depends on constraints. On flat ground, 45° balances horizontal and vertical motion equally. On a downslope, the ground falling away changes the optimization; you do not need as much vertical velocity because the slope provides vertical descent. Nature finds the angle that maximizes the goal (range along slope) given the constraints (gravity + sloped landing). Engineers designing ski jumps use this physics to set ramp angles for maximum safety and distance.*

REAL Example 22

A student throws a ball with the same velocity both uphill and downhill on identical slopes (same angle magnitude, opposite directions). The ball travels much farther down the slope than up the slope.

- Explain why the downhill throw travels farther even though the slope angles have the same steepness.
- Explain why the optimal throwing angle is different for uphill versus downhill throws.

Solution

- The difference comes from flight time:

For downhill: The ground falls away, so the ball has more flight time before landing. The projectile gets extra time from the descending slope, and gravity helps the descent.

For uphill: The ground rises to meet the ball, cutting flight time short. The ball must fight both gravity and the rising slope.

Consequently, the same initial velocity and slope steepness, but downhill range is significantly larger than that of uphill because of dramatically different flight times.

- The optimal angle must balance vertical velocity against the slope direction:

For downhill: Ground falls away, so the aim is flatter angle which is **less than 45°** to maximize horizontal velocity. This is because, gravity and the falling slope together provide sufficient vertical descent time. This is justified by optimal angle formula, $\theta_{\text{optimal}} = 45^\circ + \frac{\beta}{2}$; for downhill β is negative, so optimal angle $< 45^\circ$.

For uphill: Ground rises, so the aim is steeper angle which is **greater than 45°** to give more vertical velocity to overcome both gravity and the rising slope. Again, this is justified by optimal angle formula, $\theta_{\text{optimal}} = 45^\circ + \frac{\beta}{2}$; for uphill β is positive, so optimal angle $> 45^\circ$.

Making Sense of the Answer: To make sense the difference, think in this way: downhill, you are working **with** gravity and the falling ground. Uphill, you are working **against** both. It is like the difference between running downhill versus uphill at the same speed; going down is much easier and you cover more ground. The physics quantifies this intuition precisely.

Think Like a Physicist: This problem demonstrates broken symmetry. The setup looks symmetric (same slope angle, just opposite directions), but gravity breaks the symmetry. In physics, apparent symmetries do not always produce symmetric results; you must identify what breaks the symmetry. Here, gravity's downward direction means "downhill" and "uphill" are fundamentally different, even if the slope angles match.

HOT Example 23

An agricultural irrigation system uses a water pump to spray water onto a sloped field. The nozzle is positioned at the bottom of the field. The field slopes upward at 15° to the horizontal. The water leaves the nozzle at 12m/s at an adjustable angle θ above horizontal. Taking $g = 9.8\text{ m/s}^2$:

- Calculate the nozzle angle that maximizes the distance up the slope where water lands.
- Calculate the maximum distance up the slope that can be irrigated.
- If the field extends 25m up the slope, determine whether this irrigation system can adequately water the entire field.
- Without performing further calculations, suggest possible modifications to reach the full 25m if needed.

Solution

- (a) The optimal angle is given by:

$$\theta_{\text{optimal}} = 45^\circ + \frac{\beta}{2}$$

Where, $\beta = 15^\circ$ (uphill, positive)

$$\theta_{\text{optimal}} = 45^\circ + \frac{15^\circ}{2} = 52.5^\circ$$

The optimal nozzle angle is 52.5° above the horizontal.

- (b) Maximum distance up the slope is found by range formula along the incline:

$$L = \frac{u^2}{g \cos^2 \beta} (\sin(2\theta - \beta) - \sin \beta)$$

For optimal angle, ($\theta = 52.5^\circ$), L becomes maximum.

Substituting values:

$$L_{\text{max}} = \frac{(12\text{m/s})^2}{9.8\text{ m/s}^2 \times \cos^2(15^\circ)} (\sin(2 \times 52.5^\circ - 15^\circ) - \sin 15^\circ) = 11.7\text{m}$$

The maximum distance up slope is 11.7m .

- (c) Field extends 25m , but system reaches only 11.7m :

$$\text{Shortfall} = 25\text{m} - 11.7\text{m} = 13.3\text{m}$$

Hence, the irrigation system **cannot** adequately water the entire field. It falls short by 13.3m .

- (d) Possible modifications are:

- Increase the water velocity** by using higher-power pressure pump.
- Reposition the nozzle** by moving it partway up the slope.
- Use multiple nozzles** placed at different positions along the slope.

Making Sense of the Answer: The 52.5° angle is steeper than 45° because of the upward slope. The 11.7m range is limited because we are throwing uphill, so the water must overcome both gravity and the 15° rising ground.

Think Like a Physicist: This demonstrates practical engineering constraints. The physics clearly shows the system is inadequate for a 25m field because no angle adjustment can extend 12m/s water beyond 11.7m on a 15° upslope. Physics sets fundamental limits; engineering works within or around them.

APPLICATIONS OF PROJECTILE MOTION

The parabolic paths we have studied mathematically appear throughout the natural and engineered world. From ancient battlefields to modern sports arenas, from water fountains to spacecraft trajectories, projectile motion shapes our physical reality. Here are some of the countless ways these principles manifest in real life:

1. Basketball and free throws

Professional players intuitively understand the 45° rule and how height affects range. Shot tracking technology now analyzes launch angle, velocity, and arc height in real-time, using projectile equations to predict whether a shot will score before the ball reaches the hoop. The optimal arc for a free throw is typically 52° due to the elevated hoop.

2. Artillery and military ballistics

For centuries, armies have relied on projectile motion to aim cannons and mortars. The maximum range at 45° on level ground was discovered empirically by gunners long before Newton formalized the mathematics. Modern artillery computers instantly solve the projectile equations, accounting for wind, air resistance, and terrain elevation to hit targets kilometers away.

3. Irrigation and Water Distribution

Agricultural sprinklers use projectile motion principles to distribute water evenly across fields. Engineers design nozzle angles and pressures to achieve specific ranges, often placing systems on slopes where the inclined plane formulas determine optimal configurations. The parabolic water arcs you see from fountains and garden sprinklers are pure projectile motion.

4. Long jump and athletic performance

Olympic long jumpers launch at angles between 20° – 25° (not 45° !) because they carry substantial horizontal velocity from their run-up. Sports scientists use projectile analysis to optimize takeoff angles for maximum distance. The same principles apply to triple jump, javelin throw, shot put, and discus, each requiring different optimal angles based on initial conditions.

5. Ski jumping and winter sports

Ski jump designers use downhill projectile formulas to create safe landing zones. Jumpers launch at angles around 30° (much less than 45°) because the downward-sloping landing hill provides the vertical component naturally. The world's longest ski jumps exceed 250 metres, with flight times over 7 seconds; all predictable using our inclined plane equations.

6. Firefighting and emergency response

Fire hoses project water in parabolic arcs to reach elevated windows and distant flames. Firefighters adjust nozzle angles to maximize horizontal range or achieve specific heights, applying projectile principles under pressure. Aerial firefighting aircraft drop water or retardant, calculating release points to account for the aircraft's velocity and the target's location below.

7. Video games and animation

Every video game with jumping, throwing, or shooting implements projectile motion equations in its physics engine. Game developers use the same formulas we have studied to create realistic ball trajectories, bullet paths, and character jumps. Angry Birds, basketball simulations, and first-person shooters all run projectile calculations thousands of times per second.

8. Goalkeeper strategies in football

When a goalkeeper kicks a football from their hands, they are solving a projectile problem: what angle gives maximum distance downfield? Professional keepers typically launch at 30° – 40° , balancing range against "hang time" that allows teammates to reach the landing zone. Goalkeepers intuitively account for wind and other factors, applying projectile principles.

9. Cliff diving and extreme sports

Professional cliff divers leap from heights exceeding 25 metres, with horizontal launch velocities that determine how far from the cliff they land. Divers must account for projection from height to ensure they

clear dangerous rocks below and land safely in deep water. A miscalculation of just one degree can mean the difference between safety and catastrophe.

10. Fireworks displays

Pyrotechnic shells launched from mortars trace parabolic paths skyward before exploding at their maximum height. Display designers calculate launch angles and velocities to create patterns at specific altitudes and locations. Large displays coordinate hundreds of projectiles, each following calculated trajectories to burst at precisely timed moments.

11. Volcanic eruptions and natural phenomena

Nature's most spectacular demonstrations of projectile motion occur during volcanic eruptions, where molten rock follows parabolic trajectories reaching hundreds of metres. Geologists use these paths to estimate eruption velocities and predict hazard zones. Even simpler phenomena like water spraying from a broken pipe, obeys the same mathematical laws we have mastered.

We have explored the theory, derived the formulas, and seen how projectile motion shapes the world around us. Now it is time to put these principles to work through a diverse collection of problems that will test your understanding and sharpen your problem-solving skills. Ready? Let us tackle some projectiles in miscellaneous worked examples!

MISCELLANEOUS WORKED EXAMPLES ON PROJECTILE MOTION

Example 24

- (a) Two projectiles are launched simultaneously from the same point with the same velocity but at complementary angles. Explain why they land at the same horizontal distance.
- (b) A goalkeeper kicks a ball at 20m/s at 35° above horizontal. Taking $g = 9.8 \text{ m/s}^2$:
 - (i) Calculate the maximum height reached by the ball.
 - (ii) Calculate the time the ball stays in the air.
 - (iii) Calculate the horizontal distance the ball travels.
 - (iv) Explain why goalkeepers often prefer angles around 30–35° rather than the theoretical optimum of 45°.

Solution

- (a) A projectile launched at the smaller angle, has greater horizontal velocity and thus moves forward faster but stays in the air for a shorter time. A projectile launched at the larger complementary angle moves forward more slowly but remains in the air for a longer time. These effects balance, so both land at the same horizontal distance.

This is supported by range formula $R = (u^2 \sin 2\theta)/g$ which depends on $\sin 2\theta$. For complementary angles θ_1 and θ_2 where $\theta_1 + \theta_2 = 90^\circ$, we have $2\theta_1 + 2\theta_2 = 180^\circ$. This means $2\theta_2 = 180^\circ - 2\theta_1$, so $\sin 2\theta_2 = \sin(180^\circ - 2\theta_1) = \sin 2\theta_1$. Since both angles produce the same $\sin 2\theta$ value, they give identical ranges.

- (b) The solution of each part is as follows:

- (i) Maximum height:

$$H = \frac{u^2 \sin^2 \theta}{2g} = \frac{(20\text{m/s})^2 \times \sin^2 35^\circ}{2 \times 9.8\text{m/s}^2} = 6.71 \text{ m}$$

The maximum height is 6.71m.

- (ii) Time of flight:

$$T = \frac{2u \sin \theta}{g} = \frac{2 \times 20\text{m/s} \times \sin 35^\circ}{9.8\text{m/s}^2} = 2.34 \text{ s}$$

The ball stays in the air for 2.34s.

- (iii) Horizontal range:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(20\text{m/s})^2 \times \sin 70^\circ}{9.8\text{m/s}^2} = 38.4\text{m}$$

The ball travels 38.4m horizontally.

- (iv) Although 45° gives maximum range mathematically, goalkeepers prefer $30^\circ - 35^\circ$ for practical reasons. Such lower trajectories have smaller maximum height and shorter time of flight, allowing the ball to reach teammates faster. As a result, opponents have less time to intercept, and teammates can more easily judge and control the descending ball with moderate downward velocity (higher trajectories lead to very high downward velocity of the ball). Therefore, the slightly reduced range as result of choosing slight smaller angle is acceptable for the tactical advantages gained.

Example 25

- (a) A ball is thrown from the edge of a cliff. Explain why throwing it at an angle above horizontal can produce greater horizontal range than throwing it horizontally with the same velocity.
- (b) From the top of a 20m building, stone A is thrown horizontally at 15m/s while stone B is thrown at 15m/s at 30° above horizontal. Taking $g = 9.8 \text{ m/s}^2$:
- Calculate the time taken for stone A to hit the ground.
 - Calculate the time taken for stone B to hit the ground.
 - Determine which stone lands farther and by how much.

Solution

- (a) When thrown at a **moderate** angle above horizontal from a height, the projectile rises first before falling, which increases total flight time compared to horizontal throw. Although the horizontal velocity component is reduced ($u \cos \theta < u$), the increased flight time outweighs the decreased horizontal velocity. So the projectile stays in the air longer, allowing it to travel farther horizontally despite moving horizontally at a slower rate.
- (b) The solution of each part is as follows:
- (i) Time of flight for horizontal projection at height is given by:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 20\text{m}}{9.8\text{m/s}^2}} = 2.02\text{s}$$

Stone A takes 2.02s to hit the ground.

- (ii) Time of flight from height at an angle is given by:

$$t = \frac{u \sin \theta + \sqrt{u^2 \sin^2 \theta + 2gh}}{g} = \frac{15 \sin 30^\circ + \sqrt{(15 \sin 30^\circ)^2 + 2 \times 9.8 \times 20}}{9.8} = 2.93\text{s}$$

Stone B takes 2.93s to hit the ground.

- (iii) Range of stone A:
 $R_A = ut = 15\text{m/s} \times 2.02\text{s} = 30.3\text{m}$
 Range of stone B:

$$R_B = (u \cos \theta)t = (15\text{m/s} \times \cos 30^\circ) \times 2.93\text{s} = 38.1\text{m}$$

$$\text{Difference} = R_B - R_A = (38.1 - 30.3)\text{m} = 7.8\text{m}$$

Stone B lands **7.8m** farther than stone A.

Example 26

- (a) A projectile is launched at angle θ from ground level. Explain why the projectile's speed at any height h during ascent equals its speed at the same height h during descent.
- (b) A tennis ball is served from height 2.4m with velocity 25m/s horizontally. The net is 12m away and stands 0.9m high. Taking $g = 9.8 \text{ m/s}^2$:
- Determine whether the ball clears the net, and if so, by what margin.
 - Calculate the total horizontal distance the ball travels before hitting the ground.

Solution

- (a) By energy conservation, at any height h , the gravitational potential energy is the same whether the projectile is ascending or descending. Since total mechanical energy remains constant (no air resistance), the kinetic energy at height h must be identical in both cases. Equal kinetic energy means equal speed (since $KE = \frac{1}{2}mv^2$).

(b) The solution for each part is as follows:

Calculating time to reach net's horizontal position:

$$t = \frac{x}{u} = \frac{12\text{m}}{25\text{m/s}} = 0.48\text{s}$$

Time to reach net is 0.48s.

Calculating the ball's height when it reaches the net:

$$\text{Decrease in height: } s_y = \frac{1}{2}gt^2 = \frac{1}{2} \times 9.8\text{m/s}^2 \times (0.48\text{s})^2 = 1.13\text{m}$$

$$\text{Height at net} = h_0 - s_y = 2.4\text{m} - 1.13\text{m} = 1.27\text{m above ground.}$$

The ball is at height of 1.27m when it reaches the net.

Comparing the ball's height versus the net's height:

$$\text{Net height} = 0.9\text{m} < \text{Ball height (1.27m)} \text{ (the ball will clear the net)}$$

$$\text{Clearance} = 1.27\text{m} - 0.9\text{m} = 0.37\text{m}$$

(i) Hence, the ball clears the net by margin of 0.37m.

The horizontal range is given by:

$$R = ut = u \times \sqrt{\frac{2h}{g}} = 25\text{m/s} \times \sqrt{\frac{2 \times 2.4\text{m}}{\frac{9.8\text{m}}{\text{s}^2}}} = 17.5\text{m}$$

(ii) The total horizontal distance the ball travels before hitting the ground is 17.5m.

Example 27

- (a) Explain the difference between speed and velocity for a projectile, using the example of a ball at its maximum height.
- (b) A cricket ball is thrown at 22m/s at 40° above horizontal from ground level. Taking $g=9.8\text{m/s}^2$:
- Calculate the ball's speed when it is 8m above ground on its way up.
 - Calculate the ball's speed when it is 8m above ground on its way down.
 - Calculate the ball's velocity (magnitude and direction) at maximum height.
 - Explain why the velocities at 8m (ascending vs descending) are different even though speeds are equal.

Solution

- (a) Speed is a scalar quantity (magnitude only) measuring how fast an object moves. Velocity is a vector quantity (magnitude and direction) describing both speed and direction. At maximum height, a projectile has zero vertical velocity component but non-zero horizontal velocity component, so it still has speed (equals horizontal component $u\cos\theta$) even though vertical velocity is zero. The velocity vector is purely horizontal at this point.

(b) Initial components:

$$u_y = u\sin\theta = 22\text{m/s} \times \sin 40^\circ = 14.15\text{ m/s}$$

$$u_x = u\cos\theta = 22\text{m/s} \times \cos 40^\circ = 16.85\text{ m/s (constant)}$$

(i) At $h = 8\text{ m}$ (ascending), using $v^2 = u^2 - 2gh$:

$$v_y^2 = (14.15\text{m/s})^2 - 2 \times 9.8\text{m/s}^2 \times 8\text{m}; v_y = 6.59\text{m/s (upward)}$$

$$\text{Total speed: } v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(16.85\text{m/s})^2 + (6.59\text{m/s})^2} = 18.1\text{m/s}$$

Speed at 8m going up is 18.1m/s.

(ii) At $h = 8\text{ m}$ (descending):

$$v_y = -6.59\text{m/s (downward, same magnitude)}$$

$$v_x = 16.85\text{m/s (unchanged)}$$

$$\text{Speed: } v = \sqrt{(16.85\text{m/s})^2 + (-6.59\text{m/s})^2} = 18.1\text{m/s}$$

Speed at 8m coming down is 18.1m/s.

$$(iii) \quad \text{At maximum height: } v_y = 0, v_x = 16.85\text{m/s}$$

Velocity magnitude = 16.85 m/s, direction = horizontal

At maximum height: velocity is 16.85m/s horizontally.

- (iv) Although speeds are equal (18.1m/s in both cases), velocities differ because velocity includes direction. Ascending: vertical component points upward; descending: vertical component points downward. The horizontal component remains the same, but the different vertical directions make the velocity vectors different even though their magnitudes (speeds) are identical.

Example 28

- (a) A package dropped from a moving aircraft follows a curved path as seen from the ground. Explain why the pilot sees it fall straight down.
 (b) A ball is thrown at 30m/s at 50° from ground level. Taking $g = 9.8 \text{ m/s}^2$: Calculate the time interval during which the ball is below a height of 20m.

Solution

- (a) This is because the motion is observed from different reference frames. From the ground observer's viewpoint, the package has horizontal velocity (inherited from the aircraft) and vertical velocity (from gravity), creating a curved path. From the pilot's viewpoint, both aircraft and package share the same horizontal velocity, so relative to the pilot, the package has zero horizontal motion and only falls vertically.

(b) *Calculating the time of flight*

$$T = \frac{2u \sin \theta}{g} = \frac{2 \times 30\text{m/s} \times \sin 50^\circ}{9.8} = 4.69\text{s}$$

Total time of flight is 4.69s.

Calculating the times at which the ball is at height 20m

$$s_y = (u \sin \theta)t - \frac{1}{2}gt^2$$

$$20 = (30 \sin 50^\circ)t - \frac{1}{2}(9.8)t^2$$

$$20 = 23.0t - 4.9t^2$$

$$4.9t^2 - 23t + 20 = 0$$

$$t_1 = 1.15\text{s (smaller time, ascending)}$$

$$t_2 = 3.54\text{s (larger time, descending)}$$

Calculating the time interval during which the ball is above 20m

$$\text{Time interval: } \Delta t = t_2 - t_1 = 3.54\text{s} - 1.15\text{s} = 2.39\text{s}$$

The ball remains above 20m for 2.39s.

Calculating the time of flight

$$T = \frac{2u \sin \theta}{g} = \frac{2 \times 30\text{m/s} \times \sin 50^\circ}{9.8} = 4.69\text{s}$$

Total time of flight is 4.69s.

Finding difference between the time of flight and the time interval above 20m

$$\text{Time interval below 20m} = \text{Total flight time} - \text{Time interval above 20m}$$

$$= T - \Delta t = 4.69\text{s} - 2.39\text{s} = 2.3\text{s}$$

The time interval during which the ball is below a height of 20m is 2.3s.

Example 29

- (a) A stone is thrown downward at an angle below horizontal from a cliff. Explain how this affects the range compared to throwing it horizontally.
- (b) During a match, a goalkeeper attempts to pass the ball directly to a teammate positioned 50m away on level ground. However, an opposing defender stands directly between the goalkeeper and the teammate, positioned 1.5m in front of the teammate. The goalkeeper can kick the ball with a maximum velocity of 24m/s. Take 9.8m/s^2 :
- Determine whether the goalkeeper can reach the teammate with a direct kick on level ground.
 - Calculate the two possible launch angles that would make the ball land exactly at the teammate's position.
 - For each angle found in (ii), calculate the maximum height reached by the ball.
 - If the defender is 1.8m tall and remains standing upright without jumping, determine which of the angles found in (ii) would allow the ball to pass over the defender's head before reaching the teammate.

Solution

- (a) Throwing the stone downward below the horizontal gives it an initial downward velocity, so it reaches the ground in a shorter time (reduced time of flight) than if it were thrown horizontally. At the same time, its horizontal component of velocity is reduced to $u\cos\theta$, which is less than u . Therefore, the stone moves forward more slowly and for a shorter time, so its horizontal range is smaller than for a horizontal throw.

- (b) The solution for each part is as follows:

- (i) Finding maximum possible range on the level ground (at 45°):

$$R_{\max} = \frac{u^2}{g} = \frac{(24\text{m/s})^2}{9.8\text{m/s}^2} = 58.8\text{m} > 50\text{m}$$

Since the maximum horizontal distance travelled by the ball (58.8m) exceeds the horizontal separation between the goalkeeper and the teammate (50m), the goalkeeper can reach the teammate with a direct kick.

- (ii) For level ground projection:

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\sin 2\theta = \frac{Rg}{u^2} = \frac{50\text{m} \times 9.8\text{m/s}^2}{(24\text{m/s})^2} = 0.8507$$

$$2\theta = \sin^{-1}(0.8507) = 58.25^\circ \text{ or } 180^\circ - 58.3^\circ = 121.75^\circ$$

$$\text{Hence, } \theta = 29.1^\circ \text{ or } 60.9^\circ$$

The two possible angles are 29.1° and 60.9° .

- (iii) For $\theta = 29.1^\circ$:

$$H_1 = \frac{u^2 \sin^2 \theta}{2g} = \frac{(24\text{m/s})^2 \times \sin^2 29.1^\circ}{2 \times 9.8\text{m/s}^2} = 6.93\text{m}$$

At 29.1° , maximum height is 6.93m.

- For $\theta = 60.1^\circ$:

$$H_2 = \frac{u^2 \sin^2 \theta}{2g} = \frac{(24\text{m/s})^2 \times \sin^2 60.9^\circ}{2 \times 9.8\text{m/s}^2} = 22.4\text{m}$$

At 60.9° , maximum height is 22.4m.

- (iv) Since the opposing defender is 1m in front of the teammate and between the goalkeeper and the teammate, the distance from goalkeeper to defender is $50\text{m} - 1.5\text{m} = 48.5\text{m}$.

$$\text{Using } t = \frac{x}{u\cos\theta} \text{ (From } x = (u\cos\theta)t \text{)}$$

Time taken by the ball to reach the defender when $\theta = 29.1^\circ$:

$$t = \frac{48.5\text{m}}{24\text{m/s} \times \cos 29.1^\circ} = 2.31\text{s}$$

Vertical displacement for $\theta = 29.1^\circ$:

$$s_y = h = u_y - \frac{1}{2}gt^2 = (24\text{m/s} \times \sin 29.1^\circ)2.31\text{s} - \frac{1}{2} \times 9.8\text{m/s}^2 \times (2.31\text{s})^2 = 0.81\text{m} < 1.8\text{m}$$

Since the height of the ball is less than the height of the defender, the ball will not pass over the defender's head.

Time taken by the ball to reach the defender when $\theta = 60.9^\circ$:

$$t = \frac{48.5\text{m}}{24\text{m/s} \times \cos 60.9^\circ} = 4.16\text{s}$$

Vertical displacement for $\theta = 60.9^\circ$:

$$s_y = h = u_y - \frac{1}{2}gt^2 = (24\text{m/s} \times \sin 60.9^\circ)4.16\text{s} - \frac{1}{2} \times 9.8\text{m/s}^2 \times (4.16\text{s})^2 = 2.52\text{m} > 1.8\text{m}$$

Since the height of the ball is greater than the height of the defender, the ball will pass over the defender's head.

Hence, the suitable angle is 60.9° .

Alternative solution for (b)(iv)

By using trajectory equation:

$$y = x \tan \theta - \left(\frac{g}{2u^2 \cos^2 \theta} \right) x^2$$

For $\theta = 29.1^\circ$:

$$y = 48.5 \tan 29.1^\circ - \left(\frac{9.8}{2 \times 24^2 \cos^2 29.1^\circ} \right) 48.5^2 = 0.82\text{m} \approx 0.81\text{m} \text{ as before.}$$

For $\theta = 60.9^\circ$:

$$y = 48.5 \tan 60.9^\circ - \left(\frac{9.8}{2 \times 24^2 \cos^2 60.9^\circ} \right) 48.5^2 = 2.57\text{m} \approx 2.52\text{m} \text{ as before.}$$

Example 30

- (a) A projectile is launched with a certain velocity from the ground. Explain why increasing the launch angle beyond 45° reduces the horizontal range on level ground.
- (b) Ball A is thrown from ground level with velocity of 20 m/s at angle 60° above horizontal. One second later, ball B is thrown from the same point at angle 45° above horizontal. Find the magnitude of initial velocity u_B so that both balls land at the same time. Take $g = 9.8 \text{ m/s}^2$.

Solution

- (a) Increasing the angle beyond 45° directs too much velocity upward (vertical component) and not enough forward (horizontal component). Although a larger launch angle results in a greater maximum height and hence a longer time of flight, the reduced horizontal velocity means it covers less horizontal distance during that time. The optimal 45° balances vertical and horizontal components to maximize range.
- (b) Time of light of Ball A (launched at $t = 0$):

$$T_A = \frac{2u_A \sin \theta_A}{g} = \frac{2 \times 20\text{m/s} \times \sin 60^\circ}{9.8\text{m/s}^2} = 3.54\text{s}$$

Ball A lands at $t = 3.54\text{s}$.

For ball B to land at the same time as ball A, it must land at absolute time $t = 3.54\text{s}$.

Since ball B is launched at $t = 1\text{s}$, (1s delay) its flight time is:

$$T_B = 3.54\text{s} - 1\text{s} = 2.54\text{s}$$

Again, using time of flight formula:

$$T_B = \frac{2u_B \sin \theta_B}{g}$$

From which:

$$u_B = \frac{gT_B}{2\sin \theta_B} = \frac{9.8\text{m/s}^2 \times 2.54\text{s}}{2 \times \sin 45^\circ} = 17.6\text{m/s}$$

For both balls to land simultaneously, the magnitude of initial velocity u_B must be 17.6m/s.

Example 31

- (a) On an inclined plane, explain why the optimal launch angle for maximum range is **not** 45° but shifts depending on the plane's slope.
- (b) A ball is thrown at 18m/s from the bottom of a hill sloping upward at 25° . Take $g = 9.8 \text{ m/s}^2$:
- Calculate the optimal launch angle (from horizontal) for maximum range up the slope.
 - Calculate this maximum range along the slope.
 - Calculate how high above the launch point the ball lands.

Solution

- (a) On level ground, 45° balances horizontal and vertical components optimally. On an upward slope at angle β , the ground rises to meet the projectile, shortening effective flight time. To compensate, the launch must be steeper to gain more vertical velocity and extend time aloft despite the rising ground. Thus the optimal angle increases by half the slope angle:

$$\theta_{\text{optimum}} = 45^\circ + \frac{\beta}{2}$$

Conversely, on a downward slope (β negative), the ground falls away, allowing flatter trajectories, and the optimal angle decreases by half the slope angle:

$$\theta_{\text{optimum}} = 45^\circ - \frac{|\beta|}{2}$$

Hence, the slope breaks the symmetry that makes 45° optimal on level ground.

- (b) For upward slope $\beta = 25^\circ$:

$$(i) \quad \theta_{\text{optimal}} = 45^\circ + \frac{\beta}{2} = 45^\circ + 12.5^\circ = 57.5^\circ$$

Optimal angle is 57.5° above horizontal.

- (ii) Maximum range is found from range formula along the incline:

$$L = \frac{u^2}{g \cos^2 \beta} (\sin(2\theta - \beta) - \sin \beta)$$

For optimal angle, ($\theta = 57.5^\circ$), L becomes maximum.

Substituting values:

$$L_{\text{max}} = \frac{(18\text{m/s})^2}{9.8\text{m/s}^2 \times \cos^2(25^\circ)} (\sin(2 \times 57.5^\circ - 25^\circ) - \sin 25^\circ) = 23.24\text{m}$$

The maximum range along the slope is 23.24m.

- (iii) **Vertical height gain h** is found from:

$$h = L \sin \beta \quad \left(\text{since } \sin \beta = \frac{h}{L} \right)$$

Substituting values:

$$h = 23.24\text{m} \times \sin 25^\circ = 9.82\text{m}$$

The ball lands 9.82m above launch point.

Example 32

- (a) A projectile's trajectory is parabolic and symmetric. Explain why the projectile spends equal time ascending and descending when launched from and landing on level ground.

- (b) Two projectiles are launched simultaneously from the same point at the same velocity 20m/s: projectile A on level ground at 45° , and projectile B down a 30° slope at optimal angle for that slope. Taking $g = 9.8 \text{ m/s}^2$:
- Calculate the range and flight time for projectile A on level ground.
 - Calculate the optimal angle and range for projectile B on the downslope.
 - Calculate the flight time for projectile B.
 - When projectile A lands, determine how far projectile B has travelled along the slope from the launch point. Hence calculate the additional distance along the slope that projectile B will travel before it lands.

Solution

- (a) By symmetry, the upward path mirrors the downward path. Starting from ground with vertical velocity $u \sin \theta$ (upward), the ball decelerates uniformly at rate g until $v_y = 0$ at maximum height. During descent from maximum height back to ground level, it accelerates uniformly at rate g from $v_y = 0$ back to $u \sin \theta$ (downward). Since both phases involve the same change in velocity ($u \sin \theta$) and the same acceleration (g), they take equal time.
- (b) The solution for each part is as follows:
- At 45° :

$$R_A = \frac{u^2}{g} = \frac{(20\text{m/s})^2}{9.8\text{m/s}^2} = 40.8 \text{ m}$$

$$T_A = \frac{2u \sin \theta}{g} = \frac{2 \times 20\text{m/s} \times \sin 45^\circ}{9.8\text{m/s}^2} = 2.89\text{s}$$

For projectile A:

Horizontal range is 40.8m,

Flight time is 2.89s.

- Projectile B (30° downslope, $\beta = -30^\circ$):

Optimal angle:

$$\theta_{\text{optimal}} = 45^\circ + \frac{\beta}{2} = 45^\circ - \frac{30^\circ}{2} = 30^\circ$$

Using:

$$L = \frac{u^2}{g \cos^2 \beta} (\sin(2\theta - \beta) - \sin \beta)$$

Substituting values:

$$L = \frac{(20\text{m/s})^2}{9.8\text{m/s}^2 \times \cos^2(-30^\circ)} (\sin(2 \times 30^\circ + 30^\circ) - \sin(-30^\circ)) = 81.63\text{m}$$

For projectile B:

Optimal angle is 30° ,

Horizontal range is 81.63m along slope.

- Flight time of projectile B:

$$T_B = \frac{2u \sin(\theta - \beta)}{g \cos \beta} = \frac{2 \times 20\text{m/s} \times \sin(30^\circ + 30^\circ)}{9.8\text{m/s}^2 \times \cos(-30^\circ)} = 4.08\text{s}$$

Flight time of projectile B is 4.08s.

- Projectile A lands at 2.89s. At this time, projectile B is still in flight since its time of flight (4.08s) is greater than 2.89s.

Horizontal position of B at $t = 2.89\text{s}$:

$$x = (u \cos \theta)t = (20\text{m/s} \times \cos 30^\circ) \times 2.89\text{s} = 50.05\text{m}$$

Displacement along the slope, l :

From:

$$\cos \theta = \frac{x}{l}; l = \frac{x}{\cos \theta} = \frac{50.05\text{m}}{\cos 30^\circ} = 57.8\text{m}$$

When A lands, B has travelled 57.8m along slope.

Remaining distance B will travel = $81.63\text{m} - 57.8\text{m} = 23.8\text{m}$

B will travel another 23.8m along the slope before landing.

Example 33

- (a) In long jump, athletes try to jump forward as far as possible. Does jumping very high help increase the horizontal distance? Explain.
- (b) A military aircraft is flying horizontally at a constant velocity of 60m/s at a height of 200m above a defended region. At the instant the aircraft passes directly above a defence base, it releases a bomb. At the same instant, the air-defence system launches an interceptor projectile from the ground with a velocity of 90m/s at an angle θ above the horizontal in order to destroy the bomb before it reaches the ground. Neglect air resistance and take $g = 9.8 \text{ m/s}^2$; determine:
- The angle of projection θ required for the interceptor to collide with the bomb in mid-air.
 - The time after release when the interception occurs.
 - The position where the collision occurs.

Solution

- (a) The effect of jumping very high on the achieved horizontal distance depends on the reason behind that high jump. If it is caused by a greater take-off (initial) velocity at the same angle, it helps to achieve greater horizontal distance. This can be seen from the relation: $R = 4H\cot\theta$, from which if the **angle θ is fixed**, then **$\cot\theta$ is constant**. Therefore, **$R \propto H$** . So in this case, a higher jump means a longer jump. However, if the athlete jumps higher by using a larger take-off angle, the horizontal component of velocity is reduced, and the horizontal distance (range) may decrease. Therefore, jumping higher helps only when it comes from greater take-off velocity without losing too much horizontal velocity.
- (b) The solution for each part is as follows:
- Horizontal positions must be equal at collision.

For bomb, b (released from aircraft): Bomb velocity = Aircraft velocity = 60m/s (horizontally):

$$x_b = 60t$$

For interceptor, i:

$$x_i = (90\cos\theta)t$$

At collision: $x_b = x_i$

$$60t = (90\cos\theta)t$$

$$\cos\theta = \frac{60}{90} = 0.6667$$

$$\theta = \cos^{-1}(0.6667) = 48.19^\circ$$

The required angle of projection is 48.19° .

- The vertical positions must also be equal. Therefore, if the bomb has descended through a vertical distance **b**, the interceptor must have risen through a vertical distance **200 – b**.

For bomb: $u_y = 0\text{m/s}$ (aircraft was flying horizontally), $s_y = -b$ (downward, negative)

Using $s_y = -\frac{1}{2}gt^2$ (for $u_y = 0\text{m/s}$)

Substituting values:

$$-b = -\frac{1}{2} \times 9.8 \times t^2$$

$$b = 4.9t^2 \dots (i)$$

For interceptor:

$$\text{Using } s_y = u_y t - \frac{1}{2}gt^2$$

Substituting values:

$$200 - b = (90 \sin 48.19^\circ)t - \frac{1}{2} \times 9.8 \times t^2$$

$$b = 200 + 4.9t^2 - (90 \sin 48.19^\circ)t \dots (ii)$$

At collision:

Equating (i) and (ii):

$$4.9t^2 = 200 + 4.9t^2 - (90 \sin 48.19^\circ)t$$

$$t = \frac{200}{90 \sin 48.19^\circ} = 2.99s$$

The interception occurs after 2.99s.

Alternative solution for (b)(ii)

The quicker method of solving (b)(ii) is by using relative motion concept:

$$\text{Relative vertical acceleration, } a_R = a_i - a_b = -9.8\text{m/s}^2 - (-9.8\text{m/s}^2) = 0$$

And;

$$\text{Relative initial vertical velocity, } u_R = u_i - u_b = 90 \sin 48.19^\circ - 0\text{m/s} = 90 \sin 48.19^\circ$$

Then, the equation $s_y = u_y t + \frac{1}{2}a_y t^2$ becomes:

$$s_y = u_R t + \frac{1}{2}a_R t^2$$

Substituting values:

$$200 = (90 \sin 48.19^\circ)t + 0$$

$$t = \frac{200}{90 \sin 48.19^\circ} = 2.99s$$

(iii) At $t = 2.99s$:

Horizontal position:

$$x = 60\text{m/s} \times 2.99s = 179.4\text{m}$$

Vertical position:

$$y = 200 - b = 200 - (\frac{1}{2})(9.8)(2.99)^2 = 156.2\text{m}$$

They meet at position (179.4m, 156.2m); that is, **179.4m horizontally from the defence base and 156.2m above the ground.**

Having enjoyed the full combination of ideas, it is time to sharpen our thinking; the Digging Deeper Exercise is ready in the next page.

DIGGING DEEPER EXERCISE 6

EXERCISE 6A: BINDER QUESTIONS

Question 1

A stone is thrown horizontally from a cliff. Explain why the horizontal velocity remains constant throughout the flight while the vertical velocity continuously increases.

Question 2

Explain why does a projectile trajectory form a parabola rather than any other curve?

Question 3

Explain why a projectile launched from a height travels farther horizontally than the same projectile launched at the same angle and velocity from ground level.

Question 4

At the highest point of a projectile's trajectory, explain why the velocity is not zero even though the vertical velocity component is zero.

Question 5

Explain why throwing a ball downward at an angle from a cliff results in shorter horizontal range than throwing it horizontally at the same velocity.

Question 6

Two identical balls are thrown simultaneously from the same cliff: one horizontally and one at 30° above horizontal, both with the same velocity. Explain which ball hits the ground first and why.

Question 7

A ball is thrown at certain angle from ground level. Explain why increasing the initial velocity increases both the maximum height and the horizontal range.

Question 8

Explain why a projectile's horizontal range on level ground cannot exceed $\frac{u^2}{g}$ regardless of launch angle.

EXERCISE 6B: REAL QUESTIONS

Question 9

Why does water from a fountain follow a curved path?

Question 10

Explain why a fountain designed to spray water to a specific distance on level ground requires choosing between two different nozzle angles.

Question 11

A firefighter directs a water jet from a hose to extinguish flames on the upper floor of a building. The flames are at the same horizontal distance whether the water is aimed at 30° or 60° above horizontal. However, the firefighter chooses the 30° angle. Explain the practical advantages of using the lower angle despite both angles reaching the same distance.

Question 12

During a basketball practice session, Kipanga attempts several three-point shots. After many misses, he becomes frustrated.

"Sir, I'm shooting with good power, but the ball keeps hitting the front rim!" **Kipanga** complains to Mr. Akilikubwa.

Mr. Akilikubwa observes, *"Your shots are too flat; you're using too much horizontal velocity and not enough arc."*

Kipute adds, "I notice that when you shoot flatter, the ball arrives at the basket while still rising or level. But when I shoot with more arc, the ball is descending when it goes through the hoop."

"Exactly right, Kipute," says **Mr. Akilikubwa.** "That's the key difference between a good shot and a miss."

Explain why a ball arriving at the basket while still rising or traveling horizontally is more likely to miss than a ball that is descending.

Question 13

A gardener uses a hose to water plants at various distances. He notices that tilting the nozzle slightly downward from horizontal reduces the distance the water travels. Explain why aiming downward decreases the range even though it seems like it should help the water "fall" toward the target faster.

Question 14

Mountain rescue teams sometimes need to drop supplies to stranded climbers. When dropping from a hovering helicopter, the package lands some distance away from the helicopter rather than directly below. Explain why this happens and how the pilot can ensure the package lands at the desired location.

Question 15

A stone falls from a tall bridge into a river below. A observer standing on the bridge sees the stone fall straight down. However, a person on a moving boat passing under the bridge at the moment of release sees the stone follow a curved path. Explain how both observers can be correct.

Question 16

In the sport of shot put, athletes must release the shot from within a throwing circle and the shot must land in a marked sector. Why do shot putters release at angles around 35° – 40° rather than 45° , which gives maximum range in theory?

EXERCISE 6C: HOT QUESTIONS

Take $g = 9.8\text{ms}^{-2}$

Question 17

A ball is projected from the ground with velocity 25ms^{-1} . A wall 15m high stands 40m away.

- (a) Derive the expression for the height of the projectile as a function of horizontal distance.
- (b) Determine the minimum angle of projection required for the ball to just clear the wall.

Question 18

A projectile is fired with velocity 40ms^{-1} toward a target located 50m horizontally away and 30m above the launch point. Determine the possible angles of projection that allow the projectile to hit the target.

Question 19

A projectile is launched with velocity 20ms^{-1} at an angle 45° above the horizontal toward a plane inclined at 30° above the horizontal.

- (a) Determine the horizontal distance from the point of projection where the projectile strikes the plane.
- (b) Hence determine the distance measured along the slope.

Question 20

A ball is dropped from rest from a height of 40m. At the same instant another ball is projected horizontally with velocity 15ms^{-1} from a point 30m away.

- (a) Determine the time taken for the horizontally projected ball to reach the vertical line beneath the falling ball.
- (b) Determine the vertical positions of both balls at that instant.
- (c) State whether the two balls collide.

Question 21

Projectile A is fired vertically upward with velocity 25ms^{-1} . At the same instant projectile B is fired from ground 40m away toward A with velocity 30ms^{-1} at 45° . Determine whether the two projectiles collide.

Question 22

A cart moves along a straight horizontal track away from the point of projection with a constant velocity of 5ms^{-1} . At the instant the cart is at a distance x from the launch point, a projectile is fired from the ground with velocity 25ms^{-1} at an angle of 45° to the horizontal. Find the distance x from the launch point at which the cart must be located at the instant of projection so that the projectile lands in the cart.

Question 23

A projectile launched from level ground with speed u can achieve the same range R with two different launch angles θ_1 and θ_2 (where $\theta_1 > \theta_2$). Let H_1 and H_2 be the maximum heights and T_1 and T_2 the times of flight for the two trajectories.

Prove the following four results:

- (a) $\theta_1 + \theta_2 = 90^\circ$
- (b) $\frac{T_1}{T_2} = \tan\theta_1$
- (c) $\frac{H_1}{H_2} = \tan^2\theta_1$
- (d) $H_1H_2 = R^2/16$

Question 24

A projectile is launched from level ground with speed u at angle θ above the horizontal.

- (a) Show that the condition for the maximum height H to equal the horizontal range R is $\tan\theta = 4$.
- (b) Calculate the launch angle.
- (c) If $u = 30\text{ms}^{-1}$, calculate the common value of H and R .

Question 25

Two footballers stand 60m apart on level ground. At the same instant, each kicks a ball toward the other. Player A kicks at 30° above the horizontal with speed 25ms^{-1} . Player B kicks at 60° above the horizontal.

- (a) Find the speed at which Player B must kick for the two balls to collide in mid-air.
- (b) Find the time and height of the collision.
- (c) Find the horizontal distance from Player A to the collision point.

Chapter 7

UNIFORM CIRCULAR MOTION

INTRODUCTION

The Miono Secondary School bus was returning from a geography field trip when disaster struck while negotiating a sharp bend at Kilimani Hill. The driver, perhaps distracted by Kipanga's loud singing from the back seats, entered the curve at excessive speed. Physics cared nothing for good intentions or desperate braking. The tires screeched, lost their grip on the asphalt, and the bus began its inevitable journey off the road; sliding sideways like a reluctant dancer, crashing through the wooden barrier, and finally coming to rest at a precarious angle in the roadside ditch.

Silence. Then chaos. Students screamed, bags tumbled, and someone's food container burst open, sending mandazi flying through the air in parabolic arcs that would have made Chapter 6 proud!

Miraculously, no one was seriously hurt. As Mr. Akilikubwa helped the shaken students get out of the bus, **Kipanga** stood there dusty and untidy. He asked in confusion, *"Sir, the driver was turning the steering wheel. I saw him. Why didn't the bus turn?"*

Mr. Akilikubwa pointed at the dramatic skid marks etched into the road surface. The marks told an unambiguous story: they curved initially, obediently following the road's bend, then suddenly straightened into a ruler-straight line heading directly toward the ditch. *"Kipanga, the driver did turn the steering wheel. But steering alone doesn't bend a vehicle's path. Look at those marks. See how they curve, then suddenly go straight! That's the moment the tires lost their grip."*

Kipute, brushing dirt from her uniform, studied the marks with scientific curiosity. *"So the bus wanted to go straight, but the road wanted it to curve?"*

"Exactly!" **Mr. Akilikubwa's** eyes lit up despite the circumstances. *"Every object moving in a straight line wants to continue in a straight line, remember Newton's first law? To force something into a curved path requires a continuous inward force pulling it toward the center of the curve. For a car on a road, that force comes from friction between tires and asphalt."*

He walked to where the skid marks transitioned from curved to straight. *"But friction has limits. At low speeds on gentle curves, no problem. But this curve is sharp, and we were going too fast. The required inward force exceeded what friction could provide. The instant that happened..."* he snapped his fingers *"...the tires slid instead of gripped, the inward force vanished, and the bus immediately obeyed Newton's first law. Straight line. Straight into the ditch."*

Kipanga looked nervously at the bent barrier. *"So... if we'd gone even faster?"*

"We'd have slid off even earlier in the curve," **Mr. Akilikubwa** said seriously. *"This is why speed limits exist on curves, Kipanga. This is why mountain roads have those warning signs with curved arrows and recommended speeds. Engineers calculate the maximum safe speed based on the curve's radius and the expected friction. Exceed that speed, and physics becomes unforgiving."*

As they waited for the rescue vehicle, **Mr. Akilikubwa** sketched a diagram in his notebook. **"Circular motion,"** he announced, *"is nature's most deceptive phenomenon. It looks simple! Just going around in circles. But beneath that apparent simplicity lies a profound truth: maintaining a circular path requires constant work, constant force! Satellites circle Earth at precisely calculated speeds because gravity provides relentless inward pull. Clothes stick to the drum of a spinning washing machine because the drum wall provides inward force."*

Kipanga nodded slowly, understanding dawning. *"So circular motion isn't the natural state. Straight-line motion is natural. Curves require force."*

"Precisely! You've just grasped the central insight of this entire chapter." **Mr. Akilikubwa** smiled. *"And you learned it from skid marks and a ditch instead of a textbook. Perhaps this accident was the universe's way of teaching physics."*

"I would have preferred the textbook, sir," **Kipanga** muttered, picking mandazi crumbs from hair.

Welcome to circular motion, where every curve is a battle against inertia, where every turn demands its tribute of force, and where the difference between a safe journey and a roadside ditch can be calculated with precision. In this chapter, we will discover why the moon does not fall despite Earth's gravity constantly pulling it inward, why racing cars lean dramatically into turns, why your wet clothes cling to the washing machine drum during spin cycle, and why that innocent-looking curve at Kilimani Hill requires respect, attention, and most importantly, the right speed. The bus has already taught us the consequences of getting it wrong. Now let us learn the mathematics of getting it right.

UNDERSTANDING UNIFORM CIRCULAR MOTION

Circular motion is the movement of an object along a circular path. You observe it everywhere, for example: a car turning a corner, a satellite orbiting Earth, clothes spinning in a washing machine, a stone whirled on a string, the moon circling our planet, or a ball rolling around the inside of a bowl.

The most important insight about circular motion is this: it is not natural. Left undisturbed, every object moves in a straight line at constant velocity accordance with Newton's first law. To force an object into a curved path requires continuous application of force. Remove that force, and the object immediately returns to straight-line motion, as our bus at Kilimani Hill dramatically demonstrated.

When an object moves along a circular path at constant speed, we call this **uniform circular motion**. The word "uniform" refers to the speed remaining constant; the object covers equal distances in equal times around the circle. A satellite orbiting Earth at constant altitude, a car maintaining steady speed around a circular track, or a DVD spinning at constant rotation rate all exhibit uniform circular motion.

But here lies a subtle trap that catches many students: *constant speed does **not** mean constant velocity*. This distinction is crucial to understanding circular motion.

Velocity in Circular Motion

Velocity is a vector quantity possessing both magnitude (speed) and direction. In circular motion, even when speed remains constant, the direction of motion continuously changes as the object moves around the circle. The velocity vector always points **tangent to the circle** at the object's current position, that is, perpendicular to the radius at that point.

Imagine a stone tied to a string being whirled in a horizontal circle. At every instant, the stone's velocity points along the tangent to its circular path. As the stone moves around the circle, this direction continuously rotates. The speed might be constant, but the direction changes every fraction of a second, which means the velocity vector is constantly changing.

This is why, when the string breaks, the stone does not continue moving in a circle or fall straight down; it flies off tangentially, following the direction of its velocity at the instant the string snapped. The bus at Kilimani Hill did exactly the same thing: when friction failed to provide the necessary inward force, the bus continued along the tangent to the curve, which happened to point straight into the ditch.

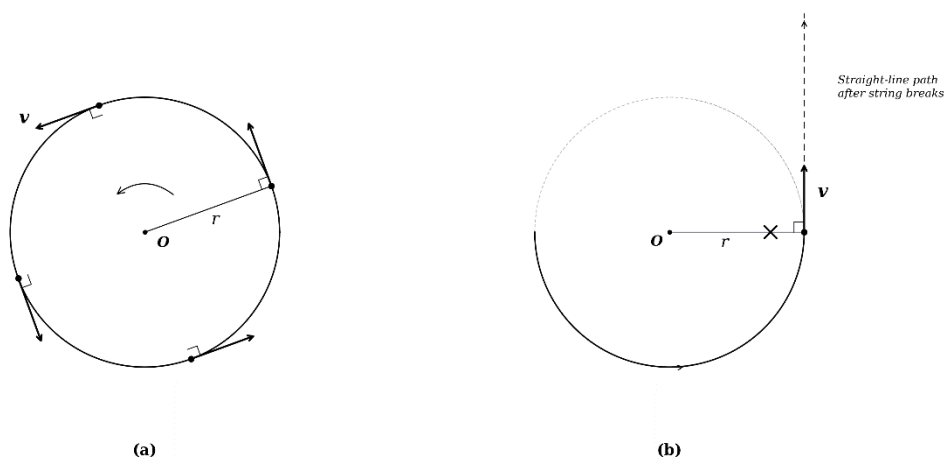


Figure: Velocity in circular motion: (a) Velocity is always tangent to the circular path and perpendicular to radius; speed is constant but direction changes continuously. (b) When string breaks, centripetal force vanishes and the object continues along the tangent (Newton's first law).

Acceleration in Circular Motion

If velocity is changing, acceleration must exist. This conclusion is inescapable. Acceleration is defined as the rate of change of velocity, and since velocity in circular motion continuously changes direction, there must be continuous acceleration.

This acceleration is always directed toward the centre of the circle. We call it **centripetal acceleration** (from Latin "centrum" meaning center and "petere" meaning to seek). The word literally means "center-seeking." Even though the object moves tangentially, the acceleration points radially inward.

This might seem strange. We usually think of acceleration as changing an object's speed, that is making it faster or slower. But acceleration can also change an object's direction while leaving speed unchanged. In uniform circular motion, the centripetal acceleration continuously deflects the velocity vector toward the center, bending the object's path into a circle while maintaining constant speed.

Centripetal Force

Newton's second law tells us that acceleration requires force: $F = ma$. If an object in circular motion experiences centripetal acceleration toward the center, there must be a force causing that acceleration. This force is called **centripetal force** which is *the inward force that maintains circular motion*.

Centripetal force is not a new type of force. It is simply whatever force happens to be pulling the object toward the center:

- **For a satellite orbiting Earth:** *gravity provides centripetal force.*
- **For a car turning a corner:** *friction between tires and road provides centripetal force.*
- **For a stone on a string:** *tension in the string provides centripetal force.*
- **For clothes in a spinning washing machine drum:** *the normal reaction from the drum wall provides centripetal force.*
- For our bus at Kilimani Hill: friction should have provided centripetal force, but the required force exceeded friction's capacity, so the bus slid off

The key insight: *circular motion requires continuous inward force. Remove or reduce that force, and circular motion immediately becomes straight-line motion along the tangent. This is not a flaw in the object or the motion; it is Newton's first law in action. Objects naturally move in straight lines. Only continuous inward force can maintain a curved path.*

With these foundational concepts established, let us explore some worked examples to deepen our understanding before we introduce the mathematical relationships in the next subtopic.

BINDER Example 1

A car moves at constant speed of 20m/s around a circular track. Explain whether the car is accelerating, and if so, in which direction.

Solution

Yes, the car is accelerating even though its speed is constant.

Acceleration is the rate of change of velocity, and velocity is a vector (magnitude and direction). Although the car's speed (magnitude of velocity) remains constant at 20m/s, the direction of the velocity continuously changes as the car moves around the circular track. At every instant, the velocity points tangent to the circle at the car's position. As the car moves, this tangent direction rotates, meaning the velocity vector is constantly changing.

Since velocity is changing, acceleration must exist. This acceleration points toward the centre of the circle; it is centripetal acceleration. The car is therefore continuously accelerating toward the centre of the circular track.

Making Sense of the Answer: *Acceleration does not only mean speeding up or slowing down. It also means changing direction. The car changes direction continuously as it goes around the circle, so it must be accelerating even though its speed stays constant.*

Think Like a Physicist: *Whenever you analyze circular motion, separate speed (scalar) from velocity (vector) in your mind. Constant speed in a circle always means changing velocity, which always means acceleration exists.*

BINDER Example 2

A stone tied to a string is whirled in a horizontal circle at constant speed. When the string suddenly breaks, the stone flies off in a straight line tangent to the circle rather than continuing to move in a circle or falling straight down. Explain why this happens.

Solution

While the string was intact, it provided tension force pulling the stone inward toward the center of the circle. This centripetal force continuously deflected the stone's velocity toward the centre, maintaining the circular path.

The moment the string broke, the centripetal force vanished. With no centripetal force acting on the stone, Newton's first law applies: an object in motion continues in a straight line at constant velocity unless acted upon by a force. The stone's velocity at the instant of breaking pointed tangent to the circle. With no force to deflect it from this direction, the stone continued moving straight along this tangent line.

Hence, the stone does not continue in a circle because circular motion requires continuous centripetal force which disappeared when the string broke. It does not fall straight down because it has horizontal velocity that persists in the absence of horizontal forces.

Making Sense of the Answer: *The tangent is the direction the stone was moving at the instant the string broke. Without the string pulling it inward, nothing deflects it from this straight-line path, so it continues moving straight ahead.*

Think Like a Physicist: *When analyzing what happens when centripetal force disappears, always identify the velocity direction at that instant. The object will continue along that tangent line because that is the direction it was already moving.*

REAL Example 3

A passenger sits in a car taking a sharp bend at speed. Many people feel as if they are being “thrown outward” against the door. Explain what is actually happening using correct physics.

Solution

The passenger is not thrown outward. Instead, the car pushes the passenger inward. This inward push provides the centripetal force needed for circular motion. However, the passenger's body wants to continue moving in a straight line (Newton's first law). So the “outward” sensation is just the passenger's inertia resisting the inward force. There is no real outward force; the real force always points toward the centre of the curve.

Making Sense of the Answer: *We feel "thrown outward" in circular motion because our bodies want to go straight while something (car door, drum wall, etc.) pushes us inward to force us into the curve. The inward push feels like we are being thrown the opposite direction, but the real force is always inward.*

Think Like a Physicist: *When you feel pushed to the outside during circular motion, recognize this as evidence of the inward centripetal force acting on you. Your sensation of being thrown outward is your inertia resisting the inward force and not an actual outward force.*

HOT Example 4

A car travels around a circular roundabout. At point A, the car is heading north. At point B, one-quarter of the way around the circle, the car is heading east. The car maintains constant speed throughout.

- Explain why the car's velocity has changed between points A and B even though its speed has not changed.
- If the car's velocity at point A is 15m/s north and at point B is 15m/s east, determine the change in velocity between these two points (magnitude and direction).
- Explain why this change in velocity requires a force, and identify the force that provides it for a car on a roundabout.
- Explain in which direction this force must act.

Solution

- Velocity is a vector quantity with both magnitude (speed) and direction. Although the car's speed remains constant at 15 m/s, the direction changes from north at point A to east at point B. Since the direction component of velocity has changed, the velocity vector has changed, even though the magnitude (speed) stayed the same.

(b) Resolving velocities

For point A: $(v_A)_x = 0$, $(v_A)_y = 15\text{m/s}$ (north taken as the positive y-direction)

For point B: $(v_B)_x = 15\text{m/s}$, $(v_B)_y = 0$ (east taken as the positive x-direction)

Finding horizontal and vertical change in velocity

$$\Delta v_x = (v_B)_x - (v_A)_x = 15\text{m/s} - 0 = 15\text{m/s}$$

$$\Delta v_y = (v_B)_y - (v_A)_y = 0 - 15\text{m/s} = -15\text{m/s}$$

Finding magnitude and direction of change in velocity

$$\text{Magnitude: } \Delta v = \sqrt{(\Delta v_x)^2 + (\Delta v_y)^2} = \sqrt{(15\text{m/s})^2 + (-15\text{m/s})^2} = 21.2\text{m/s}$$

Direction: $\theta = \tan^{-1} \left(\frac{\Delta v_y}{\Delta v_x} \right) = \tan^{-1} \left(\frac{-15}{15} \right) = -45^\circ$ (The negative sign indicates the direction is **below the positive x-axis**).

Hence, the change in velocity is 21.2m/s directed southeast (45° south of east), pointing toward the center of the roundabout.

- (c) Newton's second law states that acceleration requires a net force to act on an object: $F = ma$. Since the car's velocity changes direction as it moves around the roundabout, the car has centripetal acceleration. Therefore, a force must act on the car. For a car moving around a roundabout, this force is provided by static friction between the tyres and the road. As the car turns, the tyres tend to move sideways relative to the road surface. Static friction prevents this sideways motion and acts toward the centre of the circular path. This frictional force provides the centripetal force required to keep the car moving along the curved path.
- (d) The force must act toward the centre of the roundabout. This is because the required acceleration (centripetal acceleration) points toward the centre, and force must act in the same direction as the acceleration it produces ($F = ma$).

Making Sense of the Answer: *The change in velocity points toward the centre because the car's path is being continuously bent inward. Friction must act in that same inward direction to produce this change. Without sufficient friction (wet road, icy conditions), the car cannot generate enough inward force and slides off the curve tangentially; just like our bus at Kilimani Hill.*

Think Like a Physicist: *In circular motion problems, always remember that force and acceleration point toward the centre, while velocity points along the tangent. These are perpendicular to each other at every instant. The centripetal force continuously changes the direction of the tangent velocity vector, bending the path into a circle without changing the speed.*

With these fundamental concepts established, that circular motion requires continuous inward force, that velocity is always tangential while acceleration is always radial, and that constant speed does not mean constant velocity; we are ready to develop the mathematical relationships that allow us to calculate these quantities precisely. The next subtopic introduces angular motion and the quantitative tools we need to solve circular motion problems.

ANGULAR MOTION

In the previous topic, we established the conceptual foundation of circular motion: velocity is tangential, acceleration is radial, and continuous inward force is required. Now we develop the mathematical tools to describe and calculate circular motion quantitatively. To do this, we introduce angular quantities which is a powerful alternative way of describing circular motion that often proves more convenient than linear quantities like distance and speed.

Radian Measure

Before discussing angular motion, we must understand the proper unit for measuring angles in physics: the radian.

You are already familiar with degrees: a full circle contains 360° , a right angle is 90° , and so on. Degrees are arbitrary; there is no fundamental reason why a circle should be divided into 360 parts rather than 100 or 1000. The radian, by contrast, emerges naturally from the geometry of the circle itself. By definition:

One radian is the angle subtended at the center of a circle by an arc length equal to the radius of the circle.

To understand it properly, imagine a circle of radius r . If you measure an arc along the circumference with length exactly equal to r , the angle at the center corresponding to that arc is one radian. This definition creates a natural relationship between arc length s , radius r , and angle θ (measured in radians):

$$\theta = \frac{s}{r}$$

From this, we can derive the conversion between radians and degrees. The full circumference of a circle is $2\pi r$. This circumference subtends a complete revolution, which is 360° or, in radians:

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians}$$

Therefore: **$2\pi \text{ radians} = 360^\circ$**

In physics, we almost always use radians rather than degrees when working with circular motion. The mathematics becomes cleaner, the formulas simpler, and the relationships more transparent. Unless explicitly stated otherwise, assume **angles in circular motion are measured in radians**.

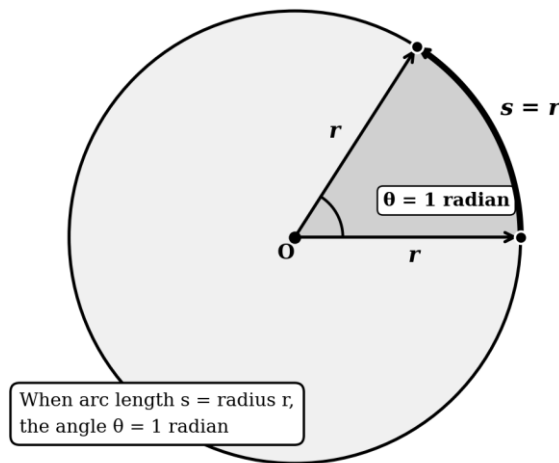


Figure: A circle of radius r with an arc of length $s = r$. The angle θ subtended at the centre O is exactly 1 radian. Since the full circumference equals $2\pi r$, a complete revolution equals 2π radians.

Angular displacement

When an object moves along a circular path from one position to another, we can describe its motion using either linear displacement (arc length travelled) or angular displacement (angle swept out).

Angular displacement (θ) is the angle through which an object has rotated about the centre of the circle. It is measured in radians.

If an object moves along an arc of length s on a circle of radius r , the angular displacement is:

$$\theta = \frac{s}{r}$$

Angular displacement is the same for all points on a rotating rigid object, regardless of how far they are from the centre. A point near the rim and a point near the hub of a wheel rotate through the same angle θ , even though the point on the rim travels a much longer arc length. This is one advantage of angular quantities as they describe the rotation of the entire object with a single number.

Angular velocity

Just as linear velocity describes how quickly linear position changes with time, angular velocity describes how quickly angular position changes with time. **Angular velocity (ω)** is the rate of change of angular displacement:

$$\omega = \frac{\theta}{t} \text{ (for motion starting from rest)}$$

Or more generally:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Angular velocity is measured in radians per second (rad/s). It tells us how many radians the object rotates through per unit time. A wheel rotating at $\omega = 10 \text{ rad/s}$ sweeps out 10 radians every second. Since **2π radians equals one complete revolution**, this wheel completes $10/(2\pi) \approx 1.59$ revolutions every second.

Angular velocity is a vector quantity, with direction along the axis of rotation, but for our purposes in uniform circular motion, we focus on its magnitude, which tells us the rate of rotation.

Relationship between linear and angular velocity

Here is the crucial connection between the linear and angular velocity of circular motion:

$$\mathbf{v} = \mathbf{r}\omega$$

This beautifully simple equation relates:

- v : linear velocity (in m/s)
- r : radius of the circular path (in m)
- ω : angular velocity (in rad/s)

Derivation:

If an object moves along an arc of length s on a circle of radius r , the angular displacement is:

$$\theta = \frac{s}{r}$$

From which:

$$s = r\theta$$

The linear velocity (v) is displacement divided by time:

$$v = \frac{s}{t} = \frac{r\theta}{t}$$

But $\frac{\theta}{t} = \omega$

Hence:

$$\mathbf{v} = \mathbf{r}\omega$$

This relationship reveals a profound truth: *all points on a rotating rigid object have the same angular velocity ω , but points farther from the centre (with greater value of r) have greater linear speed v .* A point on the outer edge of a spinning disk moves faster (covers more distance per second) than a point near the center, but both rotate through the same angle per second.

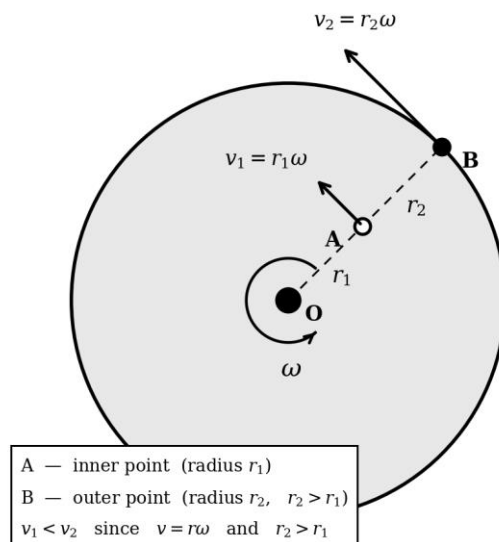


Figure: Two points A and B on a rotating rigid disk. Both rotate with the same angular velocity ω , but since $v = r\omega$, the outer point B (radius r_2) moves with greater linear velocity than the inner point A (radius r_1). The length of each velocity arrow reflects this difference.

Period and Frequency

For objects in uniform circular motion that complete regular revolutions, we can describe their motion using period and frequency.

Period (T) is the time required for one complete revolution. It is measured in seconds.

$$T = \frac{\text{Total time taken, } t}{\text{Number of revolutions, } n}$$

Frequency (f) is the number of complete revolutions per unit time. It is measured in hertz (Hz), where 1 Hz = 1 revolution per second = 1 s⁻¹.

$$f = \frac{\text{Number of revolutions, } n}{\text{Total time taken, } t}$$

Period and frequency are reciprocals of each other:

$$T = \frac{1}{f} \text{ and } f = \frac{1}{T}$$

Relating angular velocity to period and frequency

In one complete revolution, an object rotates through 2π radians. This takes time T (the period), and the angular velocity is:

$$\omega = \frac{2\pi}{T}$$

Alternatively, since $f = \frac{1}{T}$:

$$\omega = 2\pi f$$

These relationships allow us to convert easily between angular velocity, period, and frequency:

- Know the period? Calculate $\omega = \frac{2\pi}{T}$
- Know the frequency? Calculate $\omega = 2\pi f$
- Know the angular velocity? Calculate $T = \frac{2\pi}{\omega}$ or $f = \frac{\omega}{2\pi}$

Combined with $v = r\omega$, we can now express linear velocity in terms of period or frequency:

$$v = \omega r = r \left(\frac{2\pi}{T} \right) = \frac{2\pi r}{T}$$

Or

$$v = r\omega = r(2\pi f) = 2\pi r f$$

These formulas are powerful tools for analyzing circular motion problems.

Summary of key relationships

Let us consolidate the mathematical framework we have developed:

Arc length and angle:

$$s = r\theta \text{ (}\theta \text{ in radians)}$$

Angular velocity:

- $\omega = \frac{\Delta\theta}{\Delta t}$ (rad/s)
- $\omega = \frac{2\pi}{T}$
- $\omega = 2\pi f$

Linear and angular velocity:

$$v = \omega r$$

Linear velocity from period/frequency:

- $v = \frac{2\pi r}{T}$
- $v = 2\pi r f$

Period and frequency:

- $T = \frac{1}{f}$
- $f = \frac{1}{T}$

With these tools, we can solve a wide variety of circular motion problems. Let us practice applying them through worked examples.

BINDER Example 5

A wheel of radius 0.4m rotates at 5 revolutions per second. Calculate:

(a) The period of rotation. (b) The angular velocity. (c) The linear velocity of a point on the rim.

Solution

$$f = 5 \text{ rev/s} = 5\text{Hz}$$

(a) Period:

$$T = \frac{1}{f} = \frac{1}{5\text{s}^{-1}} = 0.2\text{s}$$

The period is 0.2s.

(b) Angular velocity:

$$\omega = 2\pi f = 2\pi \text{ rad} \times 5\text{s}^{-1} = 10\pi = 31.4 \text{ rads}^{-1}$$

The angular velocity is 31.4rad/s

(c) Linear velocity:

$$v = \omega r = 0.4\text{m} \times 31.4\text{rads}^{-1} = 12.6\text{m/s}$$

The linear velocity at the rim is 12.6 m/s.

Making Sense of the Answer: The wheel spins 5 times per second, so each revolution takes 0.2s. A point on the rim travels the circumference ($2\pi r = 2.51\text{m}$) in 0.2s, giving speed $v = \frac{2.51}{0.2} = 12.6\text{ms}^{-1}$, which matches our calculation.

Think Like a Physicist: Always check that your units work out correctly. Angular velocity must be in rad/s (not rev/s), and when you multiply ω (rad/s) by r (m), you get v (m/s). The "rad" unit effectively cancels because **radians are dimensionless** (they are a ratio of lengths).

BINDER Example 6

A point on the equator of the Earth moves in a circle as the Earth rotates. Taking the Earth's radius as 6400km and the period of rotation as 24 hours, calculate the linear velocity of a point on the equator.

Solution

$$\text{Using } v = \frac{2\pi r}{T};$$

$$\text{Where: } r = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}, T = 24 \text{ hours} = 24 \times 3600\text{s} = 86400\text{s}$$

Substituting values:

$$v = \frac{2\pi \times 6.4 \times 10^6 \text{m}}{86400\text{s}} = 465\text{m/s}$$

The linear velocity at the equator is 465m/s.

Making Sense of the Answer: This is about 1670 km/h, quite fast! Yet we do not feel this motion because we are moving with the Earth. This velocity is why launching rockets eastward from equatorial locations provides a "boost"; the rocket already has this velocity before its engines fire.

Think Like a Physicist: Always convert to standard SI units (metres and seconds) before calculating. Period in hours must become seconds: $24 \text{ h} \times 3600 \text{ s/h} = 86,400 \text{ s}$. Failing to convert units properly is a common source of errors in circular motion problems.

REAL Example 7

Two children sit on a merry-go-round: Amina sits 1m from the center, while Baraka sits 2m from the centre (at the outer edge). The merry-go-round rotates at constant angular velocity.

- Compare the angular velocities of the two children. Explain.
- Compare their linear velocities.

Solution

- Both children have the same angular velocity ω .** They are on the same rigid platform, so they both rotate through the same angle in the same time. When the platform completes one revolution, both Amina and Baraka complete one revolution together, even though they travel different distances.
- Using $v = r\omega$:
 Amina ($r = 1\text{m}$): $v_A = 1\omega$
 Baraka ($r = 2.0\text{m}$): $v_B = 2\omega$

Hence, **Baraka's linear velocity is twice Amina's linear velocity** (Baraka moves twice as fast as Amina in linear terms).

Making Sense of the Answer: Think of a spinning disk with dots painted on it. All dots complete one rotation together (same ω), but outer dots trace larger circles and therefore move faster (larger $v = r\omega$).

Think Like a Physicist: When analyzing rotating rigid objects, angular quantities (ω , θ) are the same for all points, but linear quantities (v , s) increase with radius. This is why ω is often more convenient than v for describing rotation; one number describes the entire object.

HOT Example 8

A DVD rotates in a player at 1800 revolutions per minute. The DVD has inner radius 2cm (where data begins) and outer radius 6cm (edge of data region). Take $\pi = 3.14$.

- Calculate the linear velocity at the inner radius.
- Calculate the linear velocity at the outer radius.
- Calculate how much faster (in percentage) the outer edge moves compared to the inner radius.
- Explain why DVD players must vary their rotation speed when reading from different parts of the disk to maintain constant linear read speed.

Solution

$$(a) f = 1800 \frac{\text{rev}}{\text{min}} \times \frac{1\text{min}}{60\text{s}} = 30\text{rev/s} = 30\text{Hz}$$

Angular velocity:

$$\omega = 2\pi f = 2 \times 3.14 \times 30 = 188.4 \text{ rad/s}$$

Linear velocity at inner radius:

$$v_{\text{inner}} = r_{\text{inner}} \times \omega = 0.02\text{m} \times 188.4\text{rad/s} = 3.77\text{m/s}$$

The linear velocity at the inner radius is 3.77m/s.

- Linear velocity at outer radius:

$$v_{\text{outer}} = r_{\text{outer}} \times \omega = 0.06\text{m} \times 188.4\text{rad/s} = 11.3\text{m/s}$$

The linear velocity at the outer radius is 11.3m/s.

$$(c) \text{ Speed difference} = v_{\text{outer}} - v_{\text{inner}} = (11.3 - 3.77)\text{m/s} = 7.53\text{m/s}$$

$$\text{Percentage increase} = \frac{\text{Speed difference}}{v_{\text{inner}}} \times 100\% = \frac{7.53}{3.77} \times 100\% = 200\%$$

The outer edge moves **200% faster** than the inner radius.

- If the angular velocity ω were constant, the linear speed $v = r\omega$ would vary dramatically from inner to outer radius. To maintain constant linear read speed:

- When reading inner tracks (small r), the player must spin the disk faster (larger ω) to achieve $v = r\omega = \text{constant}$.
- When reading outer tracks (large r), the player must spin slower (smaller ω).

Therefore, since $v = r\omega$ must stay constant, and r increases as we read outward, ω must decrease proportionally: $\omega \propto \frac{1}{r}$ for constant v .

Making Sense of the Answer: *Old vinyl records played at constant angular velocity, which meant outer grooves moved faster past the needle than inner grooves. This limited sound quality. CDs and DVDs solved this by varying angular velocity to maintain constant linear velocity, improving performance and allowing more efficient use of disk space.*

Think Like a Physicist: *The relationship $v = r\omega$ creates an inverse relationship between r and ω when v must remain constant: $v = \frac{v}{r}$. Doubling the radius requires halving the angular velocity to maintain the same linear velocity. This principle applies anywhere constant linear speed is needed despite varying radius: car transmissions, conveyor systems, and optical disk drives all use this concept.*

These angular quantities and relationships provide powerful tools for analyzing circular motion. In the next subtopic, we will use these tools to develop formulas for centripetal acceleration and centripetal force, allowing us to calculate the forces required to maintain circular motion and understand why certain speeds or radii make circular motion impossible without sufficient inward force.

CENTRIPETAL ACCELERATION AND FORCE

In the previous subtopics, we established that objects in circular motion experience acceleration toward the centre, and we developed the mathematical tools (angular quantities) to describe rotation. We have not yet answered the deeper question: *exactly how large is this centripetal acceleration? And what force is required to produce it?*

These are not merely academic questions. They are life-and-death questions.

When engineers design curved highways, they must calculate exactly how much friction is needed to keep cars on the road at various speeds. If the required force exceeds what friction can deliver, vehicles will slide off the road, just as our bus did at Kilimani Hill. When satellites are launched into orbit, mission controllers must determine precisely what velocity produces the right centripetal acceleration for a stable orbit. When you whirl a stone on a string, the tension you feel in your hand is the centripetal force, and if that tension exceeds what the string can handle, it snaps, sending the stone flying tangentially just as we saw earlier.

Understanding centripetal acceleration quantitatively transforms circular motion from an interesting concept into a powerful tool for solving real problems. To accomplish this, we need formulas. And those formulas begin with one central derivation: the magnitude of centripetal acceleration.

Deriving the Magnitude of Centripetal Acceleration

Earlier in this chapter, we established that an object in uniform circular motion is always accelerating toward the centre of the circle, even though its speed remains constant. We called this centripetal acceleration and understood its direction. Now we determine its magnitude.

The key question is: *If an object moves at constant speed v around a circle of radius r , how large is the centripetal acceleration?*

Setting up the problem

Consider an object moving at constant speed v along a circular path of radius r . At some instant, the object is at point A, moving with velocity $\mathbf{v}_A$. A short time Δt later, the object has moved to point B, where its velocity is $\mathbf{v}_B$. Both velocities have the same magnitude v (because speed is constant), but they differ in direction.

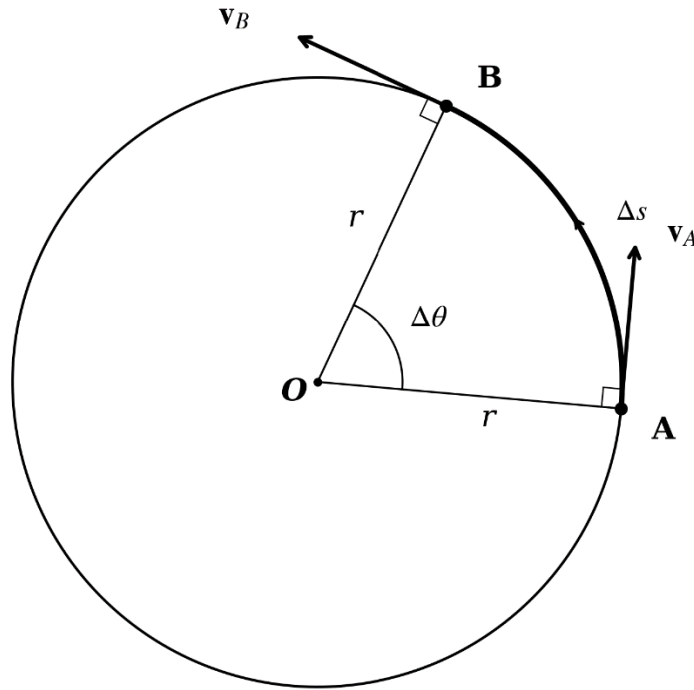


Figure (a): An object moves at constant speed along a circular path of radius r from point A to point B, sweeping through angle $\Delta\theta$ and covering arc length Δs . The velocity at each point is tangent to the circle and perpendicular to the radius.

Finding the change in velocity

The change in velocity is:

$$\Delta \mathbf{v} = \mathbf{v}_B - \mathbf{v}_A.$$

This difference $\Delta \mathbf{v}$ points toward the center of the circle.

To find $\Delta \mathbf{v}$, we use a velocity triangle. Place $\mathbf{v}_A$ and $\mathbf{v}_B$ tail-to-tail. Because both have magnitude v and the angle between them equals the angular displacement $\Delta\theta$ (the same angle the radius sweeps through), the velocity triangle is an isosceles triangle with two sides of length v and an included angle $\Delta\theta$.

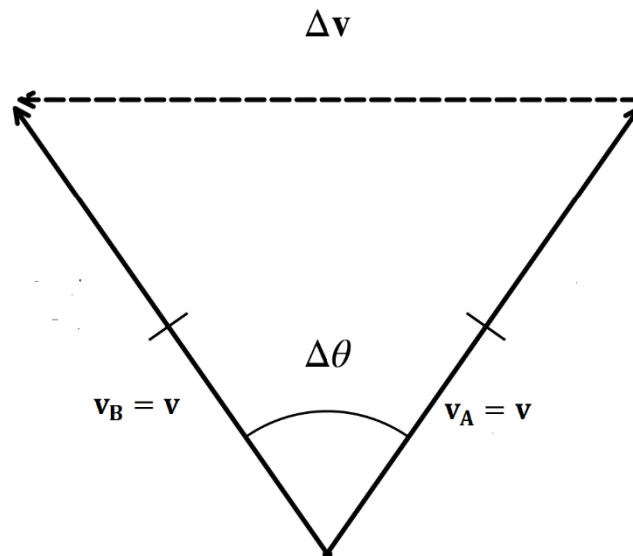


Figure (b): The velocity vectors from (a) placed tail-to-tail form an isosceles triangle. Both sides have magnitude v , and the angle between them equals $\Delta\theta$. The change in velocity, $\Delta \mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$, is shown by the dashed arrow.

For small angles $\Delta\theta$ (measured in radians), the length of the arc connecting the tips of $\mathbf{v}_A$ and $\mathbf{v}_B$ approximately equals the straight-line distance, $\Delta\mathbf{v}$.

But; arc length, s = radius, $r \times$ angle, $\Delta\theta$

Where: $s = \Delta v$, $r = v$.

Thus:

$$\Delta v \approx v\Delta\theta \text{ (for small } \Delta\theta\text{)}$$

Now, dividing both sides by Δt :

$$\frac{\Delta v}{\Delta t} \approx v \left(\frac{\Delta\theta}{\Delta t} \right)$$

As Δt approaches zero, $\frac{\Delta v}{\Delta t}$ becomes the instantaneous acceleration, and $\frac{\Delta\theta}{\Delta t}$ becomes angular velocity ω . Therefore:

$$a = v\omega$$

But we know that $v = r\omega$, so we can substitute:

$$a = (\omega r)\omega = \omega^2 r$$

This gives us one formula for centripetal acceleration:

$$\mathbf{a} = \omega^2 \mathbf{r}$$

Alternatively, we can eliminate ω using $\omega = \frac{v}{r}$:

$$a = r \left(\frac{v}{r} \right)^2 = r \left(\frac{v^2}{r^2} \right) = \frac{v^2}{r}$$

This gives us the more commonly used formula:

$$\mathbf{a} = \frac{\mathbf{v}^2}{\mathbf{r}}$$

Let us pause and appreciate what we have discovered. The centripetal acceleration depends on two factors:

- 1. Speed squared (v^2):** Doubling your speed quadruples the required centripetal acceleration. This is why *curves that feel comfortable at 30 km/h become terrifying at 60 km/h*; the required acceleration increases by a factor of four.
- 2. Inverse of radius:** Tighter curves (smaller r) require greater acceleration. This is why *a gentle highway curve can be navigated at high speed, but a sharp corner demands slowing down*.

These formulas explain why the bus at Kilimani Hill could not make the turn. The curve had a certain radius r , and the bus was traveling at speed v . The required centripetal acceleration was $a = \frac{v^2}{r}$. But friction could only provide a limited acceleration. When $\frac{v^2}{r}$ exceeded what friction could deliver, the tires lost grip and the bus slid off tangentially.

Direction of centripetal acceleration

The magnitude formulas $a = \frac{v^2}{r}$ and $a = \omega^2 r$ tell us how large the acceleration is, but they do not specify direction. We established earlier that centripetal acceleration points toward the centre of the circle, but let us see why this must be true.

The change in velocity $\Delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$ connects the tip of $\mathbf{v}_A$ to the tip of $\mathbf{v}_B$ when both are drawn from a common origin. As the time interval Δt becomes infinitesimally small, the angle $\Delta\theta$ approaches zero, and the chord connecting $\mathbf{v}_A$ and $\mathbf{v}_B$ becomes perpendicular to both velocity vectors. Since the velocity vectors are themselves tangent to the circle, $\Delta\mathbf{v}$ must be perpendicular to the tangent, which means it points along the radius.

Furthermore, since $\Delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$ represents the change needed to go from $\mathbf{v}_A$ to $\mathbf{v}_B$ as the object moves counterclockwise around the circle, and since $\mathbf{v}_B$ has rotated counterclockwise from $\mathbf{v}_A$, the difference vector points inward, toward the centre.

Therefore, *centripetal acceleration always points radially inward, toward the centre of the circle.*

This radial direction is crucial. It means centripetal acceleration is always perpendicular to velocity (which points tangentially). *A force perpendicular to velocity changes the direction of motion without changing speed* which is exactly what we need for uniform circular motion.

Centripetal force

Now we bring in Newton's second law: $\mathbf{F} = m\mathbf{a}$.

If an object of mass m experiences centripetal acceleration a toward the centre, there must be a net force F toward the centre producing that acceleration.

The centripetal force is:

$$\mathbf{F}_c = m\mathbf{a}_c = m\left(\frac{v^2}{r}\right) = m\omega^2 r$$

Let us be absolutely clear about what "centripetal force" means. *It is not a new type of force like gravity, friction, or tension. Centripetal force is simply **whatever** net force happens to be directed toward the centre of the circular path.* It is a descriptive term, not a fundamental force.

As mentioned earlier, different situations provide centripetal force in different ways:

Planetary orbits: Gravity provides centripetal force.

Car on curve: Friction between tires and road provides centripetal force.

Stone on string: Tension in the string provides centripetal force.

Satellite in orbit: Earth's gravity provides centripetal force.

Clothes in washing machine: Normal reaction force from drum wall provides centripetal force.

Electron around nucleus (classical model): Electric attraction provides centripetal force.

In each case, we simply identify what physical force acts toward the centre, then set it equal to the required centripetal force $\frac{mv^2}{r}$. This gives us an equation we can solve for unknowns like speed, radius, or the magnitude of the providing force.

What happens when centripetal force is insufficient?

The formula $F_c = \frac{mv^2}{r}$ tells us what force is **required** to maintain circular motion at speed v and radius r . *But what if the available force is less than this required amount?*

Consider our bus at Kilimani Hill. The maximum static friction force available between tires and road might be $F_{\max} = \mu_s R$, where μ_s is the coefficient of static friction and R is the normal reaction. For the bus to successfully navigate a curve of radius r , the required centripetal force $\frac{mv^2}{r}$ must not exceed this maximum:

$$\frac{mv^2}{r} \leq F_{\max}$$

If v is too large (excessive speed), or r is too small (sharp curve), or μ is too small (wet or icy road), this inequality is violated. When that happens, the available force cannot provide the necessary centripetal acceleration. The tires slip, the inward force decreases, and the vehicle slides outward; though "outward" is slightly misleading! What actually happens is that the vehicle continues along the tangent to the curve, which appears to be "outward" relative to the curve's centre.

This is why speed limits on curves exist. Engineers calculate the curve radius r and the typical friction coefficient μ for various road conditions, then determine the maximum safe speed as follows:

$$F_{\max} = \frac{mv_{\max}^2}{r}$$

For a car on a level road: $F_{\max} = \mu_s mg$

So $\mu_s mg = \frac{mv_{\max}^2}{r}$ or $\mu_s g = \frac{v_{\max}^2}{r}$

Making $v_{\max}$ the subject:

$$v_{\max} = \sqrt{\mu_s g r}$$

If you exceed this speed $\sqrt{\mu_s g r}$, you are demanding more centripetal force than friction can supply. Once this limit is crossed, the tyres can no longer maintain the necessary grip, and the vehicle will begin to slide outward. Eventually, you face the same consequence as the Miono Bus Kilimani Hill accident; a stark reminder that physics does not negotiate!

Connecting to angular velocity

Since we have two expressions for centripetal acceleration ($\frac{v^2}{r}$ and $\omega^2 r$), we similarly have two expressions for centripetal force:

$$F_c = \frac{mv^2}{r} = m\omega^2 r$$

Use whichever is more convenient for the problem at hand. Often, when dealing with rotating rigid objects (wheels, turntables, planets), angular velocity ω is given or easier to work with, making $F_c = m\omega^2 r$ more useful.

For example, every point on a rotating disk has the same ω but different v (since $v = r\omega$ increases with r). The centripetal force required increases with radius: $F_c = m\omega^2 r$. So *points farther from the centre need stronger centripetal force to maintain circular motion at the same angular velocity*. This is why *standing at the edge of a spinning merry-go-round is more dangerous than standing near the centre: **the required centripetal force is much greater at larger radii***.

Summary of key formulas

Let us consolidate what we have learned:

Centripetal acceleration:

- $a_c = \frac{v^2}{r}$ (when linear velocity v is known).
- $a_c = \omega^2 r$ (when angular velocity ω is known).
- Direction: always toward the centre (radially inward).

Centripetal force:

- $F_c = \frac{mv^2}{r}$ (when linear velocity v is known).
- $F_c = m\omega^2 r$ (when angular velocity ω is known).
- Direction: always toward the centre (radially inward).
- Not a new type of force; it is whatever net force acts toward the centre.

These formulas are the foundation for analyzing all circular motion problems. In the following subtopics, we will apply them to specific situations: objects on strings, conical pendulums, cars on curves, and objects in vertical circles. But the physics remains the same: identify what provides the centripetal force, set it equal to $\frac{mv^2}{r}$ or $m\omega^2 r$, and solve.

With this mathematical framework established, let us practice applying it through worked examples.

BINDER Example 9

A car of mass 1200kg travels around a circular track of radius 50m at constant speed 20m/s. Calculate: (a) The centripetal acceleration. (b) The centripetal force required.

Solution

(a) Centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{(20\text{m/s})^2}{50\text{m}} = 8\text{m/s}^2$$

The centripetal acceleration is 8m/s^2 toward the center of the track.

(b) Centripetal force:

$$F_c = ma_c = 1200\text{kg} \times 8\text{m/s}^2 = 9600\text{N}$$

The required centripetal force is 9600 N toward the centre.

Making Sense of the Answer: The acceleration 8m/s^2 is less than $g = 9.8\text{m/s}^2$, so it is a comfortable turn; not too extreme. The force 9600N must be provided by friction between the tires and track. If friction cannot provide this (wet track, bald tires), the car will **slide off tangentially**.

Think Like a Physicist: It is better to specify direction when stating acceleration or force. " 8m/s^2 " is incomplete; " 8 m/s^2 toward the centre" is complete. In circular motion, the inward direction is as important as the magnitude.

BINDER Example 10

A stone of mass 0.5kg is tied to a string and whirled in a horizontal circle of radius 0.8m . The stone completes 4 revolutions per second. Calculate:

- (b) The angular velocity.
- (c) The centripetal acceleration.
- (d) The tension in the string.

Solution

- (a) Using:

$$\omega = 2\pi f$$

Where: $f = 4\text{ Hz}$

Substituting:

$$\omega = 2\pi \times 4\text{s}^{-1} = 8\pi = 25.1\text{rad/s}$$

The angular velocity is 25.1 rad/s .

- (c) Using:

$$a = \omega^2 r$$

Where: $\omega = 25.1\text{ rad/s}$, $r = 0.8\text{m}$

Substituting:

$$a = (25.1\text{rad/s})^2 \times 0.8\text{m} = 504\text{m/s}^2$$

The centripetal acceleration is 504m/s^2 .

- (d) The tension in the string provides the centripetal force. Therefore:

$$T = F_c = ma$$

Where: $m = 0.5\text{ kg}$, $a = 504\text{m/s}^2$

Substituting:

$$T = 0.5\text{kg} \times 504\text{m/s}^2 = 252\text{N}$$

The tension in the string is 252N .

Making Sense of the Answer: The centripetal acceleration is about 51 times the acceleration due to gravity! This shows how rapidly the required acceleration grows with angular velocity. Even a light stone (0.5 kg) demands 252N of tension, which is equivalent to supporting a mass of about 25.7 kg . This is why strings break when you whirl objects too fast.

Think Like a Physicist: When frequency or angular velocity is given instead of linear velocity, use $a = \omega^2 r$ directly. Converting to linear velocity first and then using $a = v^2/r$ gives the same answer but takes more steps.

BINDER Example 11

The moon orbits Earth at approximately constant distance $3.84 \times 10^8\text{m}$ from Earth's centre, completing one orbit in 27.3 days. Calculate:

(a) The moon's orbital speed (b) The centripetal acceleration of the moon.

Solution

(a) The moon travels a distance equal to the circumference $2\pi r$ in time T :

$$v = \frac{2\pi r}{T}$$

Where: $T = 27.3 \text{ days} = 27.3 \text{ days} \times 24 \text{ min/day} \times 3600 \text{ sec/min} = 2.36 \times 10^6 \text{ s}$

$$r = 3.84 \times 10^8 \text{ m}$$

Substituting values:

$$T = \frac{2\pi \times 3.84 \times 10^8 \text{ m}}{2.36 \times 10^6 \text{ s}} = 1020 \text{ m/s} \approx 1.02 \text{ km/s}$$

The moon's orbital speed is approximately 1020 m/s or 1.02 km/s.

(b) Centripetal acceleration is given by:

$$a_c = \frac{v^2}{r} = \frac{(1020 \text{ m/s})^2}{3.84 \times 10^8 \text{ m}} = 0.0027 \text{ m/s}^2$$

The moon's centripetal acceleration is 0.0027 m/s^2 toward Earth's centre.

Making Sense of the Answer: This tiny acceleration (about $1/3600$ of g) is Earth's gravitational pull on the moon at that distance. It is small because the moon is far away, but it's enough to continuously deflect the moon's path into a circle instead of a straight line. Without this acceleration, the moon would fly off tangent to its orbit.

Think Like a Physicist: This example shows that circular motion does not require large accelerations; it just requires continuous acceleration toward the centre. Even tiny centripetal acceleration can maintain circular motion if applied persistently. The moon has been held in orbit by this gentle 0.0027 m/s^2 for billions of years!

REAL Example 12

During a school trip to the Dar es Salaam amusement park, **Kipanga** sits on a spinning ride that rotates faster and faster. At first he enjoys it. But as the ride speeds up, he feels increasing pressure from the seat pushing against his back. He shouts to Kipute, "Something is pushing me outward!"

Kipute, sitting next to him, shouts back, "Nothing is pushing you outward, Kipanga. Think about what is really happening!"

Explain, why Kipanga feels pushed outward even though no outward force acts on him.

Solution

When the ride rotates, Kipanga must move in a circular path. This requires a centripetal force directed toward the centre, which is provided by the seat pushing against his back. However, due to inertia (Newton's first law), Kipanga's body tends to continue in a straight line. The seat must continuously push him inward to overcome this tendency. What he feels as being "pushed outward" is actually his inertia resisting the inward push of the seat. As the ride speeds up, the required centripetal force increases (since $F_c = m\omega^2 r$), so the seat pushes harder, making the sensation feel stronger. But no outward force exists; the real force is always directed inward toward the centre.

Making Sense of the Answer: What we call "feeling pushed outward" is actually feeling the inward force that maintains circular motion. It is the same sensation you feel when a car accelerates forward and the seat pushes your back. You feel "pushed backward," but the real force is forward. In both cases, the sensation is caused by inertia resisting the applied force.

Think Like a Physicist: Never write "centrifugal force" as a real force in your solutions. The outward sensation is a consequence of inertia, not a real force. In the reference frame of the ground (an inertial frame), only the inward centripetal force exists.

REAL Example 13

Kipanga, a Form Six student preparing for his physics practical session, decided to demonstrate circular motion using a simple setup. He tied a 0.5kg stone to a string and began whirling it in a horizontal circle of radius 0.8m in the school playground. However, the string he used could withstand a maximum tension of only 50N before breaking. Curious about the limits of the experiment, Kipanga wondered how fast he could safely whirl the stone without snapping the string.

- Explain what provides the centripetal force for the stone's circular motion.
- Calculate the maximum speed at which the stone can be whirled before the string breaks.
- Explain what happens to the stone if this maximum speed is exceeded.

Solution

- In this situation, tension in the string provides the centripetal force. The string pulls the stone inward toward the centre of the circle. This inward pull continuously deflects the stone's velocity, bending its path into a circle.
- At maximum speed, tension reaches its maximum value before breaking:

$$F_c = T_{\max} = \frac{mv_{\max}^2}{r}$$

Making $v_{\max}$ the subject:

$$v_{\max} = \sqrt{\frac{rT_{\max}}{m}}$$

Substituting values:

$$v_{\max} = \sqrt{\frac{0.8\text{m} \times 50\text{N}}{0.5\text{kg}}} = 8.94\text{m/s}$$

The maximum safe speed is approximately 8.9m/s.

- If the speed exceeds 8.9 m/s, the required centripetal force mv^2/r exceeds the maximum tension the string can provide (50N). The string cannot supply the needed inward force, so it breaks. At the instant of breaking, the centripetal force vanishes. With no force to deflect its path, the stone immediately obeys Newton's first law and continues moving in a straight line along the tangent to the circular path at the breaking point.

Making Sense of the Answer: *The faster you whirl the stone, the greater the required centripetal force (because $F \propto v^2$). Eventually, you demand more force than the string cannot provide. The string does not break because of "centrifugal force throwing the stone outward." It breaks because you are asking for more inward force than it can deliver.*

Think Like a Physicist: *When analyzing circular motion, always identify what provides the centripetal force (tension, friction, gravity, normal reaction, etc.), then set it equal to the required mv^2/r or $m\omega^2r$. This equation can then be solved for any unknown: speed, radius, force, or mass.*

HOT Example 14

A car of mass 1500 kg travels around a level (unbanked) circular curve of radius 80m. The coefficient of static friction between the tyres and the dry road surface is 0.65. Take $g = 9.8 \text{ m/s}^2$.

- Calculate the maximum speed at which the car can negotiate the curve without sliding.
- After rainfall, the coefficient of static friction drops to 0.30. Calculate the new maximum safe speed.
- Determine the percentage reduction in maximum safe speed caused by the wet road.

Solution

- For a car on a level curve, friction provides the centripetal force. The car slides when the required centripetal force exceeds the maximum static friction.

At maximum speed:

$$\frac{mv_{\max}^2}{r} = \mu_s mg \text{ or } \frac{v_{\max}^2}{r} = \mu_s g$$

Making $v_{\max}$ the subject:

$$v_{\max} = \sqrt{\mu_s g r}$$

Where: $\mu_s = 0.65$, $g = 9.8 \text{ m/s}^2$, $r = 80 \text{ m}$

Substituting:

$$v_{\max} = \sqrt{0.65 \times 9.8 \text{ m/s}^2 \times 80 \text{ m}} = \mathbf{22.6 \text{ m/s}}$$

The maximum safe speed on the dry road is 22.6 m/s (about 81.3 km/h).

(b) With $\mu_s = 0.30$:

$$v_{\max} = \sqrt{\mu_s g r}$$

Substituting:

$$v_{\max} = \sqrt{0.30 \times 9.8 \text{ m/s}^2 \times 80 \text{ m}} = \mathbf{15.3 \text{ m/s}}$$

The maximum safe speed on the wet road is 15.3 m/s (about 55.2 km/h).

(c) Percentage reduction:

$$\begin{aligned} \% \text{ reduction} &= \frac{v_{\text{dry}} - v_{\text{wet}}}{v_{\text{dry}}} \times 100\% \\ \% \text{ reduction} &= \frac{(22.6 - 15.3) \text{ m/s}}{22.6 \text{ m/s}} \times 100\% = 32.3\% \end{aligned}$$

The maximum safe speed is reduced by approximately 32%.

Making Sense of the Answer: Notice that the maximum safe speed depends on the square root of μ_s . So even though friction dropped by more than half (from 0.65 to 0.30), the speed dropped by only about a third. However, the required force depends on v^2 , so even a moderate speed increase beyond the safe limit causes a dramatic increase in the force demanded from the tyres.

Think Like a Physicist: The mass cancels in the equation for maximum safe speed: $v_{\max} = \sqrt{\mu_s g r}$

This means the maximum safe speed on a level curve does not depend on the mass of the vehicle. A heavy lorry and a light car have the same maximum safe speed on the same curve, assuming identical tyres and road conditions.

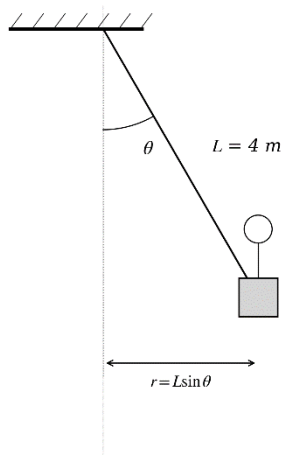
HOT Example 15

A fairground ride consists of chairs hanging from chains 4m long attached to a rotating circular platform. When rotating at constant speed, the chains make an angle of 30° with the vertical.

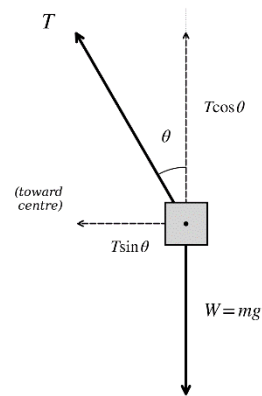
- Draw a free-body diagram for a chair with a person sitting in it during rotation.
- Explain what force provides the centripetal force for the circular motion.
- Calculate the radius of the circular path followed by the chair.
- Calculate the speed of rotation.

Solution

- Free-body diagram for chair + person (mass m):



(i) Physical setup



(ii) Free-body diagram

(b) The tension T has two components:

Vertical component: $T \cos 30^\circ$ (upward)

Horizontal component: $T \sin 30^\circ$ (toward centre of rotation)

Hence, **the centripetal force is provided by horizontal component of tension, $T \sin 30^\circ$.**

(c) From the geometry (length $L = 4.0$ m, angle $\theta = 30^\circ$):

The radius r is the horizontal distance from the rotation axis:

$$r = L \sin \theta = 4\text{ m} \times \sin 30^\circ = 2\text{ m}$$

The radius of the circular path is 2m.

(d) Vertical equilibrium (no vertical acceleration):

$$T \cos \theta = mg$$

$$T = \frac{mg}{\cos \theta}$$

Horizontally (centripetal force equation):

$$T \sin \theta = \frac{mv^2}{r}$$

Substituting $T = \frac{mg}{\cos \theta}$:

$$\left(\frac{mg}{\cos \theta} \right) \times \sin \theta = \frac{mv^2}{r}$$

Cancelling m and simplifying $\sin \theta / \cos \theta$:

$$g \tan \theta = \frac{v^2}{r}$$

From which:

$$v = \sqrt{rg \tan \theta} = \sqrt{2\text{ m} \times 9.8\text{ ms}^{-2} \times \tan 30^\circ} = 3.36\text{ m/s}$$

The speed of rotation is 3.36m/s.

Making Sense of the Answer: The tilted chain shows that the tension must both **support the combined weight of rider and chair**, and **pull the chair toward the centre of the circle**. The vertical component balances the weight mg , while the horizontal component provides the centripetal force. Because the chain is

inclined at 30° , the chair moves in a circular path of radius 2m. To maintain this angle during rotation, the ride must move at a speed of 3.36 m/s.

Think Like a Physicist: This problem shows how circular motion problems often involve both centripetal force equations and equilibrium in other directions. The vertical forces balance (no vertical acceleration), giving one equation. The horizontal force provides centripetal acceleration, giving another equation. Solving these simultaneously yields the motion parameters. This pattern appears in many circular motion scenarios: banked curves, conical pendulums, and vertical circles.

These examples demonstrate the power and versatility of the centripetal force equation. By identifying what physical force provides the centripetal force and setting it equal to the required mv^2/r , we can solve a vast array of circular motion problems. In the next subtopic, we will apply these principles specifically to horizontal circular motion, including detailed analysis of conical pendulums and banking.

HORIZONTAL CIRCULAR MOTION

In the previous subtopic, we developed the formulas for centripetal acceleration and centripetal force, and we applied them to several situations: a car on a track, a stone on a string, the moon orbiting the Earth, and even a fairground ride. Along the way, we derived the maximum safe speed on a level curve, $v_{\max} = \sqrt{\mu_s g r}$, and saw clearly why our bus at Kilimani Hill could not hold the road.

But those examples only introduced the ideas. In this subtopic, we examine horizontal circular motion in full detail. We will use free-body diagrams and Newton's second law applied carefully in two directions, just as we learned in Chapter 4. Every situation we meet here involves one central question: *what real force provides the centripetal force?* The answer changes from one situation to the next, but the method remains the same. Identify the forces, resolve them, and set the net inward component equal to $\frac{mv^2}{r}$.

We begin with the simplest case: a car negotiating a level curve.

Car on a Level (Unbanked) Curve

In the previous subtopic, we showed that the maximum safe speed on a level curve is $v_{\max} = \sqrt{\mu_s g r}$. We arrived at this result quickly by equating friction to centripetal force. Now we approach the same situation more carefully, using a proper free-body diagram and Newton's second law in both directions. This systematic method will serve us well when we meet more complex situations later.

Setting up the problem

Consider a car of mass m moving at constant speed v around a horizontal circular curve of radius r on a flat road. The road is not banked; it is perfectly level.

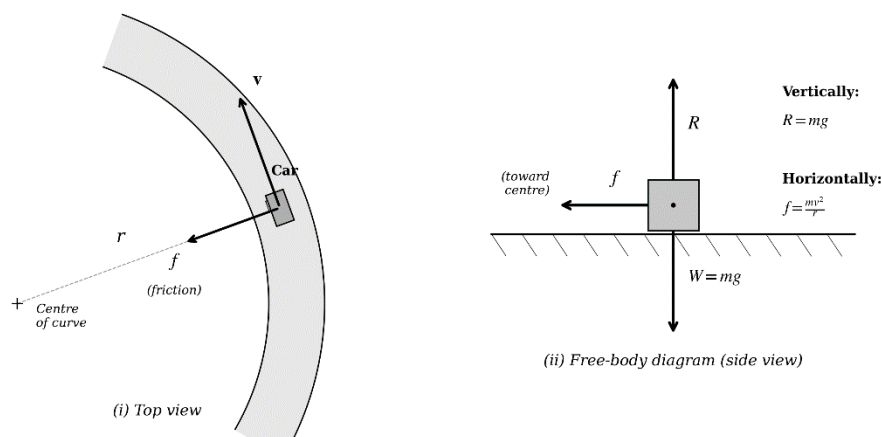


Figure: A car negotiating a level (unbanked) circular curve. (i) Top view showing velocity tangent to the path and friction directed toward the centre of the curve. (ii) Free-body diagram: the normal reaction R balances the weight mg vertically, while static friction f provides the centripetal force horizontally.

Identifying the forces

Three forces act on the car:

- (i) **Weight** ($W = mg$), acting vertically downward through the centre of mass.
- (ii) **Normal reaction** (R), acting vertically upward from the road surface, perpendicular to the road.
- (iii) **Static friction** (f), acting horizontally toward the centre of the circular path. This is the force that prevents the tyres from sliding sideways across the road surface.

Notice that *friction here acts sideways (toward the centre of the curve), not forward or backward along the direction of motion. Many students picture friction as always opposing motion along the road. In circular motion on a level surface, it is the sideways friction between tyre and road that bends the car's path into a curve.*

Applying Newton's second law

We apply Newton's second law in two perpendicular directions:

Vertically (no acceleration):

The car does not accelerate up or down, so the net vertical force is zero:

$$R - mg = 0$$

From which:

$$R = mg$$

Horizontally (toward the centre):

The only horizontal force is friction, and it acts toward the centre, providing the centripetal acceleration:

$$f = \frac{mv^2}{r}$$

This is the force that the road must supply to keep the car on the circular path. If the road cannot supply this force, the car slides.

The maximum safe speed

The maximum friction the road can provide is limited by:

$$f_{\max} = \mu_s R = \mu_s mg$$

The car remains on the curve as long as the required centripetal force does not exceed this maximum friction:

$$\frac{mv^2}{r} \leq \mu_s mg$$

At the maximum safe speed, these are equal:

$$\frac{mv_{\max}^2}{r} = \mu_s mg$$

Cancelling m and making $v_{\max}$ the subject:

$$v_{\max} = \sqrt{\mu_s gr}$$

It is important for you to understand that:

The mass m cancels completely from the equation. *This means the maximum safe speed on a level curve does not depend on the mass of the vehicle.* A fully loaded truck and an empty saloon car have exactly the same maximum safe speed, provided their tyres have the same coefficient of friction with the road. This surprises many students, but the physics is clear: *a heavier vehicle needs more centripetal force, but it also has more friction available (because $R = mg$ is larger), and these two effects cancel exactly.*

The result depends on three factors only:

- (iv) The coefficient of static friction μ_s between the tyres and the road (tyre condition, road surface, wetness).

- (v) The acceleration due to gravity g .
- (vi) The radius r of the curve (sharper curves are more dangerous).

With the ideas now set in place, let us anchor them through worked examples.

BINDER Example 16

A car moves at 15m/s around a horizontal circular curve of radius 60m on a level road. The coefficient of static friction between the tyres and the road is 0.5. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the centripetal acceleration of the car.
- (b) Determine whether the car can safely negotiate the curve at this speed.

Solution

- (a) Using:

$$a = \frac{v^2}{r}$$

Where: $v = 15\text{m/s}$, $r = 60\text{m}$

Substituting:

$$a = \frac{(15\text{m/s})^2}{60\text{m}} = 3.75\text{m/s}^2$$

The centripetal acceleration is 3.75m/s^2 .

- (b) The maximum centripetal acceleration that friction can provide is given by:

$$a_{\max} = \frac{v_{\max}^2}{r}$$

But from:

$$\begin{aligned} \frac{mv_{\max}^2}{r} &= \mu_s mg; \\ \frac{v_{\max}^2}{r} &= \mu_s g \end{aligned}$$

Hence:

$$a_{\max} = \mu_s g = 0.5 \times 9.8\text{m/s}^2 = 4.9\text{m/s}^2 > 3.75\text{m/s}^2 (\text{required})$$

Since the required centripetal acceleration (3.75m/s^2) is less than the maximum available (4.9m/s^2), the friction is sufficient. The car can safely negotiate the curve.

Alternative solution

By comparing velocities:

$$v_{\max} = \sqrt{\mu_s gr} = \sqrt{0.5 \times 9.8\text{m/s}^2 \times 60\text{m}} = 17.15\text{m/s} > 15\text{m/s} (\text{given})$$

Since the maximum allowable velocity is greater than the given velocity, the car can safely negotiate the curve.

Making Sense of the Answer: The car needs 3.75 m/s^2 of centripetal acceleration, but friction can provide up to 4.9 m/s^2 . There is a safety margin. However, if the speed increases or the road becomes wet (reducing μ_s), that margin shrinks. At some critical speed, the required acceleration equals the maximum available, and beyond that, the car slides.

Think Like a Physicist: Comparing accelerations or velocities is a quick way to check whether circular motion is possible without calculating forces. If $\frac{v^2}{r} > \mu_s g$ or $v_{\max} > v$, the car will slide.

REAL Example 17

During a road safety lesson, **Kipanga** argues: “A heavy truck is more dangerous on a curve than a small car, because the truck needs much more centripetal force and the road cannot provide it.”

Kipute disagrees: “The truck is heavier, but it also presses harder on the road, so friction is stronger too.”

Who is correct? Explain clearly.

Solution

Kipute is correct.

Explanation

A heavier vehicle does require more centripetal force to maintain circular motion, since $F_c = \frac{mv^2}{r}$ increases with mass. However, the maximum friction the road can provide also increases with mass, since $f_{\max} = \mu_s mg$.

At maximum speed:

$$\frac{mv_{\max}^2}{r} = \mu_s mg$$

The mass m appears on both sides and cancels, giving $v_{\max} = \sqrt{\mu_s gr}$, which is independent of mass.

Therefore, the maximum safe speed is the same for both vehicles, provided they have the same tyre-road friction coefficient and travel the same curve. The truck’s greater weight demands more force, but the road supplies more friction in exact proportion. The two effects cancel.

Making Sense of the Answer: This is why speed limit signs on curves do not show different limits for trucks and cars. The safe speed depends on the curve’s radius and the road condition, not on the vehicle’s mass.

Think Like a Physicist: Whenever mass cancels from a physics result, it signals something deep: the phenomenon does not depend on the amount of matter involved. This is the same reason all objects fall with the same acceleration under gravity (mass cancels in $mg = ma$). Here, mass cancels for a similar reason: both the required force and the available force are proportional to mass.

HOT Example 18

A driver travelling at 25m/s approaches a curve on a level road. The coefficient of static friction between the tyres and the road is 0.4. Take $g = 9.8\text{m/s}^2$.

- Calculate the minimum radius of curve the driver can safely negotiate at this speed.
- If the road curves with a radius of only 120m, determine whether the driver must slow down, and if so, find the maximum safe speed for that curve.
- The driver enters the 120m curve at 25m/s without slowing down. Calculate the centripetal force required and the maximum friction force available for a car of mass 1400kg. Use these values to explain what happens to the car.

Solution

- At the maximum safe speed:

$$\frac{mv_{\max}^2}{r_{\min}} = \mu_s mg$$

Making $r_{\min}$ the subject:

$$r_{\min} = \frac{v^2}{\mu_s g}$$

Where: $v = 25\text{m/s}$, $\mu_s = 0.40$, $g = 9.8\text{m/s}^2$

Substituting:

$$r_{\min} = \frac{(25\text{m/s})^2}{0.40 \times 9.8\text{m/s}^2} = 159.4\text{m}$$

The minimum safe radius is 159.4m.

- Since the actual radius (120m) is less than the minimum safe radius (159.4m), the driver **must slow down**.

The maximum safe speed for the 120m curve:

$$v_{\max} = \sqrt{\mu_s g r}$$

Substituting:

$$v_{\max} = \sqrt{0.40 \times 9.8 \text{ m/s}^2 \times 120 \text{ m}} = 21.7 \text{ m/s}$$

The maximum safe speed is 21.7m/s (about 78km/h).

So the driver must slow from 25m/s to at most 21.7m/s.

(c) Required centripetal force at 25m/s:

$$F_{\text{required}} = \frac{mv^2}{r} = \frac{1400 \text{ kg} \times (25 \text{ m/s})^2}{120 \text{ m}} = 7291.7 \text{ N}$$

Maximum friction available:

$$f_{\max} = \mu_s mg = 0.40 \times 1400 \text{ kg} \times 9.8 \text{ m/s}^2 = 5488 \text{ N} < 7291.7 \text{ N (required)}$$

The required centripetal force (7291.7N) exceeds the maximum friction (5488N) by 1803.7N. Since friction cannot supply the force needed to maintain circular motion at this speed, the tyres lose grip and the car slides outward from the curve, following a tangential path in accordance with Newton's first law.

Making Sense of the Answer: *The numbers tell the story clearly: the car demands 7292N of inward force, but the road can only offer 5488 N. The deficit of about 1804 N means the car cannot follow the curve. It is not a matter of the driver turning the steering wheel harder; the physics simply does not allow it. This is why speed limits on sharp curves are not suggestions; they are calculated from the laws of physics.*

Think Like a Physicist: *Rearranging $v_{\max} = \sqrt{\mu_s g r}$ to find $r_{\min} = \frac{v^2}{\mu_s g}$ is a useful trick. It tells you the minimum curve radius you can safely negotiate at a given speed. Highway engineers use exactly this formula when designing roads.*

With the car on the level curve now thoroughly understood, let us see what happens when a cyclist, rather than a car, must negotiate a curved path. The physics is the same, but the human body reveals it in a beautifully visible way.

Cyclist on a Curved Rough Level Road

When a car turns a corner, the physics is hidden inside the vehicle. The tyres grip, friction acts sideways, and the car follows the curve. But when a cyclist takes the same corner, something remarkable happens that the car cannot show: **the cyclist leans inward**.

This leaning is not a style choice. It is a physical necessity. A cyclist who tries to turn a corner while sitting perfectly upright will topple outward. The lean is the body's way of ensuring that the forces acting on the bicycle and rider can produce the required centripetal force while maintaining balance.

Why must the cyclist lean?

A bicycle is a narrow vehicle. Unlike a car, which has four widely spaced wheels, a bicycle makes contact with the ground along a very thin line. If the cyclist sits upright and turns, friction acts horizontally at the contact point (just as for a car), but the cyclist's weight acts vertically through the centre of mass, which is high above the ground. These two forces would then create a net *turning effect (torque)* that tips the cyclist outward.

To prevent this, the cyclist leans the entire bicycle inward. This shifts the line of action of the forces so that they balance not only in terms of net force, but also in terms of torque. The result is stable circular motion without toppling.

The forces acting on the cyclist

Consider the following diagram:

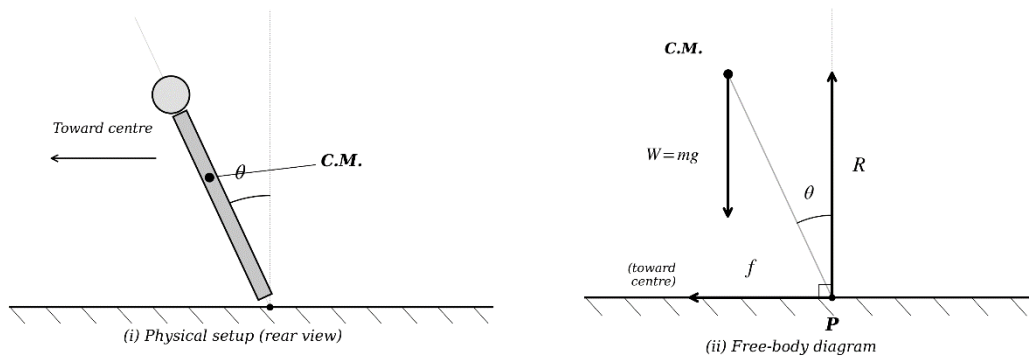


Figure: A cyclist negotiating a level circular curve, shown from behind (rear view). (i) The cyclist and bicycle lean inward at angle θ from the vertical. The centre of mass (C.M.) is displaced toward the centre of the curve. (ii) Free-body diagram: weight $W = mg$ acts downward through the centre of mass, while the normal reaction R and friction f act at the contact point P .

Two forces act on the cyclist-bicycle system:

1. **Weight** ($W = mg$), acting vertically downward through the centre of mass.
2. **Contact force from the road**, which has two components at the point of contact:
 - **Normal reaction** (R), acting vertically upward, perpendicular to the road surface.
 - **Static friction** (f), acting horizontally toward the centre of the curve.

These are the same forces as for the car on a level curve. The difference is that for the cyclist, the angle of lean θ connects them geometrically.

Applying Newton's second law

Vertically (no acceleration):

$$R = mg$$

Horizontally (toward the centre):

$$f = \frac{mv^2}{r}$$

These are identical to the equations for a car on a level curve.

Finding the angle of lean

For the cyclist to remain balanced (no toppling), the net force must pass through the contact point. This requires the resultant of weight and friction to act along the line of the bicycle frame. Geometrically, this gives:

$$\tan\theta = \frac{f}{R} = \frac{mv^2/r}{mg}$$

From which:

$$\tan\theta = \frac{v^2}{rg}$$

Where θ is the angle of lean measured from the vertical.

This equation reveals several important insights:

1. The angle of lean increases with speed. Faster turns require greater lean.
2. The angle of lean increases as the radius decreases. Sharper turns require greater lean.

3. The angle of lean does not depend on mass. A heavy cyclist and a light cyclist lean at the same angle for the same speed and radius.

Notice that this is the same expression, $\tan\theta = \frac{v^2}{rg}$, that will appear again in the conical pendulum and in banking of roads. The physics is universal; only the physical situation changes.

Maximum speed before skidding

The cyclist skids when friction reaches its maximum:

$$f_{\max} = \mu_s mg$$

At maximum speed:

$$\frac{mv_{\max}^2}{r} = \mu_s mg$$

$$v_{\max} = \sqrt{\mu_s gr}$$

This is exactly the same result as for the car.

Maximum lean angle before skidding

Since $\tan\theta = \frac{v^2}{rg}$, a larger speed requires a larger lean angle. Therefore, the maximum speed and the maximum lean angle, $\theta_{\max}$ occur at the same instant: just before skidding. The cyclist cannot lean beyond $\theta_{\max}$ because doing so would require a higher speed that demands more friction than the road can provide.

We showed earlier that the lean angle satisfies:

$$\tan\theta = \frac{f}{R} = \frac{f}{mg}$$

At maximum speed, friction reaches its limit:

$$f_{\max} = \mu_s mg$$

Substituting:

$$\tan\theta_{\max} = \frac{\mu_s mg}{mg}$$

From which:

$$\tan\theta_{\max} = \mu_s$$

Hence, at this maximum speed, the lean angle satisfies:

$$\tan\theta_{\max} = \mu_s$$

This is a remarkable result. The maximum lean angle depends only on the coefficient of static friction between the tyres and the road. It does not depend on the radius of the curve or the mass of the cyclist. A heavier cyclist, a lighter cyclist, a wide curve, a tight curve; none of these matter. Only the grip of the tyres on the road determines how far the cyclist can lean before skidding.

With the ideas laid out clearly, let us practise them through worked examples.

BINDER Example 19

A cyclist rounds a level circular curve of radius 25m at a constant speed of 10m/s. Take $g = 9.8 \text{ m/s}^2$.

- Calculate the angle at which the cyclist must lean from the vertical.
- If the cyclist doubles the speed to 20 m/s on the same curve, find the new lean angle.

Solution

- Using:

$$\tan\theta = \frac{v^2}{rg}$$

Where: $v = 10 \text{ m/s}$, $r = 25 \text{ m}$, $g = 9.8 \text{ m/s}^2$

Substituting:

$$\tan\theta = \frac{(10 \text{ m/s})^2}{25 \text{ m} \times 9.8 \text{ m/s}^2} = 0.408$$

$$\theta = \tan^{-1}(0.408) = 22.2^\circ$$

The cyclist must lean at 22.2° from the vertical.

(b) At $v = 20 \text{ m/s}$:

$$\tan\theta = \frac{(20 \text{ m/s})^2}{25 \text{ m} \times 9.8 \text{ m/s}^2} = 1.633$$

$$\theta = \tan^{-1}(1.633) = 58.5^\circ$$

The new lean angle is 58.5° .

Making Sense of the Answer: Doubling the speed more than doubled the lean angle, from 22.2° to 58.5° . This is because the lean angle depends on v^2 , not v . A 58.5° lean is extreme and very difficult to maintain in practice, which explains why cyclists slow down significantly before entering sharp curves.

Think Like a Physicist: If the calculated lean angle becomes very large (say above 45°), the required friction is enormous and the cyclist is almost certainly going to skid. In practice, lean angles beyond about 40° are dangerous on ordinary road surfaces.

REAL Example 20

Kipanga rides his bicycle fast around a sharp corner near the school. **Kipute** notices that he leans his body and the bicycle strongly toward the inside of the curve. She asks: “Why do you lean so much when turning? You don’t lean when riding straight.”

Explain why a cyclist must lean inward when negotiating a curve.

Solution

When riding in a straight line, only two forces act on the cyclist: weight downward and the normal reaction upward. These are balanced, and no lean is needed.

When turning, however, a horizontal centripetal force is required toward the centre of the curve. This force is provided by static friction at the contact point between the tyres and the road. Friction acts horizontally at ground level, while the weight acts vertically through the centre of mass, which is high above the ground.

If the cyclist remains upright, friction and weight act along different lines, creating a turning effect that would topple the cyclist outward. By leaning inward, the cyclist shifts the centre of mass so that the combined effect of weight and friction passes through the contact point. This eliminates the toppling tendency and allows stable circular motion. The faster the turn or the sharper the curve, the greater the required friction and the more the cyclist must lean to maintain balance.

Making Sense of the Answer: Leaning is not about courage or style. It is the cyclist’s only way to keep the forces balanced during a turn. A car does not need to lean because it has four widely spaced wheels that prevent toppling. A bicycle, with only two narrow contact points, must lean to achieve the same balance.

Think Like a Physicist: The lean angle satisfies $\tan\theta = \frac{v^2}{rg}$. This same expression appears in the conical pendulum and in road banking, because the underlying physics is the same: balancing a horizontal centripetal requirement against a vertical gravitational force.

HOT Example 21

A cyclist of mass 70 kg (including the bicycle) rounds a level curve of radius 30 m . The coefficient of static friction between the tyres and the rough road surface is 0.55 . Take $g = 9.8 \text{ m/s}^2$.

- Calculate the maximum speed at which the cyclist can negotiate the curve without skidding.
- Find the lean angle at maximum speed.

- (c) At maximum speed, calculate the friction force and the normal reaction.
- (d) If the road becomes wet and the coefficient of friction drops to 0.25, determine the new maximum speed and the new lean angle.

Solution

- (a) Using:

$$v_{\max} = \sqrt{\mu_s g r}$$

Where: $\mu_s = 0.55$, $g = 9.8 \text{ m/s}^2$, $r = 30 \text{ m}$

Substituting:

$$v_{\max} = \sqrt{0.55 \times 9.8 \text{ m/s}^2 \times 30 \text{ m}} = 12.7 \text{ m/s}$$

The maximum speed is 12.7 m/s (about 45.7 km/h).

- (b) At maximum speed:

$$\tan \theta_{\max} = \mu_s = 0.55$$

$$\theta_{\max} = \tan^{-1}(0.55) = 28.8^\circ$$

The lean angle at maximum speed is 28.8° .

- (c) Normal reaction:

$$R = mg = 70 \text{ kg} \times 9.8 \text{ m/s}^2 = 686 \text{ N}$$

Friction force at maximum speed:

$$f = \mu_s mg = 0.55 \times 686 \text{ N} = 377.3 \text{ N}$$

The normal reaction is 686 N and the friction force is 377.3 N.

- (e) With $\mu_s = 0.25$:

$$v_{\max} = \sqrt{0.25 \times 9.8 \text{ m/s}^2 \times 30 \text{ m}} = 8.6 \text{ m/s}$$

$$\tan \theta_{\max} = 0.25; \quad \theta_{\max} = \tan^{-1}(0.25) = 14^\circ$$

The new maximum speed on the wet road is 8.6 m/s (about 30.9 km/h) and the lean angle drops to 14° .

Making Sense of the Answer: On a wet road, the cyclist must reduce speed from 12.7 m/s to 8.6 m/s, a reduction of about 32%. The lean angle also drops sharply, from 28.8° to 14° . This makes sense: less friction means less centripetal force available, so the cyclist must go slower and lean less. Attempting the original speed on a wet road would cause the tyres to skid sideways.

Think Like a Physicist: At maximum speed, $\tan \theta_{\max} = \mu_s$. This is elegant: the maximum lean angle depends only on friction, regardless of speed, radius, or mass. It provides a quick way to estimate the danger of a curve: if the required lean angle exceeds $\tan^{-1}(\mu_s)$, skidding is inevitable.

Having explored the cyclist, we notice something interesting: the expression $\tan \theta = \frac{v^2}{rg}$ keeps appearing. It appeared for the cyclist's lean angle, and it will appear again in our next subtopic, the conical pendulum, where a mass on a string traces a horizontal circle. The angle is different, the physical setup is different, but the mathematics is remarkably the same. Let us see why.

Conical Pendulum

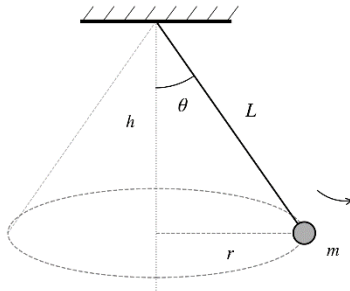
Imagine tying a small stone to the end of a string, holding the other end fixed above your head, and then setting the stone in motion so that it traces a horizontal circle. The string sweeps out the surface of a cone as the stone revolves, and the arrangement is called a **conical pendulum**.

Unlike the cyclist or the car, there is no friction here. The only forces acting on the stone are its weight and the tension in the string. Yet the stone follows a perfectly circular horizontal path. *How does it manage this without friction?* The answer lies in the geometry: the string is not vertical but inclined at an angle θ to the vertical, and it is the horizontal component of the tension that provides the centripetal force.

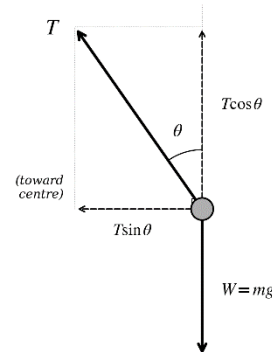
This is a beautiful example of how a single force (tension) can serve two roles simultaneously: its vertical component supports the weight, while its horizontal component maintains the circular motion.

Setting up the problem

Consider a small object of mass m attached to a string of length L , with the other end fixed to a point directly above the centre of the circular path. The object moves at constant speed in a horizontal circle of radius r . The string makes a constant angle θ with the vertical.



(i) Physical setup



(ii) Free-body diagram

Figure: A conical pendulum. (i) A mass m attached to a string of length L revolves in a horizontal circle of radius r . The string makes angle θ with the vertical and sweeps out a cone. (ii) Free-body diagram: only tension T and weight $W = mg$ act on the mass. The vertical component $T\cos\theta$ balances the weight, while the horizontal component $T\sin\theta$ provides the centripetal force.

Identifying the forces

Only two forces act on the object:

- 1) **Weight** ($W = mg$), acting vertically downward.
- 2) **Tension** (T), acting along the string, directed from the object toward the fixed point.

There is no friction, no normal reaction, and no additional horizontal force. The centripetal force must come entirely from the tension.

Applying Newton's second law

We resolve the tension into vertical and horizontal components:

Vertically (no acceleration):

The object does not move up or down, so the net vertical force is zero:

$$T\cos\theta - mg = 0$$

From which:

$$T\cos\theta = mg \quad \dots (i)$$

Horizontally (toward the centre):

The horizontal component of tension provides the centripetal force:

$$T\sin\theta = \frac{mv^2}{r} \quad \dots (ii)$$

Finding the relationship between angle and speed

Dividing equation (ii) by equation (i):

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2/r}{mg}$$

The tension T and mass m cancel:

$$\tan\theta = \frac{v^2}{rg}$$

This is the same expression we met for the cyclist's lean angle. The physics is different, the forces are different, but the geometry of balancing a horizontal centripetal requirement against a vertical gravitational force produces the same result.

It is important for you to understand that:

The angle θ does not depend on the mass of the object. A heavy stone and a light stone, whirled on the same string at the same speed, will both make the same angle with the vertical. This is because mass appears in both the centripetal force requirement (mv^2/r) and the weight (mg), and cancels when we divide.

What the angle does depend on is the speed v and the radius r . A faster speed or a smaller radius requires a larger angle.

Finding the tension

From equation (i):

$$T = \frac{mg}{\cos\theta}$$

Since $\cos\theta < 1$ for any angle $\theta > 0^\circ$, the tension is always greater than the weight. This is because the string has two duties: it must support the weight vertically and simultaneously pull the object inward. This double duty means the tension must exceed mg .

Relating radius to string length

From the geometry of the cone:

$$r = L\sin\theta$$

This connects the radius of the circular path to the string length and the angle. It is the same geometric relationship we used for the fairground chair ride in Example 15.

Deriving the period of revolution

The period T_p (we use T_p instead of the usual T because in this subtopic the symbol T is used for tension; using the same letter for both would cause confusion) can be found by combining our results. Starting from:

$$\tan\theta = \frac{v^2}{rg}$$

We substitute $v = \frac{2\pi r}{T_p}$:

$$\tan\theta = \frac{4\pi^2 r}{T_p^2 g}$$

Making T_p^2 the subject:

$$T_p^2 = \frac{4\pi^2 r}{g \tan\theta}$$

Now substituting $r = L\sin\theta$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$T_p^2 = \frac{4\pi^2 L \sin\theta}{g \times \frac{\sin\theta}{\cos\theta}} = \frac{4\pi^2 L \cos\theta}{g}$$

From which:

$$T_p = 2\pi \sqrt{\frac{L \cos\theta}{g}}$$

Understanding what this equation reveals

This is a striking result. The period depends on the string length L , the angle θ , and gravity g . It does not depend on the mass of the object.

As the angle θ increases (faster rotation), $\cos\theta$ decreases, and the period becomes shorter. The object revolves faster. In the limiting case where θ approaches 90° (string nearly horizontal), the period approaches zero and the speed becomes very large, but the tension becomes extremely large because $T = \frac{mg}{\cos\theta}$ increases without bound.

In the opposite limit, as θ approaches 0° (string nearly vertical), the period approaches $2\pi\sqrt{\frac{L}{g}}$, which is the period of a simple pendulum of the same length. This makes physical sense: at very small angles, the conical pendulum barely traces a circle and its motion resembles a simple pendulum.

With the theory complete, let us work through some examples.

BINDER Example 22

A small ball of mass 0.3 kg is attached to a string of length 0.8 m and whirled as a conical pendulum. The string makes an angle of 40° with the vertical. Take $g = 9.8 \text{ m/s}^2$.

- (a) Calculate the tension in the string.
- (b) Calculate the radius of the circular path.
- (c) Calculate the speed of the ball.

Solution

- (a) Using:

$$T\cos\theta = mg$$

Making T the subject:

$$T = \frac{mg}{\cos\theta}$$

Where: $m = 0.3 \text{ kg}$, $g = 9.8 \text{ m/s}^2$, $\theta = 40^\circ$

Substituting:

$$T = \frac{0.3 \text{ kg} \times 9.8 \text{ m/s}^2}{\cos 40^\circ} = 3.84 \text{ N}$$

The tension is 3.84 N .

- (b) Using:

$$r = L\sin\theta$$

Where: $L = 0.8 \text{ m}$, $\theta = 40^\circ$

Substituting:

$$r = 0.8 \text{ m} \times \sin 40^\circ = 0.514 \text{ m}$$

The radius is 0.514 m .

- (c) Using:

$$T\sin\theta = \frac{mv^2}{r}$$

Making v the subject:

$$v = \sqrt{\frac{T\sin\theta \cdot r}{m}}$$

Substituting:

$$v = \sqrt{\frac{3.84 \text{ N} \times 0.643 \times 0.514 \text{ m}}{0.3 \text{ kg}}} = \sqrt{4.23} = 2.06 \text{ m/s}$$

The speed is 2.06 m/s.

Making Sense of the Answer: The tension (3.84N) is greater than the weight (2.94N), as expected. The string must support the weight vertically and pull the ball inward simultaneously. At a 40° angle, the radius is about 0.51m, which is reasonable for a 0.8m string. The speed of 2.06m/s is modest, consistent with a gentle swing.

Think Like a Physicist: Always find the tension first using $T \cos \theta = mg$, because tension appears in both the vertical and horizontal equations. Once you have T , the rest follows easily.

REAL Example 23

Kipanga ties a small stone to a string and whirls it in a horizontal circle above his head. He notices that when he spins the stone faster, the string rises and the angle with the vertical increases. He asks **Kipute**: “Why does the string go higher when I spin faster? Shouldn’t gravity keep pulling it down the same way?”

Explain why the angle increases when the speed increases.

Solution

Gravity does pull the stone downward with the same force (mg) regardless of speed. However, as the speed increases, the stone requires a larger centripetal force to maintain its circular path, since the centripetal force is proportional to v^2 .

The only horizontal force available is the horizontal component of the tension: $T \sin \theta$. To increase this horizontal component, either the tension must increase, or the angle must increase, or both. In practice, as the speed increases, the string rises to a larger angle θ . A larger angle means a greater proportion of the tension is directed horizontally (larger $\sin \theta$), providing the extra centripetal force needed.

The relationship $\tan \theta = \frac{v^2}{rg}$ confirms this: increasing v increases $\tan \theta$, which means θ increases.

So gravity has not changed, but the balance between vertical support and horizontal pull has shifted. The string rises because the stone’s circular motion demands more horizontal force, and the only way to get it is by tilting the string further from the vertical.

Making Sense of the Answer: Think of it this way: at low speed, the stone barely needs any centripetal force, so the string hangs almost vertically and nearly all the tension goes toward supporting the weight. At high speed, the centripetal demand is large, so the string must tilt outward to redirect more tension horizontally.

Think Like a Physicist: The angle is a visible indicator of how much centripetal force the system demands. A large angle means the circular motion is demanding a large fraction of the available tension. This is why conical pendulums are used in some engineering speed governors: the angle directly indicates the rotation speed.

HOT Example 24

A conical pendulum has a string of length 1.2m. The period of revolution is 1.8s. Take $g = 9.8 \text{ m/s}^2$ and $\pi = 3.14$.

- Calculate the angle the string makes with the vertical.
- Calculate the radius of the circular path.
- Calculate the tension in the string if the mass is 0.5 kg.
- Determine the speed of the object.

Solution

- Using:

$$T_p = 2\pi \sqrt{\frac{L \cos \theta}{g}}$$

Making $\cos \theta$ the subject:

$$T_p^2 = 4\pi^2 \times \frac{L \cos \theta}{g}$$

$$\cos \theta = \frac{T_p^2 g}{4\pi^2 L}$$

Where: $T_p = 1.8 \text{ s}$, $g = 9.8 \text{ m/s}^2$, $L = 1.2 \text{ m}$

Substituting:

$$\cos \theta = \frac{(1.8 \text{ s})^2 \times 9.8 \text{ m/s}^2}{4 \times (3.14)^2 \times 1.2 \text{ m}} = 0.671$$

$$\theta = \cos^{-1}(0.671) = 47.9^\circ$$

The angle is 47.9° .

(b) Using:

$$r = L \sin \theta = 1.2 \text{ m} \times \sin 47.9^\circ = 1.2 \times 0.742 = 0.890 \text{ m}$$

The radius is 0.890 m .

(c) Using:

$$T = \frac{mg}{\cos \theta} = \frac{0.5 \text{ kg} \times 9.8 \text{ m/s}^2}{0.671} = 7.30 \text{ N}$$

The tension is 7.30 N .

(d) Using:

$$v = \frac{2\pi r}{T_p} = \frac{2 \times 3.14 \times 0.890 \text{ m}}{1.8 \text{ s}} = \frac{5.59 \text{ m}}{1.8 \text{ s}} = 3.1 \text{ m/s}$$

The speed is 3.1 m/s .

Making Sense of the Answer: An angle of nearly 48° is quite large; the string is closer to horizontal than vertical. This tells us the rotation is fairly fast. The tension (7.3 N) is about 1.5 times the weight (4.9 N), confirming that the string works hard to provide both vertical support and horizontal centripetal force. The speed of 3.1 m/s over a radius of 0.89 m gives a reasonable revolution time of 1.8 s .

Think Like a Physicist: When the period is given instead of the speed, use the period formula directly to find the angle. Do not try to find the speed first; it creates unnecessary steps. The formula $\cos \theta = \frac{T_p^2 g}{4\pi^2 L}$ gives the angle in one calculation.

With the conical pendulum now understood, we recognise a pattern that runs through all of horizontal circular motion. The leaning cyclist and the conical pendulum both produce the same central relationship: $\tan \theta = \frac{v^2}{rg}$. In each case, the angle θ connects the vertical force (gravity) to the horizontal force (centripetal).

This same relationship will appear once more in our next subtopic, where engineers deliberately tilt the road surface to help vehicles navigate curves safely. Welcome to banking of roads.

Banking of Roads

On a level curve, the centripetal force comes entirely from friction. But friction is unreliable. Rain reduces it. Worn tyres reduce it. Oil spills destroy it. And when friction fails, vehicles slide off the road, as we learned at Kilimani Hill.

Engineers asked a simple but powerful question: *What if the road itself could help provide the centripetal force, even without any friction at all?*

The answer is **banking**: tilting the road surface so that the outer edge is higher than the inner edge. On a banked curve, the road surface is no longer horizontal. The normal reaction, which is always perpendicular to the surface, now tilts inward. This tilted normal reaction has a horizontal component that points toward the centre of the curve, providing some or all of the centripetal force.

Banking does not eliminate friction; it reduces the dependence on friction. A well-banked curve at the correct speed requires no friction at all. At other speeds, friction assists or opposes the motion, but the banking carries the main load.

Case 1: Ideal banking (no friction)

Consider a car of mass m moving at constant speed v around a banked curve of radius r . The road surface makes an angle θ with the horizontal. We analyse the ideal case where friction is zero (perfectly smooth surface).

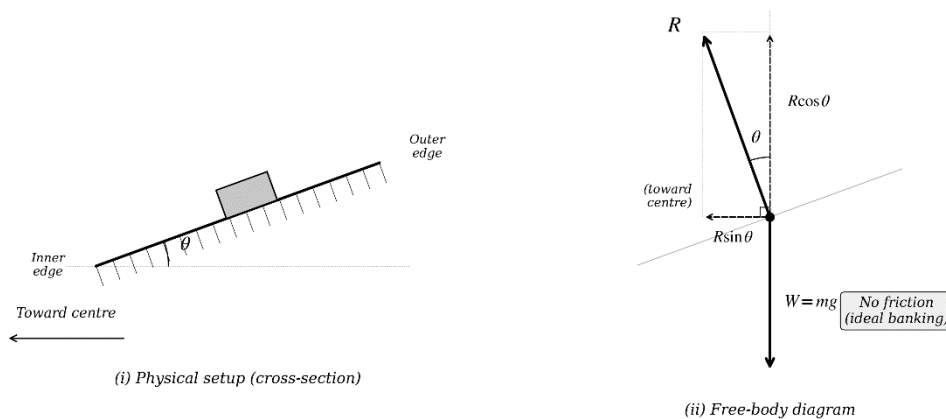


Figure: A car on a banked (tilted) curve, shown in cross-section. (i) The road surface makes angle θ with the horizontal, with the outer edge higher than the inner edge. (ii) Free-body diagram for ideal banking (no friction): the normal reaction R is perpendicular to the tilted surface. Its vertical component $R\cos\theta$ balances the weight, while its horizontal component $R\sin\theta$ provides the centripetal force.

Identifying the forces

With no friction, only two forces act on the car:

- 1) **Weight** ($W = mg$), acting vertically downward.
- 2) **Normal reaction** (R), acting perpendicular to the banked surface. Because the surface is tilted, R is no longer vertical; it tilts inward.

Resolving the normal reaction

We resolve R into two components:

- **Vertical component:** $R\cos\theta$ (upward)
- **Horizontal component:** $R\sin\theta$ (toward the centre of the curve)

Applying Newton's second law

Vertically (no acceleration):

$$R\cos\theta = mg \quad \dots (i)$$

Horizontally (toward the centre):

$$R\sin\theta = \frac{mv^2}{r} \quad \dots (ii)$$

Finding the banking angle

Dividing equation (ii) by equation (i):

$$\frac{R \sin \theta}{R \cos \theta} = \frac{mv^2/r}{mg}$$

R and m cancel:

$$\tan \theta = \frac{v^2}{rg}$$

This is the same expression we derived for the cyclist's lean angle and the conical pendulum. The physics is universal: *whenever gravity provides vertical equilibrium and a horizontal component of some force provides centripetal acceleration, this relationship emerges.*

The design speed

The equation $\tan \theta = \frac{v^2}{rg}$ can be rearranged to give the speed for which the banking angle is perfectly matched:

$$v_{\text{design}} = \sqrt{rg \tan \theta}$$

At this speed, the car negotiates the curve without any friction at all. The entire centripetal force comes from the horizontal component of the normal reaction. This is called the **design speed** of the banked curve.

It is important for you to understand that:

The design speed does not depend on the mass of the vehicle. A motorcycle, a car, and a fully loaded bus all have the same design speed on the same banked curve, because mass cancels from the equation. This is the same mass-independence we observed for the maximum safe speed on a level curve, the cyclist's lean angle, and the conical pendulum angle.

What happens at speeds other than the design speed?

If the car travels at exactly the design speed, no friction is needed.

If the car travels **faster** than the design speed, the required centripetal force exceeds what the banking alone provides. Friction must act *inward* (down the slope) to supply the extra centripetal force. Without sufficient friction, the car slides outward (up the slope).

If the car travels **slower** than the design speed, the banking provides more inward force than needed. Friction must act *outward* (up the slope) to prevent the car from sliding inward (down the slope).

At the design speed, the forces are perfectly balanced and friction is zero. This is why highway engineers choose the banking angle based on the expected speed of traffic on that section of road.

Case 2: Banked curve with friction (maximum speed)

In real life, roads are not frictionless. When a car travels faster than the design speed on a banked curve, the horizontal component of the normal reaction alone cannot supply the centripetal force required. The car tends to slide outward (up the slope). Friction therefore acts to prevent this sliding, which means friction acts **down the slope** (toward the inner edge), helping to push the car inward.

Why does friction act down the slope?

This is the point that confuses many students. On a level road, friction acts horizontally toward the centre of the curve. On a banked road, however, if the speed is too high, the vehicle tends to slide up the slope toward the outside of the curve. Friction therefore acts **down the slope** toward the centre to oppose this motion. In general, friction always acts in the direction that opposes the motion that would occur if friction were absent.

Identifying the forces

Three forces now act on the car:

- 3) **Weight** ($W = mg$), acting vertically downward.
- 4) **Normal reaction** (R), acting perpendicular to the banked surface (tilted inward from the vertical).
- 5) **Friction** (f), acting along the banked surface, directed down the slope (toward the inner edge).

Resolving the forces

Each force must be resolved into vertical and horizontal components. This is where careful thought is needed, because both R and f contribute to both directions.

Normal reaction R (perpendicular to road, tilted at angle θ from the vertical):

- Vertical component: $R\cos\theta$ (upward)
- Horizontal component: $R\sin\theta$ (toward centre)

Friction f (along the road surface, directed down the slope, at angle θ from the horizontal):

- Vertical component: $f\sin\theta$ (downward, because friction points down the slope)
- Horizontal component: $f\cos\theta$ (toward centre, because the down-slope direction has a horizontal inward component)

This is the key insight: friction has a **vertical component that acts downward**, reducing the vertical support, and a **horizontal component that acts toward the centre**, adding to the centripetal force. **Both R and f contribute to the centripetal force**, but friction slightly undermines the vertical balance.

Applying Newton's second law

Vertically (no acceleration):

The upward forces must balance the downward forces:

$$R\cos\theta = mg + f\sin\theta$$

Rearranging:

$$R\cos\theta - f\sin\theta = mg \quad \dots (i)$$

Notice that friction's vertical component ($f\sin\theta$) appears on the same side as weight. Friction acting down the slope pulls the car slightly downward, so the normal reaction must support both the weight and this downward component of friction.

Horizontally (toward the centre):

Both the horizontal component of R and the horizontal component of f point toward the centre:

$$R\sin\theta + f\cos\theta = \frac{mv^2}{r} \quad \dots (ii)$$

Banking and friction work together to provide the centripetal force.

Finding the maximum speed

The maximum speed occurs when friction reaches its limit:

$$f = \mu_s R$$

Substituting $f = \mu_s R$ into equation (i):

$$\begin{aligned} R\cos\theta - \mu_s R\sin\theta &= mg \\ R(\cos\theta - \mu_s\sin\theta) &= mg \quad \dots (iii) \end{aligned}$$

Substituting $f = \mu_s R$ into equation (ii):

$$\begin{aligned} R\sin\theta + \mu_s R\cos\theta &= \frac{mv_{\max}^2}{r} \\ R(\sin\theta + \mu_s\cos\theta) &= \frac{mv_{\max}^2}{r} \quad \dots (iv) \end{aligned}$$

Dividing equation (iv) by equation (iii) to eliminate both R and m :

$$\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta} = \frac{v_{\max}^2}{rg}$$

Dividing numerator and denominator of the left side by $\cos\theta$:

$$\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta} = \frac{v_{\max}^2}{rg}$$

Making $v_{\max}$ the subject:

$$v_{\max} = \sqrt{rg \left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta} \right)}$$

Checking the formula against known results

A good formula should reduce to simpler cases when we set certain quantities to zero. Let us verify:

- If $\theta = 0$ (level road, no banking): $\tan\theta = 0$, so the formula gives $v_{\max} = \sqrt{rg \times \mu_s} = \sqrt{\mu_s gr}$, which is our result we established earlier.
- If $\mu_s = 0$ (no friction): the formula gives $v_{\max} = \sqrt{rg \tan\theta}$, which is the design speed from Case 1.

Both limiting cases check out, giving us confidence that the derivation is correct.

Case 3: Banked curve with friction (minimum speed)

So far we have found the maximum speed. But on a banked curve, there is also a **minimum** safe speed, and this surprises many students.

When the car travels slower than the design speed, the horizontal component of the normal reaction provides more inward force than the circular motion requires. The car tends to slide **down the slope** (toward the inner edge). Friction must now act **up the slope** (toward the outer edge) to prevent this inward sliding.

Notice the reversal: at high speed, friction acts down the slope to help the car turn; at low speed, friction acts up the slope to stop the car from sliding inward. At exactly the design speed, friction is zero.

Identifying the forces

The same three forces act on the car: weight $W = mg$ (downward), normal reaction R (perpendicular to road, tilted inward), and friction f (along the road surface). But now friction acts **up the slope** instead of down.

Resolving the forces

Because friction is reversed, its components change sign compared to Case 2:

Friction f (up the slope):

- Vertical component: $f \sin\theta$ (**upward**, helping to support the weight)
- Horizontal component: $f \cos\theta$ (**away from centre**, opposing the centripetal force)

Applying Newton's second law

Vertically (no acceleration):

$$R \cos\theta + f \sin\theta = mg \quad \dots (v)$$

Compare this with equation (i) from Case 2: the friction term has changed from $-f \sin\theta$ to $+f \sin\theta$ because friction now has an upward vertical component.

Horizontally (toward the centre):

$$R \sin\theta - f \cos\theta = \frac{mv^2}{r} \quad \dots (vi)$$

Compare with equation (ii) from Case 2: the friction term has changed from $+f \cos\theta$ to $-f \cos\theta$ because friction now opposes the centripetal direction.

Finding the minimum speed

At the minimum speed, friction reaches its limit: $f = \mu_s R$.

Substituting into equations (v) and (vi):

$$R(\cos\theta + \mu_s \sin\theta) = mg \quad \dots (vii)$$

$$R(\sin\theta - \mu_s \cos\theta) = \frac{mv_{\min}^2}{r} \quad \dots \text{(viii)}$$

Dividing (viii) by (vii):

$$\frac{\sin\theta - \mu_s \cos\theta}{\cos\theta + \mu_s \sin\theta} = \frac{v_{\min}^2}{rg}$$

Dividing numerator and denominator by $\cos\theta$:

$$\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta} = \frac{v_{\min}^2}{rg}$$

Making $v_{\min}$ the subject:

$$v_{\min} = \sqrt{rg \left(\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta} \right)}$$

Understanding what this equation reveals

Compare the minimum and maximum speed formulas side by side:

$$v_{\max} = \sqrt{rg \left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta} \right)} \quad v_{\min} = \sqrt{rg \left(\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta} \right)}$$

The pattern is elegant: wherever the maximum speed formula has a +, the minimum speed formula has a -, and vice versa. This reflects the reversal of friction direction.

An important special case arises when $\tan\theta \leq \mu_s$. In this case, the numerator $(\tan\theta - \mu_s)$ becomes zero or negative, which means $v_{\min} = 0$. Physically, this means friction is strong enough to hold the car stationary on the banked surface without sliding. The car can safely travel at any speed from zero up to $v_{\max}$.

However, if the banking is steep (large θ) and friction is weak (small μ_s), then $\tan\theta > \mu_s$, and there is a real minimum speed. A car travelling too slowly on such a curve will slide down the slope toward the inner edge.

The complete picture

Every banked curve with friction has a safe speed range:

$$v_{\min} \leq v \leq v_{\max}$$

Below $v_{\min}$, the car slides inward (down the slope). Above $v_{\max}$, the car slides outward (up the slope). At the design speed $v_{\text{design}} = \sqrt{rg \tan\theta}$, friction is zero and the ride is smoothest.

The worked examples that follow will sharpen all of these ideas.

BINDER Example 25

A highway curve has a radius of 200m. The road is banked at an angle of 12° to the horizontal. Take $g = 9.8 \text{ m/s}^2$.

- Calculate the design speed of the curve.
- A car of mass 1200 kg travels at the design speed. Calculate the normal reaction acting on the car.

Solution

- Using:

$$v_{\text{design}} = \sqrt{rg \tan\theta}$$

Where: $r = 200 \text{ m}$, $g = 9.8 \text{ m/s}^2$, $\theta = 12^\circ$

Substituting:

$$v_{\text{design}} = \sqrt{200 \text{ m} \times 9.8 \text{ m/s}^2 \times \tan 12^\circ} = 20.4 \text{ m/s}$$

The design speed is 20.4 m/s (about 73.4 km/h).

- Using:

$$R \cos\theta = mg$$

Making R the subject:

$$R = \frac{mg}{\cos\theta}$$

Where: $m = 1200 \text{ kg}$, $g = 9.8 \text{ m/s}^2$, $\theta = 12^\circ$

Substituting:

$$R = \frac{1200 \text{ kg} \times 9.8 \text{ m/s}^2}{\cos 12^\circ} = \frac{11760 \text{ N}}{0.978} = 12024 \text{ N}$$

The normal reaction is 12024N.

Making Sense of the Answer: The design speed of about 73 km/h is typical for a gently banked highway curve. The normal reaction (12024N) is slightly larger than the weight (11760N) because the normal force must support the weight vertically while also providing centripetal force horizontally. This is the same effect we saw in the conical pendulum, where tension exceeded the weight.

Think Like a Physicist: At the design speed, friction is zero, so the only contact force is the normal reaction. The car feels exactly as if it were driving on a perfectly smooth surface. Passengers experience no sideways sensation because the banking angle matches the speed perfectly.

REAL Example 26

On a school trip to Iringa, the bus passes through a mountain road with several sharp curves. At one particular curve, **Kipanga** looks out the window and sees that the road is visibly tilted, with the outer edge much higher than the inner edge.

Kipanga (alarmed): “Mr. Akilikubwa! Look at this road! It is slanting badly. I think the road has been damaged by heavy rains. Should we report it to TANROADS?”

Mr. Akilikubwa (smiling): “Kipanga, if you report that road, the engineers at TANROADS will laugh. They are the ones who built it that way on purpose.”

Kipanga (confused): “On purpose? Why would anyone build a slanting road?”

Kipute (grinning): “Because physics said so, Kipanga.”

Explain why the road is deliberately tilted (banked) at curves, and what advantage this provides over a flat road.

Solution

On a level (flat) curve, the centripetal force required to keep the bus on its circular path must come entirely from friction between the tyres and the road. If friction is insufficient (due to rain, worn tyres, or excessive speed), the bus slides off the curve.

A banked road is deliberately tilted so that the outer edge is higher than the inner edge. This causes the normal reaction from the road to tilt inward. The horizontal component of this tilted normal reaction points toward the centre of the curve and provides centripetal force.

At the design speed, the banking alone supplies all the centripetal force needed, and friction is not required at all. This means the bus can safely negotiate the curve even on a wet or icy surface, provided it travels near the design speed. Thus banking greatly reduces dependence on friction and makes the curve safer under all road conditions.

The engineers at TANROADS calculated the banking angle using $\tan\theta = \frac{v^2}{rg}$, based on the expected traffic speed and the radius of the curve.

Making Sense of the Answer: Kipanga’s instinct was understandable. A tilted road looks wrong if you do not know why it is tilted. But the tilt is deliberate engineering, not damage. Every banked curve on a highway represents a careful calculation: the angle is chosen so that normal vehicles at normal speeds require little or no friction to stay on the road.

Think Like a Physicist: Banking is nature’s way of saying: “If you cannot trust friction, tilt the surface.” The same principle appears in athletics (track bicycle racing), railway curves, and aircraft turns. Whenever something must follow a curved path reliably, banking is the engineer’s first tool.

HOT Example 27

- (a) A curved section of road has a radius of 150m and is banked at an angle of 15° . The coefficient of static friction between the tyres and the road surface is 0.4. Take $g = 9.8 \text{ m/s}^2$.
- (b) Compare this maximum speed with: the design speed of the curve (no friction), and the maximum speed if the same curve were level (unbanked) with the same friction. What does this comparison reveal about the advantage of combining banking with friction?

Solution

- (a) Using the result derived for the maximum speed on a banked curve with friction:

$$v_{\max} = \sqrt{rg \left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta} \right)}$$

Where: $r = 150 \text{ m}$, $g = 9.8 \text{ m/s}^2$, $\theta = 15^\circ$, $\mu_s = 0.4$

Substituting:

$$v_{\max} = \sqrt{150 \text{ m/s} \times 9.8 \text{ m/s}^2 \times \left(\frac{\tan 15^\circ + 0.4}{1 - 0.4 \tan 15^\circ} \right)} = 33.2 \text{ m/s}$$

The maximum speed on the banked curve is 33.2 m/s (about 119.4 km/h).

- (b) Comparisons:

On a level road (no banking, $\theta = 0$), the maximum speed depends on friction alone:

$$v_{\max, \text{ level}} = \sqrt{\mu_s g r} = \sqrt{0.40 \times 9.8 \times 150} = \sqrt{588} = 24.2 \text{ m/s (about 87.3 km/h)}$$

The design speed of the banked curve (no friction):

$$v_{\text{design}} = \sqrt{rg \tan\theta} = \sqrt{150 \times 9.8 \times 0.2679} = \sqrt{393.8} = 19.8 \text{ m/s (about 71.4 km/h)}$$

So the three scenarios give:

- Banking alone (no friction): 19.8 m/s
- Friction alone (no banking): 24.2 m/s
- Banking and friction together: 33.2 m/s

The combination of banking and friction supports a maximum speed that is significantly higher than either banking or friction could achieve alone.

Making Sense of the Answer: Banking alone gives 71 km/h. Friction alone gives 87 km/h. But together they give 119 km/h, which is far more than either alone. This is not simply addition; the two effects reinforce each other through the geometry of the tilted surface. This is why highway engineers never rely on friction alone for high-speed curves; they always bank the road as well.

Think Like a Physicist: Notice that the formula has $1 - \mu_s \tan\theta$ in the denominator. If $\mu_s \tan\theta$ approaches 1, the denominator approaches zero and the maximum speed becomes very large. Physically, this means a steeply banked road with high friction can support extremely high speeds.

With banking of roads now understood, we have seen how engineers use geometry to assist circular motion. The final application in horizontal circular motion is simpler but equally instructive: an object placed on a rotating platform. Let us see what keeps it from sliding off, and what happens when the rotation becomes too fast.

Object on a Rotating Platform

Place a coin on a spinning turntable. At low speed, the coin stays in place. Increase the speed gradually, and at some critical point the coin suddenly slides outward and flies off the edge. This simple observation contains a complete physics lesson.

The coin is not glued to the turntable. The only horizontal force available to keep it moving in a circle is **static friction** between the coin and the surface. As the turntable spins faster, the required centripetal force

increases. The moment this required force exceeds the maximum static friction, the coin can no longer maintain circular motion and it slides outward.

Setting up the problem

Consider an object of mass m placed at a distance r from the centre of a horizontal turntable that rotates at angular velocity ω .

Identifying the forces

Three forces act on the object:

- (i) **Weight** ($W = mg$), acting vertically downward.
- (ii) **Normal reaction** (R), acting vertically upward. Since the surface is horizontal, R is vertical (unlike the banked road).
- (iii) **Static friction** (f), acting horizontally toward the centre of the turntable. This is the only force available to provide centripetal acceleration.

Applying Newton's second law

Vertically (no acceleration):

$$R = mg$$

Horizontally (toward the centre):

$$f = m\omega^2 r$$

Be careful: We use $m\omega^2 r$ rather than $\frac{mv^2}{r}$ here because angular velocity ω is the natural quantity for a rotating platform: every point on the turntable shares the same ω , but points at different radii have different linear speeds.

The critical condition for sliding

The object remains in place as long as friction can supply the required centripetal force:

$$m\omega^2 r \leq \mu_s mg$$

Sliding occurs when the required centripetal force exceeds the maximum friction. At the critical point:

$$m\omega_{\max}^2 r = \mu_s mg$$

Making $\omega_{\max}$ the subject:

$$\omega_{\max} = \sqrt{\frac{\mu_s g}{r}}$$

This can also be expressed as a maximum linear speed at radius r (from $v = \omega r$):

$$v_{\max} = \sqrt{\mu_s g r}$$

Or as a maximum frequency (from $\omega = 2\pi f$):

$$f_{\max} = \frac{1}{2\pi} \sqrt{\frac{\mu_s g}{r}}$$

It is important for you to understand that:

The formula $\omega_{\max} = \sqrt{\frac{\mu_s g}{r}}$ reveals something that many students find surprising: *objects farther from the centre slide off first. Why?*

At a larger radius r , the maximum angular velocity $\omega_{\max}$ is smaller. So when the turntable gradually speeds up, objects near the edge reach their sliding limit before objects near the centre. The outer objects fly off while the inner objects remain safely in place.

This makes physical sense: at the same angular velocity, an object at larger radius has greater linear speed ($v = \omega r$) and therefore needs greater centripetal force ($m\omega^2 r$). The required force is proportional to r , but the available friction ($\mu_s mg$) is the same regardless of position. So objects farther out are always the first to lose the battle against inertia.

With the theory in hand, let us explore through worked examples.

BINDER Example 28

A small coin of mass 20 g is placed 15 cm from the centre of a horizontal turntable. The coefficient of static friction between the coin and the surface is 0.35. Take $g = 9.8 \text{ m/s}^2$.

Calculate the maximum angular velocity at which the turntable can rotate before the coin slides off.

Solution

Using:

$$\omega_{\max} = \sqrt{\frac{\mu_s g}{r}}$$

Where: $\mu_s = 0.35$, $g = 9.8 \text{ m/s}^2$, $r = 0.15 \text{ m}$

Substituting:

$$\omega_{\max} = \sqrt{\frac{0.35 \times 9.8 \text{ m/s}^2}{0.15 \text{ m}}} = 4.78 \text{ rad/s}$$

The maximum angular velocity is 4.78 rad/s.

Making Sense of the Answer: At 4.78 rad/s, the turntable completes about $\frac{4.78}{2\pi} = 0.76$ revolutions per second. This is a gentle spin. The coin slides at a modest speed because friction is relatively weak ($\mu_s = 0.35$) and the coin is fairly far from the centre (15cm). Moving the coin closer to the centre would allow faster rotation before sliding.

Think Like a Physicist: Notice that mass does not appear in the formula. A heavy coin and a light coin at the same radius, with the same friction coefficient, slide off at exactly the same angular velocity. Mass cancels for the same reason as in every other horizontal circular motion problem: both the required force and the available friction are proportional to mass.

REAL Example 29

During a physics demonstration, **Mr. Akilikubwa** places several coins at different distances from the centre of a spinning turntable. He slowly increases the speed. **Kipute** observes that the coin nearest the edge slides off first, then the next one, and the coin closest to the centre is the last to slide.

Kipanga is puzzled: “All the coins are on the same turntable spinning at the same speed. Why don’t they all slide off at the same time?”

Explain why the outermost coin slides first.

Solution

All the coins share the same angular velocity ω because they are on the same rigid turntable. However, the centripetal force required to keep each coin moving in a circle is $f = m\omega^2 r$, which is proportional to the radius r .

A coin at a larger radius needs a greater centripetal force. The maximum friction available is $\mu_s mg$, which is the same for all coins (assuming equal mass and the same surface). So the coin at the largest radius demands the most force while having the same friction limit as the others. So it reaches the sliding threshold first.

As the turntable speeds up further, the sliding threshold is reached at progressively smaller radii, so the coins slide off one by one, starting from the outermost.

Making Sense of the Answer: Think of it as a competition between demand and supply. Every coin has the same friction supply ($\mu_s mg$). But the outer coins have greater centripetal demand ($m\omega^2 r$). The coin with the greatest demand loses the competition first.

Think Like a Physicist: The critical angular velocity is $\omega_{\max} = \sqrt{\frac{\mu_s g}{r}}$. Since $\omega_{\max}$ decreases as r increases, outer objects always have a lower threshold. This principle applies to any rotating system: centrifuges, spin dryers, and even planetary rings all obey the same rule.

HOT Example 30

Two coins are placed on a horizontal turntable. Coin A (mass 10g) is placed 10cm from the centre, and coin B (mass 25g) is placed 20cm from the centre. The coefficient of static friction is 0.4 for both coins. The turntable starts from rest and its angular velocity increases gradually. Take $g = 9.8 \text{ m/s}^2$.

Which coin slides first, and at what angular velocity does it happen? At that instant, what fraction of the maximum friction is being used by the other coin?

Solution

The maximum angular velocity before sliding is:

$$\omega_{\max} = \sqrt{\frac{\mu_s g}{r}}$$

For coin A ($r = 0.1\text{m}$):

$$\omega_{\max, A} = \sqrt{\frac{0.4 \times 9.8}{0.10}} = 6.26 \text{ rad/s}$$

For Coin B ($r = 0.20 \text{ m}$):

$$\omega_{\max, B} = \sqrt{\frac{0.4 \times 9.8}{0.20}} = 4.43 \text{ rad/s}$$

Since $\omega_{\max, B} < \omega_{\max, A}$, **coin B slides first** at $\omega = 4.43 \text{ rad/s}$.

At $\omega = 4.43 \text{ rad/s}$, the friction force on coin A is:

$$f_A = m_A \omega^2 r_A = 0.010 \text{ kg} \times (4.43 \text{ rad/s})^2 \times 0.10 \text{ m} = 0.0196 \text{ N}$$

The maximum friction on coin A is:

$$f_{\max, A} = \mu_s m_A g = 0.4 \times 0.01 \text{ m} \times 9.8 \text{ m/s}^2 = 0.0392 \text{ N}$$

Fraction used:

$$\frac{f_A}{f_{\max, A}} = \frac{0.0196}{0.0392} = 0.5$$

At the instant coin B slides off, coin A is using exactly **half** (50%) of its available friction. It has a comfortable safety margin.

Making Sense of the Answer: *Coin B slides first despite being heavier. Mass is irrelevant; only radius matters. Coin B is twice as far from the centre, so it needs twice the centripetal force per unit mass. At the moment coin B reaches its limit, coin A is only halfway to its own limit because it sits at half the radius. The factor of exactly one-half is not coincidence: since $r_A = \frac{1}{2} r_B$, and friction used is proportional to $\omega^2 r$, the ratio is $\frac{r_A}{r_B} = \frac{1}{2}$.*

Think Like a Physicist: *This problem tests whether you truly understand that radius, not mass, determines which object slides first. Many students instinctively expect the heavier coin to slide first because it “has more inertia.” But more mass also means more friction, and the two cancel. Only radius remains as the deciding factor.*

That brings horizontal circular motion to a close. From the simplest case of a car on a flat road, through the leaning cyclist, the conical pendulum, the banked curve, and finally the rotating platform, one principle has guided every analysis: identify the real forces, resolve them, and set the net inward component equal to $\frac{mv^2}{r}$ or $m\omega^2 r$.

In the next subtopic, we leave the horizontal plane and enter a world where gravity no longer sits quietly in the background. In vertical circular motion, gravity actively participates: it helps the centripetal force at the top and opposes it at the bottom. The speed changes, the tension changes, and new critical conditions emerge. The physics becomes richer, but the method remains the same.

VERTICAL CIRCULAR MOTION

Understanding Vertical Circular Motion

In horizontal circular motion, gravity stayed in the background. It held objects down, balanced normal reactions, and determined the weight. But it never directly helped or opposed the centripetal force. The centripetal force came from friction, tension, or the horizontal component of a normal reaction, while gravity quietly maintained vertical equilibrium.

In vertical circular motion, everything changes. The circular path is now oriented vertically, and gravity acts along the same line as the centripetal direction at certain points. At the top of the circle, gravity pulls the object **toward the centre** and helps maintain the circular path. At the bottom, gravity pulls the object **away**

from the centre and must be overcome. At the sides, gravity acts perpendicular to the radius and contributes nothing to the centripetal force.

This means the forces on the object change continuously as it moves around the loop. The tension in a string, or the normal reaction from a track, is different at every position. The speed also changes: the object slows down as it rises and speeds up as it falls, because gravity does work on it throughout the motion. Vertical circular motion is therefore **not uniform**: the speed is not constant.

Despite this complexity, the method remains exactly the same as in horizontal circular motion. At any instant, we apply Newton's second law in the radial direction (toward the centre) and set the net inward force equal to $\frac{mv^2}{r}$, where v is the speed at that particular point. The only difference is that the forces contributing to the net inward force change with position.

Forces at different positions in the vertical circle

To understand vertical circular motion, we must examine what happens at three key positions: the **bottom**, the **top**, and the **side** (at the level of the centre). Consider an object of mass m attached to a string of length r and swung in a vertical circle. At each position, we identify the forces and apply Newton's second law toward the centre.

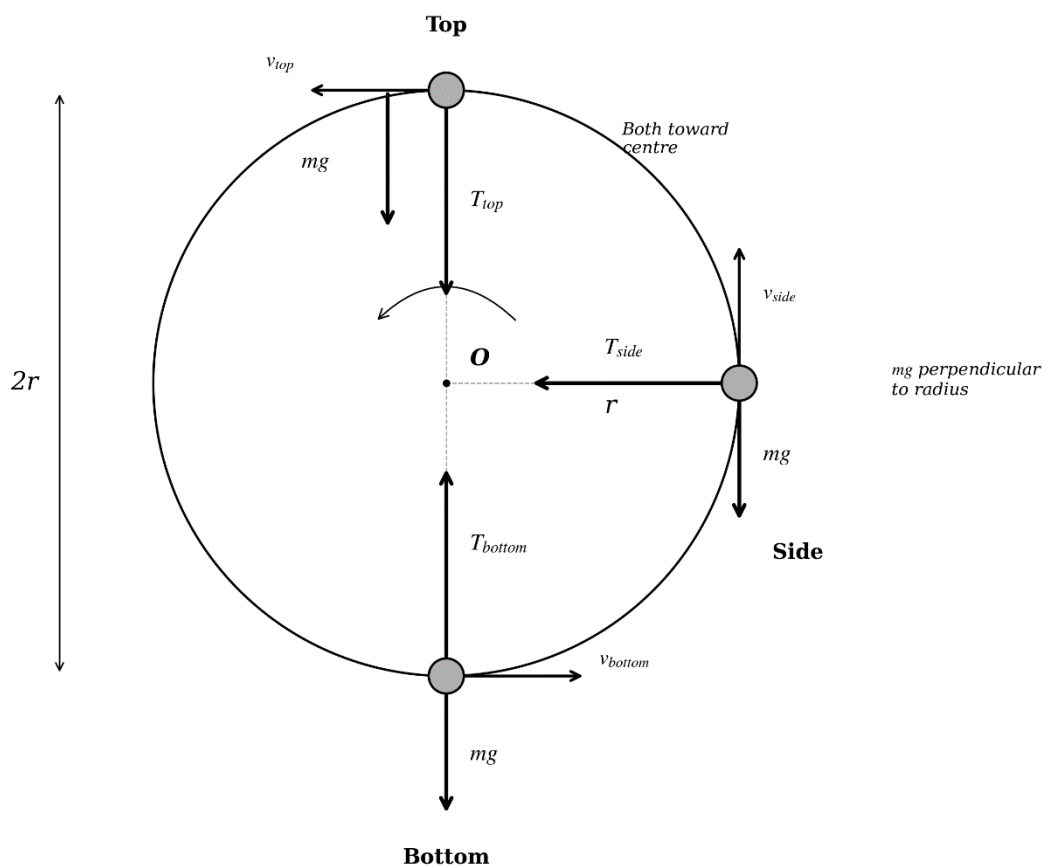


Figure: Forces on an object at three key positions in a vertical circle. At the bottom, tension acts toward the centre (upward) and weight acts away from it (downward). At the top, both tension and weight act toward the centre (downward). At the side, tension acts toward the centre (horizontally) while weight acts perpendicular to the radius (downward). The velocity is tangent to the circle at each position.

At the bottom of the circle

At the lowest point, the centre of the circle is directly above the object. The forces are:

- Tension T_{bottom} acts upward (toward the centre).
- Weight mg acts downward (away from the centre).

Applying Newton's second law toward the centre (upward at this point):

$$T_{\text{bottom}} - mg = \frac{mv_{\text{bottom}}^2}{r}$$

From which:

$$T_{\text{bottom}} = mg + \frac{mv_{\text{bottom}}^2}{r}$$

The tension at the bottom is **greater** than the weight. The string must support the weight against gravity and simultaneously provide the centripetal force. *This is the position where the tension is maximum and the string is most likely to break.*

At the top of the circle

At the highest point, the centre is directly below the object. The forces are:

- Tension T_{top} acts downward (toward the centre).
- Weight mg acts downward (also toward the centre).

Both forces point in the same direction: toward the centre. Applying Newton's second law toward the centre (downward at this point):

$$T_{\text{top}} + mg = \frac{mv_{\text{top}}^2}{r}$$

From which:

$$T_{\text{top}} = \frac{mv_{\text{top}}^2}{r} - mg$$

The tension at the top is **less** than at the bottom. Gravity now assists the centripetal force, so the string does not need to work as hard. *This is the position where the tension is minimum. If the speed is too low, the tension becomes zero or the formula gives a negative value, meaning the string goes slack and the object falls out of the circular path.*

At the side of the circle (level with the centre)

At the position level with the centre, the forces are:

- Tension T_{side} acts horizontally toward the centre.
- Weight mg acts vertically downward (perpendicular to the radius at this point).

Since weight is perpendicular to the radial direction, it does not contribute to the centripetal force. Applying Newton's second law toward the centre:

$$T_{\text{side}} = \frac{mv_{\text{side}}^2}{r}$$

At this position, the tension provides the entire centripetal force by itself, and gravity has no radial effect. However, gravity does cause the speed to change as the object moves through this position.

Summary of tension at three key positions

$$T_{\text{bottom}} = mg + \frac{mv_{\text{bottom}}^2}{r} \quad (\text{maximum})$$

$$T_{\text{side}} = \frac{mv_{\text{side}}^2}{r}$$

$$T_{\text{top}} = \frac{mv_{\text{top}}^2}{r} - mg \quad (\text{minimum})$$

Tension at a general angle

The bottom, top, and side are three special positions. *But what if you need the tension when the string makes an angle θ measured from the lowest point (the bottom)?*

At a general angle θ from the bottom, the object is at a height $h = r - r\cos\theta = r(1 - \cos\theta)$ above the lowest point. The string points from the object toward the centre, and the weight mg acts vertically downward.

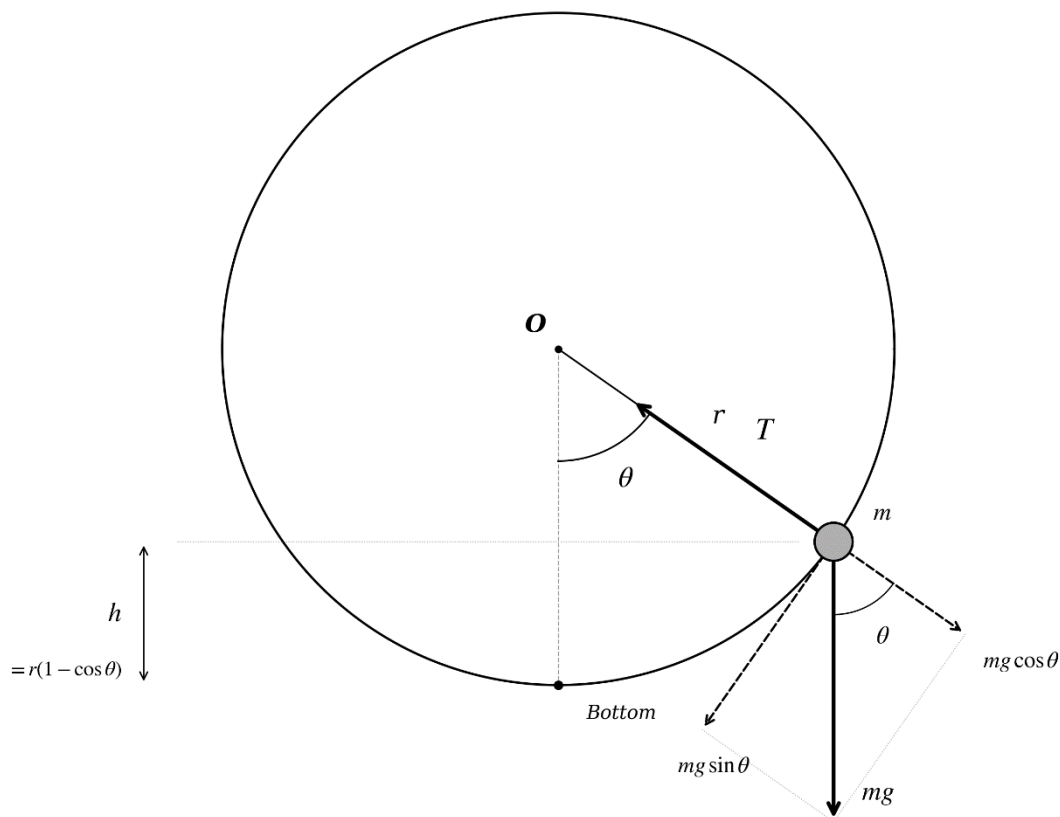


Figure: An object at a general angle θ from the bottom of a vertical circle. Tension T acts along the string toward the centre O . Weight mg acts vertically downward and is resolved into two components: $mg\cos\theta$ along the radius (away from the centre) and $mg\sin\theta$ along the tangent. The angle between mg and the outward radial direction equals θ . The height above the bottom is $h = r(1 - \cos\theta)$.

The component of the weight along the radius (toward or away from the centre) is $mg\cos\theta$. At angle θ from the bottom, this component acts away from the centre (opposing the centripetal direction).

Applying Newton's second law toward the centre:

$$T - mg\cos\theta = \frac{mv^2}{r}$$

From which:

$$T = mg\cos\theta + \frac{mv^2}{r}$$

This is the **general expression for tension at any point** in the vertical circle. Let us verify that it gives our earlier results at the three special positions:

At the **bottom** ($\theta = 0^\circ$): $\cos 0^\circ = 1$, so $T = mg + \frac{mv^2}{r}$. Correct.

At the **side** ($\theta = 90^\circ$): $\cos 90^\circ = 0$, so $T = \frac{mv^2}{r}$. Correct.

At the **top** ($\theta = 180^\circ$): $\cos 180^\circ = -1$, so $T = -mg + \frac{mv^2}{r} = \frac{mv^2}{r} - mg$. Correct.

All three special cases emerge naturally from the general formula. This confirms that the general expression is consistent.

Finding speed at any angle using energy conservation

To use the general tension formula, we need the speed v at angle θ . Using conservation of energy between the bottom and the position at angle θ :

$$\frac{1}{2}mv_{\text{bottom}}^2 = \frac{1}{2}mv^2 + mgh$$

$$\frac{1}{2}v_{\text{bottom}}^2 = \frac{1}{2}v^2 + gh$$

$$v^2 = v_{\text{bottom}}^2 - 2gh$$

Where $h = r(1 - \cos\theta)$ (you can easily deduce it from diagram by using geometry):

$$v^2 = v_{\text{bottom}}^2 - 2gr(1 - \cos\theta)$$

Substituting into the general tension formula:

$$T = mg\cos\theta + \frac{m}{r}(v_{\text{bottom}}^2 - 2gr(1 - \cos\theta))$$

$$T = mg\cos\theta + \frac{mv_{\text{bottom}}^2}{r} - 2mg(1 - \cos\theta)$$

$$\mathbf{T = \frac{mv_{\text{bottom}}^2}{r} - 2mg + 3mg\cos\theta}$$

This powerful expression gives the tension at any angle θ in terms of the speed at the bottom alone. It shows clearly that:

- Tension decreases as θ increases from 0° to 180° (because $\cos\theta$ decreases), confirming that the bottom has maximum tension and the top has minimum tension.
- The tension depends on $\cos\theta$, so it changes smoothly and continuously around the circle.

It is important for you to understand that:

The speed v is different at each position. As the object rises from bottom to top, it loses kinetic energy and gains gravitational potential energy, so it slows down. As it descends from top to bottom, it speeds up. This means $v_{\text{bottom}} > v_{\text{side}} > v_{\text{top}}$ for an object that completes the full circle.

To find the relationship between speeds at different positions, we use **conservation of energy**, which we will apply in the worked examples and in the next subtopic.

The difference between the tension at the bottom and at the top

A useful result can be obtained by subtracting the tension at the top from the tension at the bottom:

$$T_{\text{bottom}} - T_{\text{top}} = \left(mg + \frac{mv_{\text{bottom}}^2}{r} \right) - \left(\frac{mv_{\text{top}}^2}{r} - mg \right)$$

$$T_{\text{bottom}} - T_{\text{top}} = 2mg + \frac{m}{r}(v_{\text{bottom}}^2 - v_{\text{top}}^2)$$

Using conservation of energy between the bottom and top of the circle:

$$\frac{1}{2}mv_{\text{bottom}}^2 = \frac{1}{2}mv_{\text{top}}^2 + mg(2r)$$

From which:

$$v_{\text{bottom}}^2 - v_{\text{top}}^2 = 4gr$$

Substituting:

$$T_{\text{bottom}} - T_{\text{top}} = 2mg + \frac{m}{r} \times 4gr = 2mg + 4mg$$

$$\mathbf{T_{\text{bottom}} - T_{\text{top}} = 6mg}$$

This is an elegant result: regardless of the speed, the difference in tension between the bottom and the top is always exactly $6mg$. This result holds for any object completing a full vertical circle on a string, provided no energy is lost to friction or air resistance.

With the framework established, let us practise through worked examples.

BINDER Example 31

A ball of mass 0.4 kg is attached to a string of length 0.5 m and swung in a vertical circle. At the bottom of the circle, the speed is 6 m/s . Take $g = 9.8\text{ m/s}^2$.

- Calculate the tension in the string at the bottom of the circle.
- State whether this is the maximum or minimum tension in the string during the motion. Explain briefly.

Solution

- Using:

$$T_{\text{bottom}} = mg + \frac{mv_{\text{bottom}}^2}{r}$$

Where: $m = 0.4\text{ kg}$, $g = 9.8\text{ m/s}^2$, $v_{\text{bottom}} = 6\text{ m/s}$, $r = 0.5\text{ m}$

Substituting:

$$T_{\text{bottom}} = 0.4\text{kg} \times 9.8\text{ m/s}^2 + \frac{0.4\text{kg} \times (6\text{ m/s}^2)^2}{0.5\text{m}} = 32.7\text{N}$$

The tension at the bottom is 32.7N .

- This is the **maximum** tension. At the bottom, the string must support the weight and provide centripetal force simultaneously, making the tension greatest at this position.

Making Sense of the Answer: The weight is only 3.92N , but the tension is 32.7N , more than eight times the weight. The centripetal force demand (28.8N) dominates because the speed is high and the radius is small. This is why strings and ropes can snap at the bottom of a vertical swing even when the object is light.

Think Like a Physicist: At the bottom, tension and weight pull in opposite directions, so the net inward force is $T - mg$. At the top, they pull in the same direction, so the net inward force is $T + mg$. Always draw the forces first and identify which direction is “toward the centre” at each position.

BINDER Example 32

A ball of mass 0.5kg is attached to a string of length 0.8m and swung in a vertical circle. At the top of the circle, the tension in the string is 5N . Take $g = 9.8\text{ m/s}^2$.

Determine the tension in the string at the bottom of the circle.

Solution

This problem can be solved directly using the result $T_{\text{bottom}} - T_{\text{top}} = 6mg$:

$$T_{\text{bottom}} = T_{\text{top}} + 6mg$$

Substituting:

$$T_{\text{bottom}} = 5\text{N} + 6 \times 0.5\text{kg} \times 9.8\text{ m/s}^2 = 34.4\text{N}$$

The tension at the bottom is 34.4N .

Making Sense of the Answer: The tension increases by exactly $6mg = 29.4\text{ N}$ from top to bottom. This is a fixed difference that depends only on the mass and gravity, not on the speed. Even if the ball were moving faster or slower (as long as it completes the circle), this difference would be the same. The result is powerful because it allows us to find one tension from the other without knowing the speed at either point.

Think Like a Physicist: The result $T_{\text{bottom}} - T_{\text{top}} = 6mg$ is worth memorising. It provides a shortcut in many vertical circular motion problems. However, remember that it applies only when energy is conserved (no friction or air resistance) and only for motion on a string where tension provides the centripetal force.

REAL Example 33

Kipanga swings a stone tied to a string in a vertical circle. He notices that the string feels much tighter when the stone passes through the bottom of the circle than when it passes through the top. He says to **Kipute**: “The stone feels heavier at the bottom. Does its weight change during the swing?”

Help Kipute answer Kipanga’s question.

Solution

The weight of the stone (mg) does not change; gravity pulls it downward with the same force at every point in the circle. What changes is the **tension** in the string.

At the bottom, the centre of the circle is above the stone. The string must pull upward with enough force to both support the weight and provide the centripetal force toward the centre. So the tension at the bottom is

$$T_{\text{bottom}} = mg + \frac{mv_{\text{bottom}}^2}{r}, \text{ which is greater than the weight.}$$

At the top, the centre is below the stone. Both gravity and tension pull toward the centre (downward). Gravity already provides part of the centripetal force, so the string needs to supply less. The tension at the top is

$$T_{\text{top}} = \frac{mv_{\text{top}}^2}{r} - mg, \text{ which is less than at the bottom.}$$

What Kipanga feels as “heavier” is not a change in weight but an increase in tension. The string must work hardest at the bottom and easiest at the top. The difference is always exactly $6mg$.

Making Sense of the Answer: *It is the same illusion as feeling heavier when a lift accelerates upward. Your weight has not changed, but the support force (tension or normal reaction) has increased. At the bottom of a vertical circle, the string provides an upward acceleration (centripetal), making it feel like the object is heavier.*

Think Like a Physicist: *The feeling of heaviness or lightness is determined by the contact force (tension or normal reaction), not by the actual weight.*

HOT Example 34

A ball of mass 0.25kg is attached to a string of length 0.6m and set in motion in a vertical circle. The speed at the bottom of the circle is 4.5m/s . Take $g = 9.8\text{ m/s}^2$.

Determine the angle from the bottom at which the string goes slack.

Solution

The string goes slack when the tension becomes zero. Using the general expression for tension at angle θ from the bottom:

$$T = \frac{mv_{\text{bottom}}^2}{r} - 2mg + 3mg\cos\theta$$

Setting $T = 0$:

$$0 = \frac{mv_{\text{bottom}}^2}{r} - 2mg + 3mg\cos\theta$$

Making $\cos\theta$ the subject:

$$\begin{aligned} 3mg\cos\theta &= 2mg - \frac{mv_{\text{bottom}}^2}{r} \\ \cos\theta &= \frac{2}{3} - \frac{v_{\text{bottom}}^2}{3gr} \end{aligned}$$

Where: $v_{\text{bottom}} = 4.5\text{ m/s}$, $g = 9.8\text{ m/s}^2$, $r = 0.6\text{ m}$

Substituting:

$$\begin{aligned} \cos\theta &= \frac{2}{3} - \frac{(4.5\text{ m/s})^2}{3 \times 9.8\text{ m/s}^2 \times 0.6\text{ m}} = -0.481 \\ \theta &= \cos^{-1}(-0.481) = 118.8^\circ \end{aligned}$$

The string goes slack at 118.8° from the bottom.

Making Sense of the Answer: The angle is between 90° (the side) and 180° (the top). This means the ball passes through the side position successfully but loses tension before reaching the top. At 118.8° from the bottom, the ball is well above the centre of the circle. After the string goes slack, the ball follows a projectile path (parabola) rather than continuing along the circle. If the speed at the bottom were greater, the string would go slack at a larger angle (closer to the top), or not at all if the speed is sufficient to complete the full circle.

Think Like a Physicist: The formula $\cos\theta = \frac{2}{3} - \frac{v_{\text{bottom}}^2}{3gr}$ tells us directly whether the ball completes the circle. If the right side is less than -1 (meaning $\cos\theta < -1$, which is impossible), the tension never reaches zero and the ball completes the full circle. This happens when $v_{\text{bottom}}^2 > 5gr$, a result we will derive in the next subtopic.

The ideas are now in place. In the next subtopic, we examine the most important question in vertical circular motion: *what is the minimum speed needed to maintain a complete vertical circle?* This is where strings go slack, water falls from buckets, and roller coasters must be carefully engineered.

Minimum Speed and Critical Condition of Object on a String

In the previous subtopic, we derived the tension at the top of the circle as:

$$T_{\text{top}} = \frac{mv_{\text{top}}^2}{r} - mg$$

An important question arises from this equation: *what happens if the speed at the top is very small?*

As v_{top} decreases, the term $\frac{mv_{\text{top}}^2}{r}$ becomes smaller and the tension decreases. At some critical speed, the tension reaches zero. If the speed drops below this critical value, the formula gives a negative tension, but a string cannot push; it can only pull. So a negative tension is physically meaningless. It means the string has gone slack and the object has left the circular path.

Finding the minimum speed at the top

At the critical condition, the string is on the verge of going slack:

$$T_{\text{top}} = 0$$

Substituting into the equation for tension at the top:

$$0 = \frac{mv_{\text{min,top}}^2}{r} - mg$$

$$\frac{mv_{\text{min,top}}^2}{r} = mg$$

Making $v_{\text{min,top}}$ the subject:

$$v_{\text{min,top}} = \sqrt{gr}$$

At this critical speed, gravity alone provides exactly the centripetal force needed at the top of the circle. If the speed becomes any smaller, gravity would be more than enough for the circular motion at that speed. However, this would require a circular path of a smaller radius while the object cannot actually move in that smaller circle because the string fixes the radius. As a result, the object leaves the circular path and begins to fall under gravity.

Again, **you have to understand this:** The minimum speed at the top does not depend on mass. A heavy ball and a light ball on the same string both need the same minimum speed $\sqrt{gr}$ at the top to complete the circle. This is because both the centripetal force requirement ($\frac{mv^2}{r}$) and the gravitational force (mg) are proportional to mass, and mass cancels.

Finding the minimum speed at the bottom for a complete circle

Knowing the minimum speed at the top is useful, but in practice we control the speed at the bottom (where we launch or swing the object). The question becomes: *what minimum speed must the object have at the bottom so that it still has at least $\sqrt{gr}$ at the top?*

We use conservation of energy between the bottom and the top. The height difference is $2r$ (the full diameter of the circle).

$$\frac{1}{2}mv_{\text{bottom}}^2 = \frac{1}{2}mv_{\text{top}}^2 + mg(2r)$$

At the critical condition, $v_{\text{top}} = \sqrt{gr}$, so $v_{\text{top}}^2 = gr$:

$$\frac{1}{2}mv_{\text{min,bottom}}^2 = \frac{1}{2}m(gr) + mg(2r)$$

$$\frac{1}{2}v_{\text{min,bottom}}^2 = \frac{1}{2}gr + 2gr = \frac{5}{2}gr$$

$$v_{\text{min,bottom}} = \sqrt{5gr}$$

This is a key result in vertical circular motion. To complete a full vertical circle on a string, the speed at the bottom must be at least $\sqrt{5gr}$. Below this speed, the string goes slack somewhere before reaching the top.

Connecting to the general angle result

In the previous subtopic, the Think Like a Physicist section for Example 34 noted that the ball completes the circle when $v_{\text{bottom}}^2 > 5gr$. We have now derived this condition formally. The two results are consistent: $v_{\text{min,bottom}} = \sqrt{5gr}$ is the boundary between completing the circle and falling out of it.

Summary of critical speeds for a vertical circle on a string:

$$v_{\text{min,top}} = \sqrt{gr}$$

$$v_{\text{min,bottom}} = \sqrt{5gr}$$

Tension at the bottom when the ball just completes the circle

When $v_{\text{bottom}} = \sqrt{5gr}$:

$$T_{\text{bottom}} = mg + \frac{mv_{\text{bottom}}^2}{r} = mg + \frac{m(5gr)}{r} = mg + 5mg = 6mg$$

At the critical condition, the tension at the bottom is exactly $6mg$. This is a useful benchmark: if the string can withstand a tension of at least $6mg$, the object can just complete the vertical circle.

With the critical conditions established, let us put them to work.

BINDER Example 35

A ball of mass 0.3kg is attached to a string of length 0.5m and swung in a vertical circle. Take $g = 9.8\text{m/s}^2$.

- Calculate the minimum speed at the top for the ball to complete the circle.
- Calculate the minimum speed at the bottom for the ball to complete the circle.
- Find the tension in the string at the bottom when the ball has exactly the minimum speed to complete the circle.

Solution

- Using:

$$v_{\text{min,top}} = \sqrt{gr}$$

Where: $g = 9.8\text{m/s}^2$, $r = 0.5\text{m}$

Substituting:

$$v_{\min, \text{top}} = \sqrt{9.8 \text{ m/s}^2 \times 0.5 \text{ m}} = \sqrt{4.9 \text{ m}^2/\text{s}^2} = 2.21 \text{ m/s}$$

The minimum speed at the top is 2.21 m/s.

(b) Using:

$$v_{\min, \text{bottom}} = \sqrt{5gr}$$

Substituting:

$$v_{\min, \text{bottom}} = \sqrt{5 \times 9.8 \text{ m/s}^2 \times 0.5 \text{ m}} = \sqrt{24.5 \text{ m}^2/\text{s}^2} = 4.95 \text{ m/s}$$

The minimum speed at the bottom is 4.95 m/s.

(c) Using:

$$T_{\text{bottom}} = 6mg$$

Substituting:

$$T_{\text{bottom}} = 6 \times 0.3 \text{ kg} \times 9.8 \text{ m/s}^2 = 17.64 \text{ N}$$

The tension at the bottom is 17.64 N.

Making Sense of the Answer: The minimum speed at the bottom (4.95 m/s) is more than double the minimum speed at the top (2.21 m/s). This is because the ball must have enough kinetic energy at the bottom to climb a height of $2r = 1 \text{ m}$ and still arrive at the top with speed 2.21 m/s. Most of the kinetic energy at the bottom is consumed by the climb against gravity. The tension at the bottom (17.64 N) is six times the weight (2.94 N), confirming that the string is under considerable stress even at the minimum speed.

Think Like a Physicist: The ratio $\frac{v_{\min, \text{bottom}}}{v_{\min, \text{top}}} = \frac{\sqrt{5gr}}{\sqrt{gr}} = \sqrt{5} \approx 2.24$. This ratio is universal: the minimum speed at the bottom is always $\sqrt{5}$ times the minimum speed at the top, regardless of the radius or the mass.

REAL Example 36

During a school science fair, **Kipanga** fills a bucket with water and swings it in a vertical circle over his head. To everyone's amazement, the water stays inside the bucket even when the bucket is upside down at the top of the circle. **Kipute** watches carefully and then says:

"I bet if you swing it slowly, the water will fall on your head."

Kipanga grins and slows down. Sure enough, as the swing becomes slower, water splashes out of the bucket and drenches him from above. The class erupts in laughter.

Mr. Akilikubwa, wiping tears of laughter from his eyes, says: "Kipanga has just demonstrated the minimum speed condition. Now, Kipute, explain to the class: why did the water stay in the bucket at high speed, and why did it fall out when Kipanga slowed down?"

Help Kipute explain to the class.

Solution

When the bucket is at the top of the vertical circle, it is upside down. At this point, the centre of the circle is directly below. Both the weight of the water and any contact force from the bucket act downward (toward the centre).

At high speed, the water needs a large centripetal force ($\frac{mv^2}{r}$) to follow the circular path. Gravity alone provides mg , but this is not enough. The bucket base must push the water downward to supply the extra centripetal force. Since the bucket pushes down on the water, the water pushes up on the bucket (Newton's third law). The water stays in contact with the bucket and remains inside.

When Kipanga slows down, the required centripetal force decreases. At the critical speed $v = \sqrt{gr}$, gravity alone provides exactly the centripetal force needed, and the bucket base exerts zero force on the water. Below this speed, gravity provides more downward force than needed for the circular path. The water wants to fall faster than the bucket, so it separates from the bucket base and falls out, landing on Kipanga's head.

In short: the water stays in when the required centripetal acceleration ($\frac{v^2}{r}$) exceeds g , and falls out when $\frac{v^2}{r} < g$.

Making Sense of the Answer: *The bucket trick works because at high speed, the water's inertia is so large that gravity cannot pull it away from the circular path. The water "wants" to fly off tangentially but the bucket forces it to curve. At low speed, gravity dominates over inertia, and the water simply falls. The critical speed $\sqrt{gr}$ is the balance point between these two regimes.*

Think Like a Physicist: *This is exactly the same physics as the string going slack. Replace "bucket base pushing water" with "string pulling ball" and the analysis is identical. In both cases, the contact force reaches zero at $v = \sqrt{gr}$ at the top. The bucket trick is simply a dramatic demonstration of the minimum speed condition.*

HOT Example 37

A ball on a string of length 1.2m is set in motion in a vertical circle. The speed at the bottom is 6.5m/s. Take $g = 9.8\text{m/s}^2$.

Determine whether the ball completes the full circle, and if not, find the angle from the bottom at which the string goes slack.

Solution

First, check whether the speed at the bottom is sufficient to complete the circle.

The minimum speed required at the bottom is:

$$v_{\text{min,bottom}} = \sqrt{5gr} = \sqrt{5 \times 9.8\text{m/s}^2 \times 1.2\text{m}} = \sqrt{58.8\text{m}^2/\text{s}^2} = 7.67\text{m/s}$$

Since the actual speed at the bottom (6.5m/s) is less than the minimum required (7.67m/s), the ball does **not** complete the full circle. The string goes slack before the ball reaches the top.

To find the angle where the string goes slack, set $T = 0$ in the general formula:

$$\cos\theta = \frac{2}{3} - \frac{v_{\text{bottom}}^2}{3gr}$$

Substituting:

$$\begin{aligned}\cos\theta &= \frac{2}{3} - \frac{(6.5\text{m/s})^2}{3 \times 9.8\text{m/s}^2 \times 1.2\text{m}} = -0.531 \\ \theta &= \cos^{-1}(-0.531) = 122.1^\circ\end{aligned}$$

The string goes slack at 122.1° from the bottom.

Making Sense of the Answer: *The ball fails to complete the circle because 6.5m/s is well below the required 7.67m/s. It goes slack at 122.1° from the bottom, which is about 32° past the horizontal (side) position and roughly 58° short of the top. The ball makes it more than two-thirds of the way around but runs out of speed before reaching the top. After the string goes slack, the ball follows a parabolic projectile path.*

Think Like a Physicist: *This problem requires two distinct skills: first, recognising that the speed is insufficient (comparing v_{bottom} with $\sqrt{5gr}$), and second, finding exactly where the string fails (using the general tension formula with $T = 0$). A common mistake is to assume that if the ball cannot reach the top, the string must go slack at the top. In reality, it goes slack at whatever angle the tension first becomes zero, which is generally below the top.*

That brings the critical speed conditions to a close. We now know when a string goes slack and what speed is needed to prevent it. In the next subtopic, we replace the string with a track: an object moves inside a circular loop (like a roller coaster). The physics is similar, but with one important difference: the track pushes rather than pulls, and the critical condition involves the normal reaction rather than tension.

Loop-the-Loop

A roller coaster climbs slowly to the top of a tall slope, pauses for a breathless moment, then plunges downward, gaining terrifying speed. Moments later, it enters a vertical loop and the passengers find themselves upside down, pressed into their seats, screaming with a mixture of terror and delight. The car races around the inside of the loop and emerges safely at the bottom.

How does this work? Why don't the passengers fall when they are upside down? And how do engineers decide how tall the initial slope must be?

The answers come from the same physics we developed for the ball on a string, with one crucial difference. A string pulls. A track pushes. The string exerts tension directed inward along its length. The track exerts a **normal reaction** directed perpendicular to its surface. For a circular loop, the inner surface pushes the object toward the centre at every point. So in the loop-the-loop, normal reaction replaces tension as the force that works alongside gravity to maintain circular motion.

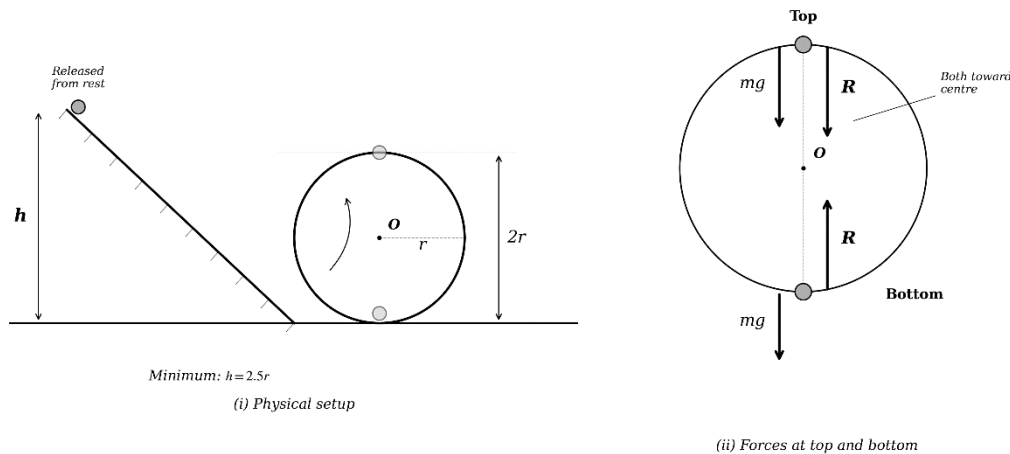


Figure: Loop-the-loop. (i) A ball is released from rest at height h on a ramp and enters a vertical circular loop of radius r . The minimum release height for a complete loop is $h = 2.5r$. (ii) Forces on the ball at the top and bottom of the inside loop. At the top, both the normal reaction R and weight mg act downward toward the centre. At the bottom, R acts upward toward the centre while mg acts downward away from it.

Forces at the top of an inside loop

At the top of a loop, the object is on the **inside** of the track. The centre of the circle is directly below. Both forces act downward (toward the centre):

- Weight mg acts downward.
- Normal reaction R from the track acts downward (the track is above the object and pushes it toward the centre).

Applying Newton's second law toward the centre (downward):

$$mg + R = \frac{mv_{\text{top}}^2}{r}$$

From which:

$$R = \frac{mv_{\text{top}}^2}{r} - mg$$

Compare this carefully with the string result: $T_{\text{top}} = \frac{mv_{\text{top}}^2}{r} - mg$. The formula is identical in structure. The only change is the symbol: R instead of T .

Forces at the bottom of the loop

At the bottom, the object is on the inside of the track. The centre is directly above. The forces oppose each other:

- Weight mg acts downward (away from centre).
- Normal reaction R acts upward (toward the centre).

Applying Newton's second law toward the centre (upward):

$$R - mg = \frac{mv_{\text{bottom}}^2}{r}$$

$$R = mg + \frac{mv_{\text{bottom}}^2}{r}$$

Again, identical in structure to $T_{\text{bottom}} = mg + \frac{mv_{\text{bottom}}^2}{r}$.

The critical condition: losing contact

For the string, the critical condition was $T = 0$ (string goes slack). For the track, the critical condition is $R = 0$ (object loses contact with the track surface).

Setting $R = 0$ at the top:

$$0 = \frac{mv_{\text{min,top}}^2}{r} - mg$$

$$v_{\text{min,top}} = \sqrt{gr}$$

The same result as for the string. At this critical speed, gravity alone provides the centripetal force at the top, and the track exerts no force on the object. Below this speed, the object separates from the track.

There is, however, a subtle but important difference. When a string goes slack, the object falls inward (toward the centre) because nothing holds it outward. When an object loses contact with the **inside** of a loop, it also falls away from the track; but now that means falling inward as well (the track is on the outside of the object's position). In both cases, the object leaves the circular path and follows a projectile trajectory.

Minimum release height for a complete loop

Roller coasters and toy loop-the-loop tracks launch the object from a height, converting gravitational potential energy into kinetic energy. The practical question is: *from what minimum height must the object be released to complete the loop?*

Let the loop have radius r , and let the object be released from rest at height h above the bottom of the loop.

Using conservation of energy between the release point and the top of the loop:

$$mgh = \frac{1}{2}mv_{\text{top}}^2 + mg(2r)$$

At the critical condition, $v_{\text{top}}^2 = gr$:

$$mgh_{\text{min}} = \frac{1}{2}m(gr) + mg(2r)$$

$$gh_{\text{min}} = \frac{1}{2}gr + 2gr = \frac{5}{2}gr$$

$$h_{\text{min}} = \frac{5}{2}r = 2.5r$$

The object must be released from a height of at least $2.5r$ above the bottom of the loop. For a loop of radius 2m , the minimum release height is 5m . Engineers always design the initial slope to be taller than this to provide a safety margin.

The worked examples that follow will bring these concepts to life and make them concrete.

BINDER Example 38

A small ball rolls without friction along a track that includes a vertical circular loop of radius 0.4m . The ball is released from rest at a height h above the bottom of the loop. Take $g = 9.8\text{m/s}^2$.

- (a) Find the minimum height from which the ball must be released to complete the loop.

- (b) At this minimum release height, find the speed of the ball at the top of the loop.
 (c) Find the normal reaction on the ball at the top of the loop at the minimum speed.

Solution

- (a) Using:

$$h_{\min} = \frac{5}{2}r = \frac{5}{2} \times 0.4\text{m} = 1\text{m}$$

The minimum release height is 1m above the bottom of the loop.

- (b) At the minimum condition, the speed at the top is:

$$v_{\min, \text{top}} = \sqrt{gr} = \sqrt{9.8\text{m/s}^2 \times 0.4\text{m}} = \sqrt{3.92\text{m}^2/\text{s}^2} = 1.98\text{m/s}$$

The speed at the top is 1.98m/s.

- (c) At the minimum speed, the normal reaction at the top is:

$$R = 0\text{N}$$

At the critical condition (minimum speed), gravity provides all the centripetal force and the track exerts no force on the ball.

Making Sense of the Answer: The release height (1m) is 2.5 times the loop radius (0.4m). The ball arrives at the top of the loop with just enough speed (1.98m/s) for gravity to maintain the circular path alone. The track barely touches the ball at this point. Any lower release height and the ball falls away from the track before reaching the top.

Think Like a Physicist: Part (c) is deceptively simple but tests deep understanding. Many students calculate R using the full formula and get zero, then doubt their answer. Trust the physics: the minimum speed condition is defined as the speed at which $R = 0$. If you set up the problem correctly and get $R = 0$ at the minimum speed, that confirms your working, not a mistake.

REAL Example 39

A toy car track includes a vertical loop. A child releases the toy car from the top of a ramp. On the first attempt, the car enters the loop, climbs partway up the inside, then falls away from the track before reaching the top. On the second attempt, the child places the car higher on the ramp, and this time it goes all the way around.

Explain why increasing the release height allows the car to complete the loop.

Solution

When the car is released from rest on the ramp, its gravitational potential energy converts into kinetic energy as it descends. A higher release point means more potential energy is available, so the car arrives at the bottom of the loop with greater speed.

As the car climbs inside the loop, it loses kinetic energy and gains potential energy. At the top of the loop, the car must still have enough speed for gravity to provide the centripetal force needed to maintain contact with the track (at minimum, $v_{\text{top}} = \sqrt{gr}$).

On the first attempt, the release height was too low. The car arrived at the bottom with insufficient speed, and its kinetic energy was exhausted before reaching the top. The car slowed to the point where gravity could no longer maintain the circular path, and it separated from the track.

On the second attempt, the higher release point provided more kinetic energy. The car reached the top of the loop with speed at or above $\sqrt{gr}$, so the track maintained contact and the car completed the loop.

The minimum release height is $2.5r$ above the bottom of the loop.

Making Sense of the Answer: Every loop-the-loop toy teaches this lesson: if the ramp is too short, the car falls. The child learns by trial and error what physics calculates precisely. The factor of 2.5 accounts for the height of the loop ($2r$) plus the extra kinetic energy needed at the top ($\frac{1}{2}r$ worth of height).

Think Like a Physicist: The minimum height of $2.5r$ is measured from the bottom of the loop, not from the ground. If the ramp connects smoothly to the loop at ground level, then $2.5r$ is the height above the ground. But if the loop is elevated, the total height above the ground would be $2.5r$ plus the height of the platform.

HOT Example 40

A ball of mass 0.2kg is released from rest at the top of a frictionless track. The track descends and enters a vertical circular loop of radius 0.5m . The release point is 2m above the bottom of the loop. Take $g = 9.8\text{m/s}^2$.

- Show that the ball completes the loop.
- Determine the speed and the normal reaction on the ball at the top of the loop.

Solution

- The minimum release height for a complete loop is:

$$h_{\min} = 2.5r = 2.5 \times 0.5\text{m} = 1.25\text{m}$$

Since the actual release height (2m) exceeds the minimum (1.25m), the ball completes the loop.

- Using conservation of energy between the release point (height $h = 2\text{m}$ above the bottom) and the top of the loop (height $2r = 1\text{m}$ above the bottom):

$$mgh = \frac{1}{2}mv_{\text{top}}^2 + mg(2r)$$

Making v_{top} the subject:

$$v_{\text{top}}^2 = 2g(h - 2r)$$

$$v_{\text{top}} = \sqrt{2g(h - 2r)}$$

Substituting:

$$v_{\text{top}} = \sqrt{2 \times 9.8\text{m/s}^2 \times (2\text{m} - 1\text{m})} = 4.43\text{m/s}$$

The speed at the top is 4.43m/s .

Normal reaction at the top:

$$R = \frac{mv_{\text{top}}^2}{r} - mg = \frac{0.2\text{kg} \times 19.6\text{m}^2/\text{s}^2}{0.5\text{m}} - 0.2\text{kg} \times 9.8\text{m/s}^2 = 5.88\text{N}$$

The normal reaction at the top is 5.88N .

Making Sense of the Answer: The release height is 2m , well above the minimum of 1.25m . The ball arrives at the top with 4.43m/s , which is more than double the minimum of $\sqrt{gr} = 2.21\text{m/s}$. As a result, the normal reaction is 5.88N , which is about three times the weight (1.96N). The ball presses firmly against the track at the top. The extra height translates directly into extra speed and extra contact force.

Think Like a Physicist: Part (a) requires a quick comparison (h versus $2.5r$), not a lengthy calculation. Recognising when a simple check answers the question saves time and shows confidence. Part (b) combines energy conservation (to find v_{top}) with Newton's second law (to find R). These two tools, used in sequence, solve almost every loop-the-loop problem.

With the loop-the-loop understood, one final scenario remains. What happens when a vehicle travels over the top of a hill? The circular path is now on the **outside** of the curve rather than the inside, and the physics flips in a surprising way. Instead of needing a minimum speed to stay on the track, there is a **maximum** speed beyond which the vehicle leaves the road. Let us see why.

Vehicle over a Hill or Hump

Every situation we have studied so far, namely the ball on a string and the loop-the-loop, involves motion on the **inside** of a circular path. The object presses against the inner surface, and the contact force points toward the centre.

But what happens when an object moves over the **outside** of a curved surface? A car cresting a hill, a motorcycle riding over a hump, or a roller coaster passing over the top of a rise; in all these cases, the object is on the outside of the curve, and the physics reverses in an important way.

On the inside of a loop, the critical question was: *what is the minimum speed to maintain contact?* Too slow, and the object falls away inward.

On the outside of a curve, the critical question flips: *what is the maximum speed before the object leaves the surface?* Too fast, and the object lifts off.

Forces at the top of a hill

Consider a vehicle of mass m travelling at speed v over the top of a circular hill of radius r . At the highest point, the centre of the circular path is directly below the vehicle.

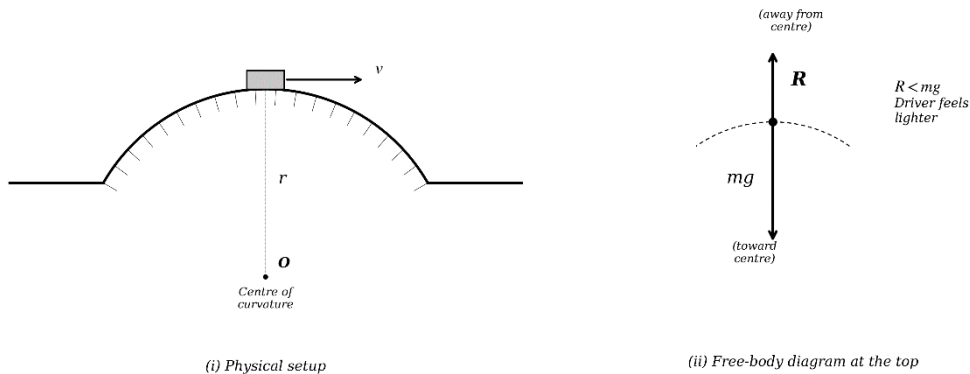


Figure: A vehicle at the top of a circular hill. (i) The centre of curvature O is below the road surface. The car travels at speed v over the crest. (ii) Free-body diagram at the top: weight mg acts downward toward the centre, while the normal reaction R acts upward away from the centre. Since $R < mg$, the driver feels lighter than normal.

The forces are:

- Weight mg acts downward (toward the centre).
- Normal reaction R acts upward (away from the centre).

This is the key difference from the inside of a loop. Here, weight acts toward the centre and the normal reaction acts away from it. They oppose each other.

Applying Newton's second law toward the centre (downward at the top of the hill):

$$mg - R = \frac{mv^2}{r}$$

From which:

$$R = mg - \frac{mv^2}{r}$$

Compare this with the top of an inside loop: $R = \frac{mv^2}{r} - mg$. The terms are reversed. On a hill, increasing speed **decreases** the normal reaction; inside a loop, increasing speed **increases** it.

Apparent weight and the sensation of lightness

The normal reaction R is what the driver feels as their "weight." At the top of a hill:

$$R = mg - \frac{mv^2}{r} = m \left(g - \frac{v^2}{r} \right)$$

Since $\frac{v^2}{r}$ is subtracted from g , the apparent weight is less than the true weight. The driver feels lighter at the top of the hill. This is the familiar stomach-dropping sensation you feel when a car goes over a bump at high speed.

The faster the car moves, the lighter the driver feels, because $\frac{v^2}{r}$ grows larger and R becomes smaller.

The critical speed: leaving the road

As speed increases, R decreases. At some critical speed, R reaches zero. The road no longer pushes the car, and the car is on the verge of lifting off the surface.

Setting $R = 0$:

$$0 = mg - \frac{mv_{\max}^2}{r}$$

$$\frac{mv_{\max}^2}{r} = mg$$

$$v_{\max} = \sqrt{gr}$$

At this speed, gravity alone provides exactly the centripetal force needed. The car follows the curved hilltop without any help from the road. Beyond this speed, the car would need more centripetal force than gravity can provide, so it lifts off and becomes briefly airborne.

Notice that $v_{\max} = \sqrt{gr}$ is the same expression as $v_{\min, \text{top}}$ for the inside loop. The critical speed has the same magnitude, but the meaning is opposite: for the inside loop it is a **minimum** (below which contact is lost), and for the hilltop it is a **maximum** (above which contact is lost).

Weightlessness at the critical speed

When $R = 0$, the driver has zero apparent weight. This is the condition of **weightlessness**. The driver is not truly weightless (gravity still acts), but there is no contact force, so they feel as though they are floating. This is the same sensation astronauts experience in orbit, where $R = 0$ and gravity provides all the centripetal force.

The worked examples bring all of this to life.

BINDER Example 41

A car of mass 1200kg travels over a hill whose crest has a radius of curvature of 50m. The speed at the top is 15m/s. Take $g = 9.8\text{m/s}^2$.

- Calculate the normal reaction on the car at the top of the hill.
- Express this as a percentage of the car's actual weight.

Solution

- Using:

$$R = mg - \frac{mv^2}{r}$$

Where: $m = 1200\text{kg}$, $g = 9.8\text{m/s}^2$, $v = 15\text{m/s}$, $r = 50\text{m}$

Substituting:

$$R = 1200\text{kg} \times 9.8\text{m/s}^2 - \frac{1200\text{kg} \times (15\text{m/s})^2}{50\text{m}} = 11760\text{N} - 5400\text{N} = 6360\text{N}$$

The normal reaction at the top is 6360N.

- The actual weight is $W = mg = 11760\text{N}$.

$$\frac{R}{W} \times 100\% = \frac{6360\text{N}}{11760\text{N}} \times 100\% = 54\%$$

The apparent weight is 54% of the actual weight.

Making Sense of the Answer: At 15m/s over a 50m-radius hill, the car loses nearly half its apparent weight. Passengers would definitely feel the "stomach drop." If the speed were higher, the effect would be more dramatic. At about 22m/s, the apparent weight would reach zero.

Think Like a Physicist: The fraction $\frac{R}{W} = 1 - \frac{v^2}{gr}$ tells you immediately how “light” the driver feels. When $\frac{v^2}{gr} = 0.5$, the driver feels half their weight. When $\frac{v^2}{gr} = 1$, the driver feels weightless. This ratio is a quick diagnostic tool.

REAL Example 42

On a family road trip, **Kipanga** notices that every time the car goes over a small hill at high speed, he feels his body lift slightly off the seat. He grabs the door handle in alarm. **Kipute**, sitting next to him, laughs: “Relax, Kipanga. You are not flying. The road just stopped pushing you as hard.”

Explain what Kipute means, and why Kipanga feels lighter at the top of the hill.

Solution

When the car is on level ground, the road pushes upward on the car with a normal reaction equal to the weight: $R = mg$. Kipanga feels his full weight pressing him into the seat.

At the top of a hill, the car follows a curved path, and the centre of curvature is below the road. Part of gravity is “used” to provide the centripetal force for this curved motion. The remaining gravity is balanced by the normal reaction, which is now less than the weight: $R = mg - \frac{mv^2}{r}$.

Since the seat pushes Kipanga upward with less force than his weight, he feels lighter. His body does not actually leave the seat (unless the speed is extremely high), but the reduced contact force creates the sensation of lifting. This is what Kipute means: the road has not disappeared; it is simply pushing with less force because some of gravity’s pull is being redirected into centripetal acceleration.

Making Sense of the Answer: The same physics explains why roller coasters are thrilling. The hills are designed so that passengers feel light (or even weightless) at the top. The sensation is entirely due to the reduced normal reaction, not any change in gravity itself.

Think Like a Physicist: What you feel as “weight” is never gravity itself; it is the contact force (normal reaction, tension, or seat force) acting on you. Gravity cannot be felt directly. This is why astronauts in orbit feel weightless even though gravity is still pulling them: there is no contact force because both the astronaut and the spacecraft are in free fall together.

HOT Example 43

A car travels along a road that passes over a circular hump of radius 40m. Take $g = 9.8\text{m/s}^2$.

At what speed does the car lose contact with the road at the top of the hump? Explain what happens to the car immediately after it loses contact.

Solution

The car loses contact when $R = 0$:

$$v_{\max} = \sqrt{gr} = \sqrt{9.8\text{m/s}^2 \times 40\text{m}} = \sqrt{392\text{m}^2/\text{s}^2} = 19.8\text{m/s}$$

The car loses contact at 19.8m/s (about 71.3km/h).

At this speed, gravity provides exactly the centripetal force needed to follow the curved road surface. Beyond this speed, gravity cannot supply enough centripetal force for the road’s curvature. The car cannot follow the tight curve of the hilltop and instead follows a gentler parabolic path (projectile motion). The car becomes briefly airborne, rising above the road surface before gravity brings it back down.

The car does not fly upward dramatically. It simply follows a trajectory with less curvature than the road, so a gap opens between the car and the road surface. When gravity eventually curves the car’s path enough to meet the road again (on the downward slope), the car lands (often with a bump).

Making Sense of the Answer: The critical speed of about 71km/h is surprisingly achievable on ordinary roads. A sharp hump with 40m radius is not unusual in hilly terrain. This is why speed bumps are effective: even a gentle hump at moderate speed creates the light sensation, and a sharper hump at higher speed can genuinely cause the car to leave the ground. Road engineers design hill crests with large radii specifically to prevent this at legal speeds.

Think Like a Physicist: After losing contact, the car is a projectile. Its horizontal velocity continues (ignoring air resistance) and gravity acts downward. The parabolic path it follows has less curvature than the circular road, so it rises above the road at the top and rejoins the road further down the slope. The landing point depends on the car's speed and the road's geometry; a problem that combines circular motion with projectile motion.

With this final subtopic, vertical circular motion is complete. From the ball on a string to the loop-the-loop and now the hilltop, we have seen how gravity and contact forces interact differently depending on whether the object is on the inside or outside of the circular path. The critical speed $\sqrt{gr}$ appears in every case, but its meaning depends on the geometry: minimum speed for the inside, maximum speed for the outside.

Throughout this chapter, we have built circular motion from first principles and applied it to idealised situations: smooth tracks, uniform strings, frictionless loops. But circular motion is not confined to textbook problems. It governs the orbits of satellites, the design of centrifuges, the banking of aircraft in flight, the spin cycle of a washing machine, and the motion of charged particles in magnetic fields. In the next subtopic, we step outside the classroom and see how the physics we have mastered finds its way into the real world.

APPLICATIONS OF CIRCULAR MOTION

The equations are powerful, but they were never meant to live only on paper. Every formula we derived in this chapter was born from nature and returns to nature. In this subtopic, we leave the idealised world of smooth strings and frictionless tracks, and see circular motion at work in the machines, vehicles, and systems that shape everyday life and advanced technology alike.

1. Satellites in Orbit

A satellite orbiting the Earth is in continuous free fall. It falls toward the Earth every moment, but because it moves sideways fast enough, the Earth's surface curves away beneath it at the same rate. The result is a circular (or nearly circular) orbit.

For a satellite at height h above the Earth's surface, the gravitational force provides the centripetal force:

$$\frac{GMm}{(R_E + h)^2} = \frac{mv^2}{R_E + h}$$

From which the orbital speed is:

$$v = \sqrt{\frac{GM}{R_E + h}}$$

Where G is the gravitational constant, M is the mass of the Earth, and R_E is the Earth's radius. The mass of the satellite cancels, confirming that all satellites at the same altitude orbit at the same speed, regardless of their mass.

For a low-Earth orbit (h much smaller than R_E), the orbital speed is approximately 7.9km/s and the period is about 90 minutes. The International Space Station orbits at this speed, completing roughly 16 orbits every day.

A special case is the **geostationary orbit**, where the satellite's period matches the Earth's rotation period (24 hours). At this altitude (about 35,786km above the equator), the satellite appears stationary relative to the ground. Communication satellites and weather satellites are placed in geostationary orbits for this reason.

2. Centrifuges

A centrifuge is a machine that spins samples at high angular velocity to separate substances of different densities. In a hospital laboratory, blood samples are spun in a centrifuge to separate red blood cells (heavier) from plasma (lighter). In industrial settings, centrifuges separate cream from milk, uranium isotopes from one another, and sediment from wastewater.

The principle relies on density difference. When the centrifuge spins, the surrounding fluid creates a pressure gradient that supplies centripetal force proportional to the **fluid's** density. A particle denser than the fluid requires more centripetal force than the fluid can supply at that radius, so it drifts outward. A particle less dense than the fluid drifts inward. This is how separation occurs.

The **relative centrifugal force (RCF)** compares the centripetal acceleration to gravitational acceleration:

$$\text{RCF} = \frac{\omega^2 r}{g}$$

A laboratory centrifuge spinning at 3000rpm with a radius of 0.15m produces an RCF of about 1500. This means particles experience an effective “gravitational pull” 1500 times stronger than normal gravity, causing separation to occur in minutes rather than the hours or days it would take under gravity alone.

3. Banking of Aircraft

When an aircraft turns in flight, there is no road to provide friction. The centripetal force must come entirely from the aircraft itself. The pilot banks (tilts) the aircraft so that the lift force, which is always perpendicular to the wings, has a horizontal component pointing toward the centre of the turn.

The analysis is identical to the banked road with no friction:

$$\tan\theta = \frac{v^2}{rg}$$

Where θ is the bank angle, v is the aircraft’s speed, and r is the radius of the turn. A faster turn or a tighter radius requires a steeper bank angle. Commercial aircraft typically bank at angles up to about 25° during normal turns, while fighter jets may bank at 60° or more during sharp manoeuvres.

At a bank angle of 60° , the passengers experience an apparent weight of $\frac{mg}{\cos 60^\circ} = 2mg$, meaning they feel twice as heavy. This is described as pulling “2g.” Fighter pilots undergo special training to withstand accelerations of up to 9g during extreme turns.

4. Washing Machine Spin Cycle

During the spin cycle of a washing machine, the drum rotates at high speed. Clothes are pressed against the inner wall of the drum, and the drum wall provides the normal reaction that acts as the centripetal force:

$$R = m\omega^2 r$$

But water in the clothes is not held by the fabric strongly enough to maintain this circular motion. When the required centripetal force exceeds the weak adhesive forces holding the water to the fabric, the water separates from the clothes and moves outward through the holes in the drum.

This is not “centrifugal force throwing water outward.” What actually happens is that the drum wall pushes the clothes inward (centripetal force), but the water, lacking sufficient inward force, cannot follow the circular path. It continues tangentially through the drum holes, following Newton’s first law. The effect is the same as the Kilimanjaro bus: when the available inward force is insufficient, the object (water, in this case) leaves the circular path.

5. Circular Motion of Charged Particles in Magnetic Fields

When a charged particle moves through a magnetic field, it experiences a force perpendicular to its velocity. This force does not change the particle’s speed (because it acts perpendicular to the motion), but it continuously changes the direction of motion. The result is circular motion.

For a particle of charge q and mass m moving at speed v perpendicular to a uniform magnetic field of strength B , the magnetic force provides the centripetal force:

$$qvB = \frac{mv^2}{r}$$

From which the radius of the circular path is:

$$r = \frac{mv}{qB}$$

This principle is used in **cyclotrons** (particle accelerators that accelerate charged particles along spiral paths), **mass spectrometers** (which separate ions by mass based on their circular path radii), and the large particle accelerators at research centres such as CERN, where protons travel in circular paths of several kilometres radius at speeds close to the speed of light.

6. Artificial Gravity in Space Stations

In the weightless environment of space, astronauts' bones weaken and muscles deteriorate over time. One proposed solution is to build a rotating space station. As the station spins, the floor (the outer wall) pushes astronauts inward, providing centripetal force. This inward push feels exactly like gravity to the astronauts standing on the inside of the outer wall.

For the artificial gravity to match Earth's gravity:

$$\omega^2 r = g$$

$$\omega = \sqrt{\frac{g}{r}}$$

A station of radius 100m would need to rotate at $\omega = \sqrt{\frac{9.8}{100}} = 0.313 \text{ rad/s}$, completing about one revolution every 20 seconds. This is slow enough to be comfortable for the inhabitants.

This concept has appeared in many science fiction films, but the physics is real and straightforward: it is nothing more than the centripetal acceleration $\omega^2 r$ mimicking the acceleration due to gravity g .

7. Speed Governors

The conical pendulum, which we have studied, has a direct engineering application: the **centrifugal governor**, for controlling steam engine speed. Two heavy balls are attached to a rotating shaft by hinged arms. As the shaft spins faster, the balls rise outward (increasing θ), just as in the conical pendulum. This outward movement is linked mechanically to a valve that reduces the steam supply, slowing the engine. When the engine slows, the balls drop, opening the valve again.

The system is self-regulating: the angle of the balls automatically adjusts to maintain a constant engine speed.

The relationship $\tan \theta = \frac{v^2}{rg}$ ensures that each angle corresponds to a specific rotation speed, providing precise feedback control.

This was one of the earliest examples of **automatic feedback control** in engineering, and it is a beautiful demonstration of the conical pendulum at work outside the physics classroom.

From satellites silently circling the Earth to the washing machine noisily spinning in the kitchen, from the blood centrifuge in Muhimbili hospital to the International Space Station floating above our heads, circular motion is everywhere. The equations we derived in this chapter are not abstract mathematics; they are the operating instructions of the universe.

And speaking of operating instructions, it is time to put everything together. In the next section, the concepts of this entire chapter: centripetal acceleration, horizontal motion, vertical motion, energy conservation, and critical conditions will stop being polite and start working together in miscellaneous worked examples. If the individual subtopics were the ingredients, the miscellaneous examples are the full meal. Sharpen your appetite, and don't forget a calculator!

MISCELLANEOUS WORKED EXAMPLES ON UNIFORM CIRCULAR MOTION

Example 44

- Explain why a satellite in a circular orbit around the Earth is accelerating even though its speed is constant.
- A satellite orbits the Earth at a height of 200km above the surface. At this altitude, the acceleration due to gravity is 9.6 m/s^2 . The radius of the Earth is 6400km. Calculate the orbital speed and the period of revolution of the satellite.

Solution

- Acceleration is defined as the rate of change of velocity. Velocity is a vector quantity that has both magnitude (speed) and direction. Although the satellite's speed is constant, its direction of motion changes continuously as it follows the circular path. This continuous change in the direction of the velocity vector constitutes a centripetal acceleration directed toward the centre of the Earth. Therefore, the satellite is accelerating even at constant speed.

(b) The orbital radius is:

$$r = R_E + h = 6400\text{km} + 200\text{km} = 6600\text{km} = 6.6 \times 10^6\text{m}$$

At this altitude, gravity provides the centripetal force. For the orbital speed:

$$\frac{mv^2}{r} = mg'$$

Where $g' = 9.6\text{m/s}^2$ is the gravitational acceleration at that height.

Making v the subject:

$$v = \sqrt{g'r} = \sqrt{9.6\text{m/s}^2 \times 6.6 \times 10^6\text{m}} = \sqrt{6.336 \times 10^7\text{m}^2/\text{s}^2} = 7960\text{m/s}$$

The orbital speed is 7960m/s.

For the period:

$$T = \frac{2\pi r}{v} = \frac{2 \times 3.14 \times 6.6 \times 10^6\text{m}}{7960\text{m/s}} = \frac{4.145 \times 10^7\text{m}}{7960\text{m/s}} = 5207\text{s}$$

$$T = \frac{5207}{60}\text{min} = 86.8\text{min}$$

The period of revolution is 5207s (about 86.8 minutes).

Example 45

- (a) A student claims: “If the net force on a body is always perpendicular to its velocity, the body cannot accelerate because the force does no work.” Explain why the student’s reasoning is only partially correct.
- (b) A particle of mass 0.05kg moves in a horizontal circle of radius 0.4m at constant speed. The centripetal force acting on the particle is 2N. Calculate the speed and the number of revolutions the particle completes per second.

Solution

- (a) The student is correct that a force perpendicular to the velocity does no work and therefore does not change the kinetic energy or the speed of the body. However, the student is wrong to conclude that the body cannot accelerate. A body can accelerate without any change in speed, provided its direction changes. This is because, acceleration is the rate of change of velocity, not the rate of change of speed. A perpendicular force changes the *direction* of the velocity without changing its magnitude.
- (b) Using:

$$F = \frac{mv^2}{r}$$

Making v the subject:

$$v = \sqrt{\frac{Fr}{m}} = \sqrt{\frac{2\text{N} \times 0.4\text{m}}{0.05\text{kg}}} = \sqrt{16\text{m}^2/\text{s}^2} = 4\text{m/s}$$

The speed is 4m/s.

The number of revolutions per second (frequency):

$$f = \frac{v}{2\pi r} = \frac{4\text{m/s}}{2 \times 3.14 \times 0.4\text{m}} = \frac{4}{2.513}\text{s}^{-1} = 1.59\text{rev/s}$$

The particle completes 1.59 revolutions per second.

Example 46

- (a) Explain why road surfaces at sharp bends are made rough rather than smooth.
- (b) A curved section of road has a radius of 120m and is banked at 18° to the horizontal. The coefficient of static friction between the tyres and the road is 0.35. Take $g = 9.8\text{m/s}^2$.

Determine the range of speeds at which a car can safely negotiate this curve without sliding.

Solution

- (a) When a vehicle negotiates a curve on a flat road, the centripetal force required to maintain the circular path is provided entirely by static friction between the tyres and the road surface. A rough surface provides a higher coefficient of static friction than a smooth surface, allowing a greater maximum centripetal force and therefore a higher safe speed on the curve. A smooth surface would provide very little friction, and vehicles would slide outward even at low speeds.
- (b) **Maximum speed** (friction acts down the slope, supplementing centripetal force):

$$v_{\max} = \sqrt{rg \left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta} \right)}$$

Where: $r = 120\text{m}$, $g = 9.8\text{m/s}^2$, $\theta = 18^\circ$, $\mu_s = 0.35$, $\tan 18^\circ = 0.3249$

$$v_{\max} = \sqrt{120\text{m} \times 9.8\text{m/s}^2 \times \frac{0.3249 + 0.35}{1 - 0.35 \times 0.3249}} = 29.9\text{m/s}$$

The maximum safe speed is 29.9m/s.

Minimum speed (friction acts up the slope, preventing inward sliding):

$$v_{\min} = \sqrt{rg \left(\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta} \right)}$$

$$v_{\min} = \sqrt{120\text{m} \times 9.8\text{m/s}^2 \times \frac{0.3249 - 0.35}{1 + 0.35 \times 0.3249}} = \sqrt{1176 \times \frac{-0.0251}{1.1137}}$$

The numerator $(\tan\theta - \mu_s) = 0.3249 - 0.35 = -0.0251$ is negative.

Since $\tan\theta < \mu_s$, the minimum speed is **zero**. Friction is strong enough to prevent the car from sliding inward down the banked surface at any speed, including rest. The car can safely travel at any speed from 0 to 29.9m/s on this curve.

The safe speed range is $0 \leq v \leq 29.9\text{m/s}$.

Example 47

- (a) Kipanga claims that a heavier person is more likely to fall off a spinning ride than a lighter person because “*more mass means more centrifugal force pulling them outward.*” Is Kipanga correct? Explain.
- (b) A spinning ride has a radius of 4m and rotates at 1.5rev/s. A child of mass 35kg and an adult of mass 80kg sit at the same radius. Calculate the centripetal force required for each person, and show that both require the same coefficient of friction to remain seated.

Solution

- (a) Kipanga is wrong in two accounts:

First, there is no outward “centrifugal force” acting on the rider. When the ride spins, friction between the rider and the seat pushes the rider inward, providing the centripetal force needed for circular motion. If friction is insufficient, the rider does not get “pulled outward” by any force. Instead, the rider’s body, obeying Newton’s first law, continues in a straight line (tangentially), which from the ride’s perspective appears as sliding outward.

Second, a heavier person is not more likely to slide off. A heavier person requires more centripetal force ($F = m\omega^2 r$), but a heavier person also has more weight, which to the same proportion, increases the normal reaction and therefore the maximum friction available ($f_{\max} = \mu mg$), providing more centripetal acceleration. Mathematically, when we set the required centripetal force equal to the maximum friction, the mass cancels:

$$m\omega^2 r = \mu mg \Rightarrow \mu = \frac{\omega^2 r}{g}$$

Hence, the coefficient of friction needed is independent of mass. Both the child and the adult slide off at the same rotation speed.

(b) Angular velocity:

$$\omega = 2\pi f = 2 \times 3.14 \times 1.5 \text{ s}^{-1} = 9.42 \text{ rad/s}$$

For the child ($m = 35 \text{ kg}$):

$$F_c = m\omega^2 r = 35 \text{ kg} \times (9.42 \text{ rad/s})^2 \times 4 \text{ m} = \mathbf{12424 \text{ N}}$$

$$\mu = \frac{F_c}{mg} = \frac{12424 \text{ N}}{35 \text{ kg} \times 9.8 \text{ m/s}^2} = \mathbf{36.22}$$

For the adult ($m = 80 \text{ kg}$):

$$F_c = 80 \text{ kg} \times (9.42 \text{ rad/s})^2 \times 4 \text{ m} = \mathbf{28397 \text{ N}}$$

$$\mu = \frac{F_c}{mg} = \frac{28397 \text{ N}}{80 \text{ kg} \times 9.8 \text{ m/s}^2} = \mathbf{36.22}$$

Both require $\mu = 36.22$, confirming that the required friction coefficient is independent of mass.

Example 48

- (a) A stone is whirled in a vertical circle on a string. State the position in the circle where the string is most likely to break, and explain why.
- (b) A stone of mass 0.2 kg is attached to a string of length 0.8 m and whirled in a vertical circle. The speed at the top of the circle is 5 m/s . The string can withstand a maximum tension of 14 N . The centre of the circle is 4 m above the ground. Take $g = 9.8 \text{ m/s}^2$.
- (i) Find the speed at the bottom of the circle.
- (ii) Calculate the tension at the bottom and determine whether the string breaks.
- (iii) If it breaks, find the time taken by stone to reach the ground and the horizontal distance covered.

Solution

- (a) The string is most likely to break at the **bottom** of the circle. At this position, the tension must support the weight and provide the centripetal force simultaneously ($T_{\text{bottom}} = mg + \frac{mv^2}{r}$), making the tension maximum.
- (b) The solution of each part is as follows:
- (i) Using conservation of energy between the top (height $2r$ above the bottom) and the bottom:

$$\frac{1}{2}mv_{\text{bottom}}^2 = \frac{1}{2}mv_{\text{top}}^2 + mg(2r)$$

$$v_{\text{bottom}}^2 = v_{\text{top}}^2 + 4gr = (5 \text{ m/s})^2 + 4 \times 9.8 \text{ m/s}^2 \times 0.8 \text{ m} = 25 + 31.36 = 56.36 \text{ m}^2/\text{s}^2$$

$$v_{\text{bottom}} = 7.51 \text{ m/s}$$

- (ii) Tension at the bottom:

$$T_{\text{bottom}} = mg + \frac{mv_{\text{bottom}}^2}{r} = 0.2 \text{ kg} \times 9.8 \text{ m/s}^2 + \frac{0.2 \text{ kg} \times 56.36 \text{ m}^2/\text{s}^2}{0.8 \text{ m}} = 16.05 \text{ N}$$

Since $16.05 \text{ N} > 14 \text{ N}$, the string breaks at the bottom.

- (iii) At the bottom, the velocity is horizontal (7.51 m/s). The height of the bottom above the ground:

$$h = 4 \text{ m} - 0.8 \text{ m} = 3.2 \text{ m}$$

The stone becomes a projectile with $u_x = 7.51 \text{ m/s}$ and $u_y = 0$.

Time to reach the ground:

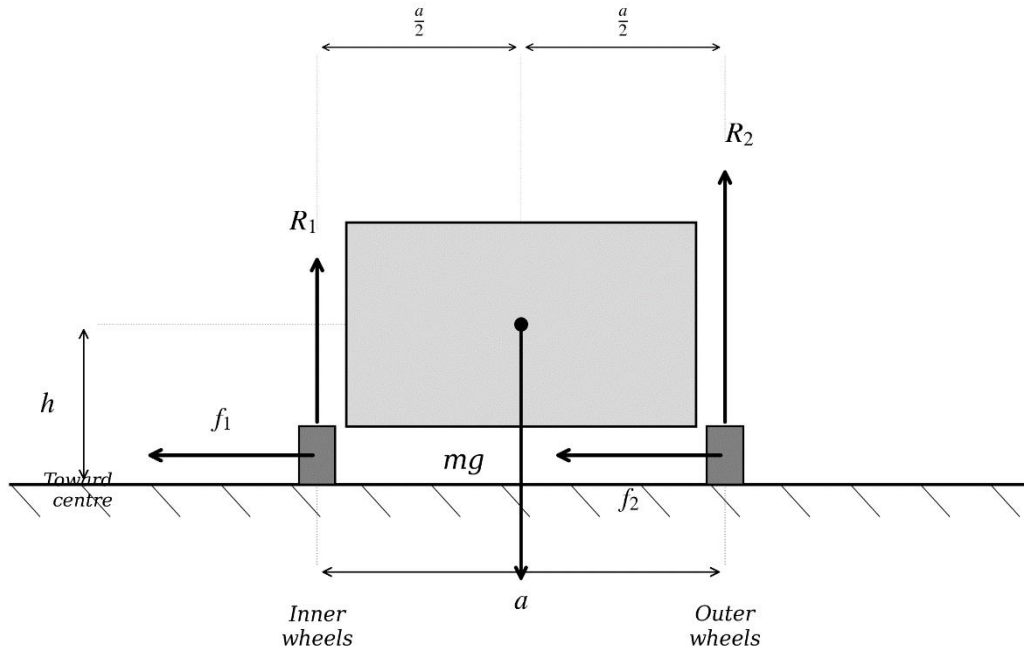
$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 3.2 \text{ m}}{9.8 \text{ m/s}^2}} = \sqrt{0.653 \text{ s}^2} = \mathbf{0.808 \text{ s}}$$

Horizontal distance:

$$x = u_x \times t = 7.51 \text{ m/s} \times 0.808 \text{ s} = \mathbf{6.07 \text{ m}}$$

Example 49

The figure below shows a rear view of a car of mass m rounding a level curved road of radius r at speed v . The lateral distance between the inner and outer wheels is a , and the centre of gravity G is at height h above the ground. R_1 and R_2 are the normal reactions on the inner and outer wheels, and f_1 and f_2 are the friction forces on the inner and outer wheels.



(a) By considering the forces and taking moments about G , derive expressions for:

- the normal reaction R_1 on the inner wheels,
- the normal reaction R_2 on the outer wheels.

(b) Hence show that:

- the maximum speed before the car overturns is $v_{\text{overturn}} = \sqrt{\frac{arg}{2h}}$,
- the maximum speed before the car skids is $v_{\text{skid}} = \sqrt{\mu_s gr}$,
- the car skids before overturning if $\mu_s < \frac{a}{2h}$, and overturns before skidding if $\mu_s > \frac{a}{2h}$.

Solution

(a) Three equations govern the system:

Vertical equilibrium:

$$R_1 + R_2 = mg \quad \dots (1)$$

Horizontal (centripetal force from friction):

$$f_1 + f_2 = \frac{mv^2}{r} \quad \dots (2)$$

Taking moments about G :

Each wheel is at horizontal distance $\frac{a}{2}$ from G . The friction forces act at the ground, a vertical distance h below G . Taking anticlockwise as positive:

R_2 acts upward at distance $\frac{a}{2}$ to the right of $G \rightarrow$ anticlockwise (positive).

R_1 acts upward at distance $\frac{a}{2}$ to the left of G $\rightarrow$ clockwise (negative).

$(f_1 + f_2)$ acts horizontally at distance h below G $\rightarrow$ clockwise (negative).

$$R_2 \times \frac{a}{2} - R_1 \times \frac{a}{2} - (f_1 + f_2) \times h = 0$$

$$\frac{a}{2}(R_2 - R_1) = (f_1 + f_2) \times h$$

Substituting (2):

$$\frac{a}{2}(R_2 - R_1) = \frac{mv^2h}{r} \quad \dots (3)$$

(i) Finding R_1 :

From (1): $R_2 = mg - R_1$. Substituting into (3):

$$\frac{a}{2}(mg - 2R_1) = \frac{mv^2h}{r}$$

$$mg - 2R_1 = \frac{2mv^2h}{ar}$$

$$\boxed{R_1 = \frac{m}{2} \left(g - \frac{2v^2h}{ar} \right)}$$

As v increases, R_1 decreases. The inner wheels progressively lose contact with the ground.

(ii) Finding R_2 :

Substituting R_1 into (1):

$$R_2 = mg - \frac{m}{2} \left(g - \frac{2v^2h}{ar} \right) = \frac{mg}{2} + \frac{mv^2h}{ar}$$

$$\boxed{R_2 = \frac{m}{2} \left(g + \frac{2v^2h}{ar} \right)}$$

As v increases, R_2 increases. The outer wheels carry progressively more load.

(c) (i) Maximum speed before overturning:

The car overturns when $R_1 = 0$:

$$0 = g - \frac{2v_{\text{overturn}}^2h}{ar}$$

$$\boxed{v_{\text{overturn}} = \sqrt{\frac{arg}{2h}}}$$

(ii) Maximum speed before skidding:

The car skids when total friction reaches its limit: $f_1 + f_2 = \mu_s(R_1 + R_2) = \mu_s mg$

$$\mu_s mg = \frac{mv_{\text{skid}}^2}{r}$$

$$\boxed{v_{\text{skid}} = \sqrt{\mu_s gr}}$$

(iii) Condition for skidding vs overturning:

Skidding occurs first if $v_{\text{skid}} < v_{\text{overturn}}$:

$$\mu_s gr < \frac{arg}{2h}$$

$$\mu_s < \frac{a}{2h} \Rightarrow \text{car skids first (safer)}$$

$$\mu_s > \frac{a}{2h} \Rightarrow \text{car overturns first (dangerous)}$$

Example 50

- (a) State two design features that make a vehicle less likely to overturn on a curved road.
- (b) A bus has its wheels 2m apart laterally and its centre of gravity is 1.2m above the ground. It rounds a level curve of radius 60m. The coefficient of static friction between the tyres and the road is 0.45. Take $g = 9.8\text{m/s}^2$.
- (i) Calculate the maximum speed before the bus overturns.
- (ii) Calculate the maximum speed before the bus skids.
- (iii) Determine which happens first, and comment on the safety of this vehicle.
- (iv) A sports car has wheels 1.8m apart and centre of gravity 0.4m above the ground. On the same road with the same friction, determine which happens first for the sports car.

Solution

- (a) (1) A low centre of gravity. (2) A wide wheel base.

- (b) The solution of each part is as follow:

- (i) For the bus ($a = 2\text{m}$, $h = 1.2\text{m}$):

$$v_{\text{overturn}} = \sqrt{\frac{arg}{2h}} = \sqrt{\frac{2\text{m} \times 60\text{m} \times 9.8\text{m/s}^2}{2 \times 1.2\text{m}}} = \mathbf{22.1\text{m/s}}$$

- (ii) Using:

$$v_{\text{skid}} = \sqrt{\mu_s gr} = \sqrt{0.45 \times 9.8\text{m/s}^2 \times 60\text{m}} = \mathbf{16.3\text{m/s}}$$

- (iii) Using:

$$\frac{a}{2h} = \frac{2\text{m}}{2.4\text{m}} = 0.833$$

Since $\mu_s(0.45) < \frac{a}{2h}(0.833)$, the bus **skids first** at 16.3m/s. This is the safer outcome. However, if the road surface improves (higher μ_s approaching 0.833), the bus could overturn instead which is a serious danger for tall vehicles.

- (iv) For the sports car ($a = 1.8\text{m}$, $h = 0.4\text{m}$):

$$\frac{a}{2h} = \frac{1.8\text{m}}{0.8\text{m}} = 2.25$$

Since $\mu_s(0.45) \ll \frac{a}{2h}(2.25)$, the sports car **skids first** with a very large safety margin against overturning. Its low centre of gravity and wide track make overturning virtually impossible under normal conditions.

Example 51

- (a) Distinguish between the apparent weight of a driver at the top of a hill and at the bottom of a valley, in terms of the direction of centripetal acceleration.
- (b) A car travels at 25m/s along a road that passes over a hill of radius 60m followed by a valley of radius 80m. Calculate the apparent weight of the driver (mass 70kg) at the top of the hill and at the bottom of the valley. Take $g = 9.8\text{m/s}^2$.

Solution

- (a) At the top of a hill, the centripetal acceleration is directed **downward** (toward the centre of curvature below the road). The normal reaction is therefore less than the weight, so the driver feels lighter.

At the bottom of a valley, the centripetal acceleration is directed **upward** (toward the centre of curvature above the road). The normal reaction is therefore greater than the weight, so the driver feels heavier.

- (b) **At the top of the hill ($r = 60\text{m}$):**

$$R_{\text{hill}} = mg - \frac{mv^2}{r} = 70\text{kg} \times 9.8\text{m/s}^2 - \frac{70\text{kg} \times (25\text{m/s})^2}{60\text{m}} = 686\text{N} - 729.2\text{N} = -43.2\text{N}$$

The result is negative. This means the car has **lost contact with the road** at the top of the hill at this speed. The driver is momentarily airborne and experiences weightlessness ($R = 0$).

Hence, the apparent weight at the top of the hill is 0N.

Checking the answer

You may check the answer by using the fact that, *if $R = 0$, the speed exceeds the critical speed for this hill.*

$$v_{\text{max}} = \sqrt{gr} = \sqrt{9.8 \times 60} = 24.2\text{m/s}$$

Since $25\text{m/s} > 24.2\text{m/s}$, the car leaves the road.

At the bottom of the valley ($r = 80\text{m}$):

$$R_{\text{valley}} = mg + \frac{mv^2}{r} = 70\text{kg} \times 9.8\text{m/s}^2 + \frac{70\text{kg} \times (25\text{m/s})^2}{80\text{m}} = 686\text{N} + 546.9\text{N} = 1232.9\text{N}$$

The apparent weight at the bottom of the valley is 1232.9N, (which is 1.80 times the actual weight (686N). The driver feels almost twice as heavy).

Example 52

- Explain why a pendulum hanging from the ceiling of a car swings outward when the car rounds a curve at constant speed.
- A small ball of mass 0.1kg is suspended by a light string from the ceiling of a car. The car rounds a horizontal curve of radius 15m at constant speed 12m/s . Find the angle the string makes with the vertical and the tension in the string. Take $g = 9.8\text{m/s}^2$.

Solution

- When the car rounds a curve, the ball must accelerate toward the centre of the curve to follow the car's circular path. The only horizontal force available to provide this centripetal acceleration is the horizontal component of the tension in the string. For the tension to have a horizontal component, the string must swing outward from the vertical. The ball does not experience an outward force; rather, the string tilts outward to provide the inward (centripetal) force needed for circular motion.
- Let θ be the angle the string makes with the vertical.

Vertically: $T\cos\theta = mg$

Horizontally (toward centre): $T\sin\theta = \frac{mv^2}{r}$

Dividing:

$$\tan\theta = \frac{v^2}{rg} = \frac{(12\text{m/s})^2}{15\text{m} \times 9.8\text{m/s}^2} = \frac{144}{147} = 0.9796$$

$$\theta = \tan^{-1}(0.9796) = 44.4^\circ$$

Tension:

$$T = \frac{mg}{\cos\theta} = \frac{0.1\text{kg} \times 9.8\text{m/s}^2}{\cos 44.4^\circ} = \frac{0.98\text{N}}{0.7147} = 1.37\text{N}$$

The string makes 44.4° with the vertical and the tension is 1.37N .

Example 53

- Explain why, for an object moving in a vertical circle on a string, the minimum speed at the top required to maintain a complete circle does not depend on the mass of the object.
- Two balls A (mass 0.3kg) and B (mass 0.5kg) are connected by a light rigid rod and move inside a smooth vertical circular track of radius 1m . At a certain instant, A is at the top and B is at the bottom.

The speed of each ball at this instant is 4m/s. Take $g = 9.8\text{m/s}^2$. Calculate the normal reaction on each ball from the track at this instant.

Solution

- (a) At the top of the vertical circle, only gravity provides the centripetal force. A heavier object requires a larger centripetal force $\left(\frac{mv^2}{r}\right)$, but it also experiences a greater gravitational force (mg). Since both the required centripetal force and the gravitational force increase in direct proportion to mass, the effect of mass change cancels. Mathematically, setting the tension to zero at the top gives: $mg = \frac{mv_{\min}^2}{r}$. The mass m appears on both sides and cancels, giving $v_{\min} = \sqrt{gr}$, which depends only on g and r . Hence, the minimum speed at the top required to maintain the circular motion is the same for all objects, regardless of their mass.
- (b) Since both balls are on a smooth track (not a string), the track pushes them. At each position, Newton's second law is applied toward the centre.

Ball A at the top (centre is below):

Both R_A and $m_A g$ act downward (toward centre):

$$R_A + m_A g = \frac{m_A v^2}{r}$$

$$R_A = \frac{m_A v^2}{r} - m_A g = \frac{0.3\text{kg} \times (4\text{m/s})^2}{1.0\text{m}} - 0.3\text{kg} \times 9.8\text{m/s}^2 = 4.8\text{N} - 2.94\text{N} = 1.86\text{N}$$

Ball B at the bottom (centre is above):

R_B acts upward (toward centre), $m_B g$ acts downward (away from centre):

$$R_B - m_B g = \frac{m_B v^2}{r}$$

$$R_B = \frac{m_B v^2}{r} + m_B g = \frac{0.5\text{kg} \times (4\text{m/s})^2}{1.0\text{m}} + 0.5\text{kg} \times 9.8\text{m/s}^2 = 8.0\text{N} + 4.9\text{N} = 12.9\text{N}$$

The normal reaction on A is 1.86N and on B is 12.9N.

Example 54

- (a) Explain why a small block placed inside a smooth rotating hemispherical bowl rises up the inner surface as the angular speed of rotation increases.
- (b) A small block of mass 0.1kg sits inside a smooth hemispherical bowl of internal radius 0.3m. The bowl rotates about its vertical axis of symmetry. The block is in equilibrium at a position where the line from the centre of the bowl to the block makes an angle of 40° with the vertical. Take $g = 9.8\text{m/s}^2$.
- (i) Show that the angular speed of the bowl is given by $\omega = \sqrt{\frac{g}{R \cos \theta}}$, where R is the radius of the bowl and θ is the angle from the vertical.
- (ii) Calculate the angular speed and the normal reaction on the block.

Solution

- (a) As the bowl spins faster, the block requires a greater centripetal force to maintain circular motion at its current radius. The only horizontal force available is the horizontal component of the normal reaction from the bowl surface. To increase this horizontal component, the block must move to a position where the bowl surface is more steeply inclined; that is, higher up the bowl where θ is larger. This increases $R \sin \theta$ (the horizontal component), providing the greater centripetal force needed.
- (b) The solution of each part is as follows:
- (i) The block moves in a horizontal circle of radius $r = R \sin \theta$. The normal reaction N acts perpendicular to the bowl surface, which at this position is directed along the line from the block toward the centre of the bowl. Resolving N :

Vertically: $N\cos\theta = mg$

Horizontally: $N\sin\theta = m\omega^2 r = m\omega^2 R\sin\theta$

Dividing:

$$\frac{\sin\theta}{\cos\theta} = \frac{\omega^2 R\sin\theta}{g}$$

The $\sin\theta$ cancels:

$$\frac{1}{\cos\theta} = \frac{\omega^2 R}{g}$$

$$\omega = \sqrt{\frac{g}{R\cos\theta}}$$

(iii) Angular speed:

$$\omega = \sqrt{\frac{9.8\text{m/s}^2}{0.3\text{m} \times \cos 40^\circ}} = 6.53\text{rad/s}$$

Normal reaction:

$$N = \frac{mg}{\cos\theta} = \frac{0.1\text{kg} \times 9.8\text{m/s}^2}{\cos 40^\circ} = \frac{0.98\text{N}}{0.766} = 1.28\text{N}$$

The angular speed is 6.53rad/s and the normal reaction is 1.28N.

Example 55

- (a) A student says: “At the top of a vertical circle, the ball is weightless because the velocity is horizontal.” Identify and correct two errors in this statement.
- (b) A ball of mass 0.25kg moves in a vertical circle of radius 0.5m on a string. The speed at the bottom is 5.5m/s. Take $g = 9.8\text{m/s}^2$. Find the angle from the bottom at which the tension in the string equals the weight of the ball.

Solution

- (a) **Error 1:** The ball is not weightless at the top. Weight (mg) acts on the ball at every point in the circle, including the top. The ball may have zero *apparent* weight (zero tension) at the critical speed, but its actual weight never changes.

Error 2: The direction of velocity does not determine whether the ball is weightless. A ball moving horizontally at the bottom of the circle is certainly not weightless. Weightlessness (zero contact force) depends on the balance between gravity and centripetal requirements, not on the direction of velocity.

- (b) Using the general expression:

$$T = \frac{mv_{\text{bottom}}^2}{r} - 2mg + 3mg\cos\theta$$

Setting $T = mg$:

$$mg = \frac{mv_{\text{bottom}}^2}{r} - 2mg + 3mg\cos\theta$$

$$3mg\cos\theta = 3mg - \frac{mv_{\text{bottom}}^2}{r}$$

$$\cos\theta = 1 - \frac{v_{\text{bottom}}^2}{3gr}$$

Substituting:

$$\cos\theta = 1 - \frac{(5.5\text{m/s})^2}{3 \times 9.8\text{m/s}^2 \times 0.5\text{m}} = 1 - \frac{30.25}{14.7} = 1 - 2.059 = -1.059$$

Since $\cos\theta = -1.059 < -1$, which is impossible, the tension **never equals the weight** at any point in the circle. The tension is always greater than mg throughout the motion.

Checking of the answer

This can be verified: at the bottom, $T_{\text{bottom}} = mg + \frac{mv^2}{r} = 0.25 \times 9.8 + \frac{0.25 \times 30.25}{0.5} = 17.58\text{N}$, which is far greater than $mg = 2.45\text{N}$.

At the top, using energy conservation: $v_{\text{top}}^2 = v_{\text{bottom}}^2 - 4gr = 30.25 - 19.6 = 10.65\text{m}^2/\text{s}^2$

$$T_{\text{top}} = \frac{mv_{\text{top}}^2}{r} - mg = \frac{0.25 \times 10.65}{0.5} - 2.45 = 2.88\text{N}$$

Even at the top, the tension (2.88N) exceeds the weight (2.45N). The speed is too high for the tension to ever drop to mg .

Example 56

- (a) Explain why a ball released from the top of a smooth hemispherical dome will eventually leave the surface before reaching the bottom.
- (b) A small ball of mass 0.15kg sits on top of a smooth hemispherical dome of radius 2m. It is given a tiny push and begins to slide down the surface. Take $g = 9.8\text{m/s}^2$.
 - (i) By considering forces along the radius and using energy conservation, show that the ball leaves the surface when $\cos\theta = \frac{2}{3}$, where θ is measured from the vertical.
 - (ii) Calculate the angle, the height above the ground, and the speed at the point where the ball leaves the dome.

Solution

- (a) As the ball slides down the dome, it speeds up (gaining kinetic energy from lost potential energy). At any point on the dome, the component of weight along the radius toward the centre ($mg\cos\theta$) must provide both the centripetal force and the normal reaction. As the ball speeds up, the centripetal force requirement ($\frac{mv^2}{r}$) increases while $mg\cos\theta$ decreases (because θ increases as the ball descends). At some angle, the required centripetal force exceeds $mg\cos\theta$, the normal reaction drops to zero, and the ball leaves the surface.
- (b) The solution of each part is as follows:
 - (i) At angle θ from the vertical, applying Newton's second law toward the centre (along the radius, inward):

$$mg\cos\theta - R = \frac{mv^2}{r}$$

The ball leaves the surface when $R = 0$:

$$mg\cos\theta = \frac{mv^2}{r} \quad \dots (1)$$

Using energy conservation from the top ($\theta = 0$, $v = 0$) to angle θ :

The ball has descended a vertical height $h = r - r\cos\theta = r(1 - \cos\theta)$

$$\begin{aligned} \frac{1}{2}mv^2 &= mgr(1 - \cos\theta) \\ v^2 &= 2gr(1 - \cos\theta) \quad \dots (2) \end{aligned}$$

Substituting (2) into (1):

$$mg\cos\theta = \frac{m \times 2gr(1 - \cos\theta)}{r}$$

$$g\cos\theta = 2g(1 - \cos\theta)$$

$$\cos\theta = 2 - 2\cos\theta$$

$$3\cos\theta = 2$$

$$\boxed{\cos\theta = \frac{2}{3}}$$

(ii) Angle:

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ$$

Height above the ground at the point of leaving:

The ball is on the *outside* of the dome. The centre of the hemisphere is at the base (ground level), and the top of the dome is at height r above the ground. At angle θ from the vertical, the ball is at height:

$$h = r\cos\theta = 2\text{m} \times \frac{2}{3} = 1.33\text{m above the ground}$$

Speed at the point of leaving, from equation (2):

$$v^2 = 2gr(1 - \cos\theta)$$

$$v = \sqrt{2gr(1 - \cos\theta)} = \sqrt{2 \times 9.8\text{m/s}^2 \times 2\text{m} \times \left(1 - \frac{2}{3}\right)} = 3.61\text{m/s}$$

The ball leaves the dome at $\theta = 48.2^\circ$ from the vertical, at a height of 1.33m above the ground, with a speed of 3.61m/s.

And with that, the miscellaneous worked examples take their final bow. Every concept in this chapter: from the gentle lean of a cyclist to the terrifying physics of overturning, from satellites silently orbiting overhead to a bucket of water that defied gravity (and soaked Kipanga), has been tested, twisted, combined, and calculated. If you survived all of that, congratulations! But do not relax just yet. The Digging Deeper Exercise is waiting on the next page, and it has no solutions to hold your hand. Just you, the physics, and a calculator. Good luck! You are ready!

DIGGING DEEPER EXERCISE 7

EXERCISE 7A: BINDER QUESTIONS

Question 1

A body moves in a circular path at constant speed. State whether each of the following is zero, constant, or changing:

- (a) speed
- (b) velocity
- (c) centripetal acceleration
- (d) centripetal force

Question 2

A body moves in a circular path at constant speed.

- (a) Explain why the body is accelerating even though its speed does not change.
- (b) State the direction of this acceleration.

Question 3

A stone tied to a string is whirled in a horizontal circle on a smooth table. The string suddenly breaks.

- (a) Describe the path of the stone immediately after the string breaks.
- (b) Explain your answer.

Question 4

Distinguish between angular velocity and linear velocity of an object in circular motion. State how they are related.

Question 5

A body moves in a circle of radius r with constant angular velocity ω .

- (a) Write the centripetal acceleration in terms of v and r .
- (b) Write the centripetal acceleration in terms of ω and r .
- (c) Explain why both expressions give the same result.

Question 6

State the real force that provides the centripetal force in each of the following situations:

- (a) a car rounding a level curve
- (b) a satellite orbiting the Earth
- (c) a ball on a string swung in a vertical circle
- (d) a car on a banked curve with no friction
- (e) a charged particle moving in a magnetic field

Question 7

Explain why the maximum safe speed of a car on a level circular curve does not depend on the mass of the car.

Question 8

In vertical circular motion on a string:

- (a) State the position where the tension in the string is maximum. Explain why.
- (b) State the position where the tension in the string is minimum. Explain why.

Question 9

Explain what is meant by the “design speed” of a banked curve, and describe what happens when a vehicle travels (a) faster than the design speed, and (b) slower than the design speed.

Question 10

A car moves over the top of a hill. Explain why the driver feels lighter at the top, and state the condition under which the car would lose contact with the road.

Question 11

Compare the critical speed condition at the top of a vertical loop for two cases:

- (a) an object on the inside of a circular loop (e.g. a ball inside a loop-the-loop track)
- (b) an object on the outside of a circular curve (e.g. a car at the top of a hill)

State how the critical speed differs in meaning between the two cases.

EXERCISE 7B: REAL QUESTIONS**Question 12**

Passengers in a bus travelling along a straight road suddenly feel thrown to the left when the bus turns sharply to the right. Explain why this happens, and state whether any force actually pushes the passengers to the left.

Question 13

On the Dar es Salaam–Morogoro highway, sharp curves have road signs showing a recommended speed limit. Explain, using physics, how the recommended speed for a curve is determined and why different curves have different recommended speeds.

Question 14

During a motorcycle race, riders lean their bodies sharply toward the inside of a bend while turning. Explain why leaning is necessary and what would happen if a rider tried to turn without leaning.

Question 15

On a school trip, the bus crosses a bridge that has a gentle hump at the centre. As the bus goes over the hump at speed, **Kipanga** grabs his seat and shouts: “*My stomach just dropped! I think we are falling!*”

Kipute laughs and says: “*We are not falling, Kipanga. The road just stopped pushing you as hard.*”

Explain what Kipute means. Why does Kipanga feel the sensation of his stomach dropping?

Question 16

Explain why a wet road is more dangerous at a curve than on a straight section of road.

Question 17

A washing machine removes water from clothes during the spin cycle by rotating the drum at high speed. Explain how the water is separated from the clothes.

Question 18

Tall vehicles such as buses and lorries are more likely to overturn on sharp curves than low vehicles such as saloon cars. Explain why.

Question 19

Kipanga watches a news report about a satellite orbiting the Earth. He turns to **Mr. Akilikubwa** and asks: “*If gravity pulls the satellite toward the Earth, why doesn’t it fall down and crash?*”

Mr. Akilikubwa smiles and says: “*It is falling, Kipanga. It just keeps missing the ground.*”

Explain what Mr. Akilikubwa means.

Question 20

Engineers build roads with banked curves rather than relying entirely on friction. Explain the advantage of banking, and state why banking alone is not sufficient for all speeds.

Question 21

A student observes that astronauts inside the International Space Station float as if there is no gravity. The student concludes that there is no gravity in space. Explain why this conclusion is incorrect.

EXERCISE 7C: HOT QUESTIONS**Question 22**

A car of mass 1200kg rounds a level curve of radius 80m at 25m/s. The coefficient of static friction is 0.60. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the centripetal force required.
- (b) Calculate the maximum friction force available.
- (c) Determine whether the car can safely negotiate the curve at this speed.

Question 23

A conical pendulum has a string of length 1.2m and the bob has mass 0.4kg. The bob moves in a horizontal circle with a period of 2s. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the angle the string makes with the vertical.
- (b) Calculate the tension in the string.
- (c) Calculate the radius of the circular path and the speed of the bob.

Question 24

A student solves the following problem: “A ball on a string moves in a vertical circle of radius 0.5m. The speed at the bottom is 4m/s. Find the tension at the top.”

The student writes: $T_{\text{top}} = \frac{mv_{\text{bottom}}^2}{r} - mg$, using the speed at the bottom directly in the tension formula at the top.

- (a) Explain the error in the student’s approach.
- (b) For a ball of mass 0.3kg, carry out the correct calculation. If the ball does not reach the top, determine the angle from the bottom at which the string goes slack. Take $g = 9.8\text{m/s}^2$.

Question 25

A curved road of radius 100m is banked at 20° . The coefficient of static friction between the tyres and the road is 0.30. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the design speed of the curve.
- (b) Determine the safe speed range.

Question 26

A ball of mass 0.2kg is attached to a string of length 0.7m and swung in a vertical circle. The speed at the bottom is 6m/s. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the tension at the bottom.
- (b) Calculate the speed and the tension at the top.
- (c) Find the angle from the bottom at which the tension equals 8N.

Question 27

A small ball rolls along a frictionless track and enters a vertical circular loop of radius 0.5m. The ball is released from rest at a height $h = 1.5\text{m}$ above the bottom of the loop. Take $g = 9.8\text{m/s}^2$.

- (a) Show that the ball completes the loop.
- (b) Calculate the speed and the normal reaction at the top of the loop for a ball of mass 0.1kg.
- (c) Calculate the normal reaction at the bottom of the loop.

Question 28

An engineer designs a level circular curve of radius 70m for a road where the expected traffic speed is 60km/h. The road surface has coefficient of static friction 0.55. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the maximum safe speed on this curve.
- (b) Determine whether the curve is safe for the expected traffic speed.
- (c) If the engineer decides to bank the road instead, calculate the banking angle required for the expected speed to be the design speed.
- (d) With this banking angle and the same friction, calculate the new maximum safe speed.

Question 29

A car of mass 1500kg travels at 20m/s over a road that has a valley (dip) of radius 50m. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the normal reaction on the car at the bottom of the valley.
- (b) Express the apparent weight of the driver (mass 75kg) as a multiple of the driver's actual weight.
- (c) Calculate the speed at which the apparent weight would be twice the actual weight.

Question 30

A stone of mass 0.3kg is whirled in a vertical circle of radius 0.9m. At the top of the circle, the speed is 4m/s. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the tension at the top.
- (b) Find the tension at the bottom.
- (c) Calculate the speed at the bottom.

Question 31

A student claims that a heavier ball needs a greater minimum speed at the bottom to complete a vertical circle than a lighter ball on the same string.

- (a) Calculate the minimum speed at the bottom for a 0.2kg ball and a 0.5kg ball, each on a string of length 0.8m. Take $g = 9.8\text{m/s}^2$.
- (b) Comment on the student's claim.

Question 32

A small ball of mass 0.1kg is suspended by a light string from the roof of a car. The car travels at constant speed around a horizontal circular track of radius 20m. The string makes an angle of 30° with the vertical. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the speed of the car.
- (b) Calculate the tension in the string.
- (c) If the speed doubles, calculate the new angle the string makes with the vertical.

Question 33

A car has wheels 1.5m apart and its centre of gravity is 0.9m above the ground. It rounds a level curve of radius 40m. The coefficient of static friction is 0.70. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the maximum speed before overturning.
- (b) Calculate the maximum speed before skidding.
- (c) Determine which happens first.
- (d) Calculate the normal reactions on the inner and outer wheels when the car travels at 15m/s.

Question 34

A ball on a string of length 1m is set in motion in a vertical circle. The speed at the bottom is 5.5m/s. Take $g = 9.8\text{m/s}^2$.

- (a) Determine whether the ball completes the full circle.
- (b) If not, find the angle from the bottom at which the string goes slack.
- (c) Find the speed at that angle.

Question 35

An engineer designs a roller coaster loop of radius 6m. The loaded car has mass 600kg. Safety regulations require that the normal reaction on the car at the top of the loop must be at least $0.5mg$. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the minimum speed at the top to meet the safety requirement.
- (b) Calculate the minimum height of the launch ramp above the bottom of the loop.
- (c) Calculate the normal reaction at the bottom of the loop when launched from this minimum height. Assess whether the structure must withstand more than $6mg$.

Question 36

A small ball sits on top of a smooth hemispherical dome of radius 3m. It is given a tiny push and slides down the surface. Take $g = 9.8\text{m/s}^2$.

- (a) Find the angle from the vertical and the speed at the point where the ball leaves the dome.
- (b) Determine the height above the ground at which the ball leaves the dome.
- (c) After leaving the dome, the ball becomes a projectile. At the point of leaving, determine the horizontal and vertical components of the velocity.

Chapter 8

GRAVITATION

INTRODUCTION

It was a quiet afternoon at Miono Secondary School. The physics class had just finished Chapter 7 on Circular Motion, and the students were packing their books. **Kipanga**, still buzzing with excitement about satellites and orbits, turned to **Mr. Akilikubwa** with a question that had been bothering him all lesson.

“Sir, you said that gravity pulls everything toward the Earth. And you said the Moon goes around the Earth in a circle. But if gravity is pulling the Moon downward, why doesn’t it just fall and crash into us?”

The class went quiet. Even **Kipute**, who usually had an answer ready, paused.

Mr. Akilikubwa smiled and picked up a tennis ball from his desk. *“Kipanga, if I throw this ball gently sideways, what happens?”*

“It curves downward and hits the ground nearby,” **Kipanga** answered.

“And if I throw it harder?”

“It goes further before hitting the ground.”

“And if I throw it harder still?”

“Even further.”

“Now,” said **Mr. Akilikubwa**, his voice dropping as though sharing a secret, *“imagine I could throw it so hard that as it falls, the Earth’s surface curves away beneath it at exactly the same rate. The ball keeps falling toward the Earth, but the ground keeps curving away. The ball falls forever but never lands. What would you call that?”*

Kipanga’s eyes widened. *“An orbit!”*

“Exactly.” **Mr. Akilikubwa** set the ball down. *“The Moon is falling toward the Earth! It has been falling for over four billion years. It just keeps missing. And the force that makes it fall, is the same force that pulls this tennis ball, is the force that holds you in your chair, the force that keeps the Earth circling the Sun; is the oldest, weakest, and most far-reaching force in the universe. Newton called it gravity. And he had the audacity to claim that it was the same force that made an apple fall from a tree.”*

Kipute leaned forward. *“The same force? An apple and the Moon?”*

“The same law,” **Mr. Akilikubwa** confirmed. *“One equation that describes both. That is the power of what you are about to learn.”*

Kipanga grinned. *“So Newton figured out the Moon is basically a very fast apple?”*

Mr. Akilikubwa laughed. *“That, Kipanga, might be the best summary of gravitation anyone has ever given.”*

In Chapter 7, we asked: *what real force provides the centripetal force for circular motion?* We found different answers for different situations: friction for cars on roads, tension for stones on strings, the normal reaction for banked curves. But we left one answer untouched, the most profound one of all: **gravity**.

Gravity is the force that holds the Moon in orbit around the Earth, the Earth in orbit around the Sun, and the Sun in orbit around the centre of the Milky Way galaxy. It is the force that shaped the solar system, controls the tides, and determines the fate of the universe itself. Yet it obeys a single, elegant law that Newton published in 1687, a law so simple that you will derive it yourself before this chapter is over.

In this chapter, we begin with the observations that led to our understanding of planetary motion: **Kepler’s three laws**. We then meet **Newton’s Law of Universal Gravitation** and use it to explain why planets move the way Kepler described. From there, we explore how gravity varies with altitude, depth, and latitude, and we introduce the concepts of **gravitational field strength** and **gravitational potential**. Finally, we apply everything to **satellites, orbits, and escape velocity**; the physics that makes space travel possible.

The mathematics builds directly on what you mastered in Chapter 7. The centripetal force equation $\frac{mv^2}{r}$ returns, but now the force supplying it is gravity: $\frac{GMm}{r^2}$. Setting these equal will unlock nearly every result in

this chapter. If you understood circular motion, gravitation will feel like a natural continuation with the same physics, played on the grandest stage imaginable.

KEPLER'S LAWS OF PLANETARY MOTION

Before Newton ever thought about gravity, a German mathematician named **Johannes Kepler** spent years staring at numbers. Not just any numbers, but the most precise astronomical measurements ever collected, recorded by the Danish astronomer **Tycho Brahe**, a man so dedicated to watching the sky that he built an entire observatory on an island. Brahe gathered mountains of data on planetary positions but died in 1601 before he could make sense of it all. Kepler inherited the data and, after years of painstaking calculation (no computers, no calculators! Just pen, paper, and stubbornness), he extracted three laws that described exactly how planets move around the Sun.

Kepler did not know *why* planets obeyed these laws. He simply showed that they did. The “why” would have to wait for Newton, who arrived on the scene about 70 years later and explained all three laws using a single equation. But before we meet Newton’s explanation, we must first understand what Kepler discovered.

Kepler’s First Law (The Law of Orbits)

The law states that:

Each planet moves in an elliptical orbit with the Sun at one focus.

Before Kepler, everyone assumed planetary orbits were perfect circles (or combinations of circles). Kepler showed that orbits are actually **ellipses**, slightly squashed circles. An ellipse has two special points called **foci** (singular: focus). The Sun sits at one focus, not at the centre.

For most planets in our solar system, the orbits are very close to circular. The Earth’s orbit, for example, deviates from a perfect circle by less than 2%. But the deviation matters: it is the reason why the Earth is slightly closer to the Sun in January than in July (this fact often surprises students who think the seasons are caused by changes in distance from the Sun. In reality, the seasons are caused by the tilt of the Earth’s axis, not the distance).

However, we will often approximate planetary and satellite orbits as circular, because this simplifies the mathematics while remaining accurate for most practical calculations.

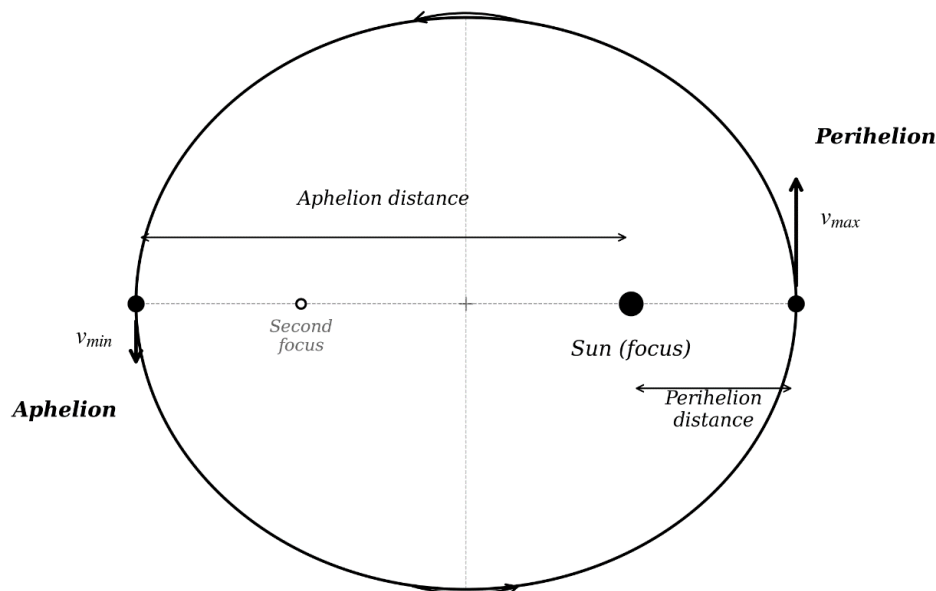


Figure: Kepler’s first law. A planet moves in an elliptical orbit with the Sun at one focus. The planet moves fastest at perihelion and slowest at aphelion.

Key terms:

Perihelion: the point in the orbit closest to the Sun. The planet moves fastest ($v_{\max}$) here.

Aphelion: the point farthest from the Sun. The planet moves slowest here ($v_{\min}$).

For orbits around the Earth (such as satellite orbits), the equivalent terms are **perigee** (closest) and **apogee** (farthest).

Kepler's Second Law (The Law of Areas)

The law states that:

A line drawn from the Sun to a planet sweeps out equal areas in equal intervals of time.

This law tells us something profound about how the speed of a planet changes as it moves along its orbit. When the planet is close to the Sun (near perihelion), the line from the Sun to the planet is short. To sweep the same area in the same time, the planet must move faster. When the planet is far from the Sun (near aphelion), the line is longer, so the planet can sweep the same area while moving more slowly.

In short: **planets speed up when closer to the Sun and slow down when farther away.**

This is not a coincidence or a design choice. It is a consequence of the **conservation of angular momentum** (this will be discussed in detail in **Chapter 9: Rotation of Rigid Bodies**). Because gravity acts along the line joining the planet and the Sun, it produces no torque about the Sun. With zero torque, angular momentum ($L = mvr$) is conserved. When r decreases, v must increase to keep L constant, and vice versa.

For simplified circular orbits (which we will mostly use in this topic), the speed is constant and the second law is automatically satisfied. This is because, with constant speed, the planet sweeps out equal angles in equal times, and since the radius is constant, equal angles means equal areas, agreeing with Kepler's second law.

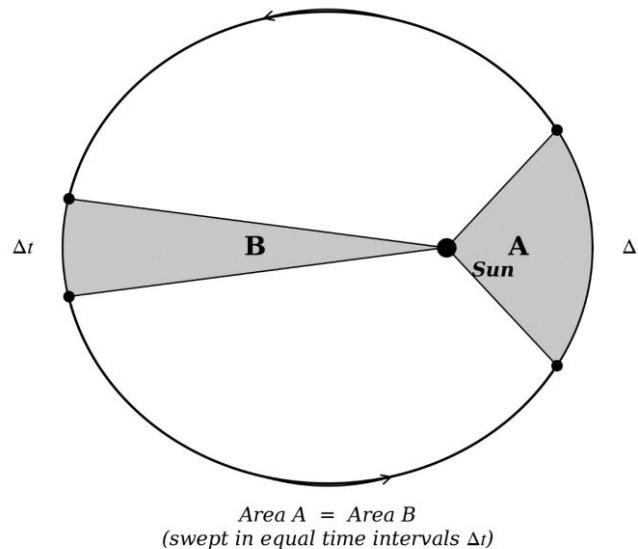


Figure: Kepler's second law. A line drawn from the Sun to the planet sweeps out equal areas in equal time intervals Δt . Sector A (near perihelion) spans a wide arc because the planet is close to the Sun and moves fast. Sector B (near aphelion) spans a narrow arc because the planet is far from the Sun and moves slowly. Despite their different shapes, both areas are equal.

Kepler's Third Law (The Law of Periods)

The law states that:

The square of the period of revolution of a planet is directly proportional to the cube of its mean distance from the Sun.

Mathematically:

$$T^2 \propto r^3$$

$$T^2 = kr^3$$

Where T is the orbital period, r is the mean orbital radius, and k is a constant that is the same for all planets orbiting the same central body.

If two planets orbit the same star, their periods and radii are related by:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

This is an extraordinarily powerful equation. It means that if you know the period and radius of one planet's orbit, you can find the radius of any other planet's orbit just by measuring its period. Kepler used this to map the entire solar system.

What the third law reveals

The third law tells us that planets farther from the Sun take longer to complete one orbit not just because they have a longer path, but also because they move more slowly. Both effects combine to make the period increase steeply with distance. For example, the Earth orbits the Sun in 1 year at a mean distance of 1AU (astronomical unit). Jupiter, at about 5.2AU, takes about 11.9 years. Neptune, at 30AU, takes about 165 years.

Later in this chapter, we will derive Kepler's Third Law from Newton's Law of Universal Gravitation and show that the constant $k = \frac{4\pi^2}{GM}$, where M is the mass of the central body. This derivation is one of the most beautiful results in all of physics: three empirical laws, discovered by painstaking observation, all explained by a single force law.

With Kepler's three laws now understood, let us practise applying them.

BINDER Example 1

- A planet moves faster at certain points in its orbit and slower at others. Using Kepler's laws, explain why this happens and identify the positions where the planet is fastest and slowest.
- The mean distance of the Earth from the Sun is $1.5 \times 10^{11}\text{m}$ and its period of revolution is $3.156 \times 10^7\text{s}$ (1 year). Calculate the constant k in Kepler's third law.

Solution

- Kepler's first law tells us that a planet's orbit is an ellipse with the Sun at one focus. This means the planet is not always at the same distance from the Sun; it is closer at one point (perihelion) and farther at another (aphelion).
Kepler's second law tells us that a line from the Sun to the planet sweeps equal areas in equal times. When the planet is near perihelion, the line from the Sun is short, so the planet must move through a larger arc to sweep the required area, and thus it moves faster. When the planet is near aphelion, the line is long, so a smaller arc is sufficient, and hence it moves slower.

Therefore, the planet is fastest at perihelion and slowest at aphelion.

- Using $T^2 = kr^3$:

$$k = \frac{T^2}{r^3} = \frac{(3.156 \times 10^7\text{s})^2}{(1.5 \times 10^{11}\text{m})^3} = 2.95 \times 10^{-19}\text{s}^2/\text{m}^3$$

Making Sense of the Answer: This constant is the same for every planet in the solar system. Mars, Jupiter, Saturn; they all give the same value of k . This universality is what makes Kepler's Third Law so powerful, and what hinted to Newton that a single force law governs all planetary motion.

Think Like a Physicist: The constant k depends on the mass of the central body (the Sun, in this case). Planets orbiting a different star would have a different k , because the star has a different mass.

REAL Example 2

Mars has a mean orbital radius of $2.28 \times 10^{11}\text{m}$. **Kipanga** claims that since Mars is about 1.5 times farther from the Sun than Earth, its year must be about 1.5 times longer than Earth's year.

Explain why Kipanga's reasoning is incorrect, and find the actual period of Mars.

Solution

Kipanga's reasoning is wrong because the period does not increase in direct proportion to the distance. Kepler's third law states $T^2 \propto r^3$, not $T \propto r$.

Using the ratio form:

$$\frac{T_{\text{Mars}}^2}{T_{\text{Earth}}^2} = \frac{r_{\text{Mars}}^3}{r_{\text{Earth}}^3}$$

$$\frac{T_{\text{Mars}}^2}{(1 \text{ year})^2} = \left(\frac{2.28 \times 10^{11} \text{ m}}{1.5 \times 10^{11} \text{ m}} \right)^3 = (1.52)^3 = 3.51$$

$$T_{\text{Mars}} = \sqrt{3.51} \text{ years} = 1.87 \text{ years}$$

Mars takes 1.87 Earth years to orbit the Sun (not 1.5 years as Kipanga predicted).

Making Sense of the Answer: A planet 1.5 times farther out has a longer path and moves more slowly. Both effects combine through the $T^2 \propto r^3$ relationship, making the period nearly twice the Earth's, not just 1.5 times.

Think Like a Physicist: The ratio form of Kepler's third law is extremely useful because it avoids the need to know the constant k or the mass of the Sun. You only need the period and radius of one known orbit to find the other.

HOT Example 3

A geostationary satellite orbits the Earth with a period of exactly 24 hours. The Moon orbits the Earth with a period of 27.3 days at a mean distance of $3.84 \times 10^8 \text{ m}$ from the Earth's centre.

Calculate the orbital radius of the geostationary satellite.

Solution

Both the satellite and the Moon orbit the Earth, so the same constant k applies. Using the ratio form:

$$\frac{T_{\text{sat}}^2}{T_{\text{Moon}}^2} = \frac{r_{\text{sat}}^3}{r_{\text{Moon}}^3}$$

Making r_{sat} the subject:

$$r_{\text{sat}}^3 = r_{\text{Moon}}^3 \times \frac{T_{\text{sat}}^2}{T_{\text{Moon}}^2}$$

Where: $T_{\text{sat}} = 24 \text{ h} = 24 \times 3600 \text{ s} = 86400 \text{ s}$, $T_{\text{Moon}} = 27.3 \text{ days} = 27.3 \times 24 \times 3600 \text{ s} = 2.359 \times 10^6 \text{ s}$,
 $r_{\text{Moon}} = 3.84 \times 10^8 \text{ m}$

$$r_{\text{sat}}^3 = (3.84 \times 10^8 \text{ m})^3 \times \frac{(86400 \text{ s})^2}{(2.359 \times 10^6 \text{ s})^2} = 7.6 \times 10^{22} \text{ m}^3$$

$$r_{\text{sat}} = \sqrt[3]{7.6 \times 10^{22} \text{ m}^3} = 4.24 \times 10^7 \text{ m}$$

The orbital radius of the geostationary satellite is $4.24 \times 10^7 \text{ m}$ (about 42400 km from the Earth's centre, or about 36,000 km above the surface).

Making Sense of the Answer: The geostationary orbit is about one-ninth of the way to the Moon. Its period (1 day) is much shorter than the Moon's (27.3 days), so it must be much closer, and Kepler's third law gives us exactly how close.

Think Like a Physicist: This problem uses the Moon as a reference orbit to find the geostationary radius, without needing the mass of the Earth or the gravitational constant G . The ratio method is powerful precisely because the unknown constants cancel.

Kepler told us *how* planets move. Three elegant laws, extracted from decades of data. But he could not answer the deeper question: *why* do they move this way? *What force compels a planet to follow an ellipse? Why does it sweep equal areas? Why does T^2 relate to r^3 and not to r^2 or r^4 ?*

The answer came from Isaac Newton, who showed that all three laws are consequences of a single universal force. In the next subtopic, we meet that force.

NEWTON'S LAW OF UNIVERSAL GRAVITATION

Kepler showed us the pattern. Three laws, elegant and precise, describing exactly how planets orbit the Sun. But Kepler could not answer the obvious question: *what force makes them do this?*

Isaac Newton answered that question with one of the boldest claims in the history of science. He proposed that the force pulling an apple to the ground and the force holding the Moon in orbit are **the same force**. Not similar forces, not analogous forces, but the *same* force, obeying the *same* law, differing only in the masses involved and the distances between them. He called it **gravitation**, and he declared it to be **universal**: every object in the universe attracts every other object. The law states that:

Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres.

Mathematically:

$$F = \frac{Gm_1m_2}{r^2}$$

Where:

F is the gravitational force of attraction between the two bodies (in N),

m_1 and m_2 are the masses of the two bodies (in kg),

r is the distance between their centres (in m),

G is the **universal gravitational constant**, with value:

$$G = 6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$$

Understanding the law

Several features of this law deserve careful attention.

1. The force is mutual

If the Earth pulls you downward with a gravitational force, you pull the Earth upward with the same force. This is Newton's third law in action. The forces are equal in magnitude and opposite in direction. The reason you accelerate more than the Earth is that your mass is vastly smaller, not that the force is different.

2. The force depends on the product of the masses

Doubling either mass doubles the force. Doubling both masses quadruples the force. If one mass is zero, the force is zero.

3. The force obeys an inverse square law

Doubling the distance reduces the force to one-quarter. Tripling the distance reduces it to one-ninth. This rapid decrease with distance is why gravitational effects from distant objects are usually negligible, even though the force technically extends to infinity.

4. The force acts along the line joining the centres

Gravity is always attractive and always acts along the straight line connecting the centres of the two masses. It never pushes; it only pulls.

The Gravitational Constant G

The constant G is extraordinarily small: $6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$. This is why you do not feel a gravitational pull toward the person sitting next to you in class, even though Newton's law says the force exists. For two people of mass 60kg each, sitting 1m apart, the gravitational force between them is about $2.4 \times 10^{-7} \text{N}$ (roughly the weight of a bacterium!).

Gravity only becomes significant when at least one of the masses is enormous, like a planet or a star. The Earth's mass ($6.0 \times 10^{24} \text{kg}$) compensates for the tiny value of G and produces the familiar gravitational pull we experience every day.

G was first measured experimentally by Henry Cavendish in 1798, more than a century after Newton proposed the law. Cavendish used a sensitive torsion balance to measure the weak gravitational attraction between lead spheres. His experiment is often described as "weighing the Earth," because knowing G allows the mass of the Earth to be calculated.

The Shell Theorem

Newton's law of gravitation is stated for **point particles**. But planets and stars are not points, they are extended spheres. *How do we apply the law to them?* Newton himself answered this question by proving what is now called the **Shell Theorem**, which has two parts:

Part 1: A uniform spherical shell attracts a particle **outside** it as if the entire mass of the shell were concentrated at its centre.

Part 2: A uniform spherical shell exerts **zero** gravitational force on a particle **inside** it.

Since a solid sphere can be thought of as a collection of concentric shells, Part 1 means that a uniform solid sphere (like a planet) attracts any external object as if all the sphere's mass were at its centre. This is the reason we are justified in using $F = \frac{Gm_1m_2}{r^2}$ with r measured from **centre to centre**, even though the masses are spread out over large volumes.

When you stand on the Earth's surface, the relevant distance in $F = \frac{Gm_E m}{r^2}$ is the distance from you to the **centre** of the Earth ($r_E = 6.4 \times 10^6 \text{ m}$), not the distance to the ground beneath your feet.

Part 2 will become important later when we study how gravity varies with depth below the Earth's surface.

Deriving Kepler's Third Law from Newton's Law

This is where the connection between Chapters 7 and 8 becomes explicit. Consider a planet of mass **m** orbiting the Sun of mass **M** in a circular orbit of radius **r**.

From Chapter 7, the planet in circular motion requires a centripetal force:

$$F_{\text{centripetal}} = \frac{mv^2}{r}$$

This centripetal force is provided by gravity:

$$F_{\text{gravity}} = \frac{GMm}{r^2}$$

Setting them equal:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

The mass of the planet m cancels:

$$v^2 = \frac{GM}{r}$$

Now, the orbital speed is related to the period by $v = \frac{2\pi r}{T}$. Substituting:

$$\begin{aligned} \left(\frac{2\pi r}{T}\right)^2 &= \frac{GM}{r} \\ \frac{4\pi^2 r^2}{T^2} &= \frac{GM}{r} \\ T^2 &= \frac{4\pi^2}{GM} r^3 \end{aligned}$$

This is Kepler's third law, with the constant identified:

$$k = \frac{4\pi^2}{GM}$$

Three observations are profound:

First: $T^2 \propto r^3$ emerges naturally from Newton's law of gravitation combined with circular motion. Kepler discovered this empirically; Newton explained *why*.

Second: The mass of the orbiting body (m) does not appear. The period depends only on the orbital radius and the mass of the central body (M). This is why *all satellites at the same altitude orbit the Earth with the same period, regardless of their mass*.

Third: the constant k depends only on G and M . If you know k (from observing any orbit), you can calculate M (the mass of the central body). This is how astronomers "weigh" the Sun, the Earth, and other planets.

Let us now put these ideas to work.

BINDER Example 4

- (a) Two identical spheres, each of mass 50kg, are placed with their centres 0.5m apart. Calculate the gravitational force between them and explain why this force is not noticeable in everyday life. Take $G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

- (b) Calculate the gravitational force between the Earth and a 70kg person standing on the surface. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$.

Solution

- (a) Using $F = \frac{Gm_1m_2}{r^2}$:

$$F = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 50\text{kg} \times 50\text{kg}}{(0.5\text{m})^2} = 6.67 \times 10^{-7}\text{N}$$

This force is about 0.00000067N. It is far too small to feel because the gravitational constant G is extremely small, and neither mass is large enough to compensate.

- (b) Using:

$$F = \frac{GM_E m}{r_E^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg} \times 70\text{kg}}{(6.4 \times 10^6\text{m})^2} = 683\text{N}$$

The gravitational force between the Earth and a 70kg person is 683N. (This is the person's weight, close to $mg = 70 \times 9.8 = 686\text{N}$; the small difference comes from rounding).

Making Sense of the Answer: *The gravitational force between everyday objects is negligible. But when one mass is as enormous as the Earth ($6 \times 10^{24}\text{kg}$), the tiny constant G is overwhelmed and the force becomes the very weight we feel every moment of our lives.*

Think Like a Physicist: *Newton's law of gravitation and the familiar formula $W = mg$ are not two different things. $W = mg$ is simply the result of applying $F = \frac{GMm}{r^2}$ at the Earth's surface, where $g = \frac{GM}{r_E^2}$. We will derive this connection explicitly in the next section.*

REAL Example 5

The Earth pulls the Moon with a gravitational force, and the Moon pulls the Earth with the same force. Yet the Moon orbits the Earth, not the other way around. Explain why both bodies experience the same force but respond so differently.

Solution

The gravitational force on the Moon due to the Earth is equal in magnitude to the force on the Earth due to the Moon. However, by Newton's second law ($a = \frac{F}{m}$), the acceleration produced depends on the mass of the body. The Moon is much less massive than the Earth, so the Moon's acceleration is much greater than the Earth's. The Moon therefore moves in a large, visible orbit, while the Earth's motion is barely noticeable.

Making Sense of the Answer: *Same force, different masses, different accelerations. A mosquito and a truck experience the same collision force during impact, but only the mosquito changes its motion dramatically. The same principle applies to the Moon and the Earth.*

Think Like a Physicist: *The phrase "the Moon orbits the Earth" is a convenient simplification. More precisely, both orbit their common centre of mass. But since the Earth is so much more massive, the centre of mass is very close to the Earth's centre, making the simplification excellent.*

HOT Example 6

- (a) The Earth orbits the Sun at a mean distance of $1.5 \times 10^{11}\text{m}$ with a period of $3.156 \times 10^7\text{s}$. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. Calculate the mass of the Sun.
- (b) A satellite orbits the Earth at a height of 300km above the surface. Calculate its orbital speed and period in minutes. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$.

Solution

- (a) From $T^2 = \frac{4\pi^2}{GM} r^3$, making M the subject:

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{4 \times (3.14)^2 \times (1.5 \times 10^{11}\text{m})^3}{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times (3.156 \times 10^7\text{s})^2} = 2.01 \times 10^{30}\text{kg}$$

The mass of the Sun is approximately $2.0 \times 10^{30} \text{ kg}$.

(b) Orbital radius: $R = r_E + h = 6.4 \times 10^6 \text{ m} + 300 \times 10^3 \text{ m} = 6.7 \times 10^6 \text{ m}$

$$F_{\text{centripetal}} = \frac{mv^2}{R}$$

This centripetal force is provided by gravity:

$$F_{\text{gravity}} = \frac{GM_E m}{R^2}$$

Equating:

$$\frac{mv^2}{R} = \frac{GM_E m}{R^2}$$

From which:

$$v^2 = \frac{GM_E}{R}$$

$$v = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 6.0 \times 10^{24} \text{ kg}}{6.7 \times 10^6 \text{ m}}} = 7726 \text{ m/s}$$

The orbital speed is 7726m/s.

$$\text{Period: } T = \frac{2\pi R}{v} = \frac{2 \times 3.14 \times 6.7 \times 10^6 \text{ m}}{7726 \text{ m/s}} = \frac{4.21 \times 10^7}{7726} = 5449 \text{ s}$$

$$T = \frac{5449}{60} \text{ min} = 90.8 \text{ min}$$

The period is 90.8min.

Making Sense of the Answer: A satellite at 300km altitude circles the Earth in about 91 minutes which is roughly the period of the International Space Station. Its speed of 7726m/s means it travels faster than a bullet. At lower altitudes, the speed would be even higher and the period shorter.

Think Like a Physicist: Notice that the satellite's mass never appeared in any calculation. Orbital speed and period depend only on the mass of the central body and the orbital radius. A feather and a truck at the same altitude would orbit at exactly the same speed, just as Galileo's insight predicts.

With Newton's law of gravitation and its connection to circular motion now established, we have the tools to explore how gravity behaves at different locations: above, on, and below the surface of the Earth. That is the subject of the next section.

ACCELERATION DUE TO GRAVITY

In Chapter 2, we introduced $g = 9.8 \text{ m/s}^2$ as a constant, which is the acceleration of any object falling freely near the Earth's surface. We used it in equation after equation without asking where it comes from. Now, with Newton's law of gravitation in hand, we can finally answer that question and discover that g is not truly constant at all.

The Origin of g

Consider an object of mass m on the surface of the Earth (mass M_E , radius r_E). The gravitational force on the object is its weight:

$$F = \frac{GM_E m}{r_E^2}$$

But we also know that weight equals mg :

$$mg = \frac{GM_E m}{r_E^2}$$

The mass m cancels:

$$g = \frac{GM_E}{r_E^2}$$

This is a profound result. The acceleration due to gravity at the surface of a planet depends only on the **mass** and **radius** of the planet, not on the mass of the falling object. A feather and a boulder experience the same gravitational acceleration. This is a fact Galileo demonstrated centuries ago, and which Newton's law now explains mathematically.

Mass and Density of the Earth

Since $g = \frac{GM_E}{r_E^2}$, we can rearrange to find the mass of the Earth:

$$M_E = \frac{gr_E^2}{G} = \frac{9.8\text{m/s}^2 \times (6.4 \times 10^6\text{m})^2}{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}} = 6.02 \times 10^{24}\text{kg}$$

The density of the Earth follows. Modelling the Earth as a sphere:

$$\rho = \frac{M_E}{\frac{4}{3}\pi r_E^3} = \frac{6.02 \times 10^{24}\text{kg}}{\frac{4}{3} \times 3.14 \times (6.4 \times 10^6\text{m})^3} = 5483\text{kg/m}^3$$

This is about 5500kg/m^3 , roughly five and a half times the density of water. Since surface rocks have a density of only about $2500 - 3000\text{kg/m}^3$, the interior of the Earth must be much denser than the surface; evidence for a heavy iron-nickel core, deduced from nothing more than g , G , and r_E .

Variation of g with Altitude

Consider the following diagram:

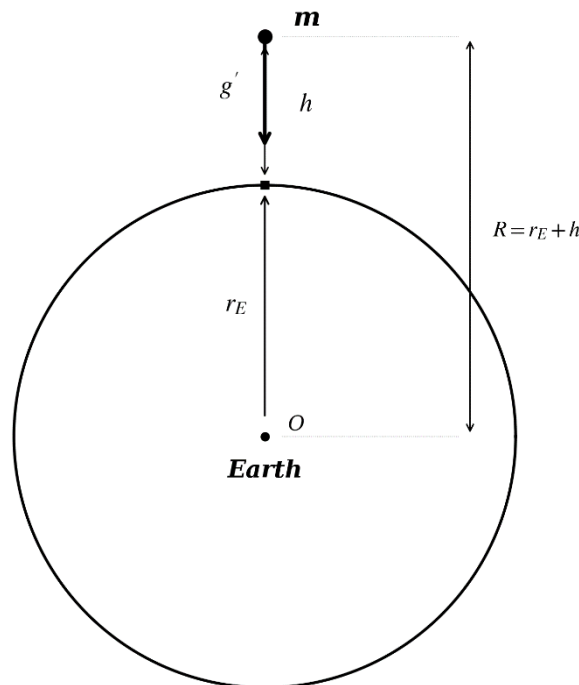


Figure: An object of mass m at height h above the Earth's surface. The distance from the centre of the Earth is $R = r_E + h$. The acceleration due to gravity g' at this height is directed toward the centre and is less than g at the surface.

At a height h above the Earth's surface, the distance from the centre of the Earth becomes:

$$R = r_E + h.$$

The acceleration due to gravity at this height is:

$$g' = \frac{GM_E}{R^2} = \frac{GM_E}{(r_E + h)^2}$$

Dividing by $g = \frac{GM_E}{r_E^2}$:

$$\frac{g'}{g} = \frac{r_E^2}{(r_E + h)^2}$$

$$g' = g \left(\frac{r_E}{r_E + h} \right)^2$$

This is the **exact** expression, valid for any height. It shows that g decreases with altitude, following an inverse square relationship with the distance from the centre (not from the surface).

Approximate formula for small heights ($h \ll r_E$):

By multiplying both the numerator and the denominator inside the bracket by $\frac{1}{r_E}$, the equation can be rewritten as:

$$g' = g \left(\frac{1}{1 + \frac{h}{r_E}} \right)^2 = g \left(1 + \frac{h}{r_E} \right)^{-2}$$

If $h \ll r_E$:

$$\frac{h}{r_E} \approx 0 \text{ (too small)}$$

Using the binomial approximation $(1 + x)^{-2} \approx 1 - 2x$ for small x :

$$g' \approx g \left(1 - \frac{2h}{r_E} \right)$$

This approximation is useful for heights up to a few hundred kilometres, where $\frac{h}{r_E}$ is still small. For larger heights, the exact formula must be used.

Variation of g with Depth

What happens below the Earth's surface? This is where Part 2 of the Shell Theorem becomes essential.

Consider an object at depth d below the surface. Its distance from the centre is $R = r_E - d$. According to the Shell Theorem, only the mass of the Earth **below** the object (within radius R) contributes to the gravitational pull. All the mass in the shell above the object exerts zero net force on it.

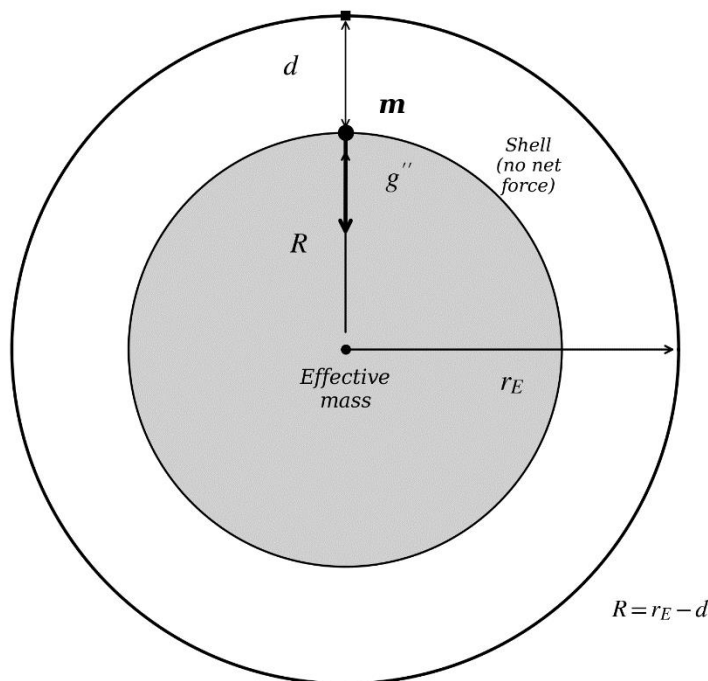


Figure: An object of mass m at depth d below the Earth's surface, at distance $R = r_E - d$ from the centre. The shaded region represents the effective mass that exerts gravitational force on the object. The outer shell (unshaded) exerts zero net force, in accordance with the Shell Theorem.

Assuming the Earth has uniform density ρ (an approximation, but one that reveals the essential physics):

The mass within radius R is:

$$M' = \rho \times \frac{4}{3} \pi R^3$$

The total mass of the Earth is:

$$M_E = \rho \times \frac{4}{3} \pi r_E^3$$

Dividing:

$$\frac{M'}{M_E} = \frac{R^3}{r_E^3}$$

The acceleration due to gravity at depth d is:

$$g'' = \frac{GM'}{R^2} = \frac{G}{R^2} \times \frac{M_E R^3}{r_E^3} = \frac{GM_E}{r_E^3} R = \frac{g}{r_E} R$$

Since $R = r_E - d$:

$$g'' = \frac{g}{r_E} (r_E - d)$$

Hence:

$$g'' = g \left(1 - \frac{d}{r_E} \right)$$

This tells us that g decreases **linearly** with depth. At the surface ($d = 0$), $g'' = g$. At the centre of the Earth ($d = r_E$), $g'' = 0$.

An object at the centre of the Earth would be pulled equally in all directions by the surrounding mass, so the net gravitational force would be zero. This is true weightlessness with genuine zero gravitational force, not the “weightlessness” of orbit, where gravity still acts but produces no contact force.

Important assumption: The derivation above assumes uniform density throughout the Earth. In reality, the Earth’s core is much denser than its mantle and crust, so the actual variation of g with depth is more complex. In fact, g initially *increases* slightly as you descend from the surface (because you move closer to the dense core), reaching a maximum at the core-mantle boundary, before decreasing to zero at the centre. However, for examination purposes, the uniform density approximation is standard.

Variation of g with Latitude

The value of g at the Earth’s surface also varies with latitude, for two reasons:

Reason 1: The Earth is not a perfect sphere. The Earth is slightly flattened at the poles and bulges at the equator (an oblate spheroid). The polar radius is about 21km shorter than the equatorial radius. Since $g \propto \frac{1}{r^2}$, the smaller radius at the poles means g is slightly larger there.

Reason 2: The Earth rotates. A body on the surface of the Earth moves in a circle (due to the Earth’s rotation) and therefore requires centripetal acceleration. Part of the gravitational pull goes toward providing this centripetal acceleration, leaving less to be felt as “weight.”

At the equator, the centripetal acceleration is maximum because the radius of the circular path (the Earth’s equatorial radius r_E) is largest. At the poles, the body is on the axis of rotation, the radius of the circular path is zero, and no centripetal acceleration is needed.

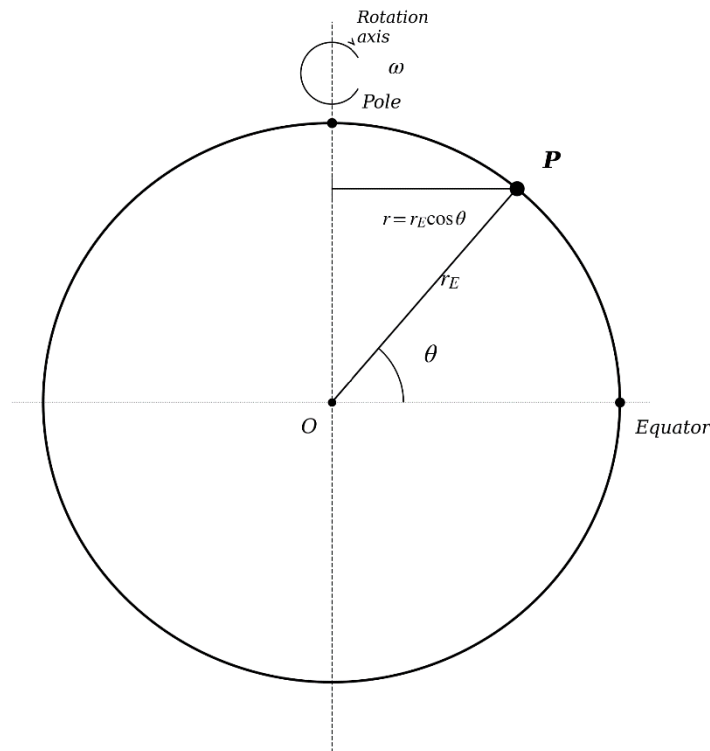


Figure: A point P on the Earth's surface at latitude θ . The Earth rotates about its polar axis with angular velocity ω . The distance from P to the rotation axis is $r = r_E \cos \theta$. As the Earth rotates, P moves in a horizontal circle of this radius, requiring centripetal acceleration $\omega^2 r_E \cos \theta$ directed toward the axis.

At latitude θ , the radius of the circular path is $r = r_E \cos \theta$, and the centripetal acceleration is $\omega^2 r_E \cos \theta$. The effective (measured) gravitational acceleration is:

$$g' = g - \omega^2 r_E \cos^2 \theta$$

At the **equator** ($\theta = 0^\circ$): $g' = g - \omega^2 r_E$ (minimum)

At the **poles** ($\theta = 90^\circ$): $g' = g$ (maximum)

The correction $\omega^2 r_E = (7.27 \times 10^{-5} \text{ rad/s})^2 \times 6.4 \times 10^6 \text{ m} = 0.034 \text{ m/s}^2$. This is small compared to 9.8 m/s^2 (about 0.3%), but it is measurable. Combined with the shape effect, the total variation in g from equator to pole is about 0.053 m/s^2 , giving $g \approx 9.78 \text{ m/s}^2$ at the equator and $g \approx 9.83 \text{ m/s}^2$ at the poles.

Summary on variation of g

As distance from the centre of the Earth **increases from zero to r_E** (moving from centre to surface): g increases linearly from 0 to g .

As distance from the centre **increases beyond r_E** (moving above the surface): g decreases as $\frac{1}{r^2}$.

The maximum value of g (under the uniform density assumption) occurs at the surface. A graph of g versus distance from the centre rises linearly to a peak at $r = r_E$ and then falls off as an inverse square curve.

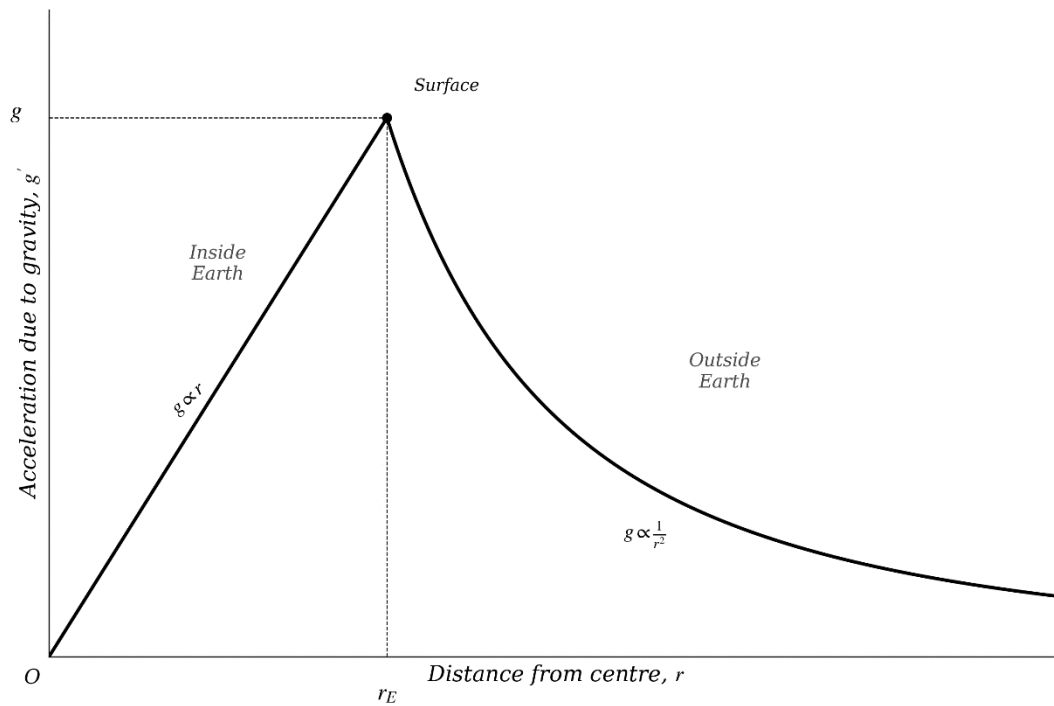


Figure: Variation of the acceleration due to gravity with distance from the centre of the Earth (assuming uniform density). Inside the Earth, g increases linearly with distance ($g \propto r$). Outside the Earth, g decreases with the inverse square of distance ($g \propto 1/r^2$). The maximum value occurs at the surface ($r = r_E$).

Let us now apply these ideas through worked examples.

BINDER Example 7

- (a) Calculate the acceleration due to gravity on the surface of Mars. Take the mass of Mars as $6.4 \times 10^{23} \text{ kg}$, the radius of Mars as $3.4 \times 10^6 \text{ m}$, and $G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.
- (b) A person weighs 700N on Earth. What would this person weigh on Mars?

Solution

- (a) Using:

$$g_{\text{Mars}} = \frac{GM_{\text{Mars}}}{r_{\text{Mars}}^2} = \frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 6.4 \times 10^{23} \text{ kg}}{(3.4 \times 10^6 \text{ m})^2} = 3.69 \text{ m/s}^2$$

The acceleration due to gravity on Mars is 3.69 m/s^2 .

- (b) The person's mass: $m = \frac{W_{\text{Earth}}}{g_{\text{Earth}}} = \frac{700 \text{ N}}{9.8 \text{ m/s}^2} = 71.4 \text{ kg}$

$$\text{Weight on Mars: } W_{\text{Mars}} = mg_{\text{Mars}} = 71.4 \text{ kg} \times 3.69 \text{ m/s}^2 = 263 \text{ N}$$

The person would weigh 263N on Mars.

Making Sense of the Answer: Mars has about 38% of Earth's surface gravity. The person feels roughly a third of their Earth weight. This is why Mars rovers can make soft landings with less braking than would be needed on Earth, but enough gravity exists to keep objects firmly on the surface.

Think Like a Physicist: Notice that Mars is both less massive **and** smaller than Earth. Less mass reduces g , but a smaller radius increases g (because you are closer to the centre). The net effect depends on which factor wins. For Mars, the mass effect dominates, giving a lower g .

BINDER Example 8

A body weighing 72N on the surface of the Earth is taken to a height equal to half the radius of the Earth above the surface. Calculate the new weight. Take $r_E = 6400 \text{ km}$.

Solution

At height $h = \frac{r_E}{2}$, the distance from the centre of the Earth becomes:

$$R = r_E + h = r_E + \frac{r_E}{2} = \frac{3}{2}r_E$$

Using the exact formula:

$$g' = g \left(\frac{r_E}{r_E + h} \right)^2 \text{ or } \frac{g'}{g} = \left(\frac{r_E}{r_E + h} \right)^2 = \left(\frac{r_E}{R} \right)^2$$

The above equation is the same as writing as:

$$\frac{mg'}{mg} = \left(\frac{r_E}{r_E + h} \right)^2 = \left(\frac{r_E}{R} \right)^2$$

Where: $mg' = W'$, $mg = W$. Thus:

$$\frac{W'}{W} = \left(\frac{r_E}{R} \right)^2 = \left(\frac{r_E}{\frac{3}{2}r_E} \right)^2 = \left(\frac{2}{3} \right)^2 = \frac{4}{9}$$

Therefore:

$$W' = \frac{4}{9} \times W = \frac{4}{9} \times 72\text{N} = 32\text{N}$$

The new weight is 32N.

Making Sense of the Answer: Moving to $1.5r_E$ from the centre reduces the weight to $\frac{4}{9}$ of the surface value which is less than half. The inverse square law is powerful: even a modest increase in distance produces a significant drop in gravitational force.

Think Like a Physicist: The approximate formula $g' \approx g \left(1 - \frac{2h}{r_E} \right)$ would give $g' = g(1 - 1) = 0$, which is clearly wrong. The approximation is only valid when $h \ll r_E$. Here $h = 0.5r_E$, which is not small, so the exact formula is essential.

REAL Example 9

A gold dealer in Dar es Salaam (near sea level) uses a precision digital scale to weigh a gold bar, and reads 500.00g. The same gold bar is taken to a dealer in Mbeya (altitude approximately 1700m) and placed on an identical scale. Explain whether the reading in Mbeya will be the same, and if not, which reading will be higher.

Solution

The reading will be slightly different. A precision scale measures the gravitational force (weight) on the object and converts it to a mass reading using a calibrated value of g .

At higher altitude, g is smaller because the object is farther from the centre of the Earth. With a lower g , the gravitational force on the gold bar is slightly less, so the scale in Mbeya reads a slightly lower value than the scale in Dar es Salaam. **The Dar es Salaam reading will be higher.**

(Using the approximate formula: $\frac{\Delta g}{g} \approx \frac{2h}{r_E} = \frac{2 \times 1700\text{m}}{6.4 \times 10^6\text{m}} = 0.00053 = 0.053\%$. The difference is tiny (about 0.27g on a 500g bar), but it is real and measurable with a precision scale.)

Making Sense of the Answer: For everyday trade this difference is negligible. But for high-precision scientific work, for gold trading in large quantities, or for pharmaceutical measurements, the variation of g with altitude matters. A scale calibrated in Dar es Salaam is not perfectly accurate in Mbeya.

Think Like a Physicist: A balance (two-pan) scale would give the same reading at both locations, because it compares masses, not weights. Both sides are affected equally by the change in g . A spring scale or digital scale, which measures force, is the one affected by altitude.

HOT Example 10

At what height above the Earth's surface is the acceleration due to gravity reduced by 36% from its surface value? Take $r_E = 6400\text{km}$.

Solution

If g is reduced by 36%, the remaining value is 64% of g :

$$g' = 0.64g$$

Using the exact formula:

$$g' = g \left(\frac{r_E}{r_E + h} \right)^2$$

Substituting $g' = 0.64g$:

$$0.64g = g \left(\frac{r_E}{r_E + h} \right)^2$$

Dividing both sides by g :

$$0.64 = \left(\frac{r_E}{r_E + h} \right)^2$$

Taking the square root of both sides:

$$\begin{aligned} \sqrt{0.64} &= \frac{r_E}{r_E + h} \\ 0.8 &= \frac{r_E}{r_E + h} \end{aligned}$$

Making $(r_E + h)$ the subject:

$$r_E + h = \frac{r_E}{0.8} = 1.25r_E$$

$$h = 1.25r_E - r_E = 0.25r_E = 0.25 \times 6400\text{km} = 1600\text{km}$$

The height is 1600km above the surface.

Making Sense of the Answer: Commercial aircraft fly at about 10km which is far too low for any noticeable change in g . Even the International Space Station at 400km experiences only about an 11% reduction. Gravity does not “switch off” in space; it simply weakens with distance.

Think Like a Physicist: Notice that $0.64 = 0.8^2$, so the distance from the centre need only increase by a factor of $\frac{1}{0.8} = 1.25$. Recognising perfect squares speeds up problem solving enormously.

HOT Example 11

- A planet has the same mass as the Earth but half the radius. Calculate the acceleration due to gravity on the surface of this planet in terms of g .
- A planet has the same density as the Earth but twice the radius. Show that the acceleration due to gravity on its surface is $2g$.

Solution

- Using $g_p = \frac{GM}{r_p^2}$, with $M = M_E$ and $r_p = \frac{r_E}{2}$:

$$g_p = \frac{GM_E}{\left(\frac{r_E}{2}\right)^2} = \frac{GM_E}{\frac{r_E^2}{4}} = 4 \times \frac{GM_E}{r_E^2} = 4 \times g$$

The surface gravity is $4g$.

- For a planet with the same density ρ but radius $r_p = 2r_E$:

The mass of the planet is:

$$M_p = \rho \times \frac{4}{3} \pi r_p^3 = \rho \times \frac{4}{3} \pi (2r_E)^3 = \rho \times \frac{4}{3} \pi \times 8r_E^3 = 8 \left(\rho \times \frac{4}{3} \pi r_E^3 \right) = 8M_E$$

The surface gravity is:

$$g_p = \frac{GM_p}{r_p^2} = \frac{G \times 8M_E}{(2r_E)^2} = \frac{8GM_E}{4r_E^2} = 2 \times \frac{GM_E}{r_E^2} = 2g$$

The surface gravity is $2g$.

Making Sense of the Answer: Part (a): same mass squeezed into a smaller sphere means the surface is closer to the centre, dramatically increasing g . Part (b): same density but larger radius means more mass, but the surface is also farther from the centre. The mass grows as r^3 while g divides by r^2 , leaving a net factor of r . Double the radius, double the g .

Think Like a Physicist: When comparing planets, always check what is held constant. Same mass, different radius: $g \propto \frac{1}{r^2}$. Same density, different radius: $g \propto r$. Same radius, different mass: $g \propto M$. Each gives a completely different scaling.

HOT Example 12

Calculate the percentage decrease in the weight of a body when it is taken to a depth of 64km below the Earth's surface. Take $r_E = 6400\text{km}$.

Solution

Using the depth formula (assuming uniform density):

$$g'' = g \left(1 - \frac{d}{r_E} \right)$$

Substituting $d = 64\text{km}$ and $r_E = 6400\text{km}$:

$$g'' = g \left(1 - \frac{64\text{km}}{6400\text{km}} \right) = g(1 - 0.01) = 0.99g$$

The fractional decrease in g (and therefore in weight) is:

$$\frac{g - g''}{g} = \frac{g - 0.99g}{g} = 0.01 = 1\%$$

The weight decreases by 1%.

Making Sense of the Answer: 64km is 1% of the Earth's radius, and the weight decreases by 1%. This is the beauty of the linear relationship: the percentage decrease in g at depth equals the percentage of the radius descended.

Think Like a Physicist: Compare altitude and depth at the same distance of 64km. At 64km altitude: $g' \approx g \left(1 - \frac{2 \times 64}{6400} \right) = g(1 - 0.02) = 0.98g$, a 2% decrease. At 64km depth: only a 1% decrease. Gravity decreases **twice as fast** with altitude as with depth (for small distances), because the altitude formula has a factor of 2 that the depth formula does not.

HOT Example 13

Calculate the effective value of the acceleration due to gravity at the equator, taking into account the Earth's rotation. Take $g = 9.8\text{m/s}^2$, $r_E = 6.4 \times 10^6\text{m}$, and the Earth completes one rotation in 24 hours.

Solution

Calculating the angular velocity of the Earth:

$$\omega = \frac{2\pi}{T} = \frac{2 \times 3.14}{24 \times 3600\text{s}} = 7.27 \times 10^{-5} \text{rad/s}$$

At the equator ($\theta = 0^\circ$, so $\cos\theta = 1$), the effective gravity is:

$$g' = g - \omega^2 r_E \cos^2\theta = g - \omega^2 r_E$$

Calculating $\omega^2 r_E$:

$$\omega^2 r_E = (7.27 \times 10^{-5} \text{ rad/s})^2 \times 6.4 \times 10^6 \text{ m} = 0.0338 \text{ m/s}^2$$

Therefore:

$$g' = 9.8 \text{ m/s}^2 - 0.034 \text{ m/s}^2 = 9.766 \text{ m/s}^2$$

The effective g at the equator is 9.766 m/s^2 , (a reduction of 0.034 m/s^2 (about 0.35%) due to the Earth's rotation).

Making Sense of the Answer: *The effect is tiny (about a third of one percent). A 70kg person would weigh about 0.24N less at the equator than at the pole. This is negligible for everyday purposes, but it matters for precision instruments and for satellite launches from equatorial sites.*

Think Like a Physicist: *If the Earth rotated fast enough that $\omega^2 r_E = g$, objects at the equator would be weightless. This would require a rotation period of about 84 minutes, which is the same as the orbital period of a low-Earth satellite. This is not a coincidence.*

The variation of g with altitude, depth, and latitude reveals that gravity is richer and more subtle than the simple constant we used in Chapter 2. In the next subtopic, we formalise this by introducing the concept of a gravitational field.

GRAVITATIONAL FIELD AND POTENTIAL

So far, we have described gravity as a force between two masses. But there is another way to think about it, one that becomes essential in more advanced physics. Instead of asking “*what force does mass A exert on mass B?*”, we ask “*what has mass A done to the space around it?*”

The answer: mass A has created a **gravitational field** which may be defined as *a region of space where any other mass placed in it will experience a gravitational force*. The field exists whether or not a second mass is present to feel it. The Earth's gravitational field fills the space around it; the Moon, satellites, and falling apples merely respond to the field that is already there.

Gravitational Field Strength

Imagine placing a small test mass m at some point in space near a large mass M . The test mass experiences a gravitational force F directed toward M . The **gravitational field strength** at that point is defined as *the gravitational force per unit mass experienced by a small test mass placed (at that point)*:

$$\mathbf{g} = \frac{\mathbf{F}}{m}$$

The SI unit of gravitational field strength is N/kg.

Since $\frac{F}{m}$ also equals acceleration (by Newton's second law), gravitational field strength is numerically equal to the acceleration due to gravity at that point. The units N/kg and m/s^2 are therefore equivalent. The two quantities: field strength (N/kg) and acceleration (m/s^2) have the same value, but they represent different physical ideas. *Field strength describes a property of the field while acceleration describes the response (resulting motion) of a mass placed in the field.*

For a point at distance r from the centre of a uniform sphere of mass M , the gravitational force on the test mass is $F = \frac{GMm}{r^2}$ (by Newton's law). Dividing by m :

$$\mathbf{g} = \frac{\mathbf{F}}{m} = \frac{\mathbf{GM}}{r^2}$$

This is the same expression we derived in the previous subtopic for the acceleration due to gravity. But now we interpret it differently: it describes a property of the field created by M , independent of whatever test mass we choose to place in it.

The gravitational field strength is a **vector** quantity. It points in the direction of the force that a test mass would experience, that is, toward the mass creating the field.

Gravitational field lines

Gravitational field lines are a visual tool for representing the direction and relative strength of a gravitational field.

Rules for field lines:

Rule 1: The direction of a field line at any point gives the direction of the gravitational field (and therefore the direction of the force on a test mass) at that point. Field lines for gravity always point **toward** the mass creating the field, because gravity is always attractive.

Rule 2: The spacing between field lines indicates the strength of the field: closer lines mean a stronger field; wider spacing means a weaker field.

How these features appear depends on the situation, as shown in the two cases below:

Radial field: Far from a spherical mass (like the Earth seen from space), the field lines point radially inward toward the centre. They spread out with distance, reflecting the inverse square decrease in field strength. This is called a **radial field**.

Uniform field: Very close to the Earth's surface (over a small area), the field lines are approximately parallel, equally spaced, and pointing vertically downward. The field is approximately **uniform** (the same strength and direction everywhere in the region). This is why we treat g as a constant in most problems near the surface.

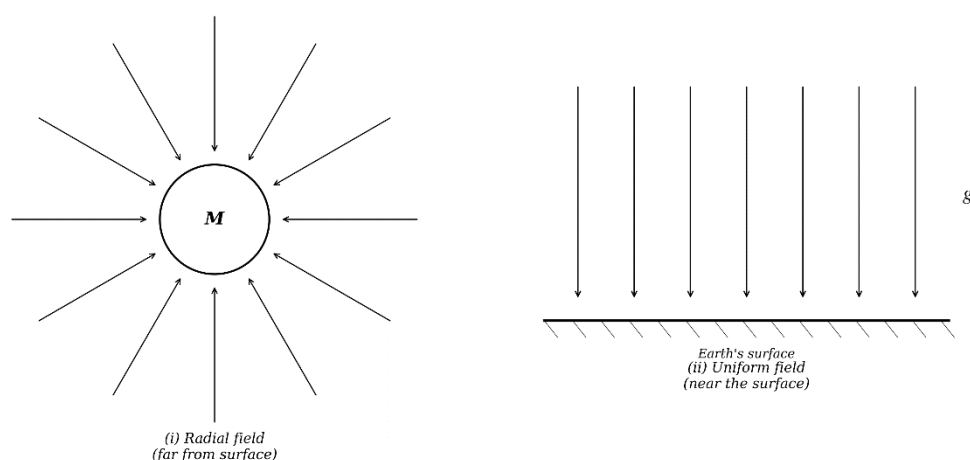


Figure: Gravitational field lines. (i) Radial field around a spherical mass M : field lines point radially inward from all directions, spreading out with distance, reflecting the inverse square decrease in field strength. (ii) Uniform field near the Earth's surface: over a small area, the field lines are parallel, equally spaced, and point vertically downward, giving an approximately constant field strength g .

Gravitational Potential

Gravitational field strength tells us about forces. But in many problems especially those involving energy, orbits, and escape velocity, we need a quantity that describes **energy** rather than force. That quantity is **gravitational potential**.

The **gravitational potential V** at a point in a gravitational field is defined as *the work done per unit mass in bringing a small test mass from infinity to that point*:

$$V = \frac{W_{\infty \rightarrow \text{point}}}{m}$$

To understand gravitational potential, consider the work required to move a mass through a gravitational field. Suppose we want to move a small mass m from a point A to a point B in the gravitational field of a large mass M . The work done against gravity depends on the positions of A and B, not on the path taken (because gravity is a **conservative force**). This means we can define a quantity at each point in the field that determines the work done in moving between any two points.

To derive the expression for V , consider bringing a test mass m from infinity to a point at distance r from the centre of mass M .

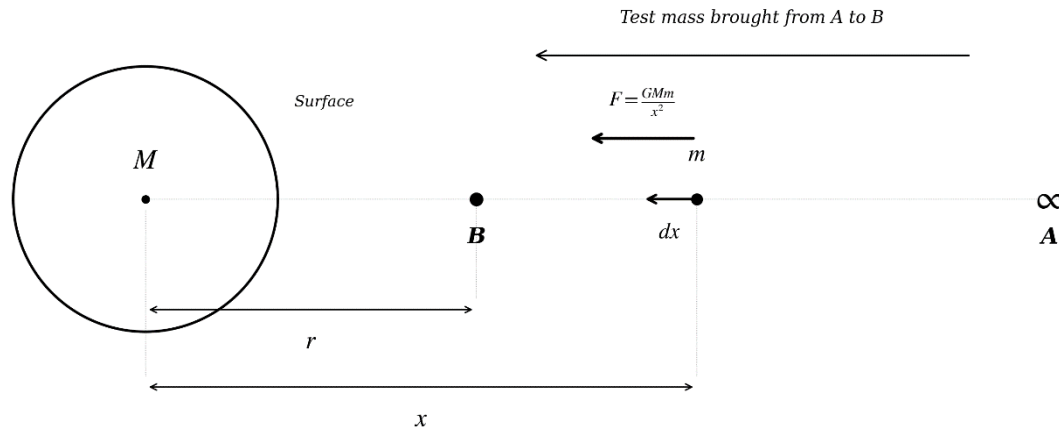


Figure: A test mass m is brought from point A (at infinity, where $V = 0$) to point B (at distance r from the centre of mass M). At an intermediate position, the test mass is at distance x from the centre and experiences a gravitational force $F = \frac{GMm}{x^2}$ directed toward M . The small displacement dx is directed inward. The total work done by gravity as the mass moves from A to B gives the gravitational potential at B.

At some intermediate position, at distance x from the centre, the gravitational force on the test mass is:

$$F = \frac{GMm}{x^2}$$

The work done by gravity in moving the test mass through a small displacement dx (inward, toward M) is $dW = Fdx$. The total work done by gravity in bringing the mass from infinity to distance r is:

$$W = \int_{\infty}^r \frac{GMm}{x^2} dx = GMm \left[-\frac{1}{x} \right]_{\infty}^r = GMm \left(-\frac{1}{r} + \frac{1}{\infty} \right) = -\frac{GMm}{r}$$

The work done per unit mass is:

$$V = \frac{W}{m} = -\frac{GM}{r}$$

Hence, the gravitational potential at distance r from the centre of a uniform sphere of mass M is given by:

$$V = -\frac{GM}{r}$$

The SI unit of gravitational potential is J/kg (joules per kilogram).

Why is the potential negative?

The negative sign deserves careful thought, because it confuses many students.

The reference point is infinity, where the potential is defined as zero (when $r = \infty$, gravitational potential formula gives $V=0$). Now consider what happens as a mass moves from infinity toward a planet. Gravity pulls the mass inward, doing positive work on it. The mass speeds up, gaining kinetic energy. But this kinetic energy comes at the expense of potential energy, which decreases. Since the potential energy at infinity was zero, and the potential energy has decreased, it must now be negative.

A more negative potential means the point is deeper inside the gravitational field. The surface of the Earth has a more negative potential than a point high above it, and the centre of the Earth has the most negative potential of all.

To move a mass upward (away from the planet), you must do work against gravity. This work increases the potential energy and makes the potential less negative. At infinity, the potential returns to zero and the mass is completely free of the gravitational field.

Think of it this way: the gravitational potential tells you how much energy per kilogram you would need to supply to completely free an object from the gravitational field and send it to infinity. At the Earth's surface, this value is about 6.25×10^7 J/kg, an enormous amount of energy per kilogram. This is why escaping the Earth's gravity requires such powerful rockets.

To move a mass **away** from a planet (upward), work must be done **against** gravity. This increases the potential (makes it less negative). At infinity, the potential returns to zero.

Gravitational potential energy

The gravitational potential V is a property of the field at a point (energy per unit mass). When a mass m is actually placed at that point, the gravitational potential energy of the system results.

Gravitational potential energy U is the energy of a **system** of two masses due to their gravitational interaction. For a mass m at distance r from the centre of mass M :

$$U = mV = -\frac{GMm}{r}$$

The gravitational potential energy is negative for the same reason the gravitational potential is negative: the system is bound, and energy must be supplied to separate the masses to infinity.

It is important to note that this formula replaces the familiar $U = mgh$ from earlier chapters. The formula mgh is an approximation that assumes g is constant, which is valid only near the Earth's surface over **small** height changes. The general formula $U = -\frac{GMm}{r}$ works at all distances and accounts for the variation of g with position.

The connection between the two can be seen as follows. For a given height h above the surface, the change in potential energy is:

$$\Delta U = \text{PE at } h - \text{PE at earth surface (} h = 0 \text{)}$$

But:

$$\text{PE at } h = -\frac{GMm}{r_E + h}$$

$$\text{PE at earth surface} = -\frac{GMm}{r_E}$$

It follows that:

$$\Delta U = -\frac{GMm}{r_E + h} - \left(-\frac{GMm}{r_E}\right) = GMm \left(\frac{1}{r_E} - \frac{1}{r_E + h}\right) = GMm \left(\frac{h}{r_E(r_E + h)}\right)$$

For $h \ll r_E$, the denominator becomes approximately r_E^2 . Thus:

$$\Delta U \approx \frac{GMmh}{r_E^2}$$

But:

$$\frac{GM}{r_E^2} = g$$

Hence, over small height changes:

$$\Delta U \approx mgh$$

So mgh is indeed a special case of the general formula, valid when the height is small compared to the Earth's radius.

Relationship Between Field Strength and Potential

Gravitational field strength and gravitational potential are two descriptions of the same field, one based on force and the other on energy. They are connected by a precise mathematical relationship.

Gravitational field strength and gravitational potential are two descriptions of the same field, one based on force and the other on energy. They are connected by a precise mathematical relationship.

Recall that the potential is defined through work: $V = -\frac{GM}{r}$. If we differentiate this with respect to r :

$$\frac{dV}{dr} = \frac{d}{dr} \left(-\frac{GM}{r} \right) = \frac{GM}{r^2}$$

But we know that $\frac{GM}{r^2} = g$. Therefore:

$$\frac{dV}{dr} = g$$

However, there is a sign to consider. The field strength points in the direction of *decreasing* potential (toward the mass, where V becomes more negative). Since r increases **outward** (away from the mass), and g points **inward** (opposite to displacement, r), the correct relationship including direction is:

$$g = -\frac{dV}{dr}$$

In words: the gravitational field strength at any point is equal to the negative of the rate of change of potential with distance at that point. The field points in the direction of steepest decrease of potential, and its magnitude equals the rate at which the potential decreases.

This is analogous to a ball on a hill. The ball accelerates in the direction of steepest descent (where height decreases fastest), and the steeper the slope, the greater the acceleration. In gravitation, the “height” is the potential, the “acceleration” is the field strength, and the “steepness” is $\frac{dV}{dr}$.

On a graph of V against r , the gradient (slope) at any point gives the field strength at that distance. At the Earth’s surface, the slope is steep (strong field). Far from the Earth, the curve flattens (weak field). At infinity, the slope is zero and the field vanishes.

Variation of potential with distance

Outside the Earth ($r > r_E$), the potential follows:

$$V = -\frac{GM_E}{r}$$

This implies that the potential is increasing (becoming less negative) as r increases and approaching zero as $r \rightarrow \infty$.

At the surface ($r = r_E$):

$$V_{\text{surface}} = -\frac{GM_E}{r_E}$$

Inside the Earth (assuming uniform density), the potential continues to decrease toward the centre. It can be shown that the potential inside a uniform sphere is:

$$V = -\frac{GM_E}{2r_E} \left(3 - \frac{r^2}{r_E^2} \right)$$

At the centre ($r = 0$):

$$V_{\text{centre}} = -\frac{3GM_E}{2r_E}$$

This is 1.5 times more negative than at the surface. The centre of the Earth is the deepest point in the gravitational potential “well.”

At the surface ($r = r_E$):

$$V_{\text{surface}} = -\frac{GM_E}{r_E}$$

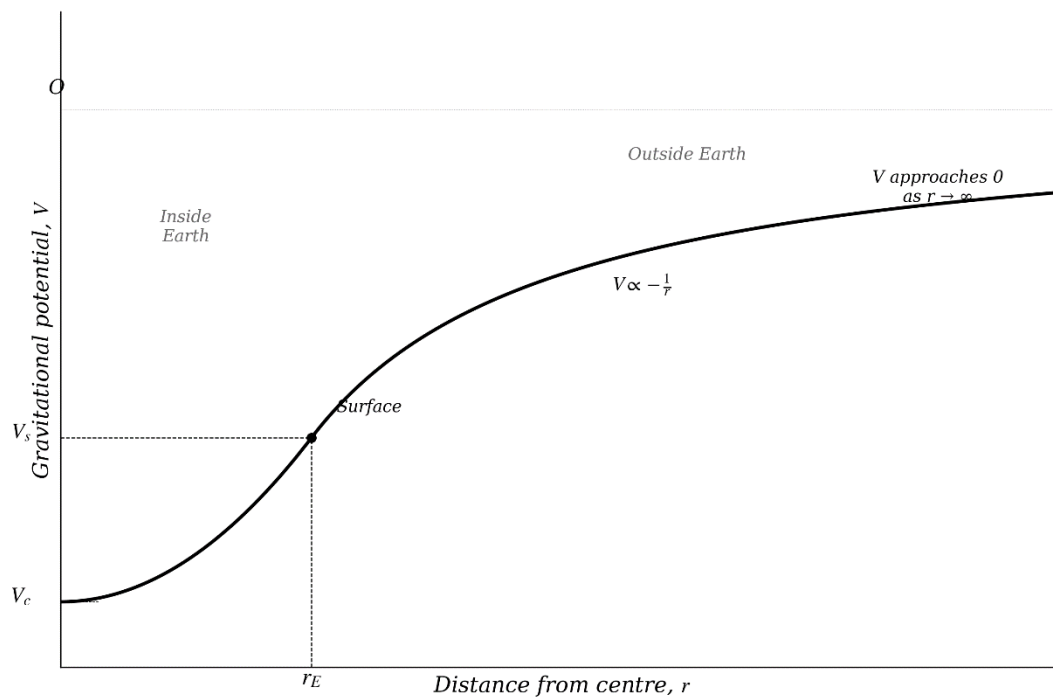


Figure: Variation of gravitational potential V with distance r from the centre of the Earth. Inside the Earth, V decreases from V_s at the surface to $V_c = -\frac{3GM_E}{2r_E}$ at the centre. Outside the Earth, V increases (becomes less negative) as $1/r$, approaching zero as $r \rightarrow \infty$. The potential is negative everywhere, reflecting the work needed to escape the gravitational field.

With this solid theoretical foundation in place, we are now ready to see these ideas come alive through carefully chosen worked examples.

BINDER Example 14

- (a) Calculate the gravitational field strength at a height of 200km above the Earth's surface. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.
- (b) Compare this with the value at the surface.

Solution

- (a) The distance from the centre of the Earth is:

$$R = r_E + h = 6.4 \times 10^6\text{m} + 200 \times 10^3\text{m} = 6.6 \times 10^6\text{m}$$

$$g = \frac{GM_E}{R^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{(6.6 \times 10^6\text{m})^2} = \frac{4.0 \times 10^{14}\text{Nm/kg}}{4.356 \times 10^{13}\text{m}^2} = 9.19\text{N/kg}$$

- (b) At the surface: $g_{\text{surface}} = \frac{4.0 \times 10^{14}}{(6.4 \times 10^6)^2} = \frac{4.0 \times 10^{14}}{4.096 \times 10^{13}} = 9.77\text{N/kg} > 9.19\text{N/kg}$

The field strength at 200km altitude is smaller than the value at the surface (about 94% of the surface value).

Making Sense of the Answer: 200km seems like a great height, but it is only about 3% of the Earth's radius. Since g decreases as $\frac{1}{r^2}$ from the centre, a 3% increase in r produces only about a 6% decrease in g .

Think Like a Physicist: This result explains why the "weightlessness" of astronauts in orbit is not due to being far from Earth. They are in a gravitational field almost as strong as the one you are sitting in right now. Their apparent weightlessness comes from the absence of a contact force, not the absence of gravity.

REAL Example 15

A well is dug to a depth of 3km below the Earth's surface. Explain whether the gravitational potential at the bottom of the well is higher or lower (more negative or less negative) than at the surface.

Solution

The gravitational potential at the bottom of the well is **lower** (more negative) than at the surface. Moving from the surface toward the centre of the Earth means moving deeper into the gravitational field. Gravity does positive work as the object descends, meaning the object loses potential energy. Since potential energy decreases, and $U = mV$, the potential V also decreases (becomes more negative).

To climb back out of the well, work must be done against gravity, which increases the potential back to its surface value.

Making Sense of the Answer: *Think of the gravitational field as a “well” or valley. The surface is already deep in the well (large negative potential). Going deeper makes the potential even more negative. Going upward (increasing altitude) makes it less negative, approaching zero at infinity.*

Think Like a Physicist: *The sign convention is consistent: potential decreases in the direction of the gravitational force (downward), and increases in the opposite direction (upward). This matches $g = -\frac{dV}{dr}$; the field points in the direction of decreasing potential.*

HOT Example 16

- Calculate the gravitational potential at the surface of the Earth. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.
- Calculate the gravitational potential at a height of 36000km above the surface (geostationary orbit altitude).
- Find the work done per unit mass in moving an object from the Earth’s surface to the geostationary orbit.

Solution

- (a) The gravitational potential is given by:

$$V_{\text{surface}} = -\frac{GM_E}{r_E} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{6.4 \times 10^6\text{m}} = -6.25 \times 10^7\text{J/kg}$$

The gravitational potential at the surface is $-6.25 \times 10^7\text{J/kg}$.

- (b) The distance from the centre to the geostationary orbit:

$$R = r_E + h = 6.4 \times 10^6\text{m} + 36.0 \times 10^6\text{m} = 42.4 \times 10^6\text{m}$$

$$V_{\text{geo}} = -\frac{GM_E}{R} = -\frac{4.0 \times 10^{14}\text{Nm/kg}}{42.4 \times 10^6\text{m}} = -9.43 \times 10^6\text{J/kg}$$

The gravitational potential at the geostationary orbit is $-9.43 \times 10^6\text{J/kg}$.

- (c) The **work done per unit mass** equals the change in gravitational potential:

$$\frac{W}{m} = V_{\text{geo}} - V_{\text{surface}} = (-9.43 \times 10^6\text{J/kg}) - (-6.25 \times 10^7\text{J/kg})$$

$$\frac{W}{m} = -9.43 \times 10^6\text{J/kg} + 6.25 \times 10^7\text{J/kg} = 5.31 \times 10^7\text{J/kg}$$

The work done is $5.31 \times 10^7\text{J/kg}$.

Making Sense of the Answer: *The positive answer confirms that work must be done to move the object upward (against gravity). For a 1000kg satellite, this is $5.31 \times 10^{10}\text{J}$ ($5.31 \times 10^7 \times 1000$). This enormous energy requirement is why rocket launches are so expensive and why most of a rocket’s mass is fuel.*

Think Like a Physicist: *The work done equals the difference in potential, not the potential itself. This is why the reference point (infinity) does not matter for practical calculations; only the difference between two points matters, and the reference cancels out.*

HOT Example 17

There is a point along the line joining the centres of the Earth and the Moon where the gravitational field strength due to both bodies is zero. Calculate the distance of this point from the centre of the Earth. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $M_M = 7.35 \times 10^{22}\text{kg}$, and the Earth-Moon distance $D = 3.84 \times 10^8\text{m}$.

Solution

Let the point be at distance x from the centre of the Earth. Then it is at distance $(D - x)$ from the centre of the Moon.

At this point, the gravitational field strength due to the Earth (pointing toward Earth) equals the field strength due to the Moon (pointing toward Moon):

$$\frac{GM_E}{x^2} = \frac{GM_M}{(D - x)^2}$$

G cancels:

$$\frac{M_E}{x^2} = \frac{M_M}{(D - x)^2}$$

Taking square roots of both sides:

$$\frac{\sqrt{M_E}}{x} = \frac{\sqrt{M_M}}{D - x}$$

Cross-multiplying:

$$\begin{aligned}(D - x)\sqrt{M_E} &= x\sqrt{M_M} \\ D\sqrt{M_E} &= x\sqrt{M_M} + x\sqrt{M_E} = x(\sqrt{M_M} + \sqrt{M_E}) \\ x &= \frac{D\sqrt{M_E}}{\sqrt{M_E} + \sqrt{M_M}}\end{aligned}$$

Substituting:

$$x = \frac{3.84 \times 10^8 \text{ m} \sqrt{6.0 \times 10^{24} \text{ kg}}}{\sqrt{6.0 \times 10^{24} \text{ kg}} + \sqrt{7.35 \times 10^{22}}} = 3.46 \times 10^8 \text{ m}$$

The point of zero gravitational field is $3.46 \times 10^8 \text{ m}$ from the centre of the Earth (about 90% of the way to the Moon).

Making Sense of the Answer: The neutral point is much closer to the Moon than to the Earth, because the Earth is far more massive. The Moon's weaker field can only match the Earth's strong field at a point very close to the Moon, where the Moon's inverse-square field is large enough to compensate for its smaller mass.

Think Like a Physicist: This is a point of zero field, not zero potential. The potential at this point is not zero, it is still negative (both the Earth and Moon contribute negative potential). A spacecraft at this point feels no net gravitational force, but it is not free from the gravitational influence of either body.

With the concepts of gravitational field strength and potential now in place, we have two complementary descriptions of gravity: one based on force (field strength) and one based on energy (potential). Both will be essential in the next section, where we apply them to the physics of orbits and escape velocity.

SATELLITES AND ORBITAL MOTION

Everything we have built in this chapter now comes together. Newton's law of gravitation provides the force. Circular motion from Chapter 7 provides the framework. Gravitational potential from the previous section provides the energy tools. In this section, we combine all three to understand how satellites orbit, what determines their speed and period, how much energy they carry, and what makes the geostationary orbit so special.

A **satellite** is any object that orbits another object under the influence of gravity. The Moon is a natural satellite of the Earth. The Earth is a natural satellite of the Sun. **Artificial satellites** are man-made objects placed into orbit around the Earth (or other bodies) for communication, navigation, weather monitoring, scientific research, and military purposes.

Orbital Velocity

Consider a satellite of mass m in a circular orbit of radius R around the Earth (mass M_E). The orbital radius R is measured from the centre of the Earth, so $R = r_E + h$, where h is the height above the surface.

The satellite is in circular motion. From Chapter 7, circular motion requires a centripetal force directed toward the centre:

$$F_{\text{centripetal}} = \frac{mv^2}{R}$$

This centripetal force is provided entirely by gravity:

$$F_{\text{gravity}} = \frac{GM_E m}{R^2}$$

Setting them equal:

$$\frac{mv^2}{R} = \frac{GM_E m}{R^2}$$

The mass of the satellite m cancels from both sides. This is a result of deep significance: **the orbital velocity does not depend on the mass of the satellite**. A communication satellite of 500kg and a space station of 400000kg at the same altitude orbit at exactly the same speed.

Solving for v :

$$v^2 = \frac{GM_E}{R}$$

$$v = \sqrt{\frac{GM_E}{R}}$$

Since $R = r_E + h$, the orbital velocity decreases as the height increases. A satellite in a low orbit moves faster than one in a high orbit. This makes physical sense: *a low satellite is deeper in the gravitational field, where the pull is stronger, so it needs a higher speed to avoid falling inward.*

For a satellite very close to the surface ($h \ll r_E$, so $R \approx r_E$), and using $GM_E = gr_E^2$:

$$v = \sqrt{\frac{gr_E^2}{r_E}} = \sqrt{gr_E} = \sqrt{9.8\text{m/s}^2 \times 6.4 \times 10^6\text{m}} = \mathbf{7920\text{m/s} \approx 7.9\text{km/s}}$$

This is the minimum orbital velocity for any satellite near the Earth's surface: about 7.9km/s, or roughly 28500km/h. Faster than a bullet, faster than any aircraft, and roughly 23 times the speed of sound.

Orbital Period

The period T is the time for one complete orbit. Since the satellite travels a distance of $2\pi R$ (the circumference) at speed v :

$$T = \frac{2\pi R}{v}$$

Substituting $v = \sqrt{\frac{GM_E}{R}}$:

$$T = \frac{2\pi R}{\sqrt{\frac{GM_E}{R}}} = 2\pi R \times \sqrt{\frac{R}{GM_E}} = 2\pi \sqrt{\frac{R^3}{GM_E}}$$

$$T = 2\pi \sqrt{\frac{R^3}{GM_E}}$$

Squaring both sides gives $T^2 = \frac{4\pi^2}{GM_E} R^3$, which is Kepler's third law. We have now come full circle (no pun intended): the law Kepler discovered empirically follows directly from Newton's gravitation combined with circular motion.

The period increases with orbital radius. Low satellites orbit quickly; high satellites orbit slowly. For a satellite near the surface ($R \approx r_E$): Again using $GM_E = gr_E^2$:

$$T = 2\pi \sqrt{\frac{(r_E)^3}{gr_E^2}} = 2\pi \sqrt{\frac{r_E}{g}} = 2 \times 3.14 \times \sqrt{\frac{6.4 \times 10^6 \text{m}}{9.8 \text{m/s}^2}} = 5074 \text{s} \approx 85 \text{min}$$

The shortest possible orbital period for satellite near the Earth's surface is about 85 minutes. Any orbit with a longer period must be at a greater height.

Energy of a Satellite in Orbit

A satellite in circular orbit possesses two forms of energy: kinetic energy (because it moves) and gravitational potential energy (because it is in a gravitational field). Understanding the relationship between these energies reveals something beautiful and surprising.

Kinetic energy (for simplicity, we will use **K** to denote it and not the usual **KE**):

$$K = \frac{1}{2}mv^2$$

From the orbital velocity result,

$$v^2 = \frac{GM_E}{R}$$

Thus:

$$K = \frac{1}{2}m \times \frac{GM_E}{R} = \frac{GM_E m}{2R}$$

Notice that KE is **positive** (as it must be, since kinetic energy is always positive).

Gravitational potential energy:

We have derived that:

$$U = -\frac{GM_E m}{R}$$

Notice that U is **negative** (the satellite is attracted to the Earth).

Total mechanical energy:

$$E = K + U = \frac{GM_E m}{2R} - \frac{GM_E m}{R} = -\frac{GM_E m}{2R}$$

$$E = -\frac{GM_E m}{2R}$$

The total energy is **negative**. This is the hallmark of a bound (attraction) system. A satellite with negative total energy cannot escape to infinity; it is gravitationally attracted (bound) to the Earth. To free the satellite completely (send it to infinity with zero residual speed), you would need to supply energy equal to:

$$|E| = \frac{GM_E m}{2R}$$

The remarkable relationships:

Looking at the three energies together:

$$K = \frac{GM_E m}{2R}, \quad U = -\frac{GM_E m}{R}, \quad E = -\frac{GM_E m}{2R}$$

Several elegant patterns emerge:

1. The kinetic energy is exactly half the magnitude of the potential energy: $K = -\frac{1}{2}U$
2. The total energy equals the negative of the kinetic energy: $E = -K$
3. The total energy equals half the potential energy: $E = \frac{1}{2}U$

These are not coincidences. They are direct consequences of the inverse square nature of gravity combined with circular motion. They hold for any circular orbit, around any central body.

What happens when a satellite gains or loses energy?

If a satellite gains energy (for example, by firing its engines), the total energy E becomes less negative. Since $E = -\frac{GM_E m}{2R}$, a less negative E means a larger R . The satellite moves to a **higher orbit**. Counterintuitively, it also moves **slower** (since $v = \sqrt{\frac{GM_E}{R}}$ and R has increased).

If a satellite loses energy (for example, through atmospheric drag), E becomes more negative, R decreases, and the satellite drops to a **lower orbit** where it moves **faster**. This is the paradox of orbital mechanics: air resistance (drag) slows a satellite down in one sense (it loses energy), but speeds it up in another sense (it moves to a lower, faster orbit). Eventually, if enough energy is lost, the satellite spirals inward and burns up in the atmosphere.

The Geostationary Orbit

Among all possible orbits, one is particularly useful: the orbit where the satellite's period exactly matches the Earth's rotation period of 24 hours. A satellite in this orbit appears to remain stationary above a fixed point on the equator, as seen from the ground. This is called a **geostationary orbit** (also known as a **geosynchronous equatorial orbit** or a **parking orbit**).

For a satellite to be geostationary, three conditions must be met:

Condition 1: The period must be exactly 24 hours (86400s).

Condition 2: The orbit must be circular (otherwise the satellite would appear to drift back and forth during the day).

Condition 3: The orbit must lie in the equatorial plane (otherwise the satellite would appear to move north and south during the day).

The radius of the geostationary orbit can be found from Kepler's third law as follows:

$$T^2 = \frac{4\pi^2}{GM_E} R^3$$

Making R the subject:

$$R^3 = \frac{GM_E T^2}{4\pi^2}$$

$$R = \sqrt[3]{\left(\frac{GM_E T^2}{4\pi^2}\right)}$$

Substituting $G = 6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$, $M_E = 6.0 \times 10^{24} \text{kg}$, and $T = 86400 \text{s}$:

$$R = \sqrt[3]{\left(\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times (86400)^2}{4 \times \pi^2}\right)} = 4.23 \times 10^7 \text{m} = 42300 \text{km}$$

The height above the surface:

$$h = R - r_E = 42300 \text{km} - 6400 \text{km} = 35900 \text{km} \approx 36000 \text{km}$$

Every geostationary satellite orbits at approximately 36000km above the equator. There is no choice in this; the altitude is fixed by the physics. This is why geostationary orbit positions are valuable and strictly regulated internationally.

The orbital speed in this orbit:

$$v = \frac{2\pi R}{T} = \frac{2 \times 3.14 \times 4.23 \times 10^7 \text{ m}}{86400 \text{ s}} = 3074 \text{ m/s} \approx 3.1 \text{ km/s}$$

This is much slower than the 7.9 km/s of a low-orbit satellite, consistent with the rule that higher orbits have lower speeds.

Why geostationary satellites are so useful:

Because a geostationary satellite appears fixed above one point on the equator, a ground antenna can be pointed at it permanently without tracking. This makes it ideal for television broadcasting, weather observation, and telecommunications. The satellite dishes you see on rooftops across Tanzania are all pointed at geostationary satellites hovering above the equator.

Energy Required to Launch a Satellite

To place a satellite into orbit, we must do two things:

First: lifting it from the surface to the orbital height (increasing its potential energy), and

Second: accelerating it to the orbital speed (giving it kinetic energy).

The total energy required is the difference between the total energy in orbit and the total energy at rest on the surface.

On the surface (at rest):

$$E_{\text{surface}} = K_{\text{surface}} + U_{\text{surface}} = 0 + \left(-\frac{GM_E m}{r_E} \right) = -\frac{GM_E m}{r_E}$$

In orbit (at radius R):

$$E_{\text{orbit}} = -\frac{GM_E m}{2R}$$

The energy that must be supplied is:

$$\Delta E = E_{\text{orbit}} - E_{\text{surface}} = -\frac{GM_E m}{2R} - \left(-\frac{GM_E m}{r_E} \right)$$

$$\Delta E = GM_E m \left(\frac{1}{r_E} - \frac{1}{2R} \right)$$

This is the minimum energy required, ignoring air resistance and other losses. In practice, much more energy is needed because rockets must also lift the fuel itself, overcome air resistance, and account for inefficiencies.

Let us now put all of these ideas to work.

BINDER Example 18

A satellite orbits the Earth at a height of 400 km above the surface (approximately the altitude of the International Space Station).

- Calculate its orbital speed.
- Calculate its orbital period in minutes.

Take $M_E = 6.0 \times 10^{24} \text{ kg}$, $r_E = 6.4 \times 10^6 \text{ m}$, and $G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

Solution

- The orbital radius:

$$R = r_E + h = 6.4 \times 10^6 \text{ m} + 400 \times 10^3 \text{ m} = 6.8 \times 10^6 \text{ m}$$

The orbital speed:

$$v = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 6.0 \times 10^{24} \text{ kg}}{6.8 \times 10^6 \text{ m}}} = 7672 \text{ m/s}$$

The orbital speed is 7672 m/s (about 7.67 km/s).

(b) The orbital period:

$$T = \frac{2\pi R}{v} = \frac{2 \times 3.14 \times 6.8 \times 10^6 \text{m}}{7672 \text{m/s}} = 5567 \text{s}$$

Converting to minutes:

$$T = \frac{5567 \text{s}}{60} = 92.8 \text{min}$$

The period is about 92.8 minutes.

The ISS completes roughly 15.5 orbits every day, experiencing 15 sunrises and 15 sunsets in 24 hours.

Making Sense of the Answer: The speed (7.67km/s) and period (93 minutes) match the well-known values for the ISS. At this speed, the station crosses the entire width of Tanzania in about 2 minutes!

Think Like a Physicist: Compare with the minimum values: 7.9km/s and 85 minutes near the surface. At 400km, the speed is slightly lower and the period slightly longer, exactly as the formulas predict. The differences are small because 400km is only about 6% of the Earth's radius.

REAL Example 19

Kipanga asks **Mr. Akilikubwa**: “Sir, if a satellite is always falling toward the Earth, why doesn't it lose energy and crash?”

Mr. Akilikubwa replies: “Kipanga, it **is** losing energy in a sense. But look at what happens when it does.”

Explain what Mr. Akilikubwa means. Describe what happens to the speed, altitude, and period of a satellite when it gradually loses energy due to atmospheric drag.

Solution

When a satellite loses energy through atmospheric drag, its total energy $E = -\frac{GM_E m}{2R}$ becomes more negative. A more negative E corresponds to a smaller orbital radius R , so the satellite drops to a lower orbit.

At this lower orbit, the orbital speed $v = \sqrt{\frac{GM_E}{R}}$ is higher (because R is smaller), and the period $T = 2\pi \sqrt{\frac{R^3}{GM_E}}$ is shorter.

This is the orbital paradox that Mr. Akilikubwa is hinting at: drag removes energy from the satellite, yet the satellite speeds up. The satellite spirals inward into ever-lower, ever-faster orbits, until it eventually enters the dense atmosphere and burns up.

The paradox resolves when you realise that the satellite gains more kinetic energy from falling to a lower orbit than it loses to drag. The potential energy lost exceeds the kinetic energy gained plus the energy lost to drag.

Making Sense of the Answer: A satellite slowed by drag does not simply slow down and stop, the way a car does when you take your foot off the accelerator. Instead, it trades altitude for speed, spiralling inward faster and faster until it meets its fiery end in the atmosphere.

Think Like a Physicist: The relationships $K = -\frac{1}{2}U$ and $E = -K$ are the key. When the satellite drops to a lower orbit, $|U|$ increases, K increases (to half of $|U|$), and E decreases (becomes more negative). The satellite has less total energy but more kinetic energy.

HOT Example 20

- Calculate the kinetic energy, potential energy, and total energy of a 200kg satellite orbiting the Earth at a height of 600km.
- How much energy must be supplied to move this satellite from its current orbit to a new orbit at 1200km above the surface?

Take $M_E = 6.0 \times 10^{24} \text{kg}$, $r_E = 6.4 \times 10^6 \text{m}$, and $G = 6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2}$.

Solution

- Orbital radius at 600km:

$$R_1 = r_E + h_1 = 6.4 \times 10^6 \text{ m} + 600 \times 10^3 \text{ m} = 7 \times 10^6 \text{ m}$$

Kinetic energy:

$$K = \frac{GM_E m}{2R_1} = \frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 6 \times 10^{24} \text{ kg} \times 200 \text{ kg}}{2 \times 7 \times 10^6 \text{ m}} = 5.717 \times 10^9 \text{ J} = 5.72 \text{ GJ}$$

Potential energy:

$$U = -\frac{GM_E m}{R_1} = -\frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 6 \times 10^{24} \text{ kg} \times 200 \text{ kg}}{7 \times 10^6 \text{ m}} = -1.143 \times 10^{10} \text{ J} = -11.43 \text{ GJ}$$

Total energy:

$$E_1 = K + U = 5.72 \text{ GJ} + (-11.43 \text{ GJ}) = -5.72 \text{ GJ}$$

You can easily check the correctness of the answers:

Check: $K = -\frac{1}{2}U$? $\frac{11.43}{2} = 5.72$. Confirmed.

Check: $E = -K$? $-5.72 = -(5.72)$. Confirmed.

(b) Orbital radius at 1200km:

$$R_2 = 6.4 \times 10^6 \text{ m} + 1200 \times 10^3 \text{ m} = 7.6 \times 10^6 \text{ m}$$

Total energy in the new orbit:

$$E_2 = -\frac{GM_E m}{2R_2} = -\frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 6 \times 10^{24} \text{ kg} \times 200 \text{ kg}}{2 \times 7.6 \times 10^6 \text{ m}} = -5.266 \times 10^9 \text{ J} = -5.27 \text{ GJ}$$

Energy that must be supplied:

$$\Delta E = E_2 - E_1 = (-5.27 \text{ GJ}) - (-5.72 \text{ GJ}) = 0.45 \text{ GJ} = 4.5 \times 10^8 \text{ J}$$

The satellite needs $4.5 \times 10^8 \text{ J}$ (about 450MJ) of additional energy to move from the 600km orbit to the 1200km orbit.

Making Sense of the Answer: The higher orbit has less negative total energy (it is less tightly bound). The difference, 450MJ, is the energy the satellite's engines must provide. Notice that moving 600km higher requires far less energy than the initial launch from the surface, because the satellite already has most of the energy it needs.

Think Like a Physicist: The explicit checks ($K = -\frac{1}{2}U$ and $E = -K$) are not just bookkeeping. They confirm that the orbit is genuinely circular. If these relationships did not hold, it would mean the orbit is elliptical or the calculation contains an error. Always verify these relationships when working with orbital energies.

HOT Example 21

Calculate the minimum energy required to launch a 1000kg satellite from the Earth's surface into a circular orbit at a height of 300km. Ignore air resistance and the Earth's rotation.

Take $M_E = 6.0 \times 10^{24} \text{ kg}$, $r_E = 6.4 \times 10^6 \text{ m}$, and $G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

Solution

The satellite starts at rest on the surface. Its initial energy:

$$E_{\text{surface}} = 0 + \left(-\frac{GM_E m}{r_E} \right) = -\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 1000}{6.4 \times 10^6}$$

$$E_{\text{surface}} = -\frac{4.002 \times 10^{17} \text{ J}}{6.4 \times 10^6} = -6.253 \times 10^{10} \text{ J}$$

The orbital radius:

$$R = 6.4 \times 10^6 \text{ m} + 300 \times 10^3 \text{ m} = 6.7 \times 10^6 \text{ m}$$

The total energy in orbit:

$$E_{\text{orbit}} = -\frac{GM_E m}{2R} = -\frac{4.002 \times 10^{17} \text{ J}}{2 \times 6.7 \times 10^6 \text{ m}} = -\frac{4.002 \times 10^{17}}{1.34 \times 10^7} = -2.987 \times 10^{10} \text{ J}$$

Minimum energy required:

$$\begin{aligned}\Delta E &= E_{\text{orbit}} - E_{\text{surface}} = (-2.987 \times 10^{10} \text{ J}) - (-6.253 \times 10^{10} \text{ J}) \\ \Delta E &= 3.266 \times 10^{10} \text{ J} = 32.7 \text{ GJ}\end{aligned}$$

The minimum energy required is 32.7GJ.

Making Sense of the Answer: 32.7GJ for a 1000kg satellite is an enormous amount of energy. For comparison, a litre of rocket fuel (kerosene) releases about 34MJ, so this launch would need at least 960 litres of fuel for the payload alone. In reality, the fuel-to-payload ratio for orbital launches is typically 20: 1 or higher, because the rocket must also lift the fuel, the engines, and the structure.

Think Like a Physicist: Most of the launch energy goes into kinetic energy, not potential energy. The potential energy change is $GM_E m \left(\frac{1}{r_E} - \frac{1}{R} \right) \approx 2.93 \text{ GJ}$, while the kinetic energy needed is $\frac{GM_E m}{2R} \approx 29.9 \text{ GJ}$. About 91% of the energy goes into making the satellite move fast enough to stay in orbit, and only 9% goes into lifting it to the correct height. This is why reaching orbit is fundamentally about speed, not altitude.

With the physics of satellites and orbits now complete, one question remains: *what if we want to escape the Earth's gravity entirely?* Not orbit, but leave. That is the subject of the next and penultimate subtopic of this topic.

Escape velocity

In the previous section, we studied satellites that orbit the Earth, bound by gravity in circular paths. Their total energy is negative, and they remain trapped in the gravitational field forever (unless something changes their energy). But what if we do not want to orbit? What if we want to *leave*?

Every child who has thrown a ball into the air has watched it slow down, stop, and fall back. Throw it harder, and it goes higher before returning. Throw it harder still, and it goes higher still. The natural question arises: *is there a speed so great that the ball never comes back? A speed at which the object rises forever, escaping the Earth's gravitational pull entirely?*

The answer is yes. That speed is called the **escape velocity**.

Derivation

Consider a body of mass m on the surface of the Earth (mass M_E , radius r_E). Suppose the body is projected vertically upward with speed v_E . We want to find the minimum speed for which the body never returns.

“Never returns” means the body reaches infinity. At infinity, the gravitational potential energy is zero (by our convention). The minimum escape condition corresponds to the body arriving at infinity with zero residual speed (just barely escaping, with no energy left over). So at infinity, both the kinetic energy and the potential energy are zero, giving a total energy of zero.

By conservation of mechanical energy, the total energy at the surface must equal the total energy at infinity:

$$\begin{aligned}K_{\text{surface}} + U_{\text{surface}} &= K_{\infty} + U_{\infty} \\ \frac{1}{2}mv_E^2 + \left(-\frac{GM_E m}{r_E} \right) &= 0 + 0 \\ \frac{1}{2}mv_E^2 &= \frac{GM_E m}{r_E}\end{aligned}$$

The mass m cancels (escape velocity does not depend on the mass of the escaping body):

$$\begin{aligned}\frac{1}{2}v_E^2 &= \frac{GM_E}{r_E} \\ v_E &= \sqrt{\frac{2GM_E}{r_E}}\end{aligned}$$

Using $GM_E = gr_E^2$, this can also be written as:

$$v_E = \sqrt{2gr_E}$$

Substituting values for the Earth ($g = 9.8\text{m/s}^2$, $r_E = 6.4 \times 10^6\text{m}$):

$$v_E = \sqrt{2 \times 9.8\text{m/s}^2 \times 6.4 \times 10^6\text{m}} = \mathbf{11200\text{m/s}}$$

The escape velocity from the Earth's surface is approximately 11.2km/s, or about 40300km/h.

The elegant connection: Escape velocity and orbital velocity

Recall that the orbital velocity near the surface is $v_{\text{orb}} = \sqrt{gr_E}$. The escape velocity is $v_E = \sqrt{2gr_E}$. Dividing:

$$\frac{v_E}{v_{\text{orb}}} = \frac{\sqrt{2gr_E}}{\sqrt{gr_E}} = \sqrt{2}$$

$$v_E = \sqrt{2} \times v_{\text{orb}} \approx \mathbf{1.414 \times v_{\text{orb}}}$$

The escape velocity is exactly $\sqrt{2}$ times the orbital velocity at the same radius. This relationship holds at any distance from any spherical body, not just at the Earth's surface. A satellite in circular orbit at any altitude needs only a 41.4% $((1.414 - 1) \times 100\%)$ increase in speed to escape the Earth entirely. This is a remarkably small margin, and it has deep implications for space travel.

Escape velocity from any height

If a body is already at height h above the surface (at distance $R = r_E + h$ from the centre), the escape velocity from that point is:

$$v_E = \sqrt{\frac{2GM_E}{R}} = \sqrt{\frac{2GM_E}{r_E + h}}$$

So the escape velocity decreases with altitude, because the body starts higher in the gravitational potential well and needs less energy to reach the top.

Alternatively, using $GM_E = gr_E^2$:

$$v_E = r_E \sqrt{\frac{2g}{r_E + h}}$$

Key features of escape velocity

Several important points are worth highlighting.

1. It does not depend on mass

A grain of sand and a spacecraft need the same escape velocity to leave the Earth. This is because both the kinetic energy and the gravitational potential energy are proportional to m , so m cancels in the energy equation. Of course, the *energy* required does depend on mass ($\frac{1}{2}mv_E^2$ is larger for a heavier object), but the *speed* does not.

2. It does not depend on the direction of projection

The derivation used energy conservation, which is a scalar equation. It does not matter whether the body is launched vertically, horizontally, or at any angle. As long as the initial speed equals the escape velocity and the body does not hit the Earth on the way out, it will escape. (In practice, launching vertically is the most common choice because it gets above the dense atmosphere quickly.)

3. It depends on the mass and radius of the planet.

A more massive planet has a higher escape velocity (stronger gravitational pull). A larger planet (same mass but greater radius r) has a lower escape velocity (the surface is farther from the centre). For a planet of mass M and radius r :

$$v_E = \sqrt{\frac{2GM}{r}}$$

Escape Velocities of Different Bodies

The escape velocity varies enormously across the solar system. The Moon, with its small mass and radius, has an escape velocity of only about 2.4km/s. Jupiter, the most massive planet, has an escape velocity of about 60km/s. The Sun's escape velocity from its surface is about 617km/s.

Why the Moon has no atmosphere

This concept directly explains one of the most striking differences between the Earth and the Moon. The Earth has a thick atmosphere; the Moon has essentially none. The reason is escape velocity.

Gas molecules in an atmosphere are in constant random motion. Their average speed depends on the temperature: hotter gases move faster. At the **high temperatures** found on the Moon's surface (which can reach over 100°C in direct sunlight), the average speed of light gas molecules such as hydrogen and helium exceeds the Moon's escape velocity of 2.4km/s. These molecules simply fly off into space and never return. Over billions of years, even heavier gas molecules gradually escape, leaving the Moon with no atmosphere.

The Earth's escape velocity (11.2km/s) is much higher than the average speed of atmospheric gas molecules at terrestrial temperatures (typically 0.3 to 0.6km/s for nitrogen and oxygen). So the Earth retains its atmosphere comfortably.

Let us now apply these ideas through worked examples.

BINDER Example 22

- Calculate the escape velocity from the surface of the Moon. Take the mass of the Moon as $7.35 \times 10^{22}\text{kg}$, the radius of the Moon as $1.74 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.
- Compare this with the escape velocity from the Earth (11.2km/s) and explain why the Moon has no atmosphere while the Earth does.

Solution

- Using $v_E = \sqrt{\frac{2GM}{r}}$:

$$v_E = \sqrt{\frac{2 \times 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 7.35 \times 10^{22}\text{kg}}{1.74 \times 10^6\text{m}}} = 2374\text{m/s}$$

The escape velocity from the Moon is 2374m/s (about 2.4km/s).

(b) Comparison:

The Earth's escape velocity (11.2km/s) is greater (about 4.7 times greater) than the Moon's (2.4km/s).

Explanation:

Gas molecules move very fast at ordinary temperatures, but on Earth their speeds are still much lower than the escape velocity, so they remain held by gravity and the atmosphere is retained. On the Moon, however, gravity is much weaker, giving a much lower escape velocity. Because the Moon also experiences very high surface temperatures (over 100°C) in sunlight, some gas molecules gain enough speed to escape into space. Over a very long time, this gradual loss means the Moon has lost all atmospheric gases, leaving the Moon with no atmosphere.

Making Sense of the Answer: *The Moon's low escape velocity is a direct consequence of its small mass. A body with 81 times less mass than the Earth simply cannot hold onto fast-moving gas molecules. This single number, 2.4km/s, explains why astronauts on the Moon need spacesuits.*

Think Like a Physicist: *Not all molecules in a gas move at the same speed. Even if the average speed is below the escape velocity, some molecules in the high-speed tail of the distribution will exceed it and escape. Over geological time, this slow leakage is enough to strip the Moon of any atmosphere it once had.*

REAL Example 23

Kipanga watches a science fiction film where a character escapes from a planet by jumping very hard. He asks Kipute: “Is that even possible? Could you ever jump fast enough to escape the Earth?”

Explain whether it is theoretically possible, and discuss why rockets are needed in practice.

Solution

Theoretically, if a person could jump at 11.2km/s (40300km/h), they would escape the Earth because Escape velocity does not depend on mass. So even a person would need the same speed as a heavier rocket. However, this is completely impossible in practice as the speed of man is far below the escape velocity (the fastest a human can move their body is about 10m/s (a sprint), which is more than 1000 times too slow).

Rockets are needed because they provide a sustained thrust over many minutes, gradually building up speed. A rocket does not need to reach escape velocity all at once; it accelerates continuously until the required speed is achieved. It also needs to overcome air resistance and lift its own fuel, which is why real launches require far more energy than the theoretical minimum.

In the film, the escape might be plausible only if the planet were very small and had very low surface gravity, giving a correspondingly low escape velocity.

Making Sense of the Answer: *Escape velocity is about the speed needed, not the force of a single push. A cannon could theoretically launch a projectile at escape velocity in one shot, but the enormous acceleration would destroy any payload (and any passenger). Rockets solve this by applying a gentler acceleration over a longer time.*

Think Like a Physicist: *The derivation of escape velocity assumes no further force acts after the initial launch (pure projectile). A rocket, by contrast, burns fuel continuously, so it does not need to start at escape velocity. It can leave the Earth at any speed, as long as it keeps accelerating. The escape velocity formula gives the minimum speed for an unpowered escape.*

HOT Example 24

- (a) Show that the escape velocity from the surface of a planet can be written as $v_E = \sqrt{2}v_{\text{orb}}$, where v_{orb} is the orbital velocity near the surface.
- (b) A satellite orbits the Earth at a height of 500km. Calculate the orbital velocity and the escape velocity at this altitude. By what percentage must the satellite’s speed increase to escape the Earth?

Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Solution

- (a) The orbital velocity near the surface is:

$$v_{\text{orb}} = \sqrt{\frac{GM}{r}}$$

The escape velocity from the surface is:

$$v_E = \sqrt{\frac{2GM}{r}}$$

Dividing v_E by v_{orb} :

$$\frac{v_E}{v_{\text{orb}}} = \frac{\sqrt{\frac{2GM}{r}}}{\sqrt{\frac{GM}{r}}} = \sqrt{\frac{2GM/r}{GM/r}} = \sqrt{2}$$

Therefore $v_E = \sqrt{2}v_{\text{orb}}$.

- (b) Orbital radius at 500km:

$$R = r_E + h = 6.4 \times 10^6\text{m} + 500 \times 10^3\text{m} = 6.9 \times 10^6\text{m}$$

Orbital velocity:

$$v_{\text{orb}} = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24} \text{kg}}{6.9 \times 10^6 \text{m}}} = 7616 \text{m/s}$$

Escape velocity at this altitude:

$$v_E = \sqrt{2} \times v_{\text{orb}} = 1.414 \times 7616 \text{m/s} = 10,769 \text{m/s}$$

Percentage increase needed:

$$\frac{v_E - v_{\text{orb}}}{v_{\text{orb}}} \times 100\% = \frac{10769 \text{m/s} - 7616 \text{m/s}}{7616 \text{m/s}} \times 100\% = 41.4\%$$

The satellite needs a 41.4% increase in speed to escape.

Note: This is exactly $(\sqrt{2} - 1) \times 100\% = 41.4\%$, and it is the same at every altitude. A satellite anywhere in a circular orbit is always just 41.4% of extra speed away from freedom.

Making Sense of the Answer: 41.4% may seem like a small margin, but in energy terms it is not. Since kinetic energy scales as v^2 , a 41.4% increase in speed requires a 100% increase in kinetic energy (since $(\sqrt{2})^2 = 2$). The satellite must exactly double its kinetic energy to escape.

Think Like a Physicist: The result $v_E = \sqrt{2}v_{\text{orb}}$ connects beautifully to the energy relationships learned earlier. For a circular orbit, $E = -K$, so the total energy is the negative of the kinetic energy. To escape, the total energy must become zero, which requires adding energy equal to $|E| = K$. This doubles the kinetic energy, which increases the speed by a factor of $\sqrt{2}$. The physics is perfectly self-consistent.

We have now completed the core theory of gravitation: from Kepler's empirical laws, through Newton's universal force, the variation of gravity with position, the concepts of field and potential, the energy of orbits, and finally escape velocity. In the next section, we gather the applications of these ideas into a single, coherent picture.

APPLICATIONS OF GRAVITATION

The equations we have derived in this chapter were not built for textbooks. They were built for the universe. From the motion of planets to the design of satellite networks, from measuring the mass of distant stars to explaining why the ocean tides rise and fall, gravitation touches nearly every branch of physics and engineering. In this section, we step outside the classroom and see how the physics we have mastered finds its way into the real world.

1. Weighing the Earth and other Celestial bodies

One of the most remarkable applications of Newton's law of gravitation is that it allows us to "weigh" objects we can never place on a scale. If we can observe any orbit around a body (its period and radius), Kepler's third law immediately gives us the mass of the central body:

$$M = \frac{4\pi^2 R^3}{GT^2}$$

This is how the mass of the Earth was first determined (using the Moon's orbit), how the mass of the Sun is known (using the Earth's orbit), and how the masses of distant planets are measured (using their moons or spacecraft flybys). We do not need to touch the object; we only need to watch something orbit it.

Even the mass of a distant star can be estimated if it has a companion star or planet whose orbital parameters can be measured. Gravitation turns every orbit into a weighing scale.

2. Communication satellites and the geostationary belt

Modern telecommunications depend heavily on geostationary satellites. As we showed earlier, these satellites orbit at approximately 36000km above the equator with a period of exactly 24 hours, appearing stationary above a fixed point on the ground.

A single geostationary satellite can “see” about one-third of the Earth’s surface. Three satellites, spaced 120° apart around the equator, can cover almost the entire globe (except for small regions near the poles). This is the basis of global satellite communication, television broadcasting, and weather monitoring.

The geostationary orbit is sometimes called the **Clarke orbit**, after the science fiction writer Arthur C. Clarke, who first proposed the idea of geostationary communication satellites in 1945, more than a decade before the first satellite (Sputnik) was launched. Clarke’s prediction, based entirely on the physics of this chapter, turned out to be one of the most commercially valuable ideas in the history of technology.

Because all geostationary satellites must orbit at the same altitude and in the same plane, the available “slots” are limited. The International Telecommunication Union (ITU) regulates the allocation of these orbital positions to prevent interference between neighbouring satellites.

3. Low Earth Orbit (LEO) satellites

Not all satellites need to be geostationary. Satellites in **low Earth orbit** (typically 200 to 2000km altitude) move much faster and have much shorter periods (about 90 to 130 minutes). They pass over different parts of the Earth’s surface on each orbit, making them ideal for applications that require global coverage over time rather than a fixed view.

Earth observation satellites photograph the surface with high resolution, monitoring deforestation, urban growth, agricultural productivity, and natural disasters. **Weather satellites** in polar orbits scan the entire Earth twice a day. The **International Space Station** orbits at about 400km and serves as a laboratory for research in microgravity.

Global navigation systems such as GPS (United States), GLONASS (Russia), Galileo (European Union), and BeiDou (China) use constellations of satellites at medium altitudes (about 20000km). By receiving signals from at least four satellites simultaneously, a GPS receiver on the ground can calculate its position to within a few metres. The physics that makes this possible is precisely the orbital mechanics of this chapter.

4. Tides

The tides are one of the oldest known effects of gravitation. The Moon’s gravitational pull on the Earth is not uniform: it is stronger on the side of the Earth closest to the Moon and weaker on the far side. This difference in gravitational pull (called the **tidal force**) stretches the Earth slightly along the Earth-Moon line, creating two bulges in the oceans. As the Earth rotates beneath these bulges, coastal areas experience two high tides and two low tides each day.

The Sun also contributes to tides, but because it is much farther away, its tidal effect is about half that of the Moon. When the Sun and Moon are aligned (during new moon and full moon), their tidal effects add together, producing exceptionally high tides called **spring tides**. When they are at right angles (during quarter moons), the tides are weaker, called **neap tides**.

Tides affect not only the oceans. The Earth’s solid crust also deforms slightly (by about 25cm) under tidal forces, and tidal forces from Jupiter cause intense volcanic activity on its moon Io, the most volcanically active body in the solar system.

5. Determining the density of the Earth

Earlier, we showed that $g = \frac{GM_E}{r_E^2}$, which gives $M_E = \frac{gr_E^2}{G}$. Once the mass is known, the density follows from:

$$\rho = \frac{M_E}{\frac{4}{3}\pi r_E^3} = \frac{3g}{4\pi G r_E}$$

This result is significant because it reveals that the Earth’s average density (about 5500kg/m^3) is much higher than the density of surface rocks (2500 to 3000kg/m^3). Without ever drilling to the centre, physicists deduced that the Earth must have a very dense core, most likely composed of iron and nickel. This conclusion, drawn from nothing more than g , G , and r_E , was later confirmed by seismic studies.

6. Space travel and interplanetary missions

Every interplanetary mission uses the gravitational physics of this chapter. Launching a spacecraft from Earth requires overcoming the Earth’s gravitational potential well, which demands enormous energy. Once in space, the spacecraft can use the gravity of other planets to change its speed and direction, a technique called a **gravitational slingshot** (or gravity assist).

In a gravity assist, a spacecraft approaches a planet, is deflected by the planet's gravity, and leaves with a different speed and direction. No fuel is burned during the manoeuvre; the energy comes from the planet's orbital motion. The Voyager spacecraft, launched in 1977, used gravity assists from Jupiter and Saturn to reach speeds high enough to leave the solar system entirely. Voyager 1 is currently the most distant human-made object, over 24 billion kilometres from Earth, still sending data back using a 23-watt radio transmitter (about the power of a refrigerator light bulb).

7. Apparent weightlessness in orbit

We have mentioned weightlessness several times in this chapter, but it deserves a clear, consolidated explanation because it is one of the most widely misunderstood concepts in physics.

An astronaut in orbit is not beyond the reach of gravity. As we calculated in Example 14, the gravitational field at the altitude of the ISS is still about 94% of its surface value. The astronaut floats not because gravity is absent, but because both the astronaut and the spacecraft are falling toward the Earth at exactly the same rate. There is no relative acceleration between the astronaut and the spacecraft, so there is no contact force (no normal reaction from the floor, no tension from a support). With zero contact force, the astronaut feels weightless.

This is exactly the same physics as the momentary weightlessness at the top of a hill (in Chapter 7), but sustained continuously. In orbit, the object is always at the “top of the hill” because the circular path ensures that gravitational acceleration exactly equals centripetal acceleration at every point.

True weightlessness (zero gravitational force) would require being infinitely far from all masses, which is physically impossible. What astronauts experience in orbit is more accurately called **free fall** or **microgravity**.

From satellites silently circling the Earth to the tides rising along the coast of Dar es Salaam, from GPS guiding a driver through Dodoma to the Voyager spacecraft sailing beyond the edge of the solar system, gravitation is the invisible thread connecting all of it. The equations we derived in this chapter are not abstract mathematics; they are the operating instructions for the largest structures in the universe.

And with that, the core of gravitation is complete. In the next section, the concepts of this entire chapter will come together in miscellaneous worked examples, where the physics will stop being polite and start working together. If the individual sections were the instruments, the miscellaneous examples are the orchestra. Tune up, and let us play.

MISCELLANEOUS WORKED EXAMPLES ON GRAVITATION

Example 25

- Explain why all objects, regardless of their mass, fall with the same acceleration in a gravitational field (in the absence of air resistance).
- Two stars orbit each other with a period of 2.0 years. The mean distance between their centres is 3.0×10^{11} m. Calculate the combined mass of the two-star system. Take $G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

Solution

- The gravitational force on an object of mass m near a planet of mass M is $F = \frac{GMm}{r^2}$. By Newton's second law, the acceleration is $a = \frac{F}{m} = \frac{GM}{r^2}$. The mass m of the falling object appears in both the gravitational force (which pulls it) and the inertia (which resists acceleration), and these two effects cancel exactly. Thus the acceleration depends only on M and r , not on m .
- For a binary star system, Kepler's third law gives:

$$T^2 = \frac{4\pi^2 R^3}{G(M_1 + M_2)}$$

Making the combined mass the subject:

$$M_1 + M_2 = \frac{4\pi^2 R^3}{GT^2}$$

Converting the period: $T = 2.0 \text{ years} = 2.0 \times 365.25 \times 24 \times 3600 \text{ s} = 6.312 \times 10^7 \text{ s}$

Substituting:

$$M_1 + M_2 = \frac{4 \times (3.14)^2 \times (3.0 \times 10^{11} \text{m})^3}{6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2} \times (6.312 \times 10^7 \text{s})^2} = 4.01 \times 10^{30} \text{kg}$$

The combined mass of the two stars is approximately $4.0 \times 10^{30} \text{kg}$.

Example 26

- (a) Explain why the gravitational potential at any point near a planet is always negative.
- (b) Calculate the gravitational potential at a point 1000km above the Earth's surface and the work done in moving a 50kg object from the surface to this point. Take $M_E = 6.0 \times 10^{24} \text{kg}$, $r_E = 6.4 \times 10^6 \text{m}$, and $G = 6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2}$.

Solution

- (a) Gravitational potential is defined with the reference point at infinity, where $V = 0$. To reach any point near a planet from infinity, gravity does positive work on the object (it pulls the object inward). This means the object loses potential energy as it approaches the planet. Since the potential energy at infinity is zero and the object has less potential energy at any point near the planet, the potential must be negative. The closer to the planet, the more negative the potential becomes.

- (b) At height $h = 1000 \text{km} = 1.0 \times 10^6 \text{m}$:

$$R = r_E + h = 6.4 \times 10^6 \text{m} + 1.0 \times 10^6 \text{m} = 7.4 \times 10^6 \text{m}$$

$$V_{1000} = -\frac{GM_E}{R} = -\frac{4.002 \times 10^{14} \text{J}}{7.4 \times 10^6 \text{m}} = -5.408 \times 10^7 \text{J/kg}$$

At the surface:

$$V_{\text{surface}} = -\frac{GM_E}{r_E} = -\frac{6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2} \times 6.0 \times 10^{24} \text{kg}}{6.4 \times 10^6 \text{m}} = -6.253 \times 10^7 \text{J/kg}$$

The work done in moving the 50kg object from the surface to this height:

$$W = m(V_{1000} - V_{\text{surface}}) = 50 \text{kg} \times ((-5.408 \times 10^7) - (-6.253 \times 10^7)) \text{J/kg}$$

$$W = 50 \text{kg} \times 8.45 \times 10^6 \text{J/kg} = 4.225 \times 10^8 \text{J} = 422.5 \text{MJ}$$

Example 27

- (a) **Kipanga** declares confidently to the class: "If the Earth suddenly stopped rotating, we would all fly off into space!" **Kipute** shakes her head, showing disagreement. **Mr. Akilikubwa** hides a smile. Who is correct? Explain why.
- (b) Calculate the angular velocity at which the Earth would need to rotate for objects at the equator to become weightless. Express this as a multiple of the current angular velocity of the Earth, and find the corresponding length of a day. Take $g = 9.8 \text{m/s}^2$ and $r_E = 6.4 \times 10^6 \text{m}$.

Solution

- (a) Kipute is correct. Kipanga has the physics backwards. The Earth's rotation currently **reduces** the effective gravity at the equator (because part of the gravitational pull goes toward providing centripetal acceleration for the rotation). If the Earth stopped rotating, no centripetal acceleration would be needed, and the effective gravity would actually **increase** slightly. Everyone would feel heavier, not lighter. There would be no tendency to fly off at all.

Flying off into space would require the Earth to rotate much **faster**, not to stop. Objects would leave the surface only if the required centripetal acceleration exceeded g , so that gravity could no longer hold them down.

- (b) At the equator, objects become weightless when the centripetal acceleration equals g :

$$\omega^2 r_E = g$$

Making ω the subject:

$$\omega = \sqrt{\frac{g}{r_E}} = \sqrt{\frac{9.8 \text{ m/s}^2}{6.4 \times 10^6 \text{ m}}} = \sqrt{1.531 \times 10^{-6} \text{ s}^{-2}} = 1.237 \times 10^{-3} \text{ rad/s}$$

The required angular velocity for weightless is $1.237 \times 10^{-3} \text{ rad/s}$.

The current angular velocity of the Earth:

$$\omega_{\text{current}} = \frac{2\pi}{24 \times 3600 \text{ s}} = \frac{6.283}{86400 \text{ s}} = 7.272 \times 10^{-5} \text{ rad/s}$$

The ratio:

$$\frac{\omega}{\omega_{\text{current}}} = \frac{1.237 \times 10^{-3}}{7.272 \times 10^{-5}} = 17.0$$

The Earth would need to rotate 17 times faster than it currently does.

The corresponding length of a day:

$$T = \frac{2\pi}{\omega} = \frac{6.283}{1.237 \times 10^{-3} \text{ s}^{-1}} = 5079 \text{ s} = \frac{5079}{60} \text{ min} = 84.7 \text{ min}$$

The length of a day would be about 84.7 minutes.

This is the same as the orbital period of a satellite near the Earth's surface, which is not a coincidence: at this rotation speed, the surface itself moves at orbital velocity, and objects sitting on the equator are effectively in orbit.

Example 28

- (a) A satellite in a higher orbit moves more slowly than one in a lower orbit, even though more energy was needed to place it there. Explain why this is not a contradiction.
- (b) Two satellites A and B orbit the Earth in circular orbits. Satellite A has orbital radius $2R$ and satellite B has orbital radius $8R$. Find the ratio of:
 - (i) their orbital speeds,
 - (ii) their orbital periods,
 - (iii) their total mechanical energies.

Solution

- (a) The orbital speed is determined by the balance between gravity and centripetal force: $v = \sqrt{\frac{GM}{R}}$. A higher orbit has a larger R , which gives a smaller v . The satellite moves more slowly because gravity is weaker at greater distances, so less speed is needed to maintain the circular path.

The total energy, however, is $E = -\frac{GMm}{2R}$. A higher orbit has a larger R , giving a less negative (larger) total energy. More energy was indeed needed to place it there. The apparent contradiction dissolves when you realise that kinetic energy and total energy are different quantities. A higher satellite has less kinetic energy but much more potential energy, and the increase in potential energy outweighs the decrease the kinetic energy, giving a higher total energy overall.

- (b) Let the orbital radius of A be $R_A = 2R$ and that of B be $R_B = 8R$.
 - (i) Orbital speed ratio:

$$\frac{v_A}{v_B} = \frac{\sqrt{\frac{GM}{2R}}}{\sqrt{\frac{GM}{8R}}} = \sqrt{\frac{8R}{2R}} = \sqrt{4} = 2$$

Satellite A moves 2 times faster than satellite B.

- (ii) Period ratio, using $T = 2\pi\sqrt{\frac{R^3}{GM}}$:

$$\frac{T_A}{T_B} = \sqrt{\frac{R_A^3}{R_B^3}} = \sqrt{\frac{(2R)^3}{(8R)^3}} = \sqrt{\frac{8}{512}} = \sqrt{\frac{1}{64}} = \frac{1}{8}$$

Satellite A has a period 8 times shorter than satellite B.

(iii) Total energy ratio (assuming both satellites have the same mass m):

$$\frac{E_A}{E_B} = \frac{-\frac{GMm}{2 \times 2R}}{-\frac{GMm}{2 \times 8R}} = \frac{\frac{1}{4R}}{\frac{1}{16R}} = \frac{16}{4} = 4$$

The total energy of satellite A is 4 times the total energy of satellite B.

Understand this: Since both energies are negative, this means A has a more negative energy (it is more tightly bound to the Earth). This is consistent with A being in a lower orbit.

Example 29

- (a) Explain why the approximate formula $g' \approx g \left(1 - \frac{2h}{r_E}\right)$ works well for small heights but fails for large heights.
- (b) At what height above the Earth's surface is the gravitational field strength exactly half the surface value? Express your answer in terms of r_E and in kilometres. Take $r_E = 6400\text{km}$.

Solution

- (a) The exact formula is: $g' = g \left(\frac{r_E}{r_E + h}\right)^2$. The approximate formula is obtained by writing this as $g \left(1 + \frac{h}{r_E}\right)^{-2}$ and applying the binomial expansion, which gives $g \left(1 - \frac{2h}{r_E} + \dots\right)$. The higher-order terms (containing $\frac{h^2}{r_E^2}$ and beyond) are dropped. When h is much smaller than r_E , these dropped terms are negligible and the approximation is accurate. When h is comparable to r_E , the dropped terms become significant and the approximation gives increasingly wrong results. For example, at $h = \frac{r_E}{2}$, the approximate formula gives $g' = 0$, while the exact formula gives $g' = 0.44g$.
- (b) Setting $g' = \frac{1}{2}g$ in the exact formula:

$$\frac{1}{2}g = g \left(\frac{r_E}{r_E + h}\right)^2$$

Dividing both sides by g :

$$\frac{1}{2} = \left(\frac{r_E}{r_E + h}\right)^2$$

Taking the square root:

$$\frac{1}{\sqrt{2}} = \frac{r_E}{r_E + h}$$

Making $(r_E + h)$ the subject:

$$\begin{aligned} r_E + h &= r_E \sqrt{2} \\ h &= r_E(\sqrt{2} - 1) = 0.4142r_E \end{aligned}$$

In kilometres:

$$h = 0.4142 \times 6400\text{km} = 2651\text{km}$$

The gravitational field strength is halved at a height of $0.4142r_E$, or approximately 2651km above the surface.

Example 30

- (a) A student claims: “Since the gravitational field strength is zero at the centre of the Earth, the gravitational potential must also be zero there.” Identify and correct the error in this reasoning.
- (b) Calculate the gravitational potential at the surface and at the centre of the Earth, assuming uniform density. Show that the potential at the centre is 1.5 times more negative than at the surface. Take $M_E = 6.0 \times 10^{24} \text{ kg}$, $r_E = 6.4 \times 10^6 \text{ m}$, and $G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

Solution

- (a) The student is confusing two different quantities. Gravitational field strength (g) is the force per unit mass, while gravitational potential (V) is the work done per unit mass in bringing a mass from infinity to that point. They are related by $g = -\frac{dV}{dr}$, which means that g is zero when the *rate of change* of V is zero, not when V itself is zero. At the centre of the Earth, V is at its minimum (most negative) value, and the gradient of V is zero (like the bottom of a valley, where the slope is flat but the height is at its lowest). Zero field strength means the potential has reached a turning point, not that the potential is zero.
- (b) At the surface:

$$V_{\text{surface}} = -\frac{GM_E}{r_E} = -\frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 6.0 \times 10^{24} \text{ kg}}{6.4 \times 10^6 \text{ m}} = -6.25 \times 10^7 \text{ J/kg}$$

To find the potential at the centre, we derive it from first principles. Inside the Earth (assuming uniform density), the gravitational field strength at distance r from the centre is $g_{\text{inside}} = \frac{GM_E}{r_E^3} r$. Since $g = -\frac{dV}{dr}$, we can find the potential difference between the centre and the surface by integrating from the centre ($r = 0$) to the surface ($r = r_E$):

$$\begin{aligned} V_{\text{surface}} - V_{\text{centre}} &= \int_0^{r_E} g_{\text{inside}} \, dr = \int_0^{r_E} \frac{GM_E}{r_E^3} r \, dr \\ &= \left[\frac{GM_E}{r_E^3} \times \frac{r^2}{2} \right]_0^{r_E} = \frac{GM_E}{r_E^3} \times \frac{r_E^2}{2} = \frac{GM_E}{2r_E} \end{aligned}$$

Making V_{centre} the subject:

$$V_{\text{centre}} = V_{\text{surface}} - \frac{GM_E}{2r_E} = -\frac{GM_E}{r_E} - \frac{GM_E}{2r_E} = -\frac{2GM_E + GM_E}{2r_E} = -\frac{3GM_E}{2r_E}$$

Numerically:

$$V_{\text{centre}} = -\frac{3}{2} \times \frac{GM_E}{r_E} = \frac{3}{2} \times (-6.25 \times 10^7 \text{ J/kg}) = -9.375 \times 10^7 \text{ J/kg}$$

The ratio:

$$\frac{V_{\text{centre}}}{V_{\text{surface}}} = \frac{-9.375 \times 10^7 \text{ J/kg}}{-6.25 \times 10^7 \text{ J/kg}} = 1.5$$

The potential at the centre is 1.5 times more negative than at the surface.

Understand: The factor $\frac{3}{2}$ arises naturally from the integration: the potential difference between centre and surface is $\frac{GM_E}{2r_E}$, which when subtracted from $V_{\text{surface}} = -\frac{GM_E}{r_E}$ gives $-\frac{3GM_E}{2r_E}$. So the potential at the centre is indeed 1.5 times more negative than at the surface, confirming that the centre is the deepest point in the gravitational potential well.

Example 31

- (a) Explain why the escape velocity from a planet does not depend on either the mass or the direction of projection of the escaping object.

- (b) A planet has a mass 4 times the Earth's mass and a radius 2 times the Earth's radius. Calculate the escape velocity from this planet in terms of the Earth's escape velocity and hence express your answer in km/h.

Solution

- (a) **Independence from mass:** The escape velocity is derived from conservation of energy: $\frac{1}{2}mv_E^2 = \frac{GMm}{r}$. The mass m of the escaping object appears on both sides and cancels. The kinetic energy needed to escape is proportional to m , but so is the gravitational potential energy that must be overcome. These two effects scale identically, leaving the required speed independent of mass.

Independence from direction: The energy equation is a scalar equation. It involves v^2 (which is the square of the speed, not the velocity), so only the magnitude of the speed matters, not its direction. Whether the object is launched vertically, horizontally, or at any angle, the energy balance is the same. As long as the initial speed equals v_E and the object does not collide with the surface, it will escape.

- (b) The escape velocity from a planet of mass M and radius r is $v_E = \sqrt{\frac{2GM}{r}}$.

For the given planet: $M_p = 4M_E$ and $r_p = 2r_E$.

$$v_p = \sqrt{\frac{2G \times 4M_E}{2r_E}} = \sqrt{\frac{4}{2} \times \frac{2GM_E}{r_E}} = \sqrt{2} \times \sqrt{\frac{2GM_E}{r_E}} = \sqrt{2} \times v_{E, \text{Earth}}$$

The escape velocity from this planet is $\sqrt{2}$ times the Earth's escape velocity:

$$v_p = \sqrt{2} \times 11.2 \text{ km/s} = 1.414 \times 11.2 \text{ km/s} = \mathbf{15.8 \text{ km/s}}$$

Example 32

- (a) Explain two advantages of placing a communication satellite in a geostationary orbit rather than a low Earth orbit.
- (b) A telecommunications company plans to place an 800 kg satellite in geostationary orbit. Take $M_E = 6.0 \times 10^{24} \text{ kg}$, $r_E = 6.4 \times 10^6 \text{ m}$, and $G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

Calculate:

- the orbital radius,
- the orbital speed,
- the total energy of the satellite in orbit,
- the minimum energy required to launch the satellite from the Earth's surface to this orbit.

Solution

- (a) **First advantage:** A geostationary satellite remains fixed above one point on the equator, so ground antennas can be permanently pointed at it without tracking. This greatly simplifies the ground equipment needed for communication and broadcasting.

Second advantage: A single geostationary satellite can see about one-third of the Earth's surface continuously, providing uninterrupted coverage over that region. A low-orbit satellite passes overhead for only a few minutes at a time, requiring a constellation of many satellites and complex handover systems to maintain continuous coverage.

- (b) The solution of each part is as follows:

- (i) Using $T = 24 \text{ h} = 86400 \text{ s}$:

$$R^3 = \frac{GM_E T^2}{4\pi^2} = \frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 6.0 \times 10^{24} \text{ kg} \times (86400 \text{ s})^2}{4 \times (3.14)^2} = 7.567 \times 10^{22} \text{ m}^3$$

$$R = \sqrt[3]{(7.567 \times 10^{22} \text{ m}^3)} = 4.23 \times 10^7 \text{ m} = \mathbf{42300 \text{ km}}$$

- (ii) Orbital speed:

$$v = \frac{2\pi R}{T} = \frac{2 \times 3.14 \times 4.23 \times 10^7 \text{m}}{86400 \text{s}} = 3075 \text{m/s} = \mathbf{3.08 \text{km/s}}$$

(iii) Total energy in orbit:

$$E_{\text{orbit}} = -\frac{GM_E m}{2R} = -\frac{6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2} \times 6.0 \times 10^{24} \text{kg} \times 800 \text{kg}}{2 \times 4.23 \times 10^7 \text{m}}$$

$$E_{\text{orbit}} = -\frac{3.202 \times 10^{17} \text{J}}{8.46 \times 10^7 \text{m}} = \mathbf{-3.784 \times 10^9 \text{J} = -3.78 \text{GJ}}$$

(iv) Total energy at the surface (at rest):

$$E_{\text{surface}} = -\frac{GM_E m}{r_E} = -\frac{3.202 \times 10^{17} \text{J}}{6.4 \times 10^6 \text{m}} = -5.003 \times 10^{10} \text{J} = -50.03 \text{GJ}$$

Minimum energy required:

$$\Delta E = E_{\text{orbit}} - E_{\text{surface}} = (-3.78 \text{GJ}) - (-50.03 \text{GJ}) = 46.25 \text{GJ}$$

The minimum energy required to launch the 800kg satellite to geostationary orbit is approximately 46.3GJ.

Example 33

- (a) Explain why the gravitational field strength is zero at the neutral point between the Earth and the Moon, but the gravitational potential is not zero at that point.
- (b) Two masses $M_1 = 4.0 \times 10^{24} \text{kg}$ and $M_2 = 1.0 \times 10^{24} \text{kg}$ have their centres separated by a distance of $5.0 \times 10^8 \text{m}$. Find the position of the neutral point measured from M_1 . Take $G = 6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2}$.

Solution

- (a) At the neutral point, the gravitational field due to the Earth (pointing toward Earth) is equal in magnitude to the field due to the Moon (pointing toward Moon). Since these fields point in opposite directions and are equal, they cancel, giving zero net field.

However, gravitational potential is a scalar, not a vector. It has no direction and cannot cancel by opposition. Both the Earth and the Moon contribute negative potential at every point in the space between them. At the neutral point, the total potential is the sum of two negative values, which is itself negative. Zero field means the forces balance; it does not mean the energies cancel.

- (b) Let the neutral point be at distance x from M_1 . Then it is at distance $(5.0 \times 10^8 \text{m} - x)$ from M_2 .

At the neutral point, the field strengths are equal:

$$\frac{GM_1}{x^2} = \frac{GM_2}{(5 \times 10^8 - x)^2}$$

G cancels:

$$\frac{M_1}{x^2} = \frac{M_2}{(5 \times 10^8 - x)^2}$$

Taking the square root of both sides:

$$\frac{\sqrt{M_1}}{x} = \frac{\sqrt{M_2}}{5 \times 10^8 - x}$$

Cross-multiplying:

$$(5 \times 10^8 - x)\sqrt{M_1} = x\sqrt{M_2}$$

$$5 \times 10^8 \sqrt{M_1} = x(\sqrt{M_1} + \sqrt{M_2})$$

$$x = \frac{5 \times 10^8 \text{m} \times \sqrt{M_1}}{\sqrt{M_1} + \sqrt{M_2}}$$

Substituting:

$$x = \frac{5 \times 10^8 \text{m} \times \sqrt{4 \times 10^{24} \text{kg}}}{\sqrt{4 \times 10^{24} \text{kg}} + \sqrt{1 \times 10^{24} \text{kg}}} = 3.33 \times 10^8 \text{m}$$

The neutral point is $3.33 \times 10^8 \text{m}$ from M_1 .

Example 34

- (a) Explain why a ball thrown upward from the Moon's surface would reach a much greater height than the same ball thrown with the same speed from the Earth's surface.
- (b) A ball is thrown vertically upward from the surface of the Moon with a speed of 20m/s . Take $g_{\text{Moon}} = 1.6 \text{m/s}^2$.
 - (i) Calculate the maximum height reached.
 - (ii) If the same ball is instead thrown horizontally at 20m/s from the edge of a crater 50m deep on the Moon, calculate the time to reach the bottom and the horizontal distance covered.

Solution

- (a) The maximum height reached by a ball thrown upward with speed u is $H = \frac{u^2}{2g}$. Since the Moon's gravitational acceleration (1.6m/s^2) is about 6 times weaker than the Earth's (9.8m/s^2), the same initial speed produces about 6 times greater height on the Moon. The weaker gravity decelerates the ball more slowly, allowing it to travel much farther before stopping.

(b)

- (i) Using $v^2 = u^2 - 2g_{\text{Moon}}H$, and setting $v = 0$ at maximum height:

$$0 = (20 \text{m/s})^2 - 2 \times 1.6 \text{m/s}^2 \times H$$

$$H = \frac{(20 \text{m/s})^2}{2 \times 1.6 \text{m/s}^2} = 125 \text{m}$$

Make sense...! On Earth, the same throw would reach $\frac{(20 \text{m/s})^2}{2 \times 9.8 \text{m/s}^2} = 20.4 \text{m}$. The Moon height is about 6 times greater, as expected.

- (ii) The ball is thrown horizontally, so the initial vertical velocity is zero. On the Moon, the vertical acceleration is $g_{\text{Moon}} = 1.6 \text{m/s}^2$.

Time to fall 50m :

$$h = \frac{1}{2} g_{\text{Moon}} t^2$$

$$t = \sqrt{\frac{2h}{g_{\text{Moon}}}} = \sqrt{\frac{2 \times 50 \text{m}}{1.6 \text{m/s}^2}} = \sqrt{62.5 \text{s}^2} = 7.91 \text{s}$$

The time to reach the bottom is 7.91s .

Horizontal distance:

$$x = u \times t = 20 \text{m/s} \times 7.91 \text{s} = 158.1 \text{m}$$

The horizontal distance covered is 158.1m .

Do you see....? On the Moon, the ball takes 7.91s to reach the bottom (compared to 3.19s on Earth) and covers 158.1m horizontally (compared to 63.9m on Earth). The weaker gravity gives the ball much more time in the air, and therefore a much greater range.

Example 36

- (a) Explain why a planet with the same density as Earth but twice the radius would have a surface gravity of $2g$, not $4g$.
- (b) Planet X has a radius 3 times that of Earth and a surface gravity of 27m/s^2 . Calculate the density of Planet X in terms of the Earth's density. Take $g_{\text{Earth}} = 9.8 \text{m/s}^2$.

Solution

- (a) For a planet with the same density ρ but radius $2r_E$, the mass is:

$$M = \rho \times \frac{4}{3}\pi(2r_E)^3 = 8\rho \times \frac{4}{3}\pi r_E^3 = 8M_E$$

The mass increases by a factor of 8 (because volume scales as r^3). The surface gravity is:

$$g = \frac{GM}{r^2} = \frac{G \times 8M_E}{(2r_E)^2} = \frac{8GM_E}{4r_E^2} = 2 \times \frac{GM_E}{r_E^2} = 2g_{\text{Earth}}$$

The mass increases 8-fold but the radius squared increases 4-fold, so the net effect is $\frac{8}{4} = 2$. Hence, for same density, g varies linearly with radius ($g \propto r$), not quadratically.

- (b) Using $g = \frac{GM}{r^2}$ and $M = \rho \times \frac{4}{3}\pi r^3$:

$$g = \frac{G\rho \times \frac{4}{3}\pi r^3}{r^2} = \frac{4}{3}\pi G\rho r$$

For Earth: $g_E = \frac{4}{3}\pi G\rho_E r_E$

For Planet X: $g_X = \frac{4}{3}\pi G\rho_X \times 3r_E$

Dividing:

$$\frac{g_X}{g_E} = \frac{\rho_X \times 3r_E}{\rho_E \times r_E} = \frac{3\rho_X}{\rho_E}$$

$$\frac{\rho_X}{\rho_E} = \frac{g_X}{3g_E} = \frac{27\text{m/s}^2}{3 \times 9.8\text{m/s}^2} = 0.918$$

Hence $\rho_X = 0.918\rho_E$

The density of Planet X is approximately 0.92 times the Earth's density.

Example 36

- (a) Explain the difference between gravitational potential and gravitational potential energy.
- (b) A 500g object is at a point in the Earth's gravitational field where the gravitational potential is $-4.0 \times 10^7 \text{J/kg}$. Take $M_E = 6.0 \times 10^{24} \text{kg}$, $r_E = 6.4 \times 10^6 \text{m}$, and $G = 6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2}$.
- (i) Calculate the gravitational potential energy of the object at this point.
- (ii) Calculate the distance of this point from the centre of the Earth and the gravitational field strength at this point.

Solution

- (a) Gravitational potential (V) is a property of the gravitational field at a particular point. It is defined as the work done per unit mass in bringing a test mass from infinity to that point. Its unit is J/kg , and it does not depend on the mass of any object placed there.

Gravitational potential energy (U) is a property of the system of two masses. It is the actual energy stored in the gravitational interaction when a specific mass m is placed at that point: $U = mV$. Its unit is J . Unlike potential, it depends on the mass of the object.

In short: potential describes the field; potential energy describes what happens when a specific mass enters that field.

- (b) The solution of each part is as follows:

- (i) Converting mass: $m = 500\text{g} = 0.5\text{kg}$

$$U = mV = 0.5\text{kg} \times (-4.0 \times 10^7 \text{J/kg}) = -2.0 \times 10^7 \text{J} = -20\text{MJ}$$

- (ii) From $V = -\frac{GM_E}{r}$, making r the subject:

$$r = -\frac{GM_E}{V} = -\frac{6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24} \text{kg}}{-4.0 \times 10^7 \text{J/kg}}$$

$$r = \frac{4.002 \times 10^{14} \text{Nm/kg}}{4.0 \times 10^7 \text{J/kg}} = 1.0 \times 10^7 \text{m} = 10000 \text{km}$$

The height above the surface:

$$h = r - r_E = 10000 \text{km} - 6400 \text{km} = \mathbf{3600 \text{km}}$$

The gravitational field strength at this point:

$$g = \frac{GM_E}{r^2} = \frac{4.002 \times 10^{14} \text{Nm/kg}}{(1 \times 10^7 \text{m})^2} = \frac{4.002 \times 10^{14}}{1 \times 10^{14}} = \mathbf{4.0 \text{N/kg}}$$

Alternatively, using $g = -\frac{V}{r} = \frac{4.0 \times 10^7 \text{J/kg}}{1.0 \times 10^7 \text{m}} = \mathbf{4.0 \text{N/kg}}$. Both methods agree.

Example 37

- Explain why a geostationary satellite must orbit in the equatorial plane and cannot be placed in a polar orbit with a 24-hour period.
- A satellite is observed to have an orbital period of 8 hours. Calculate its orbital radius and height above the Earth's surface. Take $M_E = 6.0 \times 10^{24} \text{kg}$, $r_E = 6.4 \times 10^6 \text{m}$, and $G = 6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$.

Solution

- A geostationary satellite must appear stationary above a fixed point on the Earth's surface. For this to happen, the satellite must rotate with the Earth at exactly the same angular velocity and in the same direction. If the satellite were in a polar orbit (passing over the poles), it would orbit in a plane that does not rotate with the Earth. Even if its period were 24 hours, it would appear to trace a figure-of-eight path as seen from the ground, moving north and south during the day, and would not remain fixed above one point. Only an orbit in the equatorial plane keeps the satellite directly above the same point on the equator at all times.
- Converting the period: $T = 8 \text{h} = 8 \times 3600 \text{s} = 28800 \text{s}$

Using Kepler's third law:

$$R^3 = \frac{GM_E T^2}{4\pi^2} = \frac{6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24} \text{kg} \times (28800 \text{s})^2}{4 \times (3.14)^2}$$

$$R^3 = \frac{4.002 \times 10^{14} \text{Nm/kg} \times 8.294 \times 10^8 \text{s}^2}{39.48} = \frac{3.319 \times 10^{23}}{39.48} = 8.407 \times 10^{21} \text{m}^3$$

$$R = \sqrt[3]{(8.407 \times 10^{21} \text{m}^3)} = 2.032 \times 10^7 \text{m} = 20320 \text{km}$$

The orbital radius is 20320km.

Height above the surface:

$$h = R - r_E = 20320 \text{km} - 6400 \text{km} = 13920 \text{km}$$

The satellite orbits at a height of 13920km above the surface.

Example 38

- A satellite in a circular orbit fires its engines briefly to increase its speed. Explain what happens to the shape of the orbit, the altitude, and the period immediately after the burn.
- A 300kg satellite is in a circular orbit at 600km altitude. It needs to be moved to a circular orbit at 1000km altitude. Calculate the total energy in each orbit and the energy that must be supplied by the engines. Take $M_E = 6.0 \times 10^{24} \text{kg}$, $r_E = 6.4 \times 10^6 \text{m}$, and $G = 6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$.

Solution

- Immediately after a brief speed increase, the satellite has more kinetic energy than required for its current circular orbit. The orbit is no longer circular; it becomes **elliptical**, with the burn point as the lowest point (perigee) of the new ellipse. The satellite climbs to a **higher altitude** on the opposite side

of the orbit (apogee). The **period increases** because the elliptical orbit has a larger semi-major axis. If a second burn is performed at the apogee to increase speed again, the orbit can be circularised at the higher altitude.

(b) Orbital radius at 600km:

$$R_1 = r_E + 600 \times 10^3 \text{m} = 6.4 \times 10^6 \text{m} + 6.0 \times 10^5 \text{m} = 7.0 \times 10^6 \text{m}$$

Total energy in the lower orbit:

$$E_1 = -\frac{GM_E m}{2R_1} = -\frac{6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2} \times 6.0 \times 10^{24} \text{kg} \times 300 \text{kg}}{2 \times 7.0 \times 10^6 \text{m}}$$

$$E_1 = -\frac{1.201 \times 10^{17} \text{J}}{1.4 \times 10^7 \text{m}} = -8.576 \times 10^9 \text{J} = -8.58 \text{GJ}$$

Orbital radius at 1000km:

$$R_2 = r_E + 1000 \times 10^3 \text{m} = 6.4 \times 10^6 \text{m} + 1.0 \times 10^6 \text{m} = 7.4 \times 10^6 \text{m}$$

Total energy in the higher orbit:

$$E_2 = -\frac{GM_E m}{2R_2} = -\frac{1.201 \times 10^{17} \text{J}}{2 \times 7.4 \times 10^6 \text{m}} = -8.115 \times 10^9 \text{J} = -8.12 \text{GJ}$$

Energy supplied by the engines:

$$\Delta E = E_2 - E_1 = (-8.12 \text{GJ}) - (-8.58 \text{GJ}) = 0.46 \text{GJ} = 4.6 \times 10^8 \text{J} = 460 \text{MJ}$$

The engines must supply approximately 460MJ of energy.

Make sense...! This is a modest amount compared to the total orbital energy, because the satellite is only being raised by 400km.

You have survived all subtopics of gravitation. You have derived orbital velocities, escaped planets, weighed the Sun, and sent a spacecraft to Mars (on paper, at least). The miscellaneous examples gave you a taste of what happens when the ideas combine. Now the Digging Deeper Exercise takes over in the next page, and it has no worked solutions to hold your hand. Just you, the physics, and the knowledge that G is a very small number. Good luck. You are ready.

DIGGING DEEPER EXERCISE 8

EXERCISE 8A: BINDER QUESTIONS

Question 1

A heavy iron ball and a light wooden ball are released from the same height above the ground in a vacuum. Explain why both balls reach the ground at the same time.

Question 2

Newton's law of gravitation states that every mass attracts every other mass. Explain why you do not feel a gravitational pull toward the person sitting next to you in class, even though the force exists.

Question 3

The Earth exerts a gravitational force on the Moon, and the Moon exerts an equal force on the Earth (Newton's third law). Explain why the Moon orbits the Earth rather than both bodies moving equally.

Question 4

Explain why the gravitational potential at every point near a planet is negative, and explain what it means physically for the potential to become "more negative" as you move closer to the planet.

Question 5

Explain why the acceleration due to gravity decreases with altitude above the Earth's surface but also decreases with depth below the surface, even though the reasons are different in each case.

Question 6

Explain why all satellites at the same altitude orbit the Earth with the same speed and the same period, regardless of their mass.

Question 7

The total mechanical energy of a satellite in circular orbit is negative. Explain what this means physically, and explain what would happen if the total energy became zero, and if it became positive.

Question 8

Explain why the escape velocity from a planet does not depend on the direction in which the object is launched.

Question 9

Explain why the Moon has no atmosphere while the Earth retains a thick one, using the concept of escape velocity.

Question 10

A satellite in circular orbit gradually loses energy due to atmospheric drag. Explain why the satellite speeds up rather than slows down as it loses energy.

EXERCISE 8B: REAL QUESTIONS

Question 11

A precision digital scale in a gold shop in Dar es Salaam reads 100.000g for a gold sample. The same sample is taken to a shop in Iringa (altitude about 1600m) and placed on an identical scale. Explain whether the reading in Iringa will be the same, higher, or lower, and whether the actual mass of the gold has changed.

Question 12

Astronauts aboard the International Space Station (orbiting at about 400km altitude) are often shown floating inside the spacecraft. Many people believe this is because there is no gravity in space. Explain why this belief is incorrect and describe the real reason for the apparent weightlessness.

Question 13

Tanzania's coastline experiences two high tides and two low tides each day. Explain why there are two high tides per day rather than just one.

Question 14

Kipanga says: *"If I dug a tunnel straight through the centre of the Earth and jumped in, I would fall faster and faster until I reached the centre, and then I would be crushed by the enormous gravity there."*

Kipute replies: *"You have at least two things wrong, Kipanga."*

Identify and explain the two errors in Kipanga's reasoning.

Question 15

Explain why rockets are launched from sites as close to the equator as possible, and why they are usually launched in the eastward direction.

Question 16

A geostationary satellite and a low-orbit satellite both orbit the Earth. The geostationary satellite is much higher and moves much more slowly. Explain why the geostationary satellite has more total energy (less negative) than the low-orbit satellite, even though it moves more slowly.

Question 17

Explain why the value of g at the Earth's surface varies from about 9.78m/s^2 at the equator to about 9.83m/s^2 at the poles, giving both reasons for this variation.

Question 18

Kipanga looks at the night sky and asks Mr. Akilikubwa: *"Sir, if the Sun suddenly disappeared, would the Earth immediately fly off in a straight line, or would it keep orbiting for a while?"*

Mr. Akilikubwa replies: *"That is a much deeper question than you realise, Kipanga."*

Explain what would happen to the Earth's motion if the Sun's gravitational field suddenly vanished, and explain why the Earth would move in a straight line rather than a curve.

Question 19

The Voyager 1 spacecraft, launched in 1977, used gravity assists from Jupiter and Saturn to gain enough speed to leave the solar system. Explain how a spacecraft can gain speed by flying past a planet, even though no fuel is burned during the encounter.

Question 20

Kepler's third law states that $T^2 \propto r^3$ for planets orbiting the Sun. Explain why this relationship only holds for objects orbiting the same central body, and explain what would change if the central body had a different mass like Earth.

EXERCISE 8C: HOT QUESTIONS

Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$, $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6400\text{km}$, and $g = 9.8\text{m/s}^2$ unless otherwise stated.

Question 21

The Moon orbits the Earth at a mean distance of $3.84 \times 10^8\text{m}$ with a period of 27.3 days.

- Use this data to calculate the mass of the Earth.
- Calculate the orbital speed of the Moon.

- (c) Using your value from (a) and the known radius of the Earth, calculate g at the Earth's surface and compare with the accepted value of 9.8m/s^2 .

Question 22

- (a) Calculate the gravitational potential at the surface of the Earth.
- (b) A 2.0kg object is lifted from the surface to a height where the potential is $-4.0 \times 10^7\text{J/kg}$. Calculate the work done against gravity.
- (c) Calculate the gravitational field strength at the height found in (b).

Question 23

A student calculates the height at which g is halved using the approximate formula:

$$\frac{1}{2}g = g\left(1 - \frac{2h}{r_E}\right),$$

and gets:

$$h = \frac{r_E}{4} = 1600\text{km}$$

- (a) Use the exact formula to find the correct height.
- (b) Calculate the percentage error introduced by the student's use of the approximation.
- (c) Why the approximation fails here.

Question 24

- (a) Calculate the escape velocity from the surface of a planet whose mass is $3.2 \times 10^{23}\text{kg}$ and whose radius is $2.4 \times 10^6\text{m}$.
- (b) Show that the escape velocity from this planet's surface is $\sqrt{2}$ times the orbital velocity near the surface.
- (c) A probe is launched from the surface of this planet at 5000m/s . Determine whether the probe will escape the planet, orbit it, or fall back.

Question 25

Two masses of $5.0 \times 10^{24}\text{kg}$ and $2.0 \times 10^{24}\text{kg}$ are separated by a distance of $6.0 \times 10^8\text{m}$.

- (a) Find the position of the neutral point (where the net gravitational field is zero) measured from the larger mass.
- (b) Calculate the gravitational potential at the neutral point.

Question 26

A student calculates the energy required to launch a satellite into orbit by computing only the kinetic energy in orbit: $E_{\text{launch}} = \frac{1}{2}mv^2$. The satellite has mass 500kg and orbits at 400km altitude.

- (a) Calculate the student's answer in GJ.
- (b) Calculate the correct minimum launch energy.
- (c) Explain why the student's answer is wrong, and determine whether it overestimates or underestimates the true value.

Question 27

The planet Mars has mass $6.4 \times 10^{23}\text{kg}$ and radius $3.4 \times 10^6\text{m}$.

- (a) Calculate the acceleration due to gravity on the surface of Mars.
- (b) Calculate the escape velocity from the surface of Mars.
- (c) A ball is thrown vertically upward on Mars with a speed of 15m/s . Calculate the maximum height reached and the total time of flight.

Question 28

An engineer designs a satellite system for weather monitoring. The satellite must orbit at an altitude that gives it a period of exactly 6 hours.

- (a) Calculate the orbital radius and altitude of this satellite.
- (b) Calculate the orbital speed.
- (c) Calculate the minimum energy required to launch a 400kg satellite into this orbit from the Earth's surface.
- (d) If the satellite costs Tsh 50 billion and the launch fuel costs Tsh 200,000 per megajoule, estimate the fuel cost and comment on whether it is a significant fraction of the total cost.

Question 29

A planet has uniform density $\rho = 5000\text{kg/m}^3$ and radius $r = 5.0 \times 10^6\text{m}$.

- (a) Calculate the mass of the planet.
- (b) Calculate g at the surface.
- (c) Calculate g at a depth of $2.0 \times 10^6\text{m}$ below the surface.
- (d) Calculate the gravitational potential at the surface and at the centre.

Question 30

A satellite is in a circular orbit at 500km altitude.

- (a) Calculate its orbital speed and escape velocity at this altitude.
- (b) By what factor must the satellite's speed be increased to escape the Earth from this orbit?
- (c) By what factor must the satellite's kinetic energy be increased to escape?
- (d) If the satellite has mass 800kg, calculate the energy that must be supplied to escape from this orbit.

Question 31

A space agency plans to place a 1200kg satellite in geostationary orbit.

- (a) Calculate the orbital radius, speed, and period of the geostationary orbit.
- (b) Calculate the total energy of the satellite in the geostationary orbit.
- (c) Calculate the minimum energy required to launch the satellite from the Earth's surface.
- (d) If the rocket's overall efficiency (fuel energy to useful orbital energy) is 5%, and rocket fuel has an energy density of 3.0MJ/kg, calculate the mass of fuel required.

Question 32

The Earth orbits the Sun at $1.5 \times 10^{11}\text{m}$ with a period of 1 year. Jupiter orbits at $7.78 \times 10^{11}\text{m}$.

- (a) Calculate the orbital period of Jupiter in years.
- (b) A spacecraft is sent from Earth to Jupiter along a Hohmann transfer orbit. Calculate the transfer time.

Question 33

At what angular velocity would the Earth need to rotate so that a person at the equator would feel 75% of their normal weight?

- (a) Calculate this angular velocity.
- (b) Express it as a multiple of the Earth's current angular velocity.
- (c) Find the length of a "day" at this rotation speed.

Question 34

A body is projected vertically upward from the Earth's surface with a speed equal to half the escape velocity.

- (a) Show that the maximum height reached is $\frac{r_E}{3}$.
- (b) Calculate this height in kilometres.

EXAMINATION QUESTIONS FOR PART TWO

Question 1

- (a) A heavy crate is dragged across a rough warehouse floor by two ropes pulled by two workers. One pulls northward, the other pulls eastward. The crate moves in a direction between north and east. Account for the direction of the crate's motion.
- (b) A goalkeeper catches a fast-moving football and is pushed backward slightly. A spectator says the ball *"transferred its energy to the goalkeeper."* Discuss whether this description is physically complete, and identify what quantity is actually conserved during the catch.
- (c) A crane supports a 200kg load using two cables. One cable makes 40° with the horizontal and the other makes 60° with the horizontal, both on the same side of the vertical. Take $g = 9.8\text{m/s}^2$. Calculate the tension in each cable.

Question 2

- (a) A motorcycle stunt rider loops inside a vertical circular track. At the topmost point, the rider is upside down yet does not fall. Give a physical account of why the rider remains on the track.
- (b) When a heavy lorry and a small car collide head-on, the occupants of the small car are usually more seriously injured. Both vehicles experience the same magnitude of force during the collision. Account for the greater injury in the small car.
- (c) A 0.06kg tennis ball travelling horizontally at 40m/s is struck by a racket and returned in the opposite direction at 55m/s. The contact time is 0.008s. Calculate the average force exerted by the racket on the ball.

Question 3

- (a) A javelin thrower releases the javelin at an angle to the horizontal. After release, the javelin continues to rise for a while before curving downward. Explain why the javelin rises initially and what causes it to eventually descend.
- (b) A precision scale calibrated in a laboratory at Iringa (altitude 1600m) is transported to a laboratory in Bagamoyo (near sea level). A scientist notices that the same standard mass gives a slightly different reading. Account for this discrepancy.
- (c) A particle is in equilibrium under the action of three forces: $F_1 = 10\text{N}$ acting along the positive x-axis, $F_2 = 15\text{N}$ acting at 120° to F_1 , and a third force F_3 . Calculate the magnitude and direction of F_3 .

Question 4

- (a) A satellite in low Earth orbit completes one revolution approximately every 90 minutes. An observer claims the satellite is "not affected by gravity" because it does not fall. Discuss why this claim is physically incorrect.
- (b) During a road accident investigation, investigators estimate the speed of a vehicle before impact by measuring the length of skid marks on the road. Explain the physical principle behind this method.
- (c) A ball is kicked from the edge of a flat rooftop 15m above the ground with a velocity of 20m/s at 30° above the horizontal. Take $g = 9.8\text{m/s}^2$. Calculate the speed of the ball just before it hits the ground.

Question 5

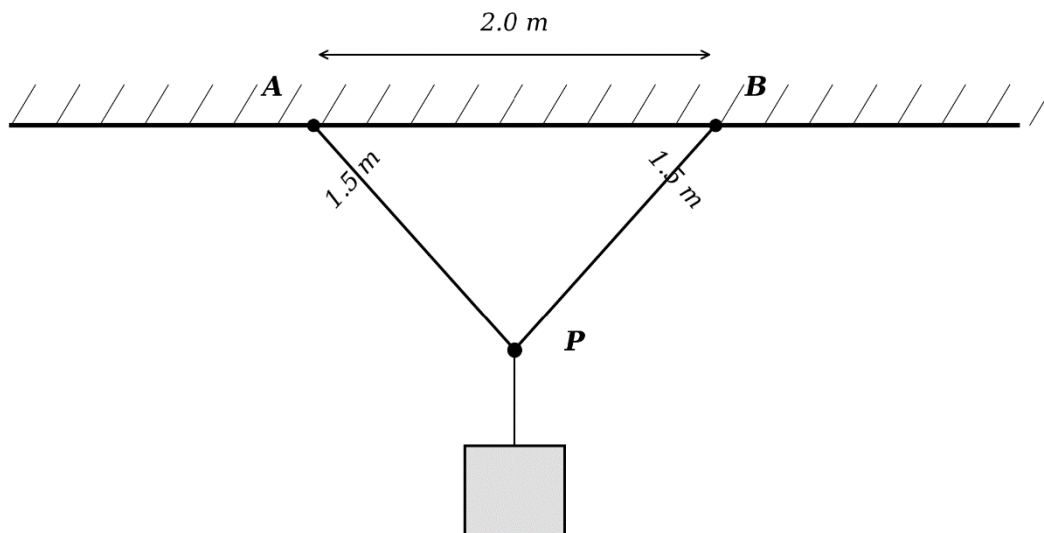
- (a) A coin placed on a spinning turntable stays in place at low speed but flies off when the turntable spins too fast. Explain this observation.
- (b) An athlete throws a discus at an angle to the horizontal. Explain why the range of the discus depends on the angle of projection, and identify the angle that gives maximum range on level ground.
- (c) A 0.50kg object moves at 4.0m/s on a smooth horizontal surface and collides head-on with a 0.30kg object moving at 3.0m/s in the opposite direction. After the collision, the 0.50kg object moves at 1.0m/s in its original direction. Calculate the velocity of the 0.30kg object after the collision and determine whether kinetic energy is conserved.

Question 6

- (a) A stone tied to a string is whirled in a horizontal circle. When the string breaks, the stone flies off in a straight line tangent to the circle, not outward. Account for this observation.
- (b) Explain why the gravitational force between two ordinary objects (such as two books on a table) exists but cannot be felt or measured without extremely sensitive instruments.
- (c) A mortar fires a shell at 100m/s at 53° to the horizontal from level ground. Take $g = 9.8\text{m/s}^2$. Calculate the horizontal distance at which the shell is at a height of 100m for the first time.

Question 7

- (a) A car travels over a hump-backed bridge and the driver feels momentarily lighter at the top. Give a physical explanation for this sensation.
- (b) An ice hockey puck slides a great distance across the ice after being struck, yet a football kicked with similar effort on a grass field stops much sooner. Both objects were given a similar initial push. Account for the difference in the distances covered.
- (c) Two ropes are attached to a ceiling at points 2.0m apart. A 15kg mass hangs from the junction of two ropes, each 1.5m long. Take $g = 9.8\text{m/s}^2$. Calculate the tension in each rope.

**Question 8**

- (a) A planet moves faster at perihelion and slower at aphelion. Explain why this change in speed occurs, using the concept of angular momentum conservation.
- (b) Explain why a satellite does not need to fire its engines to remain in orbit, yet an aeroplane must continuously run its engines to stay aloft.
- (c) A 3.0kg body moving at 8.0m/s to the right collides elastically and head-on with a 1.0kg body at rest. Calculate the velocity of each body after the collision.

Question 9

- (a) Explain why the escape velocity from a planet depends on the planet's mass and radius but not on the mass of the escaping object.
- (b) A child on a playground roundabout (merry-go-round) moves to the outer edge and finds it harder to hold on. Account for this observation.
- (c) A projectile is launched from ground level at 35m/s at 50° above the horizontal. Take $g = 9.8\text{m/s}^2$. Calculate the total time the projectile spends above a height of 20m .

Question 10

- (a) Two ropes attached to a ring pull in different directions. A third rope is attached such that the ring remains stationary. Explain what condition the three forces must satisfy and how the third force is determined.
- (b) Water from a hosepipe follows a curved path after leaving the nozzle, while a laser beam travels in a straight line through the same space. Account for this difference.
- (c) A 800kg car rounds a flat circular bend of radius 50m. The coefficient of static friction between the tyres and the road is 0.45. Take $g = 9.8\text{m/s}^2$. Calculate the maximum safe speed for the car on this bend.

Question 11

- (a) A satellite in a higher orbit moves more slowly than one in a lower orbit, yet more energy was needed to place it there. Explain why this is not contradictory.
- (b) In a cricket match, a batsman swings through the ball rather than just blocking it. Explain how the follow-through affects the ball's speed after being struck.
- (c) A 5.0kg object hangs from two strings. One string makes 30° with the ceiling and the other makes 45° with the ceiling. Take $g = 9.8\text{m/s}^2$. Calculate the tension in each string.

Question 12

- (a) Explain why a bullet fired horizontally from a gun and a bullet dropped from the same height at the same instant hit the ground at the same time (neglecting air resistance).
- (b) A car entering a roundabout at moderate speed does not skid, but the same car going too fast may skid outward. Explain the role of friction in each case.
- (c) The Moon orbits the Earth at a mean distance of $3.84 \times 10^8\text{m}$. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$ and $M_E = 6.0 \times 10^{24}\text{kg}$. Calculate the speed at which the Moon must travel to maintain its orbit.

Question 13

- (a) A truck moving at constant velocity on a highway has zero acceleration. A student concludes that no forces act on it. Explain why this conclusion is incorrect.
- (b) A long jumper takes off at an angle to the horizontal. Explain why the jumper's centre of mass follows a parabolic path regardless of the body movements performed during flight.
- (c) A 0.025kg bullet is fired horizontally into a 4.975kg wooden pendulum bob hanging at rest. The bob swings upward to a height of 0.10m. Take $g = 9.8\text{m/s}^2$. Calculate the initial speed of the bullet.

Question 14

- (a) A ball thrown vertically upward with speed less than the escape velocity always returns to the ground. Explain why, using energy considerations.
- (b) During a tug-of-war, neither team moves for several seconds despite both teams pulling very hard. Account for the force situation during this period.
- (c) A 0.80kg ball is attached to a 1.2m string and swung in a vertical circle. Calculate the minimum speed at the top of the circle for the ball to complete the circular motion. Take $g = 9.8\text{m/s}^2$.

Question 15

- (a) Explain why two satellites at the same altitude but with different masses orbit the Earth at exactly the same speed.
- (b) Explain why there is a maximum height from which a person can jump safely onto the ground, and why landing on a soft surface increases this safe height.
- (c) A satellite orbits a planet at a height where the gravitational field strength is 3.2N/kg . The planet has a radius of $5.0 \times 10^6\text{m}$ and a mass of $4.0 \times 10^{24}\text{kg}$. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. Calculate the altitude of the satellite above the planet's surface.

Question 16

- (a) Two balls of different masses are dropped simultaneously from the same height. In the absence of air resistance, they hit the ground at the same time. Explain why mass does not affect the time of fall.
- (b) A roundabout (merry-go-round) at a playground spins at constant angular velocity. A child sitting near the centre feels a gentle outward pull, while another child at the edge feels a strong outward pull. Account for this difference.
- (c) A shell explodes at the highest point of its parabolic trajectory into two equal fragments. One fragment falls vertically from rest. Determine the direction and speed of the other fragment immediately after the explosion, given that the shell had a horizontal velocity of 200m/s at the point of explosion.

Question 17

- (a) Explain why a banked road allows a car to round a curve at high speed even when the road surface is icy and friction is negligible.
- (b) A large asteroid approaching the Earth has a negative gravitational potential energy with respect to the Earth. Explain what this negative value physically means.
- (c) A stone is projected at 25m/s at 37° above the horizontal from the top of a 45m cliff. Take $g = 9.8\text{m/s}^2$. Calculate the time taken for the stone to reach the ground below.

Question 18

- (a) A person travelling in a car notices that when the car suddenly brakes, their body lurches forward but the seatbelt holds them back. Using the concept of impulse, explain how the seatbelt protects the person.
- (b) A tennis player follows through with the racket after hitting the ball. Explain how this follow-through increases the speed of the ball.
- (c) A 2.0kg mass moves in a horizontal circle on the inner surface of a smooth conical bowl. The half-angle of the cone is 50° (measured from the vertical axis). Take $g = 9.8\text{m/s}^2$. Calculate the speed of the mass if the radius of the circular path is 0.30m.

Question 19

- (a) A ball that just barely completes a vertical circle on a string is released at the top. Explain why it appears to “float” momentarily before falling.
- (b) Explain why a very high coefficient of friction between the tyres and the road can be both an advantage and a disadvantage for a car negotiating a level curved road.
- (c) A truck has its wheels 1.8m apart laterally and its centre of gravity 1.0m above the ground. It rounds a level curve of radius 50m. Special high-grip tyres give a coefficient of static friction of 0.95. Take $g = 9.8\text{m/s}^2$.
 - (i) Calculate the maximum speed before skidding.
 - (ii) Calculate the maximum speed before overturning.
 - (iii) Determine which happens first, and explain why the high-grip tyres make this vehicle more dangerous on this curve, not less.

Question 20

- (a) Explain why a geostationary satellite appears stationary to an observer on Earth, even though it is moving at over 3km/s.
- (b) A fire hose directs a powerful jet of water at a burning building. Explain why the firefighter holding the hose is pushed backward.
- (c) A ball of mass 0.4kg is attached to a string and swung in a vertical circle of radius 0.8m. The speed at the lowest point is 6.0m/s. Take $g = 9.8\text{m/s}^2$. Calculate the tension in the string at the lowest point.

Question 21

- (a) Explain why the acceleration due to gravity on the surface of a planet depends on both the mass and the radius of the planet.

- (b) When a gun is fired, the bullet moves forward at high speed while the gun recoils backward at a much lower speed. Explain why the speeds are different even though the forces on the bullet and the gun are equal.
- (c) A stone is projected horizontally at 12m/s from a bridge and hits the water 2.4s later. Take $g = 9.8\text{m/s}^2$. Calculate the speed at which the stone enters the water.

Question 22

- (a) Explain why all geostationary satellites must orbit at the same altitude above the Earth.
- (b) A student drops a heavy ball and a light ball simultaneously from the same height. In the absence of air resistance, both hit the ground at the same time. Explain why.
- (c) A 0.05kg bullet travelling at 300m/s strikes a 2.95kg stationary block on a smooth surface and embeds itself. The block then slides up a rough incline of 30° with coefficient of kinetic friction 0.25. Calculate how far up the incline the block travels before stopping.

Question 23

- (a) Explain why the gravitational potential energy of a satellite in orbit is negative while its kinetic energy is positive.
- (b) A sprinter pushes backward on the starting blocks at the beginning of a race. Explain how this causes the sprinter to accelerate forward.
- (c) Two ropes support a 40kg traffic light at a junction. One rope makes 35° and the other makes 55° with the horizontal. Take $g = 9.8\text{m/s}^2$. Calculate the tension in each rope.

Question 24

- (a) Explain why a body moving in a circular path at constant speed requires a net force, even though its speed is not changing.
- (b) A lorry loaded with goods is more difficult to stop than the same lorry when empty, even if both are travelling at the same speed. Explain why, using the concept of momentum.
- (c) A mortar fires a shell at 80m/s at 50° to the horizontal from the edge of a cliff 60m above flat ground. Take $g = 9.8\text{m/s}^2$. Calculate the horizontal distance from the base of the cliff to where the shell lands.

Question 25

- (a) Explain why Kepler's second law (equal areas in equal times) is a consequence of the conservation of angular momentum.
- (b) Passengers in a car lean sideways when the car goes around a sharp bend. Explain why this happens.
- (c) A 3.0kg object moving east at 5.0m/s collides with a 2.0kg object moving north at 7.5m/s. The objects stick together. Calculate the speed and direction of the combined object after the collision.

Question 26

- (a) Explain why the gravitational field strength is zero at the centre of the Earth but the gravitational potential is not zero there.
- (b) A person standing on a weighing scale inside a freely falling lift would observe a reading of zero. Explain why.
- (c) A rocket of total mass 5000kg (including fuel) is launched vertically. The exhaust gases are expelled at 800m/s relative to the rocket at a rate of 20kg/s. Take $g = 9.8\text{m/s}^2$. Calculate the initial acceleration of the rocket.

Question 27

- (a) Explain why a projectile launched at 45° to the horizontal achieves the maximum range on level ground.
- (b) Explain why the period of a satellite in orbit does not depend on the mass of the satellite.

- (c) A block of mass 4.0kg on a smooth horizontal surface is connected by a light string over a frictionless pulley to a hanging mass of 1.0kg. Take $g = 9.8\text{m/s}^2$. Calculate the acceleration of the system and the tension in the string.

Question 28

- (a) Explain why the total mechanical energy of a satellite in circular orbit is negative.
- (b) A cricket ball hit into the air reaches the same height whether the batsman hits it gently or hard, as long as the vertical component of the initial velocity is the same. Explain why.
- (c) A force of F at angle θ above the horizontal is just sufficient to move a block of mass m at constant velocity on a rough horizontal surface with coefficient of kinetic friction μ . Show that $F = \frac{\mu mg}{\cos\theta + \mu \sin\theta}$.

Question 29

- (a) Explain the physical difference between mass and weight, and why an astronaut in orbit has mass but appears to have no weight.
- (b) Explain why a car with a higher centre of mass is more likely to overturn on a curved road than a car with a lower centre of mass.
- (c) A planet has a surface gravity of 4.5m/s^2 and a radius of $4.8 \times 10^6\text{m}$. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. Calculate the minimum speed at which a spacecraft must be launched from this planet to escape its gravitational field.

Question 30

- (a) Explain why a ball kicked off a cliff follows a parabolic path, while the same ball rolling along a flat field follows a straight path.
- (b) In the sport of ice curling, a heavy stone slides across the ice with very little friction. If two identical stones collide elastically and head-on, the first stone stops and the second moves off at the first stone's original speed. Explain why this outcome conserves both momentum and kinetic energy.
- (c) The Moon orbits the Earth at a mean distance of $3.84 \times 10^8\text{m}$. Take $M_E = 6.0 \times 10^{24}\text{kg}$ and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. Calculate the orbital period of the Moon in days.

Question 31

- (a) Explain why it is impossible for a body to escape from a planet if its total mechanical energy is negative.
- (b) During a rainstorm, a person walking forward notices that the rain appears to come at an angle from the front, even though the rain is actually falling vertically. Explain this observation.
- (c) A car of mass 1200kg rounds a banked curve of radius 80m. The banking angle is 20° . Take $g = 9.8\text{m/s}^2$. Calculate the design speed of the curve.

Question 32

- (a) Explain why the value of g at the equator is slightly less than at the poles.
- (b) Explain why a parachutist reaches terminal velocity and stops accelerating, even though gravity continues to act.
- (c) A 0.2kg ball travelling at 8m/s to the right bounces off a wall and returns at 6m/s to the left. The contact time with the wall is 0.05s. Calculate the average force exerted by the wall on the ball.

Question 33

- (a) Explain why the relationship $v_{\text{escape}} = \sqrt{2} \times v_{\text{orbital}}$ holds at any distance from a planet, not just at the surface.
- (b) A bus driver brakes gradually to stop rather than braking suddenly. Explain the advantage of gradual braking for the passengers, using the impulse-momentum theorem.

- (c) A 60kg skier starts from rest at the top of a 25° slope and skis down for 200m. The coefficient of kinetic friction between the skis and the snow is 0.10. Take $g = 9.8\text{m/s}^2$. Calculate the speed of the skier at the bottom.

Question 34

- (a) Explain why a stone thrown at an angle to the horizontal has its minimum speed at the highest point of its trajectory.
- (b) Explain why a moving satellite does not need any fuel to maintain its orbit (in the absence of atmospheric drag).
- (c) Two bodies of masses $m_1 = 3.0\text{kg}$ and $m_2 = 5.0\text{kg}$ are connected by a compressed spring and held together on a smooth horizontal surface. When released, m_1 moves to the left at 4.0m/s . Calculate the energy stored in the spring before release.

Question 35

- (a) Explain why the approximate formula $g' \approx g\left(1 - \frac{2h}{r_E}\right)$ gives a percentage error that increases rapidly with height.
- (b) Explain why a head-on collision between two cars moving at 60km/h each is not the same as one car hitting a rigid wall at 120km/h .
- (c) A stone is thrown from ground level at 30m/s at 37° above the horizontal. At what time after launch is the speed of the stone equal to 25m/s ? Take $g = 9.8\text{m/s}^2$.

Question 36

- (a) Explain why the escape velocity from the surface of the Moon is much lower than from the surface of the Earth.
- (b) A cricket fielder catches a fast-moving ball by drawing their hands backward. Explain how this technique reduces the force on the fielder's hands.
- (c) A 1500kg car enters a vertical loop of radius 12m at a speed of 20m/s at the bottom. Take $g = 9.8\text{m/s}^2$. Using energy conservation, calculate the speed at the top and hence determine whether the car completes the loop.

Question 37

- (a) A car travels at constant speed over a hump-backed bridge. Explain why the driver feels lighter at the top of the bridge than on flat road.
- (b) A bullet embeds itself in a wooden block, and a large amount of kinetic energy is lost. Explain where this "lost" kinetic energy goes.
- (c) A stone is projected at 40m/s at an angle of 35° above the horizontal from the top of a cliff 70m above flat ground. Take $g = 9.8\text{m/s}^2$. Calculate the speed of the stone just before it strikes the ground.

Question 38

- (a) A spacecraft uses a "gravity assist" to gain speed by passing close to a planet, without using any fuel. Explain how the spacecraft gains kinetic energy in this manoeuvre.
- (b) Two equal forces act on an object, yet the object does not accelerate. Explain how this is possible.
- (c) A 0.40kg ball attached to a string of length 0.60m is whirled in a vertical circle. At the top of the circle, the tension in the string is 5.0N . Take $g = 9.8\text{m/s}^2$. Calculate the speed of the ball at the top.

Question 39

- (a) Explain why a satellite in geostationary orbit must orbit at a specific altitude and cannot be placed at any other height.
- (b) A goalkeeper dives to catch a fast ball and rolls on the ground after catching it. Explain how rolling helps protect the goalkeeper from injury.

- (c) Three concurrent forces act on a bolt at the origin: $F_1 = 25\text{N}$ along the positive x-axis, $F_2 = 18\text{N}$ at 130° from the positive x-axis, and $F_3 = 12\text{N}$ at 240° from the positive x-axis. Calculate the magnitude and direction of the equilibrant.

Question 40

- (a) An object is projected horizontally from the top of a tall building. Explain why the horizontal distance covered increases with the height of the building, even though the initial horizontal speed is unchanged.
- (b) Explain why a person standing on a bathroom scale inside a freely falling lift would observe a reading of zero.
- (c) A 2.0kg object moving at 6.0m/s to the east collides with a 3.0kg object moving at 4.0m/s to the north. The objects stick together after the collision. Calculate the velocity and direction of the combined object.

Question 41

- (a) Explain why the acceleration due to gravity on the surface of the Moon (1.6m/s^2) is much less than on the surface of the Earth (9.8m/s^2), even though the Moon's radius is only about $\frac{1}{3.7}$ of the Earth's.
- (b) A football is kicked into the air. At the highest point, the velocity of the ball is not zero. Explain why.
- (c) A 1500kg car travels at 25m/s around a flat circular bend of radius 60m . Take $g = 9.8\text{m/s}^2$. Calculate the minimum coefficient of static friction between the tyres and the road required to prevent the car from skidding.

Question 42

- (a) Explain why the total mechanical energy of a satellite in circular orbit is negative, and what this negative value physically signifies.
- (b) A car moving at constant speed on a straight road uses more fuel than the same car parked with its engine idling. Explain why, even though neither car is accelerating.
- (c) A ball is thrown horizontally at 15m/s from the top of a tower. It hits the ground at an angle of 60° below the horizontal. Take $g = 9.8\text{m/s}^2$. Calculate the height of the tower.

Question 43

- (a) Why does launching a golf ball upward increase its horizontal distance?
- (b) A coin placed on a smooth rotating turntable slides off when the turntable spins too fast. Explain why the coin slides off and why it moves outward rather than in any other direction.
- (c) A satellite orbits the Earth at a height where the gravitational field strength is 4.9N/kg . Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. Calculate the altitude of the satellite above the Earth's surface.

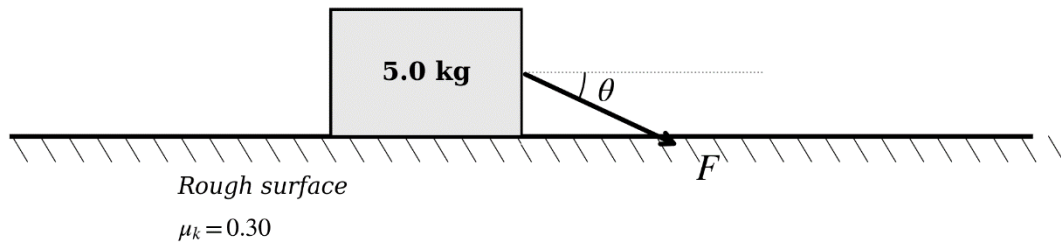
Question 44

- (a) Why is a rope thrown outward from a cliff rather than dropped straight down when trying to reach someone below?
- (b) A large ship and a small rowing boat collide at low speed. The rowing boat is severely damaged while the ship is barely affected. Explain this difference.
- (c) A body of mass 0.5kg is attached to a string of length 1.0m and swung in a vertical circle. Calculate the minimum speed at the bottom of the circle for the body to just complete the full circle. Take $g = 9.8\text{m/s}^2$.

Question 45

- (a) A planet at perihelion moves faster than at aphelion. Explain why the planet's speed changes during its orbit, even though no tangential force appears to act on it.
- (b) A truck carrying sand has a small hole in its floor. As the truck moves, sand leaks out at a steady rate. If the engine continues to provide the same driving force, explain what happens to the truck's acceleration over time.

- (c) A force F acts at angle θ below the horizontal on a 5.0kg block resting on a rough horizontal surface. The coefficient of kinetic friction is $\mu_k = 0.30$. If $F = 40\text{N}$ and $\theta = 25^\circ$, calculate the acceleration of the block. Take $g = 9.8\text{m/s}^2$.

**Question 46**

- (a) Explain why two satellites at the same altitude orbit the Earth at exactly the same speed, regardless of their different masses.
- (b) A volleyball player “cushions” the ball by moving their arms backward as the ball arrives. Explain how this technique reduces the force on the arms.
- (c) A ball is projected from ground level at 28m/s at 40° above the horizontal. Take $g = 9.8\text{m/s}^2$. Calculate the velocity (magnitude and direction) of the ball 2.0s after launch.

Question 47

- (a) A person weighs slightly less at the equator than at the North Pole. Explain the two physical reasons for this difference.
- (b) A tennis ball is served at very high speed. The racket is in contact with the ball for only a few milliseconds. Explain why a short contact time results in a large force on the ball.
- (c) A 0.020kg bullet travelling at 350m/s embeds itself in a 2.48kg stationary wooden block on a smooth surface. The block then slides onto a rough patch of floor with coefficient of kinetic friction 0.40 . Take $g = 9.8\text{m/s}^2$. Calculate the distance the block slides on the rough patch before stopping.

Question 48

- (a) Explain why the gravitational field strength decreases as one moves deeper below the Earth’s surface, and why it is zero at the centre.
- (b) A car enters a roundabout too fast and the rear of the car swings outward. Explain what causes this to happen.
- (c) A particle is held in equilibrium by three coplanar forces. One force of 50N acts vertically downward and another of 30N acts horizontally to the right. Calculate the magnitude and direction of the third force.

Question 49

- (a) Explain why a satellite that gradually loses energy due to atmospheric drag moves to a lower orbit and speeds up, rather than slowing down and drifting outward.
- (b) A person catches a cricket ball bare-handed without pulling the hands back. Explain why this is painful.
- (c) A 1200kg car travels at 20m/s on a banked curve of radius 100m . The banking angle is 15° . Take $g = 9.8\text{m/s}^2$. Calculate the friction force on the car and state whether it acts up or down the slope.

Question 50

- (a) Explain why it is dangerous for a car to travel at high speed on a wet curved road, even if the same speed is safe on a dry road.
- (b) A comet moves in a highly elliptical orbit around the Sun. It moves very fast near the Sun and very slowly when far away. Explain why the comet’s speed varies so dramatically.

- (c) A 2.0kg mass hangs from a spring balance inside a lift. The balance reads 24.0N when the lift is accelerating. Take $g = 9.8\text{m/s}^2$. Determine whether the lift is accelerating upward or downward, and calculate the magnitude of the acceleration.

Question 51

- (a) Explain why a pilot at the bottom of a vertical loop feels much heavier than normal, while at the top of the loop the pilot may feel almost weightless.
- (b) A loaded minibus takes much longer to stop than an empty one travelling at the same speed, even though both have similar brakes. Explain why.
- (c) A planet has a radius twice that of Earth and a density half that of Earth. Calculate the gravitational field strength on the surface of this planet, in terms of g .

Question 52

- (a) Two forces of 10N act on an object. Explain why the resultant can have any value from 0 to 20N depending on the angle between the forces.
- (b) An astronaut on the Moon can throw a ball much farther than on Earth with the same effort. However, the astronaut cannot run much faster on the Moon than on Earth. Explain the difference.
- (c) A 0.10kg ball moving at 10m/s strikes a stationary 0.10kg ball in an elastic head-on collision. Calculate the velocities of both balls after the collision.

Question 53

- (a) Explain why a satellite in a lower orbit completes more orbits per day than a satellite in a higher orbit.
- (b) A helicopter hovers at a constant height above the ground. Explain why the helicopter's engines must work continuously even though it is not accelerating.
- (c) A 3.0kg block on a smooth 30° incline is connected by a light string over a smooth pulley at the top of the incline to a 2.0kg block hanging vertically. Take $g = 9.8\text{m/s}^2$. Calculate the acceleration of the system.

Question 54

- (a) Explain why a person pushing a heavy box on a rough floor finds it easier to push horizontally than to push downward at an angle.
- (b) During a collision between two vehicles, crumple zones at the front of the car deform. Explain how this deformation protects the occupants.
- (c) A 1400kg car travels at 18m/s over the crest of a hill with radius of curvature 40m. Take $g = 9.8\text{m/s}^2$. Calculate the normal reaction on the car at the crest, and state whether the car maintains contact with the road.

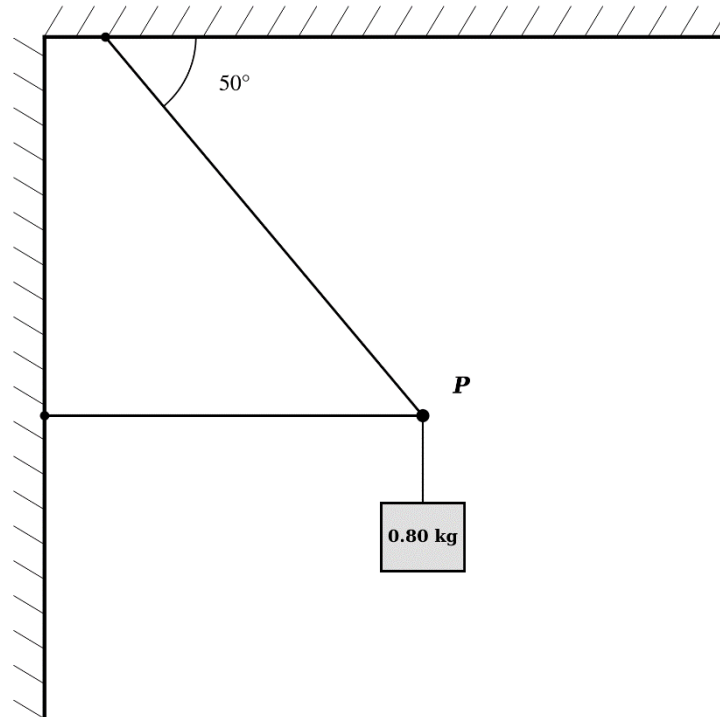
Question 55

- (a) Explain why the gravitational potential energy of a satellite becomes less negative (increases) as the satellite moves to a higher orbit.
- (b) A student observes that a freely falling ball near the top of a tall building and a satellite orbiting the Earth are both "falling under gravity." Explain the key difference between these two situations.
- (c) A 5.0kg mass and a 3.0kg mass are connected by a light string passing over a smooth pulley. The system is released from rest. Take $g = 9.8\text{m/s}^2$. Calculate the distance each mass travels in the first 2.0s.

Question 56

- (a) A rifle bullet travels in an almost flat trajectory over short distances, while a thrown ball follows a clearly curved arc. Both are projectiles. Explain why their paths appear so different.
- (b) A road is banked at an angle designed for a particular speed. Explain what happens to a car travelling at this exact design speed on the banked curve.

- (c) A body of mass 0.80kg is suspended from two strings. One string is horizontal and attached to a wall. The other is attached to the ceiling and makes 50° with the horizontal. Take $g = 9.8\text{m/s}^2$. Calculate the tension in each string.

**Question 57**

- Two identical trucks, one loaded and one empty, travel at the same speed toward a junction. Explain why the loaded truck needs to start braking earlier than the empty truck to stop at the same point.
- Explain why the gravitational field of the Earth is approximately uniform near the surface but clearly non-uniform at the scale of satellite orbits.
- A ball is projected from ground level and lands 60m away on the same level. The maximum height reached is 20m . Take $g = 9.8\text{m/s}^2$. Calculate the initial speed and angle of projection.

Question 58

- Explain why a pendulum swings more slowly on the Moon than on Earth, even though the length of the string is the same.
- A spacecraft in orbit around the Earth fires its engines to speed up briefly. Explain what happens to the orbit immediately after the burn.
- A 0.030kg bullet travelling at 400m/s passes through a 1.5kg stationary wooden plank and emerges at 100m/s . Calculate the speed of the plank after the bullet passes through it and the kinetic energy lost.

Question 59

- Explain why a long jumper achieves a greater distance by running fast before jumping rather than jumping from a standing position.
- The value of g inside the Earth decreases linearly with depth (assuming uniform density), while above the surface it decreases as $\frac{1}{r^2}$. Explain why the two patterns are different.
- A satellite has a period of 12 hours orbiting a planet of mass $5.0 \times 10^{24}\text{kg}$. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. Calculate the orbital radius.

Question 60

- (a) Explain why a planet's escape velocity can be calculated from its mass and radius alone, without knowing anything about the object that is escaping.
- (b) A car travelling on a flat road enters a dip (valley) at speed. Explain why the driver feels heavier at the bottom of the dip.
- (c) A 4.0kg body slides from rest down a smooth curved ramp of height 3.0m onto a rough horizontal floor with $\mu_k = 0.50$. Take $g = 9.8\text{m/s}^2$. Calculate how far the body slides on the floor before stopping.

Question 61

- (a) A bucket of water is swung in a vertical circle. At the top, the water does not spill. Explain why, and state what happens if the speed at the top is too low.
- (b) A hammer thrower spins several times before releasing the hammer. Explain why more spins lead to a longer throw.
- (c) A ball is dropped from a height of 5m onto a hard floor. It bounces repeatedly. The coefficient of restitution between the ball and floor is 0.80. Calculate the height reached after the third bounce and the total distance travelled by the ball before the fourth bounce.

Question 62

- (a) Explain why a heavy lorry moving at low speed can have the same momentum as a light car moving at high speed, and discuss which one is more dangerous in a collision.
- (b) A person weighs 700N at the surface of the Earth. Explain qualitatively how this weight would change if the Earth suddenly shrank to half its radius without losing any mass.
- (c) A ball is thrown at 16m/s at 30° above the horizontal from the top of a building 25m tall. At the same instant, a person at ground level 40m from the base of the building starts running at constant speed to catch the ball. Take $g = 9.8\text{m/s}^2$. Calculate the speed at which the person must run.

Question 63

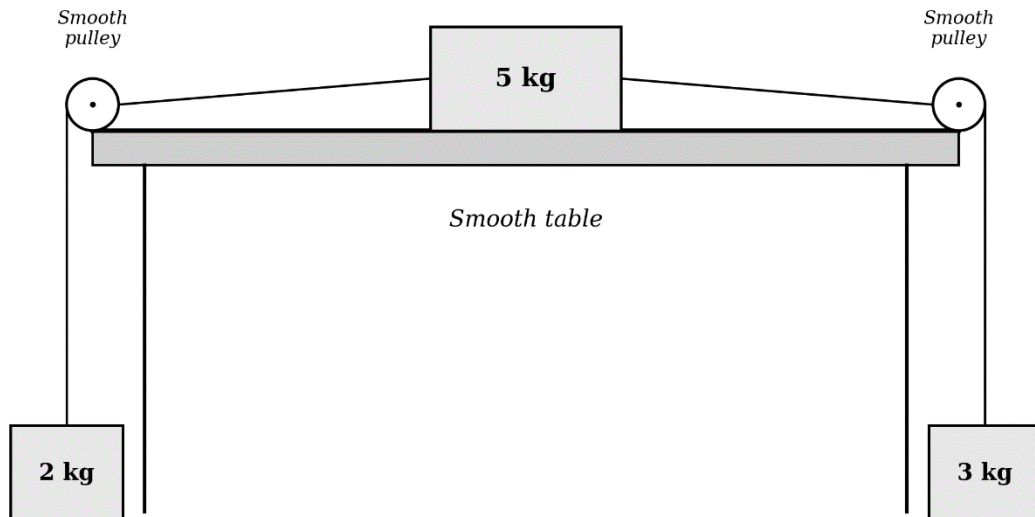
- (a) Explain why it is impossible for a single force acting alone to keep a body in equilibrium.
- (b) Explain why the path of a projectile in reality deviates from a perfect parabola.
- (c) A 3kg block travelling at 4m/s on a smooth surface collides with a 2kg block at rest. After the collision, the 3kg block moves at 1.6m/s in the same direction. Calculate the coefficient of restitution.

Question 64

- (a) A package is dropped from a moving helicopter. Explain why the package does not land directly below the helicopter (assume the helicopter continues at constant velocity).
- (b) A train moving at constant speed on a curved track tilts slightly inward. Explain why the track is designed this way.
- (c) A projectile is launched from level ground and must pass over a 15m wall located 30m away. The launch speed is 25m/s. Take $g = 9.8\text{m/s}^2$. Calculate the minimum angle of projection required to clear the wall.

Question 65

- (a) Explain why a satellite in a polar orbit can eventually photograph the entire surface of the Earth, while a satellite in an equatorial orbit cannot.
- (b) A space station rotates to simulate gravity for the astronauts inside. Explain how this artificial gravity works and why the astronauts must stand on the outer wall rather than the inner wall.
- (c) A 5kg mass on a smooth table is connected by a string passing over a smooth pulley at the edge of the table to a 3kg mass hanging freely. The 5kg mass is also connected by another string over another smooth pulley on the opposite edge to a 2kg mass hanging freely. All strings are light and the pulleys are smooth. Take $g = 9.8\text{m/s}^2$. Calculate the acceleration of the system and the tension in each string.

**Question 66**

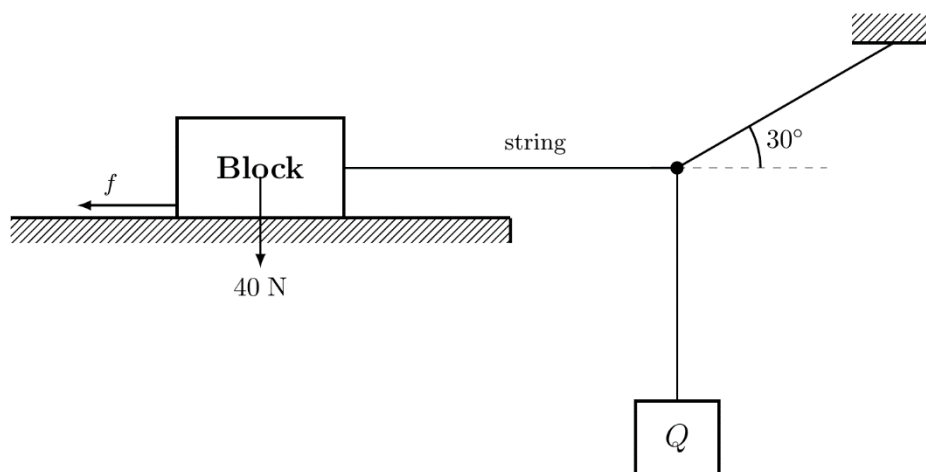
- As an aircraft prepares to land, the pilot tilts (banks) the wings. Explain how banking helps the aircraft change direction and align with the runway.
- A cricket fielder always draws his hands backward while catching a fast-moving ball. Explain why this technique reduces the chance of injury to the hands.
- A bullet of mass 12g travelling horizontally at 150ms^{-1} embeds itself in a block of wood of mass 1.188kg suspended freely by two strings. Calculate the velocity of the block immediately after impact and the kinetic energy lost in the collision.

Question 67

- For a projectile launched at a given angle above the horizontal, explain what happens to the horizontal range if the initial velocity is doubled.
- A satellite orbiting the Earth in an elliptical orbit moves faster when closer to the Earth and slower when farther away. Explain why the kinetic energy of the satellite changes as it moves between these two positions.
- A ball is kicked with an initial velocity of 8.0ms^{-1} such that it just passes over a barrier which is 2.2m high. Neglecting air resistance, calculate the horizontal velocity of the ball and the total time of flight. Take $g = 9.8\text{ms}^{-2}$.

Question 68

- The gravitational potential energy of an object at a point in a gravitational field is always negative. Explain what the negative sign signifies and why it is never zero at any finite distance from the attracting body.
- A person standing perfectly still on level ground takes a step forward. Before the step, the person's momentum was zero, but afterwards the person clearly has some momentum in the forward direction. Explain whether the conservation of momentum is violated in this situation.
- A block of weight 40N rests on a horizontal table. A light horizontal string is attached to the block and runs to a knot at the edge of the table. At the knot, two other strings are attached: one goes upward to a fixed support on the ceiling at 30° to the horizontal, and the other hangs vertically supporting a weight Q, as shown in the figure. The maximum friction force between the block and the table is 12N. Determine the maximum value of Q.

**Question 69**

- A rocket in outer space, far from any planet, fires its engines and accelerates forward even though there is no air to push against. Explain how the rocket is able to accelerate in a vacuum.
- The Earth is not a perfect sphere but is slightly flattened at the poles and bulges at the equator. Explain what causes this equatorial bulge.
- A ball is thrown upward with an initial velocity of 33ms^{-1} from a point on a hillside which slopes upward uniformly at 28° to the horizontal. The ball is launched at 65° to the horizontal. Calculate the distance up the slope at which the ball strikes the hill and the time of flight. Take $g = 9.8\text{ms}^{-2}$.

Question 70

- In a perfectly inelastic collision, the colliding bodies stick together and move as one unit after impact. Explain why kinetic energy is not conserved in such a collision even though momentum is conserved, and describe how the “lost” kinetic energy is accounted for.
- When two vehicles collide at an intersection, crash investigators use the direction of skid marks left by the combined wreckage together with momentum conservation to reconstruct the collision. Explain why the direction of the post-collision skid marks depends on the ratio of the momenta of the two vehicles before impact.
- A cyclist moving at 72kmh^{-1} is decelerating at a rate of 5ms^{-2} while travelling along a circular path of radius 100m . Calculate the magnitude and direction of the net acceleration of the cyclist.

Question 71

- A rocket in deep space ejects exhaust gases to accelerate. Explain, using Newton’s third law and conservation of momentum, why the rocket accelerates even though there is nothing external for the exhaust to push against.
- A spacecraft orbiting the Earth fires its engines to increase speed and move to a higher orbit. Explain why the gravitational potential energy of the spacecraft increases by more than the kinetic energy it gains, and discuss where the extra energy comes from.
- A cannon of mass 1300kg fires a 72kg ball in a horizontal direction with a muzzle speed of 55ms^{-1} . The cannon recoils freely. Calculate the recoil velocity of the cannon and the velocity of the ball relative to the Earth.

Question 72

- An insect is released from rest at the top of a smooth spherical bowl turned upside down. It slides over the surface. Explain why the insect must eventually lose contact with the surface, and identify the condition that determines where this happens.

- (b) Between any two massive bodies, such as the Earth and the Moon, there exists a point where the net gravitational pull on an object is zero. Explain what determines the location of this neutral point and why it lies closer to the less massive body.
- (c) A spacecraft is launched from the Earth to the Moon. The mass of the Earth is 81 times that of the Moon, and the distance from the centre of the Earth to the centre of the Moon is about 4.0×10^5 km. Calculate the distance from the centre of the Earth at which the resultant gravitational force on the spacecraft becomes zero.

Question 73

- (a) A fireman fighting a fire in a tall building must aim the nozzle of his hose well above the window he wants the water to reach. Explain why the water must be launched at an angle above the horizontal rather than aimed directly at the target.
- (b) A rifle bullet travels 200m to its target in a fraction of a second. To the naked eye, the bullet appears to travel in a perfectly straight line. Explain why the bullet's path is actually curved even though it appears straight.
- (c) A fireman stands at a horizontal distance of 38m from the edge of a burning building. He aims a stream of water at 60° above the horizontal into a first-floor window which is 20m above the ground. The water strikes this floor 2m from the outer edge. Calculate the speed at which the water leaves the nozzle of the fire hose. Take $g = 9.8\text{ms}^{-2}$.

Question 74

- (a) A body is projected at an angle of 60° to the horizontal and is required to pass just above two vertical walls each of height 7m. Explain why, for a given launch speed, there are exactly two points along the trajectory at which the projectile is at this height, and why the two walls must be placed at these two positions.
- (b) In forensic ballistics, investigators fire test bullets through stacks of identical wooden boards to estimate the bullet's initial speed. Each board reduces the bullet's speed by the same fraction. Explain why a bullet that loses one-twentieth of its speed per board is stopped by far fewer than twenty boards.
- (c) A body is projected from the ground at 60° to the horizontal with a velocity of 100ms^{-1} . It passes just above two vertical walls each of height 7m. Calculate the distance between the two walls. Take $g = 9.8\text{ms}^{-2}$.

Question 75

- (a) When a system of connected bodies is in static equilibrium, the net force on the entire system is zero. Explain why the total momentum of such a system remains constant, and discuss what happens to the momentum of individual parts if one of the connections suddenly breaks.
- (b) The escape velocity from the Earth's surface has a fixed value regardless of the direction in which the object is launched. Explain why the direction of launch does not affect the escape velocity.
- (c) A 0.060kg tennis ball travelling horizontally at 35ms^{-1} is struck by a racket and returned in the opposite direction at 45ms^{-1} . The contact time between the racket and the ball is 0.006s .
 - (i) Calculate the impulse exerted on the ball by the racket.
 - (ii) Calculate the average force exerted on the ball during contact.

Question 76

- (a) A rocket accelerates by continuously ejecting exhaust gases at high speed. Explain how the thrust force on the rocket depends on both the rate of mass ejection and the exhaust velocity, and why the acceleration of the rocket increases as fuel is consumed even if the thrust remains constant.
- (b) A rocket is launched from the Earth toward the Sun. At some point along the journey, the net gravitational force on the rocket due to the Earth and the Sun is zero. Explain why this equilibrium point lies much closer to the Earth than to the Sun.
- (c) A rocket of mass 20kg (excluding fuel) carries 180kg of fuel. The exhaust velocity is 1.6kms^{-1} .

- (i) Calculate the minimum rate of fuel consumption that enables the rocket to lift off from the ground.
- (ii) If the actual rate of fuel consumption is 2.0kg s^{-1} , determine the ultimate vertical speed gained by the rocket. Take $g = 9.8\text{ms}^{-2}$ and assume g remains constant during the burn.

Question 77

- (a) A block rests on a rough inclined plane at a small angle. As the angle is gradually increased, the block remains stationary up to a certain critical angle, then begins to slide. Explain why the block can remain stationary over a range of angles rather than sliding as soon as the plane is tilted.
- (b) A ball projected at 30° and another projected at 60° with the same speed from level ground land at the same horizontal distance. Explain why two different angles of projection can produce the same range.
- (c) A helicopter is rising vertically at a uniform velocity of 14.7ms^{-1} . When it is 196m above the ground, a ball is projected from it with a horizontal velocity of 8.5ms^{-1} relative to the helicopter.
 - (i) Calculate the time taken for the ball to reach the ground.
 - (ii) Determine the horizontal distance from the point directly below the helicopter (at the instant of projection) to where the ball hits the ground.
 - (iii) Find the speed of the ball when it hits the ground.

Question 78

- (a) Two automobiles of equal mass approach an intersection at right angles. After colliding, they stick together and leave parallel skid marks. Explain why the direction of these skid marks reveals the ratio of the speeds of the two vehicles before impact.
- (b) An elevator carrying passengers decelerates as it approaches a lower floor. Explain how the apparent weight of each passenger changes during this deceleration phase compared to when the elevator moves at constant speed.
- (c) Two automobiles of equal mass approach an intersection at right angles. One is travelling east at 13ms^{-1} and the other is travelling north at speed v . They collide and stick together, leaving skid marks at 55° north of east. The speed limit for both roads is 15.6ms^{-1} .
 - (i) Determine the speed v of the northward-moving vehicle.
 - (ii) Was the driver of the northward-moving vehicle within the speed limit? Justify your answer.

Question 79

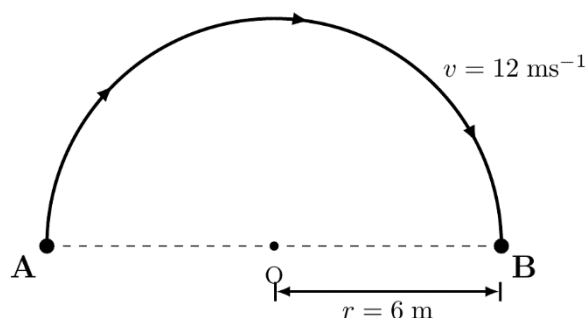
- (a) The gravitational field strength g near the Earth's surface decreases with altitude according to the inverse square law. Explain why, for most projectile motion problems near the surface, we treat g as constant.
- (b) At sea level, the gravitational field strength is 9.81Nkg^{-1} . At the summit of a certain mountain, the measured value is 9.70Nkg^{-1} . Explain how the difference between these two values can be used to estimate the height of the mountain. Identify one physical reason why the actual measured value at the summit might differ from the prediction.
- (c) A ball is kicked from the top of a building of height 1000m at a velocity of 80ms^{-1} at 60° above the horizontal. It enters a window of another building of height 400m across the street.
 - (i) Calculate the time taken for the ball to reach the window.
 - (ii) Determine the horizontal distance between the two buildings. Take $g = 9.8\text{ms}^{-2}$.

Question 80

- (a) A satellite moves in a circular orbit around the Earth under the influence of gravity alone. Explain why the satellite's speed remains constant even though a net force acts on it continuously. Relate your explanation to the work done by gravity.
- (b) A particle moves at constant speed along a semicircular path. Explain why the average velocity over the semicircular path is not equal to the speed, and discuss how this distinction is important in analysing circular motion.

(c)

- (i) A particle moves along a semicircular path AB of radius 6m with a constant speed of 12ms^{-1} . Calculate its average velocity over the semicircular path.



- (ii) Show that the angular momentum of a satellite of mass M_s revolving around the Earth of mass M_E in a circular orbit of radius r is equal to $M_s\sqrt{GM_E r}$.

Question 81

- (a) When a satellite is launched from the Earth's surface, it must acquire both a vertical velocity to reach the desired altitude and a horizontal velocity to maintain a stable orbit. Explain why simply launching a satellite vertically with a very high speed does not produce a circular orbit.
- (b) A spaceship in a circular orbit close to the Earth's surface needs additional velocity to escape the Earth's gravitational field entirely. Explain why the required additional speed is less than the orbital speed itself.
- (c) A satellite of mass 50kg is to be placed into a circular orbit 500km above the Earth's surface. After launch, the satellite reaches the orbital altitude with a vertical velocity of 2000ms^{-1} and zero horizontal velocity.
- (i) Calculate the orbital speed required for a stable circular orbit at this height.
- (ii) Determine the magnitude of the impulse needed to redirect the satellite from its vertical motion into the correct circular orbit. Take $M_E = 5.97 \times 10^{24}\text{kg}$, $R_E = 6.4 \times 10^6\text{m}$, $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Question 82

- (a) Two projectiles are launched simultaneously from different positions in the same vertical plane. Explain why the condition for the two projectiles to collide in mid-air is independent of the value of g .
- (b) A diver jumping into a swimming pool from a 10m platform experiences a much larger average force from the water than a diver jumping from the 3m platform. Explain this difference using the concepts of impulse and momentum.
- (c) Two objects A and B are projected simultaneously in the same vertical plane. Object A is projected from a height of 2m above the ground at 30° above the horizontal. Object B is projected from the ground directly below A with a velocity of 30ms^{-1} at 60° above the horizontal. If the two objects collide in mid-air, determine the time taken until they collide.

Question 83

- (a) A marksman aims a rifle directly at a small target on the same horizontal level some distance away. Explain why the bullet will always strike below the target, and discuss how the marksman should compensate for this effect.
- (b) A hunter aims a rifle directly at a monkey hanging from a tree branch. At the exact instant the gun is fired, the monkey releases the branch and begins to fall freely. Explain whether the bullet will hit the monkey.

- (c) A marksman fires a rifle at a small target 200m away on the same horizontal level. The muzzle velocity is 1000ms^{-1} .
- (i) Calculate how far below the target the bullet strikes if the rifle is aimed directly at the target.
- (ii) How high above the target the marksman must aim to hit it. Take $g = 9.8\text{ms}^{-2}$.

Question 84

- (a) A person standing at the equator experiences a slightly lower apparent weight than a person of the same mass standing at the North Pole. Explain this difference using the concept of centripetal acceleration.
- (b) A spaceship in a low circular orbit just above the Earth's atmosphere has a well-defined orbital speed. Explain why a speed increase of about 41% above the orbital speed allows the spaceship to escape the Earth entirely.
- (c)
- (i) Calculate the length of the day if the Earth were to rotate fast enough that a person standing at the equator would experience zero apparent weight. Take $R_E = 6.4 \times 10^6\text{m}$ and $g = 9.8\text{ms}^{-2}$.
- (ii) A spaceship is in a circular orbit close to the Earth's surface with orbital speed 7.9kms^{-1} . Calculate the additional speed the spaceship must acquire to escape the Earth's gravitational field.

Question 85

- (a) The gravitational potential energy of a body at the surface of a planet is negative when measured relative to infinity. Explain what this negative value means physically, and why a body must be given a minimum kinetic energy to escape the planet's gravitational pull entirely.
- (b) A rocket launched vertically from a planet's surface loses a fraction of its kinetic energy to atmospheric drag during ascent. Explain how energy conservation determines the maximum height the rocket reaches.
- (c) Take $M_E = 6 \times 10^{24}\text{kg}$, $R_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.
- (i) Calculate the work done to move a 10kg body from the Earth's surface to infinity.
- (ii) If the same body falls from rest at infinity toward the Earth, calculate its speed when it strikes the Earth's surface.
- (iii) A rocket is fired vertically from the surface of Mars with a speed of 2kms^{-1} . If 20% of its initial kinetic energy is lost to atmospheric resistance, calculate how far the rocket rises above the surface. Take $g_{\text{Mars}} = 3.7\text{ms}^{-2}$ and assume it remains constant.

Question 86

- (a) When an object moving in a circular path is suddenly released, it travels in a straight line tangent to the circle at the point of release. Explain why the object does not fly radially outward.
- (b) The driver of a truck travelling at speed v suddenly notices a wall at a distance r directly ahead. The driver has two options: brake to stop, or maintain speed and swerve in a circular arc of radius r to avoid the wall. In both cases the maximum friction force is the same. Explain which option gives the driver a better chance of avoiding a collision.
- (c) A stone is whirled in a horizontal circle of radius 1.5m by means of a string, at a height of 2m above the ground. The string breaks and the stone flies off horizontally, striking the ground at a point 10m from the point directly below the break. Calculate the centripetal acceleration of the stone just before the string broke. Take $g = 9.8\text{ms}^{-2}$.

Question 87

- (a) In a binary star system, two stars orbit their common centre of mass under their mutual gravitational attraction. Explain why each star experiences a centripetal acceleration directed toward the centre of mass, and discuss how the orbital radii of the two stars depend on their masses.

- (b) If one star in a binary system is much more massive than the other, explain how the position of the centre of mass shifts and how this changes the orbital motion of each star. Relate your answer to the Earth and Sun system.
- (c)
 - (i) Calculate the height above the Earth's surface at which the acceleration due to gravity becomes half its value at the surface. Take $R_E = 6400\text{km}$.
 - (ii) A satellite is placed in a circular orbit around Jupiter so that it always remains above the same point on Jupiter's equator. Jupiter rotates once every 9.84 hours, its mass is $1.9 \times 10^{27}\text{kg}$, and its mean radius is $6.99 \times 10^7\text{m}$. Calculate the altitude of the satellite above Jupiter's surface. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Question 88

- (a) When a stationary object explodes into several fragments, the total momentum of all fragments must be zero. Explain why individual fragments can nevertheless have large momenta, and discuss how momentum conservation constrains the directions and speeds of the fragments.
- (b) A person standing inside a lift holds a briefcase at waist height and then releases it. The briefcase does not fall to the floor but instead remains floating at the same height relative to the person. Explain what this observation reveals about the motion of the lift.
- (c) A bomb of mass M is suspended at rest from a cable. The cable is cut and the bomb simultaneously explodes into three pieces. The first piece (mass $M/2$) moves horizontally in the positive x -direction at speed V_0 . The second piece (mass $M/3$) moves at speed $2V_0$ at an angle of 37° above the positive x -axis. Determine the speed and direction of the third piece (mass $M/6$).

Question 89

- (a) A ball thrown at a certain angle reaches a maximum height during its flight. Explain how the launch angle and initial velocity together determine the maximum height, and discuss why there is a maximum horizontal range for a given speed regardless of the angle chosen.
- (b) A helicopter hovers by using its rotor blades to push air downward. Explain, using Newton's third law and the concept of momentum change per unit time, how the downward thrust of air supports the weight of the helicopter.
- (c)
 - (ii) A ball is thrown at 40ms^{-1} inside a hall whose ceiling is 25m high. Determine the maximum horizontal distance the ball can travel without hitting the ceiling. Take $g = 9.8\text{ms}^{-2}$.
 - (iii) A helicopter hovers in still air. Its rotor blades sweep a circle of radius 2m and push air vertically downward at 8ms^{-1} . Taking the density of air as 1.3kgm^{-3} , estimate the mass of the helicopter. Take $g = 9.8\text{ms}^{-2}$.

Question 90

- (a) When a person pulls an object along a rough horizontal surface by a rope inclined above the horizontal, the normal reaction force is less than the weight of the object. Explain why this occurs and discuss how the reduced normal reaction affects the friction force.
- (b) Roads and railway lines are banked on curves. Explain how banking allows a vehicle to negotiate a curve safely even without friction, and describe the role of the normal reaction force in providing the centripetal force.
- (c)
 - (i) Ambitious is pulling an object of mass 20kg across a horizontal plane at constant velocity. The force she exerts makes an angle of 30° with the horizontal. If the coefficient of sliding friction is 0.57, calculate the magnitude of the force Ambitious exerts. Take $g = 9.8\text{ms}^{-2}$.
 - (ii) A highway designed for an average speed of 72kmh^{-1} has a curve of radius 50m. Calculate the minimum banking angle so that cars at this speed do not depend on friction. Take $g = 9.8\text{ms}^{-2}$.

Question 91

- (a) Two projectiles are launched from the same point with the same speed but at different angles above the horizontal. Explain why, in general, their horizontal ranges differ, and discuss the special relationship between the two angles that produce the same range.
- (b) Gravel is dropped vertically onto a horizontal conveyor belt that moves at constant speed. Explain, using the concept of impulse, why a horizontal force must act on the gravel to accelerate it to the belt's speed.
- (c)
 - (i) Particle P is projected at angle α with velocity u_P and particle Q is projected at angle β with velocity u_Q from the same point. The vertical components of their initial velocities are equal. Find the ratio of their horizontal ranges in terms of α and β .
 - (ii) Gravel is dropped vertically onto a horizontal conveyor belt at a rate of 75kg s^{-1} . The belt moves at a constant speed of 2m s^{-1} . Calculate the force needed to keep the belt moving at constant speed, and the rate at which kinetic energy is given to the gravel.

Question 92

- (a) A satellite moves in an elliptical orbit around the Earth. Explain why the kinetic energy of the satellite changes as it moves around its orbit, relating your answer to the work done by gravity.
- (b) The Moon has no significant atmosphere while the Earth retains a thick one. Explain the reason for this difference, using the concepts of escape velocity and surface gravity.
- (c) Two stars, each of one solar mass ($2 \times 10^{30}\text{kg}$), are approaching each other for a head-on collision. When they are 10^9km apart, their speeds are negligible. Each star has a radius of 10^4km . Calculate the speed with which they collide. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Question 93

- (a) A car moves on a circular track while its speed is increasing. Explain why the resultant acceleration is not directed toward the centre of the circle, and discuss what determines the angle between the resultant acceleration and the velocity direction.
- (b)
 - (i) A projectile is thrown from the base of an inclined plane that makes 30° with the horizontal. Calculate the angle of projection, measured from the horizontal, that gives the maximum range along the surface of the incline.
 - (ii) A car moves at 30m s^{-1} on a circular track of radius 500m while its speed increases at 2m s^{-2} . Calculate the magnitude of the resultant acceleration and the angle it makes with the velocity direction.

Question 94

- (a) A projectile is fired at 45° to achieve maximum horizontal range. At the highest point of its trajectory, the velocity is entirely horizontal. Explain why the kinetic energy at the highest point is not zero, and show that it is exactly half the kinetic energy at launch.
- (b) A conveyor belt at a shopping mall carries passengers at constant speed regardless of how many people are standing on it. Explain what physical condition the driving mechanism must satisfy for the belt to maintain constant speed under varying loads.
- (c) A projectile is launched with a kinetic energy of 500J at the angle that gives maximum horizontal range. Calculate the kinetic energy of the projectile at the highest point of its trajectory.

Question 95

- (a) An object released from rest slides down a smooth inclined plane. Explain why the speed at the bottom depends only on the vertical height and not on the angle or length of the slope.
- (b) A boy throws a stone vertically upward and catches it when it returns. A student claims this is an example of projectile motion. Evaluate this claim, clearly stating the conditions that define projectile motion.

- (c) A ball is thrown from a point 2m above the ground, at a horizontal distance of 3m from a smooth vertical wall. The ball is launched at 20ms^{-1} at 30° above the horizontal toward the wall. After an elastic collision with the wall, the ball bounces back. Determine the horizontal distance from the wall to the point where the ball hits the ground. Take $g = 9.8\text{ms}^{-2}$.

Question 96

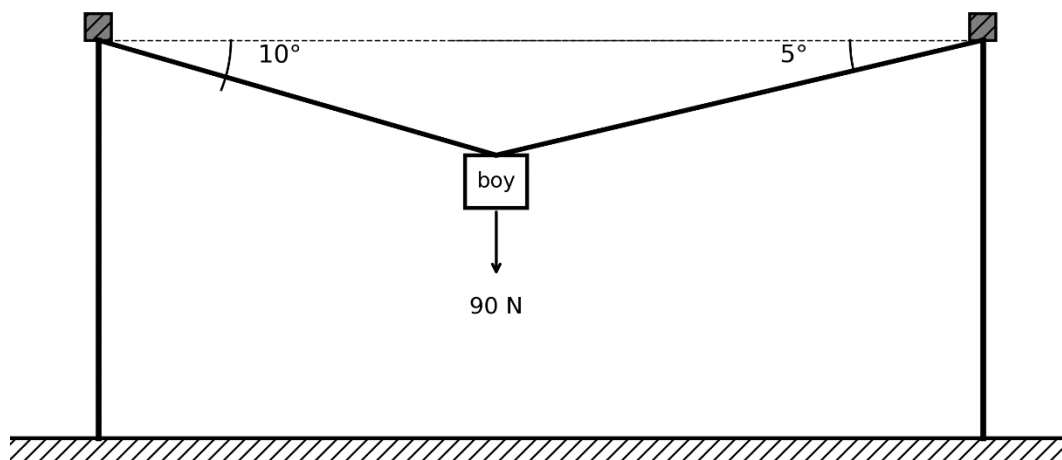
- (a) A magician pulls a tablecloth out from under a set of plates without disturbing them. The trick works best when the cloth is pulled very quickly. Explain why a fast pull succeeds while a slow pull drags the plates along.
- (b) A rocket accelerates by ejecting exhaust gases at high speed. Explain why the acceleration of the rocket increases over time even if the thrust force remains constant.
- (c) A rocket of initial total mass 100kg launches vertically and ejects mass at a constant rate of 20kgs^{-1} . The thrust force is constant at 1500N. Calculate the acceleration at launch and write an expression for the acceleration at time t . Take $g = 9.8\text{ms}^{-2}$.

Question 97

- (a) When a spacecraft uses a planet's gravity to change its speed and direction (a gravitational slingshot), the spacecraft can gain a large amount of kinetic energy even though the planet's speed barely changes. Explain why, using conservation of momentum.
- (b) A satellite in a circular orbit experiences no change in speed, yet a continuous force acts on it. Explain how this is consistent with the work-energy theorem. Contrast this with the case of a satellite in an elliptical orbit where the speed does change.
- (c) Two identical spacecraft, each of mass 500kg, are in the same circular orbit at height 400km above the Earth. They collide head-on and stick together. Calculate the orbital speed before the collision. Determine the speed of the combined wreckage immediately after the collision and discuss what happens to the wreckage. Take $M_E = 5.97 \times 10^{24}\text{kg}$, $R_E = 6.4 \times 10^6\text{m}$, $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

Question 98

- (a) Two important factors must be considered when placing a satellite into geostationary orbit. Identify these two factors and explain why each is necessary for the satellite to remain permanently above the same point on the Earth's equator.
- (b) A rope is stretched between two vertical poles. A boy weighing 90N hangs from the rope, causing it to sag. The left segment makes an angle of 10° below the horizontal and the right segment makes 5° below the horizontal. Calculate the tension in each segment of the rope.



Question 99

- (a) A stone thrown horizontally from the top of a cliff and a stone dropped vertically from the same height reach the ground at the same time. Explain what determines the horizontal distance the thrown stone travels before hitting the ground.
- (b) In a football match, a player some distance from the kicker starts running the instant the ball is kicked, hoping to reach the ball just as it lands. Explain what physical quantities the player must estimate to decide at what speed to run.
- (c) In a football game, a ball is kicked from the ground at 19.5ms^{-1} at 45° above the horizontal. A player standing 55m away in the direction of the kick starts running toward the ball at the instant it is kicked. Calculate the minimum average speed the player must maintain to reach the ball just before it hits the ground. Take $g = 9.8\text{ms}^{-2}$.

Question 100

- (a) A person standing on a moving platform throws a ball. Explain how the velocity of the platform affects the ball's trajectory as seen by a stationary observer on the ground.
- (b) A gun is kept on a straight horizontal road and used to fire a shell at a car travelling along the same road at a uniform speed. At the instant of firing, the car is some distance from the gun. Explain what conditions must be satisfied for the shell to hit the moving car.
- (c) A gun fires a shell at 45° above the horizontal at a car travelling along the same straight road at 72kmh^{-1} . At the instant of firing, the car is 500m from the gun. Calculate the minimum muzzle speed of the shell required to hit the car. Take $g = 9.8\text{ms}^{-2}$.

Question 101

- (a) Two objects connected by a string over a frictionless pulley rest on the two slopes of a rough double wedge. Explain why the coefficient of friction affects the range of conditions for which equilibrium is possible.
- (b) A ball attached to a string moves in a horizontal circle (conical pendulum). Explain why the string cannot be perfectly horizontal during this motion.
- (c) Two equal masses, each of mass m , are connected by a light string passing over a frictionless pulley at the peak of a rough double wedge. One slope makes 60° with the horizontal and the other makes 30° . The coefficient of friction μ is the same on both surfaces. If the masses move at constant velocity, find μ .

Question 102

- (a) A block rests on an inclined plane of angle θ and is held in place by a horizontal force. Explain why the normal reaction force in this case is not simply $mg\cos\theta$.
- (b) A satellite in a low circular orbit has a well-defined orbital speed. If the satellite fires a short burst of its engines to increase its speed, it moves into a higher orbit. Explain why the satellite ends up moving slower in the higher orbit despite having been given extra energy.
- (c) A block of mass m is held at rest on a rough inclined plane of angle θ by a horizontal force F . The coefficient of static friction is μ_s . Show that the maximum horizontal force that can be applied before the block begins to slide up the plane is:

$$F_{\max} = \frac{mg(\sin\theta + \mu_s\cos\theta)}{\cos\theta - \mu_s\sin\theta}$$

Question 103

- (a) In a conical pendulum, the bob moves in a horizontal circle while the string traces out a cone. Explain why the vertical component of the string tension must equal the weight of the bob.
- (b) A loaded truck negotiating a banked curve at too high a speed tends to slide outward, while the same truck moving too slowly tends to slide inward. Explain both observations, stating direction of friction in each case.
- (c) A conical pendulum consists of a 0.2kg bob attached to a 1.5m string. The string makes an angle of 25° with the vertical. Calculate the tension in the string, the speed of the bob, and the period of revolution. Take $g = 9.8\text{ms}^{-2}$.

Question 104

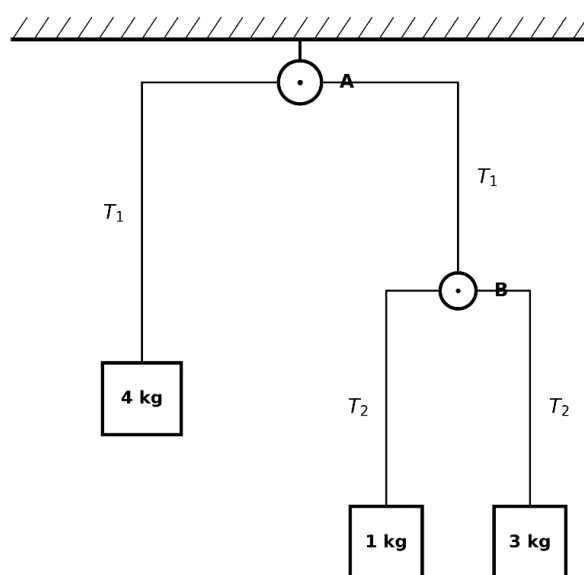
- (a) A firework rocket explodes at the highest point of its trajectory, splitting into several fragments. Explain why the centre of mass of all the fragments continues to follow the original parabolic path even after the explosion.
- (b) A cannon mounted on a wheeled carriage fires a shell horizontally. Explain, using conservation of momentum, why the carriage recoils backward, and how the recoil speed depends on the ratio of the shell's mass to the carriage's mass.
- (c) A projectile of mass 3 kg is launched from the ground at 50 ms^{-1} at 60° above the horizontal. At the highest point of its trajectory, the projectile splits into two fragments. One fragment, of mass 1 kg , falls vertically downward with zero initial speed relative to the ground. Calculate the velocity of the remaining 2 kg fragment immediately after the split, and the horizontal distance from the launch point to where it lands. Take $g = 9.8\text{ ms}^{-2}$.

Question 105

- (a) When a stone is thrown at a window, the glass shatters into many pieces. When a high-speed bullet hits the same glass, it passes through leaving a clean round hole without shattering the rest of the pane. Explain this difference using the concepts of impulse and contact time.
- (b) Imagine that gravity were to suddenly vanish everywhere. Describe what would immediately happen to objects resting on the Earth's surface, the Moon in its orbit, and the Earth's atmosphere. Explain each prediction using the relevant law of motion.
- (c) A fireman stands 20 m from the base of a burning building. The water leaves the hose at a height of 1 m above the ground at 55° to the horizontal. The water just enters an open window 18 m above the ground. Calculate the speed of the water as it leaves the hose. Take $g = 9.8\text{ ms}^{-2}$.

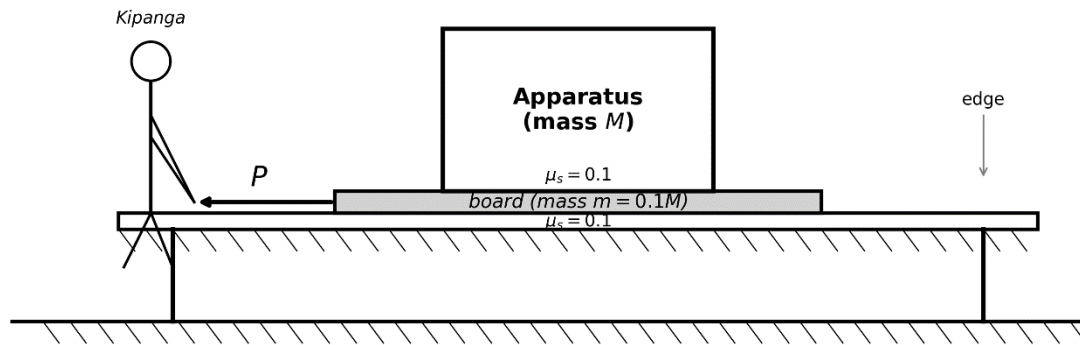
Question 106

- (a) Explain how the motion of a body whirled in a horizontal circle differs from the motion of a body whirled in a vertical circle. In your answer, discuss the variation of speed and string tension in each case.
- (b) When two objects collide and stick together, the final speed of the combined mass is less than the initial speed of the faster object. Explain why this is always the case.
- (c) The figure shows a frictionless pulley A fixed to the ceiling. On one side hangs a 4 kg mass. On the other side hangs a small frictionless pulley B from which a 1 kg mass and a 3 kg mass are suspended. All strings are light and inextensible. Find the acceleration of each mass and the tensions in the strings. Take $g = 9.8\text{ ms}^{-2}$.



Question 107

- (a) A satellite that gradually loses energy due to atmospheric drag speeds up rather than slowing down. Explain why this seemingly paradoxical result follows from the relationship between orbital speed and radius.
- (b) In Mr. Akilikubwa's laboratory, a heavy measuring apparatus of mass M sits on a thin wooden board of mass $m = 0.1M$ that rests on a long laboratory bench. The coefficient of static friction between all contact surfaces is $\mu_s = 0.1$. Kipanga wants to remove the board without lifting the apparatus. Mr. Akilikubwa tells him he must pull the board quickly, not slowly. Explain why the friction force between the board and the bench is greater than the friction force between the apparatus and the board, even though the coefficient of friction is the same at both surfaces.



- (c) Using the arrangement described in (b):
- Draw a free body diagram for the board, clearly showing all forces acting on it, and a separate free body diagram for the apparatus.
 - Calculate the minimum value of P required to cause any motion in the system. Express your answer in terms of M and g .
 - Calculate the force P required to slide the board out from under the apparatus while the apparatus remains stationary. Express your answer in terms of M and g .
 - Kipanga pulls the board out quickly and the apparatus is left at rest on the bench, 5cm from the edge. The coefficient of kinetic friction between the apparatus and the bench is also 0.1. Determine whether the apparatus slides off the edge of the bench.

PART THREE

EXTENDED MECHANICS

Having developed the principles of particle mechanics, we now extend our study to more complex systems. Extended mechanics examines the behaviour of rigid bodies and fluids, where factors such as rotation, deformation, and flow become important. This section introduces concepts related to rotational motion and fluid behaviour, allowing us to analyse real-world situations more accurately. These ideas build upon Newtonian mechanics and provide essential tools for understanding systems that cannot be treated as simple point masses. Mastery of this section will deepen your insight into rotational dynamics and fluid motion, and prepare you for more advanced applications in physics and engineering.

Chapter 9

ROTATION OF RIGID BODIES

INTRODUCTION

The trouble began, as it usually did, with Kipanga.

Mr. Akilikubwa had brought a collection of objects to class: a metre ruler, a bicycle wheel, a hard-boiled egg, and, for reasons that would become clear shortly, an uncooked egg. He placed them on the demonstration table and announced, *“Today, we leave the comfortable world of forces and straight-line motion. Today, we enter the world of rotation.”*

Kipanga, sitting at the front as usual, immediately picked up the metre ruler and balanced it horizontally on one finger at its centre. *“Like this, sir? Rotation is easy.”*

“Try spinning it,” said **Mr. Akilikubwa**.

Kipanga flicked one end. The ruler spun beautifully on his finger like a tiny helicopter blade.

“Now,” said **Mr. Akilikubwa**, moving Kipanga’s finger to one end of the ruler, *“spin it from here.”*

Kipanga tried. The ruler wobbled, resisted, and barely completed half a turn before clattering onto the desk. *“What happened? It’s the same ruler! Same mass! Why is it suddenly so stubborn?”*

“Excellent question. Kipute, would you like to explain?”

Kipute, who had been quietly spinning the bicycle wheel on its axle with impressive ease, looked up. *“I think it’s because the mass is distributed differently relative to the point of rotation. When you spin from the centre, most of the mass is close to the pivot. When you spin from the end, more of the mass is far from the pivot, and somehow that makes it harder to spin.”*

“Somehow!” **Mr. Akilikubwa** clapped his hands in delight. *“That ‘somehow’ is precisely what this chapter is about. In Chapters 2 and 3, we learned that a body’s resistance to changes in linear motion is described by its mass. Push a 10kg box, and it resists. Push a 100kg box with the same force, and it resists ten times more. Mass is the measure of linear inertia. But Kipanga just demonstrated something fascinating: the same mass can be easy to spin or difficult to spin, depending on how it is arranged around the axis. Rotation has its own kind of inertia, and mass alone does not capture it.”*

He picked up the two eggs. *“Watch carefully.”* He spun both eggs on the table. One spun smoothly and continued spinning for several seconds. The other wobbled pathetically and stopped almost immediately. *“Same size, same shape, nearly the same mass. Why does one spin well and the other refuses?”*

Kipanga pointed confidently. *“The hard-boiled one spins better because it’s solid inside. The raw one has liquid that sloshes around and... resists the spinning?”*

“Very close. The liquid inside the raw egg does not rotate with the shell immediately. It lags behind, and the internal friction between the liquid and the shell dissipates energy very quickly. The hard-boiled egg, being solid throughout, rotates as one rigid body. Every particle inside it moves together, in perfect coordination.”

He wrote two words on the board in large letters: **RIGID BODY**.

“A rigid body is an object in which the distance between every pair of particles remains constant, no matter what forces act on it. It does not bend, stretch, compress, or flow. The hard-boiled egg is approximately rigid. The raw egg is not. This chapter deals exclusively with rigid bodies, because their rotation follows clean, predictable laws.”

He paused and looked at the class with a mischievous smile. *“Now, here is the beautiful secret that will carry you through this entire chapter. You already know the physics of rotation. You just do not know that you know it.”*

The students looked puzzled.

“Every concept in linear motion, every formula, every law, has an exact twin in rotational motion. Displacement becomes angular displacement. Velocity becomes angular velocity. Acceleration becomes angular acceleration. Mass becomes moment of inertia. Force becomes torque. Momentum becomes angular momentum. Even $F = ma$ has a twin: $\tau = I\alpha$. Rotation is not a new subject. It is the same subject, wearing a different costume.”

Kipanga's eyes widened. "So if I already understand $F = ma$, I already understand rotation?"

"In principle, yes. In practice, the costume is quite convincing and the details take some getting used to. But the structure is identical. And that is why we will build this chapter as a mirror of what you already know. At every step, we will place the linear version and the rotational version side by side, so you can always see the twin."

Kipute smiled. "A twin language."

"Exactly. A twin language. Two dialects of the same physics."

He picked up the bicycle wheel and gave it a powerful spin. "By the end of this chapter, you will understand why this wheel resists being tilted while it spins. You will understand why an ice skater spins faster by pulling in her arms. You will understand why a solid sphere beats a hollow cylinder in a race down a slope, even though both are released at the same time from the same height. You will calculate the energy stored in a spinning flywheel, the torque needed to stop a rotating platform, and the speed of a ball rolling down a hill. And you will do all of it using the twin of what you already know."

He set the wheel down. It continued spinning on its axle, perfectly balanced, humming quietly.

"Welcome to Chapter 9. The world is about to start spinning."

THE BIG PICTURE: FROM LINEAR TO ROTATIONAL MOTION

Up to this point in the book, we have been remarkably successful at describing the physical world using a small set of ideas: displacement, velocity, acceleration, force, mass, momentum, energy. These ideas, forged in Chapters 2 through 8, allowed us to launch projectiles, crash vehicles, swing objects in circles, and send satellites into orbit. But all of this success was built on a quiet assumption that we are about to confront.

We treated every object as a **particle**: a point mass with no size, no shape, and no internal structure. When we said "a car accelerates" or "a ball is projected," we meant a single point moved in a certain way. We never asked *how* the car was oriented, or whether the ball was spinning. We never needed to.

That changes now.

Consider a simple experiment. Place a pencil on a smooth table and flick it at the centre. The pencil slides forward in a straight line without rotating. Now flick the same pencil at one end. The pencil both slides and spins. The same force, applied at different points, produces completely different motions. This is something that the particle model cannot explain, because a particle has no "centre" or "end." It is a point. Every force on a particle produces the same type of motion: translation along a straight line.

Real objects are not points. They have size, shape, and mass distributed over a volume. When forces act on such objects, two fundamentally different types of motion can occur:

Translation, in which every particle in the body moves the same distance in the same direction in the same time. The body shifts position without changing orientation. A box sliding across a floor is translating.

Rotation, in which every particle in the body moves in a circle about a common axis. The body changes orientation without necessarily changing position. A wheel spinning on a fixed axle is rotating.

Most real motions are a combination of both. A rolling football translates across the pitch while simultaneously rotating about its own axis. A thrown spanner tumbles through the air, rotating about its centre of mass while the centre of mass follows a parabolic projectile path. Even the Earth participates in both: it translates along its orbit around the Sun while rotating about its north-south axis once every 24 hours.

In Chapters 2 through 8, we mastered translation. In this chapter, we master rotation. And the beautiful surprise waiting for us is this: *rotation is not new physics*. It is the same physics, expressed in a different language.

The Twin Language

Every quantity we used to describe translation has an exact counterpart in rotation. Every equation we used to describe translational motion has an identical twin for rotational motion. The structure is perfectly parallel, and once you see this parallelism, the entire chapter becomes a translation exercise: take what you already know about linear motion, replace each quantity with its rotational twin, and the physics works.

Here is a preview of the twin pairs, without any formulas yet. We will develop each one rigorously in the sections that follow, but seeing the full map now will help you navigate the journey.

In translation, we describe *where* a body is using **displacement** (how far it has moved in a straight line). In rotation, the twin is **angular displacement** (how far the body has turned, measured in radians).

In translation, we describe *how fast* a body moves using **velocity** (rate of change of displacement). In rotation, the twin is **angular velocity** (rate of change of angular displacement, how fast the body spins).

In translation, we describe *how quickly the motion changes* using **acceleration** (rate of change of velocity). In rotation, the twin is **angular acceleration** (rate of change of angular velocity, how quickly the spin speeds up or slows down).

In translation, a body's resistance to change in motion is measured by its **mass**. Push a heavy box and it resists. Push a light box with the same force and it responds more readily. In rotation, the twin is **moment of inertia**: a body's resistance to change in rotational motion. Try to spin a metre ruler from its centre, and it spins easily. Try to spin the same ruler from one end, and it resists stubbornly. The mass has not changed, but the *distribution of mass relative to the axis* has changed. Moment of inertia captures this distribution, and it plays exactly the same role in rotation that mass plays in translation.

In translation, the cause of acceleration is **force**. In rotation, the twin is **torque**: the rotational equivalent of force. Torque is what makes things spin, just as force is what makes things accelerate. A force applied far from the pivot produces a large torque; the same force applied close to the pivot produces a small torque. This is why a long spanner is more effective than a short one, and why a door handle is placed at the edge farthest from the hinges, not next to them.

In translation, $F = ma$ is the master equation: force equals mass times acceleration. In rotation, the master equation is its twin: torque equals moment of inertia times angular acceleration. This single equation governs all rotational dynamics, just as $F = ma$ governs all translational dynamics.

In translation, the quantity of motion is **momentum** ($p = mv$). In rotation, the twin is **angular momentum**: the product of moment of inertia and angular velocity. Angular momentum is conserved when no external torque acts, just as linear momentum is conserved when no external force acts. This is why an ice skater spins faster when she pulls her arms inward: by reducing her moment of inertia, she must increase her angular velocity to keep angular momentum constant.

In translation, the energy of motion is **kinetic energy** ($\frac{1}{2}mv^2$). In rotation, the twin is **rotational kinetic energy**: $\frac{1}{2}$ times the moment of inertia times the square of the angular velocity. A spinning flywheel stores energy in exactly the same way that a moving car stores energy, but through rotation rather than translation.

In translation, the rate of doing work is **power** (Fv). In rotation, the twin is **rotational power**: torque times angular velocity.

Every pair follows the same pattern: replace displacement with angular displacement, velocity with angular velocity, mass with moment of inertia, force with torque, and the physics transfers perfectly.

Why rotation needs its own treatment

A reasonable student might ask: if rotation is just the same physics in a different costume, *why do we need a separate chapter? Why not simply apply $F = ma$ to every particle in a spinning object and add up the results?*

In principle, that approach would work. A spinning disc is nothing more than a vast number of tiny particles, each obeying $F = ma$. If we could track every particle individually, we would not need any rotational concepts at all.

In practice, this is impossible. A solid disc contains roughly 10^{25} atoms. Writing $F = ma$ for each one and solving the resulting system of equations would take longer than the age of the universe. The rotational quantities (angular velocity, moment of inertia, torque, angular momentum) are not new physics; they are powerful *shortcuts*. They compress the behaviour of 10^{25} particles into a single equation that a Form Five student can solve in two minutes.

This is the real reason rotation needs its own language: not because the physics is different, but because describing rotation in the language of translation is hopelessly inefficient. The twin language is a labour-saving device of extraordinary power.

What does “rigid” mean, and why does it matter?

In the introduction, Mr. Akilikubwa distinguished between the hard-boiled egg (rigid) and the raw egg (not rigid). This distinction is essential.

A **rigid body** is an idealised object in which the distance between every pair of particles remains fixed, regardless of the forces applied. A rigid body does not bend, stretch, compress, or flow. When one part of a rigid body moves, every other part moves in a perfectly predictable way, because the shape never changes.

Real objects are never perfectly rigid. A steel beam bends slightly under load. A rubber ball deforms when squeezed. Even a diamond compresses by a tiny amount when pressed. However, for most everyday objects under normal forces, the deformations are so small that treating the object as rigid introduces negligible error. Throughout this chapter, we assume all objects are perfectly rigid unless stated otherwise.

The rigidity assumption is what makes rotational physics tractable. If a body is rigid, then knowing how any one particle moves tells us how *every* particle moves. If the body rotates through an angle θ , every particle rotates through the same angle θ . If the body has angular velocity ω , every particle has the same angular velocity ω . The entire motion of the body is captured by a single variable (the angle), rather than by 10^{25} separate position vectors. This is an enormous simplification, and it is the foundation upon which the entire chapter rests.

The road ahead

With the big picture in place, here is the journey we will take through this chapter.

We begin with the **language of rotation**: angular displacement, angular velocity, angular acceleration, and the kinematic equations that connect them. These are the rotational twins of the kinematic equations you learned in Chapter 2.

Next, we identify the **centre of mass**: the special point about which a free body naturally rotates, and through which the net external force effectively acts.

Then comes the **heart of the chapter**: moment of inertia. We will derive it from first principles and calculate it for every standard shape. This is the most challenging subtopic, and also the most rewarding.

Armed with moment of inertia, we discover two powerful **shortcut theorems** that let us find moments of inertia for new axes without repeating long integrations.

We then introduce **radius of gyration**, a neat way to express how mass is distributed in a single number.

We then move from description to cause: **torque**, the thing that makes objects rotate, including the special case of a couple.

With torque established, we then develop the **energy and power** of rotation.

The crown jewel of the chapter follows: **rolling motion**, where translation and rotation combine and the twin languages merge into one.

We conclude with **angular momentum and its conservation**, one of the most profound principles in all of physics.

Finally, we lay out the full **comparison table**, followed by real-world **applications** that bring the physics to life.

The world has always been spinning. After this chapter, you will finally understand why.

ANGULAR KINEMATICS: THE LANGUAGE OF ROTATION

You already speak half of this language. In Chapter 7 (Uniform Circular Motion), we introduced the radian as the natural unit for measuring angles, defined angular displacement $\theta = \frac{s}{r}$ and established angular velocity $\omega = \frac{d\theta}{dt}$ as the rate of rotation. We derived the relationship $v = \omega r$ connecting linear and angular speed, and we used the period $T = \frac{2\pi}{\omega}$ and frequency $f = \frac{1}{T}$ to describe objects moving at constant speed around a circle.

But Chapter 7 dealt exclusively with **uniform** circular motion, where the angular velocity never changed. A satellite in a stable orbit, a stone on a string whirled at constant speed, a car rounding a bend at steady velocity: in all of those situations, ω was constant.

Real rotating objects are rarely so well-behaved. A fan speeds up when switched on and slows down when switched off. A car engine increases its rotational speed from idle to full throttle. A potter's wheel gradually loses speed to friction. A gymnast's somersault starts slow and becomes faster as she tucks. In every one of these situations, the angular velocity is changing.

To describe changing angular velocity, we need a new quantity: **angular acceleration**. And with angular acceleration in hand, we can build a complete set of kinematic equations for rotation, exactly mirroring the equations of Chapter 2.

Quick reference: What we already know from chapter 7

Quantity	Definition	Unit
Angular displacement, θ	Angle swept, $\theta = \frac{s}{r}$	rad
Angular velocity, ω	Rate of rotation, $\omega = \frac{\Delta\theta}{\Delta t}$	rad/s
Linear velocity from angular velocity	$v = \omega r$	m/s
Period, T	Time for one revolution, $T = \frac{2\pi}{\omega}$	s
Frequency, f	Revolutions per second, $f = \frac{\omega}{2\pi}$	Hz
Centripetal acceleration, a_c	$a_c = \omega^2 r = \frac{v^2}{r}$	m/s ²

Conversion reminder: $\omega = 2\pi f = \frac{2\pi N}{60}$, where N is the speed in rpm (revolutions per minute).

All of this remains valid. We now extend it.

Angular Acceleration

If the angular velocity of a body changes with time, the body has an **angular acceleration**, denoted by α .

The **average angular acceleration** over a time interval is:

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

The **instantaneous angular acceleration** is:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Angular acceleration is measured in **radians per second squared** (rad/s²).

The sign convention follows naturally: if we take anticlockwise as positive, then a positive α means the rotation is speeding up in the anticlockwise direction (or slowing down in the clockwise direction). A negative α means the rotation is slowing down in the anticlockwise direction (or speeding up clockwise).

Angular acceleration is the rotational twin of linear acceleration. In Chapter 2, acceleration told us how rapidly an object's velocity was changing. Here, angular acceleration tells us how rapidly an object's angular velocity is changing. A flywheel that spins up from rest has a positive α . A grinding wheel that is switched off and gradually slows has a negative α (deceleration). A fan spinning at constant speed has $\alpha = 0$, which is exactly the uniform circular motion of Chapter 7.

Tangential Acceleration

In Chapter 7, a particle moving in a circle at constant ω had only centripetal acceleration $a_c = \omega^2 r$, directed toward the centre. This acceleration changed the *direction* of the velocity but not its magnitude.

When ω is changing, a second component of acceleration appears. Differentiating $v = \omega r$ with respect to time:

$$\frac{dv}{dt} = \frac{d(wr)}{dt} = r \frac{dw}{dt}$$

But:

$$\frac{dv}{dt} = a, \frac{dw}{dt} = \alpha$$

Thus:

$$a = \alpha r$$

The acceleration, a , acts along the **tangent** to the circular path and changes the *speed* of the particle. It is therefore known as **tangential acceleration, a_t** . Hence:

$$a_t = \alpha r$$

Tangential acceleration exists only when $\alpha \neq 0$.

A particle undergoing non-uniform circular motion therefore has two perpendicular components of acceleration:

- 1) **Centripetal:** $a_c = \omega^2 r$, directed toward the centre (changes direction).
- 2) **Tangential:** $a_t = \alpha r$, directed along the tangent (changes speed).

The total linear acceleration has magnitude:

$$a = \sqrt{a_t^2 + a_c^2}$$

When $\alpha = 0$ (uniform circular motion), a_t vanishes ($a_t = 0$) and only the centripetal acceleration remains ($a = \sqrt{0 + a_c^2} = a_c$). This is precisely the situation we studied in Chapter 7. Non-uniform circular motion is the general case, and uniform circular motion is the special case where α happens to be zero.

The Rotational Kinematic Equations

In Chapter 2, we derived four equations that describe straight-line motion under constant acceleration. Because the mathematical structure of rotation is identical, we can write down the rotational equivalents immediately by replacing every linear quantity with its angular twin.

The correspondence is:

Linear quantity	Rotational twin
Displacement, s	Angular displacement, θ
Initial velocity, u	Initial angular velocity, ω_0
Final velocity, v	Final angular velocity, ω
Acceleration, a	Angular acceleration, α
Time, t	Time, t (no twin needed)

The four equations, side by side:

Linear (constant a)	Rotational (constant α)
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s = \left(\frac{u+v}{2}\right)t$	$\theta = \left(\frac{\omega_0 + \omega}{2}\right)t$

These equations are valid only when the angular acceleration α is constant, just as the linear equations are valid only when the linear acceleration a is constant.

Notice that time t appears unchanged in both columns. Time does not have a “rotational twin” because time is the same for all observers regardless of the type of motion being described.

If you mastered Chapter 2, you already understand these equations. The physics is identical; only the symbols have changed. A flywheel starting from rest and spinning up at constant angular acceleration behaves mathematically in exactly the same way as a car starting from rest and accelerating at constant linear acceleration.

A note on angular quantities in rigid bodies:

We emphasized in Chapter 7 that all points on a rigid body share the same ω . This extends naturally: all points on a rigid body also share the same α , because if the angular velocity changes by the same amount for every point (rigidity demands this), then the rate of change α must be the same for all.

However, the linear quantities v , a_t , and a_c differ from point to point because they all depend on r (the distance from the rotation axis). A point far from the axis has a larger linear velocity, a larger tangential acceleration, and a larger centripetal acceleration than a point close to the axis, even though both share the same ω and α . This is precisely why angular quantities are more convenient than linear quantities for describing rotation: one value of ω describes the entire body, while the linear velocity varies from particle to particle.

Angular Displacement in the n th Second

There is one more kinematic result that appears frequently in examinations: the angular displacement turned through during a specific second of motion, say the n th second.

Derivation:

Using:

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

The total angular displacement from $t = 0$ to $t = n$ is:

$$\theta_n = \omega_0 n + \frac{1}{2} \alpha n^2$$

The total angular displacement from $t = 0$ to $t = (n - 1)$ is:

$$\theta_{n-1} = \omega_0 (n - 1) + \frac{1}{2} \alpha (n - 1)^2$$

The angular displacement during the n th second alone is the difference:

$$\begin{aligned} \theta_{nth} &= \theta_n - \theta_{n-1} = \omega_0 n + \frac{1}{2} \alpha n^2 - \omega_0 (n - 1) - \frac{1}{2} \alpha (n - 1)^2 \\ &= \omega_0 (n - (n - 1)) + \frac{1}{2} \alpha (n^2 - (n - 1)^2) \\ &= \omega_0 (1) + \frac{1}{2} \alpha (n^2 - n^2 + 2n - 1) \\ &= \omega_0 + \frac{1}{2} \alpha (2n - 1) \\ \theta_{nth} &= \omega_0 + \frac{1}{2} \alpha (2n - 1) \end{aligned}$$

This is the rotational twin of $s_{nth} = u + \frac{1}{2} a (2n - 1)$ for translational motion. It gives the angle turned through during one specific second, not the total angle from the start.

Key points:

- 1) The formula gives the angular displacement during the n th second, not at time n . For example, for $n = 3$, it gives the angle turned through between $t = 2s$ and $t = 3s$.

- 2) If $\alpha = 0$ (uniform angular velocity), $\theta_{\text{nth}} = \omega_0$ for every second. Each second contributes the same angle, as expected.
- 3) If $\omega_0 = 0$ (starting from rest), $\theta_{\text{nth}} = \frac{1}{2}\alpha(2n - 1)$. The angles in successive seconds are in the ratio 1: 3: 5: 7: ..., which is the familiar odd-number sequence.
- 4) The unit of θ_{nth} is radians (not radians per second). Although the formula contains ω_0 (which has units rad/s), the implicit factor is $\omega_0 \times 1\text{s}$, giving radians. Similarly, $\frac{1}{2}\alpha(2n - 1)$ has units $\text{rad/s}^2 \times \text{s} = \text{rad/s}$, but again the implicit $\times 1\text{s}$ gives radians. This is the same dimensional subtlety as in the translational version.

Before the ideas tangle any further, let us straighten them out using a few worked examples.

BINDER Example 1

- (a) Convert the following angular speeds to rad/s:
(i) 3000rpm, (ii) 8rev/s.
- (b) A disc starts from rest and reaches an angular velocity of 40rad/s in 8s with constant angular acceleration. Calculate the angular acceleration, the total angle turned through, and the total number of revolutions made by disc.

Solution

- (a) The solution of each part is as follows:
(i) Using $\omega = \frac{2\pi N}{60}$ and substituting $N = 3000\text{rpm}$:

$$\omega = \frac{2\pi \times 3000}{60} \text{rad/s} = 100\pi \text{rad/s} = 314 \text{rad/s}$$

- (i) Using $\omega = 2\pi f$ and substituting $f = 8\text{rev/s}$:

$$\omega = 2\pi \times 8 \text{rad/s} = 16\pi \text{rad/s} = 50.3 \text{rad/s}$$

- (d) Given: $\omega_0 = 0$ (starts from rest), $\omega = 40\text{rad/s}$, $t = 8\text{s}$.

Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t}$$

Substituting:

$$\alpha = \frac{40 \text{rad/s} - 0}{8\text{s}} = 5 \text{rad/s}^2$$

For the angle, using $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ and substituting:

$$\theta = 0 + \frac{1}{2} \times 5 \text{rad/s}^2 \times (8\text{s})^2 = \frac{1}{2} \times 5 \times 64 = 160 \text{rad}$$

Converting to revolutions: $\frac{160 \text{rad}}{2\pi \text{rad/rev}} = 25.5 \text{rev}$

Making Sense of the Answer: The disc completes about 25.5 revolutions while spinning up from rest to 40rad/s over 8 seconds. This is reasonable: it accelerates gently to a modest angular speed, so a few dozen revolutions is expected. If the answer had been 0.5 revolutions or 5000 revolutions, we would know something had gone wrong.

Think Like a Physicist: Notice how similar this calculation feels to a Chapter 2 problem where a car starts from rest and accelerates uniformly. The structure is identical: find the acceleration from the first equation, then find the displacement from the second. The only difference is the symbols: ω instead of v , θ instead of s , α instead of a . This is the twin language in action, and it will serve you throughout the entire chapter.

BINDER Example 2

A grinding wheel rotates at 3000rpm. A brake is applied, and the wheel decelerates uniformly to rest in 15s.

- (a) Calculate the angular deceleration.
 (a) Calculate the total number of revolutions the wheel makes while coming to rest.

Solution

- (a) First, converting the initial angular speed to rad/s:

$$\omega_0 = \frac{2\pi \times 3000}{60} \text{ rad/s} = 314.2 \text{ rad/s}$$

The wheel comes to rest, so $\omega = 0$.

Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t}$$

Substituting:

$$\alpha = \frac{0 - 314.2 \text{ rad/s}}{15 \text{ s}} = -20.9 \text{ rad/s}^2$$

The negative sign confirms that the wheel is decelerating.

The magnitude of the angular deceleration is 20.9 rad/s^2 .

- (b) Using $\theta = \frac{1}{2}(\omega_0 + \omega)t$ and substituting:

$$\theta = \frac{1}{2}(314.2 + 0) \text{ rad/s} \times 15 \text{ s} = \frac{1}{2} \times 314.2 \times 15 = 2357 \text{ rad}$$

Converting to revolutions: $\frac{2357 \text{ rad}}{2\pi \text{ rad/rev}} = 375 \text{ rev}$

Making Sense of the Answer: The wheel was spinning very fast (3000rpm is 50 turns per second) and took 15 seconds to stop, so a few hundred revolutions during braking is reasonable. The deceleration of about 21 rad/s^2 means the wheel loses 21 rad/s of angular speed every second, which brings 314 rad/s to zero in exactly 15s. Everything checks out.

Think Like a Physicist: Compare this to a car braking from 100 km/h (27.8 m/s) to rest. The structure is identical: initial velocity, zero final velocity, constant deceleration, find the deceleration and the stopping distance. If you can solve the car problem, you can solve the wheel problem, because the equations are the same. Only the symbols differ.

REAL Example 3

A traditional Tanzanian potter's wheel is spun by hand. The potter gives the wheel a push that sets it spinning at 5 rev/s , then begins shaping the clay. Friction gradually slows the wheel at a constant rate. After 20s, the wheel has slowed to 2 rev/s , and the potter must push it again.

- (a) Explain why the wheel does not stop instantly when friction begins acting on it.
 (b) Calculate the angular deceleration of the wheel.
 (c) If the potter does not push again, calculate the total number of revolutions the wheel would make from the initial push until it stops.

Solution

- (a) The wheel has mass distributed at a distance from the rotation axis, so it possesses rotational inertia (a concept we will develop fully later). Because of this inertia, the wheel resists changes to its angular velocity, just as a heavy box on a surface resists changes to its linear velocity. Friction provides a constant decelerating torque, but the wheel's rotational inertia ensures that the angular velocity decreases gradually rather than dropping to zero instantly. The heavier the wheel and the farther its mass is from the axis, the longer it takes to stop.

- (b) Converting to rad/s:

$$\omega_0 = 2\pi \times 5 = 10\pi \text{ rad/s} = 31.42 \text{ rad/s}$$

$$\omega = 2\pi \times 2 = 4\pi \text{ rad/s} = 12.57 \text{ rad/s}$$

Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{12.57\text{rad/s} - 31.42\text{rad/s}}{20\text{s}} = \frac{-18.85\text{rad/s}}{20\text{s}} = -0.943\text{rad/s}^2$$

The angular deceleration of the wheel is 0.943rad/s^2 .

- (c) From the initial push ($\omega_0 = 31.42\text{rad/s}$) to stop ($\omega = 0$), using $\omega^2 = \omega_0^2 + 2\alpha\theta$ and making θ the subject:

$$\theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{0 - (31.42\text{rad/s})^2}{2 \times (-0.943\text{rad/s}^2)} = \frac{-987.2\text{rad}^2/\text{s}^2}{-1.886\text{rad/s}^2} = 523.5\text{rad}$$

Converting to revolutions: $\frac{523.5\text{rad}}{2\pi\text{rad/rev}} = 83.3\text{rev}$

The total number of revolutions is 83.3rev.

Making Sense of the Answer: The wheel makes about 83 revolutions over roughly 33 seconds before stopping. Most of these revolutions occur early, when the wheel is spinning fast: during the first 20 seconds, the wheel makes about 70 revolutions, but during the last 13 seconds, it makes only about 13. This pattern mirrors what we saw in Chapter 2 for a decelerating car: most of the distance is covered early, when the speed is still high.

Think Like a Physicist: Part (c) asks for the total angle from start to stop. Rather than splitting the motion into two stages (first 20 seconds, then the remaining time) and adding the angles separately, we used the equation $\omega^2 = \omega_0^2 + 2\alpha\theta$, which connects the initial state directly to the final state without needing time at all. Whenever a problem gives you initial and final angular velocities plus angular acceleration, but does not ask for time, this third kinematic equation is the most efficient tool.

REAL Example 4

Kipanga watches a ceiling fan after the power is switched off. He counts 15 complete revolutions before the fan stops, and times the process at 12s.

- Calculate the initial angular velocity of the fan in rpm.
- Calculate the angular deceleration.
- How fast was the fan spinning (in rpm) after the first 4s?

Solution

- (a) Total angle: $\theta = 15\text{rev} \times 2\pi\text{rad/rev} = 30\pi\text{rad} = 94.25\text{rad}$

The fan starts at ω_0 and comes to rest ($\omega = 0$) in $t = 12\text{s}$.

Using $\theta = \frac{1}{2}(\omega_0 + \omega)t$ and making ω_0 the subject:

$$\omega_0 = \frac{2\theta}{t} - \omega = \frac{2 \times 94.25\text{rad}}{12\text{s}} - 0 = 15.7\text{rad/s}$$

Converting: $\frac{15.7\text{rad/s}}{2\pi} = 2.5\text{rev/s} = 2.5 \times 60\text{rpm} = 150\text{rpm}$

The initial angular velocity is 150rpm.

- (b) Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 15.7\text{rad/s}}{12\text{s}} = -1.31\text{rad/s}^2$$

The angular deceleration has magnitude 1.31rad/s^2 .

- (c) At $t = 4\text{s}$, using $\omega = \omega_0 + \alpha t$ and substituting:

$$\omega = 15.7\text{rad/s} + (-1.31\text{rad/s}^2) \times 4\text{s} = 15.7\text{rad/s} - 5.24\text{rad/s} = 10.5\text{rad/s}$$

Converting: $\frac{10.5\text{rad/s}}{2\pi} \times 60 = 100\text{rpm}$

After 4 seconds, the fan was spinning at 100rpm.

Making Sense of the Answer: An initial speed of 150rpm (2.5 turns per second) is typical for a ceiling fan on medium setting. After 4 seconds (one-third of the total 12-second stopping time), the fan has lost exactly one-third of its angular speed (50rpm out of 150rpm). This is exactly what constant angular deceleration predicts: the angular velocity decreases uniformly with time, so equal fractions of time produce equal drops in speed.

Think Like a Physicist: We were given two pieces of information (15 revolutions and 12 seconds) and needed to find two unknowns (ω_0 and α). This is a two-equation, two-unknown problem, just like in Chapter 2. The key strategy is to choose the equation that has only one unknown first. Since both θ and t are known and $\omega = 0$, the fourth equation $\theta = \frac{1}{2}(\omega_0 + \omega)t$ contains only ω_0 as unknown. Once ω_0 is found, any other equation gives α . Choosing the right equation first, the one with fewer unknowns, is a powerful strategy in both linear and rotational kinematics.

BINDER Example 5

A wheel starts from rest and accelerates uniformly at 4rad/s^2 .

- Calculate the angle turned through during the 1st, 2nd, 3rd, and 4th seconds.
- Show that these angles are in the ratio 1: 3: 5: 7.
- Calculate the total angle after 4s and verify that it equals the sum of the individual seconds.

Solution

- (a) Using $\theta_{\text{nth}} = \omega_0 + \frac{1}{2}\alpha(2n - 1)$ with $\omega_0 = 0$ and $\alpha = 4\text{rad/s}^2$:

$$\text{1st second (n = 1): } \theta_1 = 0 + \frac{1}{2} \times 4 \times (2 \times 1 - 1) = 2 \times 1 = 2\text{rad}$$

$$\text{2nd second (n = 2): } \theta_2 = 0 + \frac{1}{2} \times 4 \times (2 \times 2 - 1) = 2 \times 3 = 6\text{rad}$$

$$\text{3rd second (n = 3): } \theta_3 = 0 + \frac{1}{2} \times 4 \times (2 \times 3 - 1) = 2 \times 5 = 10\text{rad}$$

$$\text{4th second (n = 4): } \theta_4 = 0 + \frac{1}{2} \times 4 \times (2 \times 4 - 1) = 2 \times 7 = 14\text{rad}$$

The angles are 2, 6, 10, and 14 radians respectively.

- (b) The ratio is 2: 6: 10: 14 = 1: 3: 5: 7. *Shown.*

This is the odd-number ratio, characteristic of uniform acceleration from rest. It holds for any value of α .

- (c) Total from the formula: $\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2} \times 4 \times (4)^2 = 32\text{rad}$

Sum of individual seconds: $2\text{rad} + 6\text{rad} + 10\text{rad} + 14\text{rad} = 32\text{rad}$. *Verified.*

Making Sense of the Answer: The wheel turns only 2rad in the 1st second but 14rad in the 4th second, seven times more. This is because the wheel is faster in the 4th second than in the 1st. Uniform acceleration means the angular velocity increases steadily, so the wheel covers more angle in each successive second. The total of 32rad is about 5.1 revolutions in 4 seconds.

Think Like a Physicist: The odd-number ratio (1: 3: 5: 7) is one of the most elegant results in kinematics, first discovered by Galileo for falling bodies. It applies equally to translational and rotational motion with uniform acceleration from rest. If an examination gives you angles in successive seconds and asks you to prove uniform acceleration, check for this ratio. Conversely, if you know the acceleration is uniform and the body starts from rest, you can predict the ratio without any calculation.

HOT Example 6

A car engine crankshaft accelerates uniformly from 1200rpm to 4800rpm in 6s. A small pulley of radius 0.15m is attached to the crankshaft and rotates with it.

- Calculate the angular acceleration.
- Calculate the number of revolutions made during this acceleration.
- At the instant the crankshaft reaches 4800rpm, calculate the tangential and centripetal accelerations of a point on the rim of the pulley.

Solution

(a) Converting to rad/s:

$$\omega_0 = \frac{2\pi \times 1200}{60} = 40\pi \text{ rad/s} = 125.7 \text{ rad/s}$$

$$\omega = \frac{2\pi \times 4800}{60} = 160\pi \text{ rad/s} = 502.7 \text{ rad/s}$$

Using $\omega = \omega_0 + \alpha t$ and making α the subject:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{502.7 \text{ rad/s} - 125.7 \text{ rad/s}}{6 \text{ s}} = 62.8 \text{ rad/s}^2$$

(b) Using $\theta = \frac{1}{2}(\omega_0 + \omega)t$ and substituting:

$$\theta = \frac{1}{2}(125.7 + 502.7) \text{ rad/s} \times 6 \text{ s} = \frac{1}{2} \times 628.4 \times 6 = 1885 \text{ rad}$$

Converting to revolutions: $\frac{1885 \text{ rad}}{2\pi \text{ rad/rev}} = 300 \text{ rev}$

It made 300 revolutions.

(c) The tangential acceleration changes the speed of the rim point.

Using $a_t = \alpha r$ and substituting:

$$a_t = 62.8 \text{ rad/s}^2 \times 0.15 \text{ m} = 9.42 \text{ m/s}^2$$

The tangential acceleration is 9.42 m/s^2 .

The centripetal acceleration changes the direction of the rim point.

Using $a_c = \omega^2 r$ at $\omega = 502.7 \text{ rad/s}$ and substituting:

$$a_c = (502.7 \text{ rad/s})^2 \times 0.15 \text{ m} = 37906 \text{ m/s}^2$$

The centripetal acceleration is 37906 m/s^2 .

Have you noticed this? The centripetal acceleration is very huge, approximately 3870g!

Making Sense of the Answer: The centripetal acceleration is nearly 4000 times the acceleration due to gravity. This is why engine components must be carefully balanced and manufactured from strong materials: even a tiny mass imbalance at these rotational speeds produces enormous centripetal forces. The tangential acceleration (9.42 m/s^2 , roughly equal to g) is modest by comparison because the engine takes a full 6 seconds to change speed. But the centripetal acceleration is huge because it depends on ω^2 , and ω at 4800rpm is very large.

Think Like a Physicist: At high angular velocities, centripetal acceleration completely dominates over tangential acceleration because $a_c \propto \omega^2$ while $a_t \propto \alpha$, which is independent of ω . This is why high-speed rotating machinery is dangerous and why turbine blades, centrifuge rotors, and jet engine fans are engineered with extreme precision. This problem bridges Chapter 7 (centripetal acceleration) and Chapter 9 (angular acceleration), showing that non-uniform circular motion has both components acting simultaneously.

HOT Example 7

A flywheel initially spinning at 600rpm is brought to rest by a constant braking torque. During the braking, the flywheel makes exactly 400 revolutions.

- Calculate the angular deceleration.
- Calculate the time taken to stop.
- Calculate the angular velocity (in rpm) of the flywheel when it has completed 300 of the 400 revolutions.

Solution

(a) Converting the given quantities:

$$\omega_0 = \frac{2\pi \times 600}{60} = 20\pi \text{ rad/s} = 62.83 \text{ rad/s}$$

$\omega = 0$ (comes to rest)

$$\theta = 400 \text{ rev} \times 2\pi \text{ rad/rev} = 800\pi \text{ rad} = 2513 \text{ rad}$$

Using $\omega^2 = \omega_0^2 + 2\alpha\theta$ and making α the subject:

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta} = \frac{0 - (62.83 \text{ rad/s})^2}{2 \times 2513 \text{ rad}} = -0.786 \text{ rad/s}^2$$

The angular deceleration is 0.786 rad/s^2 .

(b) Using $\omega = \omega_0 + \alpha t$ and making t the subject:

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 62.83 \text{ rad/s}}{-0.786 \text{ rad/s}^2} = 79.9 \text{ s}$$

The flywheel takes approximately 80 seconds (1 minute 20 seconds) to stop.

(c) After 300 revolutions: $\theta_{300} = 300 \text{ rev} \times 2\pi \text{ rad/rev} = 600\pi \text{ rad} = 1885 \text{ rad}$

Using $\omega^2 = \omega_0^2 + 2\alpha\theta$ and substituting:

$$\omega^2 = (62.83 \text{ rad/s})^2 + 2 \times (-0.786 \text{ rad/s}^2) \times 1885 \text{ rad} = 985 \text{ rad}^2/\text{s}^2$$

$$\omega = \sqrt{985 \text{ rad}^2/\text{s}^2} = 31.4 \text{ rad/s}$$

Converting: $\frac{31.4 \text{ rad/s}}{2\pi} \times 60 = 300 \text{ rpm}$

The angular velocity is 300 rpm.

Making Sense of the Answer: After 300 out of 400 revolutions (75% of the total angle), the flywheel still has 31.4 rad/s out of an initial 62.83 rad/s, which is 50% of the original speed. This reveals an important insight: the flywheel has covered 75% of its total angle but lost only 50% of its speed. This makes perfect sense because a faster-spinning wheel covers more angle per second, so most of the 400 revolutions are completed while the speed is still high, and the last 100 revolutions take disproportionately long as the wheel crawls to a stop.

Think Like a Physicist: Part (c) is a “mid-journey” question: what is the velocity after a certain fraction of the total displacement? This type of question appears frequently in both linear and rotational kinematics. The third equation ($\omega^2 = \omega_0^2 + 2\alpha\theta$) is the ideal tool because it directly connects velocity to displacement without involving time. Notice how cleanly the answer 300 rpm emerges: after 300 of 400 revolutions, the speed is exactly half the original 600 rpm. Verifying such clean patterns is a powerful way to check your work during an examination.

Having given the worked examples their full say, we can now welcome the next subtopic and see what fresh ideas it brings along.

CENTRE OF MASS

We have described how a thrown spanner tumbles through the air, rotating about some internal point while that point itself follows a smooth parabolic path. A thrown boomerang does the same. So does a spinning baton tossed by a cheerleader. In every case, there is *one special point in the body that moves as though the entire mass of the body were concentrated there and all external forces acted on it*. That point is the **centre of mass**.

The centre of mass is the “spokesperson” for the whole body. When we said “the ball was projected at 20 m/s” in Chapter 6 (Projectile Motion), we were really saying “the centre of mass of the ball was projected at 20 m/s.” When we wrote $F = ma$ in Chapter 3, we were really describing the acceleration of the centre of mass. Every time we treated an object as a particle in Chapters 2 through 8, we were unknowingly tracking its centre of mass. Now we make this explicit.

Why Centre of Mass Matters for Rotation

When a rigid body is free to move (not constrained by a pivot or axle), it naturally rotates about its centre of mass. Throw a ruler in the air and watch it spin: it rotates about a point near the middle, not about one end. This happens because the centre of mass is the point about which the mass is, in a sense, “balanced.”

When a body is pivoted at a fixed point (such as a door on its hinges or a wheel on its axle), the rotation axis is determined by the pivot, and the centre of mass orbits around that pivot. But when no pivot is imposed, the body chooses its own axis, and that axis passes through the centre of mass.

Understanding where the centre of mass is located is therefore essential before we can calculate moment of inertia, because moment of inertia depends on the distance of each particle from the rotation axis, and the most natural rotation axis passes through the centre of mass.

Deriving the Centre of Mass: Two Particles

Let us begin with the simplest possible case: two particles on a line. Consider two particles of masses m_1 and m_2 placed on a light, rigid rod at positions x_1 and x_2 along the x-axis. We want to find the **balance point**, the single point at which we could support the rod and have it remain perfectly horizontal.

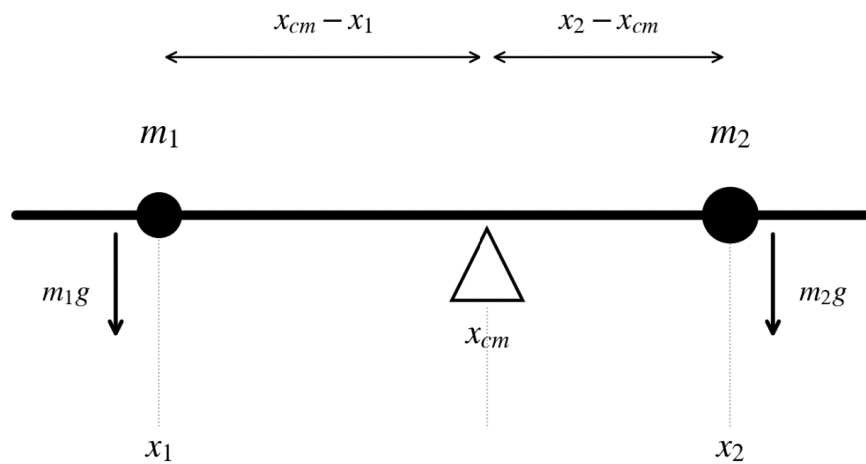


Figure: Two particles of masses m_1 and m_2 on a light rigid rod, balanced at the centre of mass position x_{cm} . The weight m_1g acts downward at x_1 and the weight m_2g acts downward at x_2 . For balance, the clockwise turning effect $m_1g(x_{cm} - x_1)$ must equal the anticlockwise turning effect $m_2g(x_2 - x_{cm})$.

Suppose the balance point is at position x_{cm} . For the rod to balance, the turning effect (about x_{cm}) of the weight on one side must equal the turning effect of the weight on the other side. Since the weight of each particle is $m_i g$ and the perpendicular distance from the balance point is $|x_i - x_{cm}|$, the balance condition is:

$$m_1 g(x_{cm} - x_1) = m_2 g(x_2 - x_{cm})$$

The factor g cancels from both sides (which is why the balance point does not depend on the strength of gravity):

$$m_1(x_{cm} - x_1) = m_2(x_2 - x_{cm})$$

Expanding:

$$m_1 x_{cm} - m_1 x_1 = m_2 x_2 - m_2 x_{cm}$$

Collecting terms with x_{cm} on the left:

$$m_1 x_{cm} + m_2 x_{cm} = m_1 x_1 + m_2 x_2$$

$$(m_1 + m_2)x_{cm} = m_1 x_1 + m_2 x_2$$

Making x_{cm} the subject:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

This is the centre of mass of a two-particle system. It is a **weighted average** of the positions, where each position is weighted by its mass. The centre of mass lies closer to the heavier particle, which makes physical sense: a heavier particle has more “pull” on the balance point.

Notice that the derivation required only the balance of turning effects. The cancellation of g means that the centre of mass is a purely geometric-and-mass property of the system. It does not depend on gravity at all, even though we used gravity to motivate it. The same formula applies in deep space, where there is no gravity, because centre of mass can also be derived from Newton’s laws applied to internal forces (the result is the same).

Generalising to Many Particles

The derivation extends naturally to any number of particles. For n particles on a line, requiring the net turning effect about x_{cm} to be zero gives:

$$\sum_{i=1}^n m_i (x_i - x_{cm}) = 0$$

Expanding:

$$\sum m_i x_i - x_{cm} \sum m_i = 0$$

Making x_{cm} the subject:

$$x_{cm} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}$$

For particles distributed in two or three dimensions, we apply the same formula independently in each direction:

$$x_{cm} = \frac{\sum m_i x_i}{M} \quad y_{cm} = \frac{\sum m_i y_i}{M} \quad z_{cm} = \frac{\sum m_i z_i}{M}$$

where $M = \sum m_i$ is the total mass of the system. Each coordinate of the centre of mass is calculated independently of the others.

Centre of Mass for Continuous Bodies

For a continuous body (a solid rod, a disc, a sphere), we cannot sum over individual particles because there are infinitely many of them. Instead, we imagine dividing the body into infinitesimally small elements, each of mass dm , at position x . The sum becomes an integral:

$$x_{cm} = \frac{\int x dm}{M} = \frac{1}{M} \int x dm$$

where $M = \int dm$ is the total mass of the body. Similarly for y_{cm} and z_{cm} .

To evaluate this integral, we express dm in terms of the body’s geometry. For a uniform body (constant density), the mass element is proportional to the length, area, or volume element:

For a rod: $dm = \lambda dx$, where $\lambda = \frac{M}{L}$ is the mass per unit length.

For a plate: $dm = \sigma dA$, where $\sigma = \frac{M}{A}$ is the mass per unit area.

For a solid: $dm = \rho dV$, where $\rho = \frac{M}{V}$ is the mass per unit volume.

Derivation for a uniform rod:

Consider a uniform rod of mass M and length L , lying along the x -axis from $x = 0$ to $x = L$. The linear mass density is $\lambda = \frac{M}{L}$. A small element at position x has mass $dm = \lambda dx$.

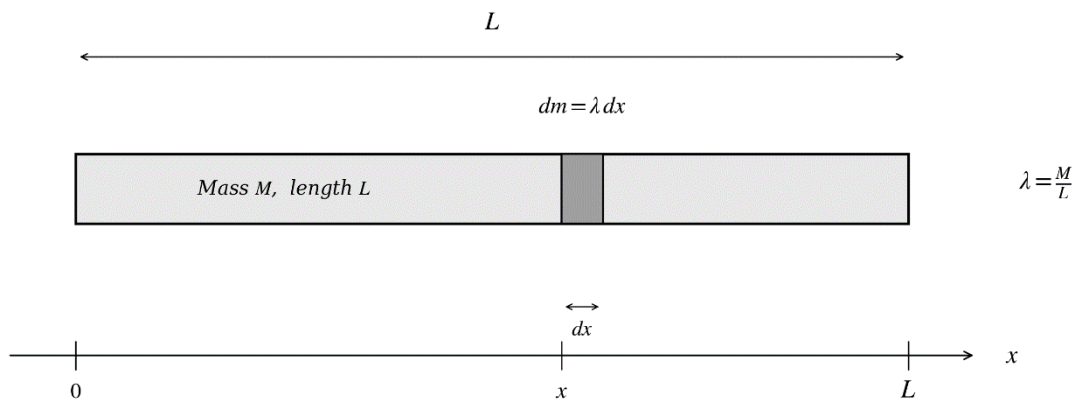


Figure: A uniform rod of mass M and length L , lying along the x -axis from $x = 0$ to $x = L$. A small element of width dx at position x has mass $dm = \lambda dx$, where $\lambda = \frac{M}{L}$ is the linear mass density. Integrating the contribution of all such elements gives the centre of mass of the rod.

Substituting into the integral:

$$x_{\text{cm}} = \frac{1}{M} \int_0^L x \lambda dx = \frac{\lambda}{M} \int_0^L x dx$$

$$= \frac{\lambda}{M} \left[\frac{x^2}{2} \right]_0^L = \frac{\lambda}{M} \times \frac{L^2}{2} = \frac{M/L}{M} \times \frac{L^2}{2} = \frac{L}{2}$$

The centre of mass of a uniform rod is at $\frac{L}{2}$, exactly at the midpoint. This confirms what symmetry tells us, but the integration method is essential for non-uniform bodies and for the moment of inertia derivations in the next subtopic.

The symmetry shortcut

For most standard uniform shapes, we do not need to perform the integration because the centre of mass can be found by **symmetry**:

If a uniform body has a geometric centre (a point, line, or plane of symmetry), the centre of mass lies at that geometric centre.

This powerful result gives us the centre of mass for all the common shapes without any calculation:

Body	Centre of mass location
Uniform rod	Midpoint of the rod
Uniform rectangular plate	Intersection of the diagonals
Uniform circular disc	Centre of the disc
Uniform sphere (solid or hollow)	Centre of the sphere
Uniform cylinder	Midpoint of the axis
Uniform ring (hoop)	Centre of the ring (not on the ring itself!)
Uniform triangular plate	Intersection of the medians ($\frac{1}{3}$ from base to apex)

Notice the ring: its centre of mass is at the geometric centre, which is a point in empty space. There is no material at the centre of mass! This is perfectly valid. The centre of mass is a mathematical point, not necessarily a physical location within the body.

CENTRE OF MASS OF COMPOSITE BODIES

Many real objects are not simple shapes but combinations of simple shapes. To find the centre of mass of a composite body, treat each component as a particle located at its own centre of mass, with mass equal to the component's total mass. Then apply the discrete particle formula.

For example, to find the centre of mass of an L-shaped plate, divide it into two rectangles. Find the centre of mass and mass of each rectangle. Then treat these as two particles and use the weighted average formula.

Similarly, to find the centre of mass of a body with a hole cut out (such as a disc with a smaller disc removed), treat the original body as one positive mass and the removed piece as a negative mass at the position of its own centre of mass.

Before these ideas start wandering off in search of their own centre of mass, let us pin them down with some worked examples.

BINDER Example 8

Three particles of masses 2kg, 3kg, and 5kg are placed at coordinates (0,0), (4,0), and (2,3) respectively, where all distances are in metres.

- Calculate the position of the centre of mass.
- Explain why the centre of mass lies closest to the 5kg particle.

Solution

- Total mass: $M = 2 + 3 + 5 = 10\text{kg}$

Using $x_{\text{cm}} = \frac{\sum m_i x_i}{M}$ and substituting:

$$x_{\text{cm}} = \frac{2\text{kg} \times 0\text{m} + 3\text{kg} \times 4\text{m} + 5\text{kg} \times 2\text{m}}{10\text{kg}} = 2.2\text{m}$$

Using $y_{\text{cm}} = \frac{\sum m_i y_i}{M}$ and substituting:

$$y_{\text{cm}} = \frac{2\text{kg} \times 0\text{m} + 3\text{kg} \times 0\text{m} + 5\text{kg} \times 3\text{m}}{10\text{kg}} = 1.5\text{m}$$

The centre of mass is at (2.2m, 1.5m).

- The centre of mass formula is a weighted average, and the 5kg particle carries the largest weight in the average. Because its mass is 50% of the total, it exerts the strongest pull on the centre of mass. As a result, the point (2.2,1.5) lies closest to the 5kg particle at (2,3) and farthest from the lightest particle at (0,0).

Making Sense of the Answer: The centre of mass (2.2,1.5) lies inside the triangle formed by the three particles. If it had fallen outside this triangle, we would know an error had been made, because the weighted average of positions must always lie within the region spanned by the masses.

Think Like a Physicist: In two dimensions, we apply the one-dimensional formula separately in each direction. The x-coordinate of the centre of mass depends only on the x-coordinates and masses, and similarly for y. The two calculations are completely independent, which makes the problem simple to set up and easy to check.

Now that the arithmetic feels comfortable, let us see how the centre of mass behaves in the real world, where things are rarely sitting politely on coordinate axes.

REAL Example 9

A see-saw is 4m long and has a pivot at its centre. Kipanga (50kg) sits at one end, and Kipute (35kg) sits at the other end.

- Explain why the see-saw tilts toward Kipanga.
- Where should Kipute sit (measured from the pivot) so that the see-saw balances?

Solution

- (a) The see-saw tilts toward Kipanga because the centre of mass of the system (Kipanga + Kipute) does not lie above the pivot. Since Kipanga is heavier than Kipute and both sit at equal distances from the pivot, the centre of mass lies on Kipanga's side. Because gravity acts through the centre of mass, the system rotates until the centre of mass is as low as possible, which means Kipanga's side goes down.
- (b) For the see-saw to balance, the centre of mass of the two (Kipanga and Kipute) must lie directly above the pivot. Taking the pivot as the origin, Kipanga sits at $x_1 = -2\text{m}$ (one end) and Kipute sits at $x_2 = d$ (unknown distance on the other side).

Setting $x_{\text{cm}} = 0$ (centre of mass at the pivot):

$$0 = \frac{50\text{kg} \times (-2\text{m}) + 35\text{kg} \times d}{50\text{kg} + 35\text{kg}}$$

The denominator is non-zero, so the numerator must be zero:

$$\begin{aligned} 50 \times (-2) + 35d &= 0 \\ -100 + 35d &= 0 \end{aligned}$$

Making d the subject:

$$d = \frac{100}{35}\text{m} = 2.86\text{m}$$

Kipute should sit 2.86m from the pivot to balance the see-saw.

Making Sense of the Answer: Kipute (35kg) is lighter than Kipanga (50kg), so she must sit farther from the pivot to compensate. Indeed, $2.86\text{m} > 2.0\text{m}$: the lighter person needs a larger distance to produce the same turning effect as the heavier person. This is the same principle used in beam balances and traditional market scales in Tanzania, where a smaller weight can balance a larger one by being placed farther from the pivot.

Think Like a Physicist: This is the familiar balance condition $m_1d_1 = m_2d_2$ from O-level. The centre of mass approach gives the same answer but is more general, because it works for any number of objects in any arrangement, not just two objects on a straight beam. The balance condition is simply the special case of "centre of mass lies at the pivot."

REAL Example 10

A 12kg uniform rectangular table top measures 2.0m by 1.0m. A 4kg book is placed at one corner.

- (a) Explain why the centre of mass of the table top alone is at its geometric centre.
- (b) Calculate the position of the centre of mass of the system (table + book).

Solution

- (a) The table top is uniform, meaning its mass is evenly distributed across its area. Because of this even distribution, the table top has two lines of symmetry (one along each pair of parallel edges). The centre of mass must lie on both lines of symmetry simultaneously, and the only point satisfying this is the intersection of the two lines: the geometric centre of the rectangle. Therefore the centre of mass of the table top alone is at its centre, at (1.0m, 0.5m).
- (b) Setting up coordinates with one corner at the origin: the table's centre of mass is at (1.0m, 0.5m) with mass 12kg. The book is at the corner (0,0) with mass 4kg.

Total mass: $M = 12 + 4 = 16\text{kg}$

$$\begin{aligned} x_{\text{cm}} &= \frac{12\text{kg} \times 1.0\text{m} + 4\text{kg} \times 0\text{m}}{16\text{kg}} = \frac{12\text{kgm}}{16\text{kg}} = 0.75\text{m} \\ y_{\text{cm}} &= \frac{12\text{kg} \times 0.5\text{m} + 4\text{kg} \times 0\text{m}}{16\text{kg}} = \frac{6\text{kgm}}{16\text{kg}} = 0.375\text{m} \end{aligned}$$

The centre of mass of the system is at (0.75m, 0.375m).

Making Sense of the Answer: Without the book, the centre of mass would be at (1.0,0.5). Adding the 4kg book at the corner (0,0) pulls the centre of mass toward that corner. The shift is modest because the book (4kg) is much lighter than the table (12kg). If the book were much heavier (say 100kg), the centre of mass would be pulled almost all the way to the corner.

Think Like a Physicist: We treated the entire 12kg table as a single particle at its centre of mass, and the 4kg book as a particle at its corner. This “collapse everything to its centre of mass” strategy is the power of this concept: no matter how complex a body is, once you know where its centre of mass is, you can replace it with a single particle for purposes of finding the system’s centre of mass. This simplification will prove invaluable when we deal with composite bodies.

The theory has been polite so far, dealing with well-behaved particles and tidy rectangles. Time to give it a proper workout.

HOT Example 11

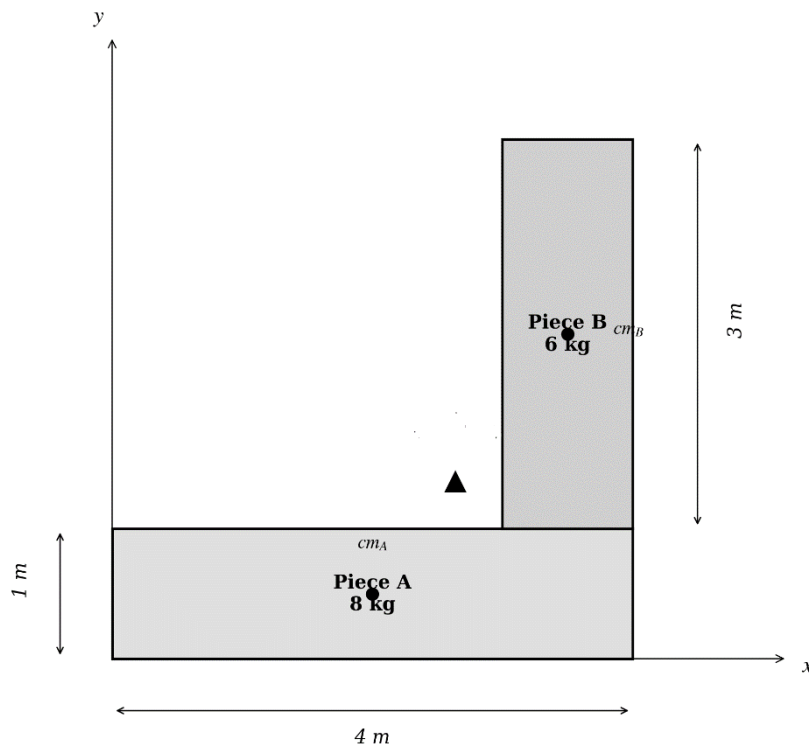
A uniform L-shaped plate is made by joining two uniform rectangular pieces:

Piece A: 4m long, 1m wide, mass 8kg (horizontal, from $x = 0$ to $x = 4$ m, from $y = 0$ to $y = 1$ m)

Piece B: 1m wide, 3m tall, mass 6kg (vertical, from $x = 3$ m to $x = 4$ m, from $y = 1$ m to $y = 4$ m)

Calculate the position of the centre of mass of the L-shaped plate.

Solution



The strategy is to find the centre of mass of each piece separately, then combine them as if they were two particles.

Centre of mass of Piece A (by symmetry of a rectangle): (2.0m, 0.5m), mass = 8kg

Centre of mass of Piece B (by symmetry of a rectangle): (3.5m, 2.5m), mass = 6kg

Total mass: $M = 8 + 6 = 14$ kg

Using $x_{\text{cm}} = \frac{\sum m_i x_i}{M}$ and substituting:

$$x_{\text{cm}} = \frac{8\text{kg} \times 2.0\text{m} + 6\text{kg} \times 3.5\text{m}}{14\text{kg}} = 2.64\text{m}$$

Using $y_{\text{cm}} = \frac{\sum m_i y_i}{M}$ and substituting:

$$y_{\text{cm}} = \frac{8\text{kg} \times 0.5\text{m} + 6\text{kg} \times 2.5\text{m}}{14\text{kg}} = 1.36\text{m}$$

The centre of mass of the L-shaped plate is at (2.64m, 1.36m).

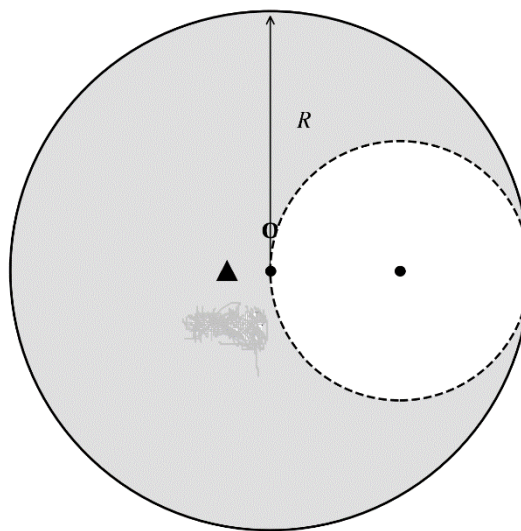
Making Sense of the Answer: The centre of mass at (2.64, 1.36) lies near the junction of the two pieces, which is where the L-shape is “thickest.” Notice that this point falls in the empty concave region of the L, not on the plate itself. This is perfectly valid: the centre of mass is a mathematical point, not necessarily a physical location within the material. A boomerang, a ring, and an L-bracket all have their centres of mass in empty space.

Think Like a Physicist: We solved this without any integration. We simply treated each rectangular piece as a particle at its own centre of mass, then combined them using the discrete particle formula. This “divide and conquer” strategy works for any composite body: break it into simple shapes whose centres of mass you know by symmetry, then combine. The same strategy will reappear when we use the Parallel Axis Theorem to find moments of inertia of composite bodies.

HOT Example 12

A uniform circular disc of mass M and radius R has a smaller circular hole of radius $\frac{R}{2}$ cut out of it. The centre of the hole is at a distance $\frac{R}{2}$ from the centre of the original disc (the hole is tangent to the centre). Find the position of the centre of mass of the remaining piece.

Solution



We treat this as a subtraction problem. The key insight is:

$$(\text{Full disc}) = (\text{Remaining piece}) + (\text{Removed piece})$$

Therefore, the centre of mass of the full disc can be expressed in terms of the centres of mass of the remaining piece and the removed piece. We know the full disc and the removed piece; we solve for the remaining piece.

Setting up coordinates with the centre of the original disc at the origin:

Full disc: mass M , centre of mass at $x = 0$.

The removed piece is a disc of radius $\frac{R}{2}$. Since the disc is uniform, mass is proportional to area. The area of the removed piece is $\pi \left(\frac{R}{2}\right)^2 = \frac{\pi R^2}{4}$, which is $\frac{1}{4}$ of the full disc's area πR^2 . Therefore the removed piece has mass $\frac{M}{4}$, and its centre of mass is at $x = \frac{R}{2}$.

The remaining piece has mass $M - \frac{M}{4} = \frac{3M}{4}$, and its centre of mass is at x_{cm} (what we want to find).

Applying the centre of mass condition for the full disc:

$$M \times 0 = \frac{3M}{4} \times x_{\text{cm}} + \frac{M}{4} \times \frac{R}{2}$$

$$0 = \frac{3M}{4} x_{\text{cm}} + \frac{MR}{8}$$

Making x_{cm} the subject:

$$\frac{3M}{4} x_{\text{cm}} = -\frac{MR}{8}$$

$$x_{\text{cm}} = -\frac{MR}{8} \times \frac{4}{3M} = -\frac{R}{6}$$

The centre of mass of the remaining piece is at a distance $\frac{R}{6}$ from the centre of the original disc, on the opposite side from the hole.

Making Sense of the Answer: The negative sign tells us the centre of mass shifted away from the hole, which is physically correct: removing mass from one side of a symmetric body pulls the centre of mass toward the heavier (opposite) side. The shift is $\frac{R}{6}$, which is relatively small because the removed piece ($\frac{1}{4}$ of the total mass) is not dominant. If we had removed half the disc, the shift would be much larger.

Think Like a Physicist: The subtraction method is elegant: instead of integrating over the complicated remaining shape, we used the fact that (full disc) = (remaining piece) + (removed piece). Since we know the centre of mass of both the full disc and the removed piece, we solve for the unknown algebraically. This technique is extremely useful for any body with a hole, and it completely avoids integration. Remember this trick, because it appears frequently in examinations.

And with that, we now know where every object's centre of mass lives, whether it is sitting politely at the geometric centre of a symmetric body, hiding between two unequal masses on a rod, lurking in the empty space inside an L-shaped bracket, or quietly shifting sideways when someone cuts a hole in a perfectly good disc. The centre of mass has been found, pinned down, and put to work. But knowing where a body rotates is only half the story. The next question is far more interesting: *how stubbornly does it resist being rotated?* Two objects of the same mass can behave very differently when you try to spin them, as Kipanga discovered with his metre ruler in the introduction. The answer lies in a quantity called the **moment of inertia**, and it is waiting for us in the next subtopic with a stack of integrals and a mischievous smile.

MOMENT OF INERTIA: THE ROTATIONAL MASS

If mass is the measure of how stubbornly an object resists being pushed, moment of inertia is the measure of how stubbornly it resists being spun. And as Kipanga discovered with his metre ruler, stubbornness in rotation depends not only on how much mass there is, but on where that mass is sitting relative to the axis of rotation.

This is the heart of the chapter. Everything before this section was preparation: we learned the language of rotation, and we found out where the natural rotation axis is. Now we answer the central question: *given a body and an axis, how do we calculate its resistance to rotation?*

The answer is the **moment of inertia**, often denoted by **I**. It plays exactly the same role in rotational dynamics that mass plays in translational dynamics. In $F = ma$, mass tells you how much force is needed to produce a given acceleration. In the rotational twin $\tau = I\alpha$, moment of inertia tells you how much torque is needed to produce a given angular acceleration. The larger the moment of inertia, the harder it is to spin the object (or to stop it once spinning).

But moment of inertia is richer than mass, and more interesting. The mass of an object is a single fixed number: a 5kg brick is 5kg whether you push it sideways, lift it upward, or throw it across the room. Moment of inertia, however, depends on the axis. The same metre ruler has a small moment of inertia about an axis through its centre (easy to spin) and a large moment of inertia about an axis through one end (hard to spin). The mass has not changed. The distribution of that mass relative to the axis has changed, and that changes everything.

This subtopic is long. There is no way around it. We will derive the moment of inertia for eight standard shapes, each requiring a separate integration. If that sounds terrifying, take comfort in two facts. First, every derivation follows the same recipe. Once you understand the first one, the rest are variations on the same

theme. Second, at the end of this section, we will collect all eight results into a single summary table that you can use as a reference for the rest of the chapter (and the rest of your physics career). The individual derivations teach you *why* the formulas are what they are. The table lets you *use* them efficiently.

Let us begin.

What Moment of Inertia Means Physically

Imagine a collection of small masses $m_1, m_2, m_3, \dots$ arranged at various distances $r_1, r_2, r_3, \dots$ from a fixed rotation axis.

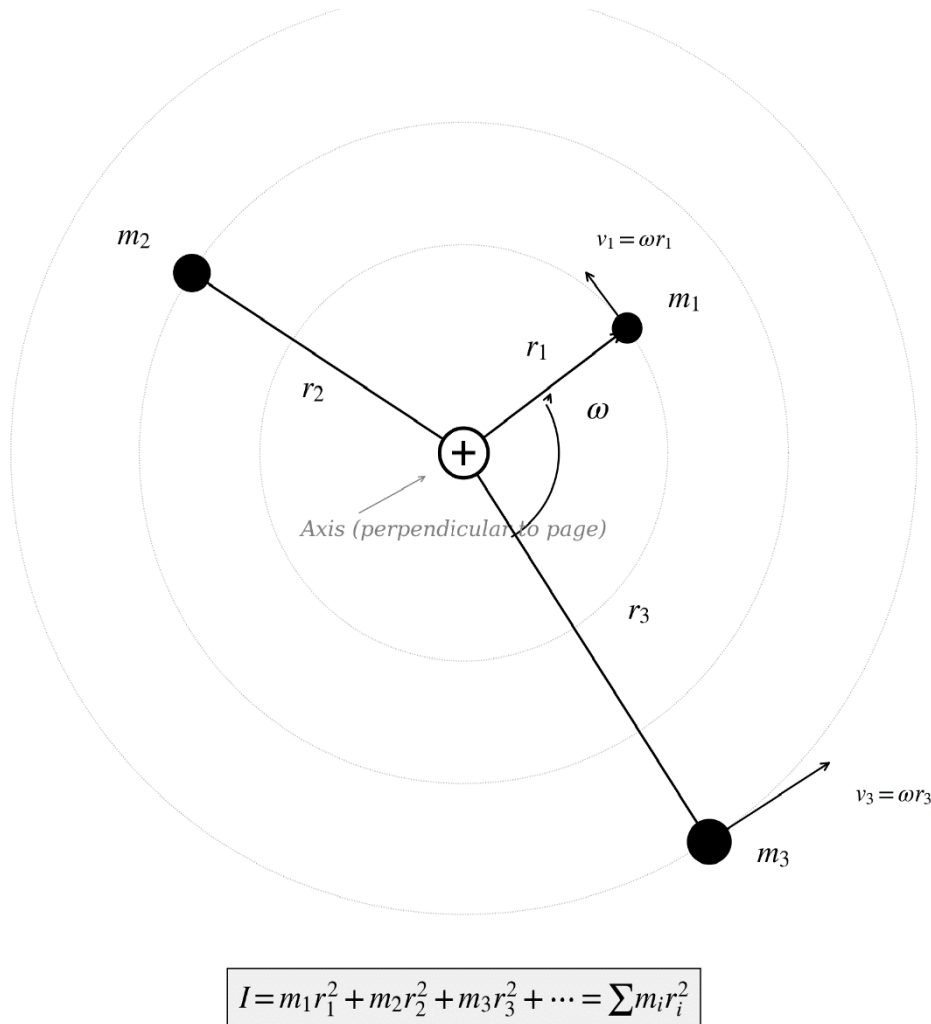


Figure: Three particles of masses m_1, m_2 , and m_3 rotate about a common axis (perpendicular to the page) with angular velocity ω . Each particle moves in a circle of radius r_i . The particle farther from the axis (m_3) has a larger tangential velocity ($v_3 = \omega r_3$) and contributes more to the moment of inertia because I depends on r^2 .

When the system rotates with angular velocity ω , each mass moves in a circle. The linear velocity of the i -th mass is $v_i = \omega r_i$ (from Chapter 7), and its kinetic energy is:

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

The total kinetic energy of the entire system is the sum over all particles:

$$K_{\text{total}} = \sum \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} \omega^2 \sum m_i r_i^2$$

Since ω is the same for all particles in a rigid body (they all rotate together), we can factor it out. What remains inside the sum depends only on the masses and their distances from the axis. We give this sum a name:

$$I = \sum m_i r_i^2$$

This is the **moment of inertia** of the system about the given axis. It has units of $\text{kg} \cdot \text{m}^2$.

With this definition, the total rotational kinetic energy becomes:

$$KE = \frac{1}{2} I \omega^2$$

Compare this with the translational kinetic energy $KE = \frac{1}{2} mv^2$. The structure is identical: mass m is replaced by moment of inertia I , and velocity v is replaced by angular velocity ω . The twin language strikes again.

Why r^2 , not r ?

You might wonder: *why does distance appear squared?* The answer comes from the kinetic energy. Each particle's kinetic energy is $\frac{1}{2} mv^2 = \frac{1}{2} m(\omega r)^2 = \frac{1}{2} m\omega^2 r^2$. The r^2 arises because the linear velocity itself is proportional to r (particles farther from the axis move faster), and kinetic energy depends on velocity squared. As a result, a mass at double the distance contributes **four times** as much to the moment of inertia, not twice as much. This is why the distribution of mass matters so dramatically: even a small mass placed far from the axis can dominate the moment of inertia.

This is exactly what Kipanga experienced. When he spun the ruler from the centre, most of the mass was close to the axis, and the r^2 factors were small. When he spun it from one end, half the mass was far from the axis, and the r^2 factors were large. Same mass, same ruler, very different moment of inertia.

For a continuous body, where the mass is distributed smoothly rather than concentrated in discrete particles, we replace the sum with an integral:

$$I = \int r^2 dm$$

where r is the perpendicular distance from the element dm to the rotation axis. This integral is the workhorse of this section. We will evaluate it for each of the standard shapes in the subsections that follow.

The recipe for every derivation is the same:

- 5) Choose a coordinate system and define the rotation axis.
- 6) Identify a thin element of the body at distance r from the axis.
- 7) Express the mass of the element dm in terms of the body's geometry and density.
- 8) Write $dI = r^2 dm$.
- 9) Integrate over the entire body.
- 10) Simplify and express the result in terms of total mass M and the relevant dimension.

Once you have this recipe in your head, every derivation becomes a cooking exercise: same method, different ingredients, different result. And if cooking integrals sounds unappetizing, remember that by the time you have cooked eight of them, you will be a master chef! Or at least a master physicist, which is arguably more useful.

Before we begin the derivations, let us warm up with a simple discrete example to make sure the concept is solid.

BINDER Example 13

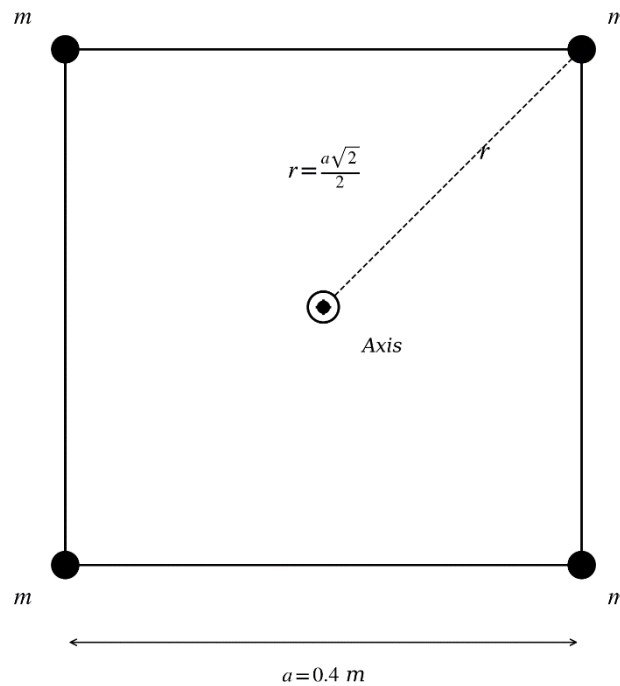
Four particles, each of mass 0.5kg, are connected by light rigid rods to form a square of side 0.4m. The system rotates about an axis through the centre of the square, perpendicular to the plane of the square.

- (a) Calculate the moment of inertia of the system about this axis.
- (b) If the system rotates at 10rad/s, calculate the rotational kinetic energy.
- (c) Explain why moving the masses to the corners of a larger square (say, 0.8m side) would quadruple the moment of inertia, even though the total mass is unchanged.

Solution

- (a) Each particle is at a corner of the square. The distance from the centre of the square to each corner is half the diagonal:

$$r = \frac{\text{diagonal}}{2} = \frac{0.4\sqrt{2}}{2} \text{ m} = 0.283 \text{ m}$$



Since all four particles are at the same distance from the axis:

$$I = \sum m_i r_i^2 = 4 \times 0.5 \text{ kg} \times (0.283 \text{ m})^2 = 4 \times 0.5 \times 0.080 \text{ kgm}^2 = 0.16 \text{ kgm}^2$$

The moment of inertia is 0.16 kgm^2 .

(b) Using $KE = \frac{1}{2} I \omega^2$ and substituting:

$$KE = \frac{1}{2} \times 0.16 \text{ kgm}^2 \times (10 \text{ rad/s})^2 = 8.0 \text{ J}$$

The rotational kinetic energy is 8.0 J .

(c) If the side length doubles from 0.4 m to 0.8 m , the distance from the centre to each corner also doubles (0.283 m to 0.566 m). Because moment of inertia depends on r^2 , doubling r quadruples r^2 , and hence quadruples the moment of inertia. The total mass ($4 \times 0.5 = 2 \text{ kg}$) has not changed at all. This demonstrates the key principle: moment of inertia is not just about how much mass there is, but about how far that mass is from the axis. Spreading mass outward has a dramatic effect because of the r^2 dependence.

Making Sense of the Answer: The moment of inertia of 0.16 kgm^2 is modest because both the masses (0.5 kg each) and the distances from the axis (0.28 m) are small. Doubling the square size to 0.8 m would give $I = 0.64 \text{ kgm}^2$, four times larger, even though not a single gram has been added. This is the essence of the r^2 factor: geometry matters as much as mass.

Think Like a Physicist: This example used discrete masses, so we could compute I by simple addition. For continuous bodies (rods, discs, spheres), the “particles” are infinitely small elements dm , and the sum becomes an integral. But the idea is identical: add up $r^2 dm$ for every bit of mass. The integral is just a sum with better manners.

With the concept established, we now march into the derivations. Take a deep breath, keep your pen steady, and remember: each one follows the same recipe. If you can survive the first, the rest are just different flavours of the same meal.

Uniform Rod: Axis Through the Centre

This is our first derivation using integration, and it is the simplest. If you can follow this one, every derivation that follows uses the same logic with slightly different geometry. Think of it as learning to ride a bicycle: wobble through this one, and the rest become easier.

Consider a uniform rod of mass M and length L . The axis of rotation passes through the centre of the rod, perpendicular to it. We want to find the moment of inertia I about this axis.

Setting up the integral

We place the rod along the x -axis with the centre at the origin, so the rod extends from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$. The axis of rotation is at $x = 0$.

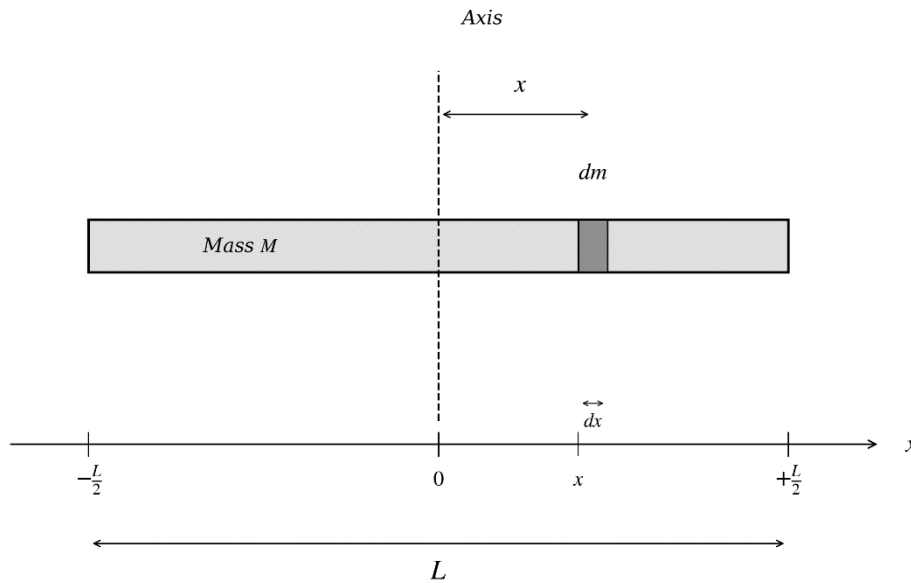


Figure: A uniform rod of mass M and length L with the rotation axis through the centre, perpendicular to the rod. The rod extends from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$. A small element of mass dm at position x is at distance x from the axis.

The rod is uniform, so the linear mass density (mass per unit length) is constant:

$$\lambda = \frac{M}{L}$$

We divide the rod into thin elements. A typical element is a tiny slice at position x , with width dx . Its mass is:

$$dm = \lambda dx = \frac{M}{L} dx$$

The distance of this element from the rotation axis is simply $|x|$, but since we square it in the moment of inertia formula ($r^2 = x^2$), the sign does not matter.

The contribution of this element to the moment of inertia is:

$$dI = x^2 dm = x^2 \times \frac{M}{L} dx$$

Performing the integration

To find the total moment of inertia, we integrate over the entire rod from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$:

$$I = \int_{-L/2}^{+L/2} x^2 \times \frac{M}{L} dx$$

The constant $\frac{M}{L}$ comes out of the integral:

$$I = \frac{M}{L} \int_{-L/2}^{+L/2} x^2 dx$$

The integral of x^2 is $\frac{x^3}{3}$:

$$I = \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{+L/2}$$

Evaluating the limits:

$$I = \frac{M}{L} \times \frac{1}{3} \left(\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right)$$

$$\begin{aligned}
 &= \frac{M}{L} \times \frac{1}{3} \left(\frac{L^3}{8} - \left(-\frac{L^3}{8} \right) \right) \\
 &= \frac{M}{L} \times \frac{1}{3} \times \frac{2L^3}{8} \\
 &= \frac{M}{L} \times \frac{L^3}{12} \\
 &I = \frac{1}{12} ML^2
 \end{aligned}$$

This is the moment of inertia of a uniform rod about an axis through its centre, perpendicular to the rod.

Making Sense of the Result: The result $\frac{1}{12} ML^2$ tells us several things. First, I is proportional to M : a heavier rod is harder to spin, which makes sense. Second, I is proportional to L^2 : a longer rod is much harder to spin (doubling the length quadruples I), because the mass extends farther from the axis and the r^2 factor amplifies this effect. Third, the numerical coefficient $\frac{1}{12}$ is specific to this shape and this axis. Different shapes and different axes give different coefficients, as we will see in the following subsections.

As a quick check: if $L = 0$ (a point mass at the axis), then $I = 0$. That is correct, because a point mass on the axis has zero distance from the axis and contributes nothing to the moment of inertia. If all the mass were concentrated at the ends ($\pm \frac{L}{2}$), we would get $I = 2 \times \frac{M}{2} \left(\frac{L}{2} \right)^2 = \frac{ML^2}{4}$, which is larger than $\frac{ML^2}{12}$. This confirms that a continuous rod, with its mass spread between the ends and the centre, has a smaller moment of inertia than the same mass placed only at the ends. Mass near the axis contributes less, pulling the average down.

Now let us cook the next dish. Same recipe, different ingredient.

Uniform Rod: Axis Through One End

This is the situation Kipanga struggled with in the introduction. The ruler is now pivoted at one end, and we expect a larger moment of inertia because more mass is farther from the axis.

Consider the same uniform rod of mass M and length L , but now the axis passes through one end, perpendicular to the rod.

Setting up the integral

This time we place the rod along the x -axis from $x = 0$ (where the axis is) to $x = L$. Everything else is the same as before.

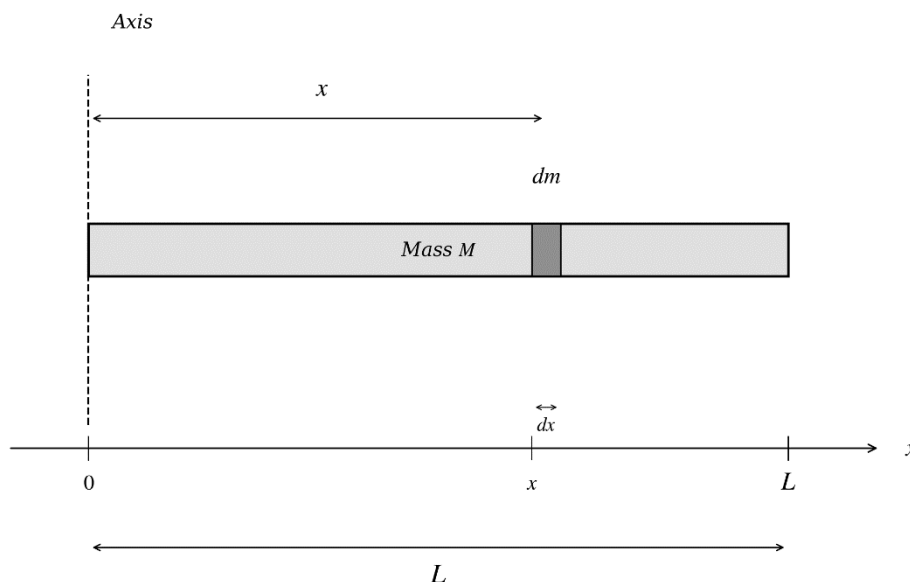


Figure: The same uniform rod, but with the rotation axis through one end, perpendicular to the rod. The rod extends from $x = 0$ to $x = L$. A small element of mass dm at position x is at distance x from the axis. All mass is on one side, resulting in a larger moment of inertia.

The mass element at position x is still $dm = \frac{M}{L} dx$, and its distance from the axis is x .

$$dI = x^2 dm = x^2 \times \frac{M}{L} dx$$

Performing the integration

$$I = \frac{M}{L} \int_0^L x^2 dx = \frac{M}{L} \left[\frac{x^3}{3} \right]_0^L = \frac{M}{L} \times \frac{L^3}{3}$$

$$I = \frac{1}{3} ML^2$$

This is the moment of inertia of a uniform rod about an axis through one end, perpendicular to the rod.

Making Sense of the Result: Comparing the two results: $\frac{1}{3} ML^2$ (end) versus $\frac{1}{12} ML^2$ (centre). The end-axis moment of inertia is exactly four times larger! This makes physical sense: when the axis is at the end, all the mass is on one side, extending all the way to a distance L from the axis. When the axis is at the centre, the farthest mass is only $\frac{L}{2}$ away. Since I depends on r^2 , moving the axis from the centre to the end has a dramatic effect.

This is exactly what Kipanga experienced. The same ruler, the same mass, but four times harder to spin from the end than from the centre. Physics does not care about your muscles; it cares about where the mass sits relative to the axis.

Two rods down. Let us leave the one-dimensional world and step into the plane.

Ring (Thin Hoop): Axis Through the Centre, Perpendicular to the Plane

After two rod derivations, this one will feel like a holiday. It is the simplest moment of inertia of all, and the reason is beautiful: every particle in a thin ring is at exactly the same distance from the axis.

Consider a thin ring (hoop) of mass M and radius R . The axis of rotation passes through the centre of the ring, perpendicular to its plane.

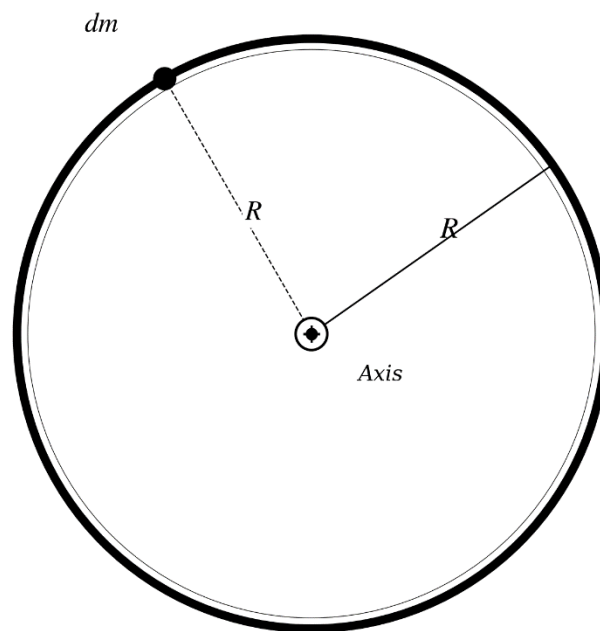


Figure: A thin ring (hoop) of mass M and radius R , with the rotation axis through the centre, perpendicular to the plane. Every element of mass dm lies at the same distance R from the axis.

The derivation

Every tiny piece of mass in the ring sits on the circumference, at exactly the same distance R from the axis. This makes the integral remarkably simple.

Using the general definition $I = \int r^2 dm$, and noting that $r = R$ for every element of mass in the ring, R is a constant and comes out of the integral. Integrating from zero mass to the full mass M :

$$I = \int_0^M r^2 dm = \int_0^M R^2 dm = R^2 \int_0^M dm = R^2 [m]_0^M = R^2 M$$

$$I = MR^2$$

That is it. No complicated integration, no tricky limits, no algebraic gymnastics. The moment of inertia of a ring is simply MR^2 .

When the distance from the axis is the same for every element, the integral collapses to a simple product. The ring is the one shape in this section where the integration is trivial, because the crucial variable (the distance from the axis) does not vary across the body.

Making Sense of the Result: *This is the largest possible moment of inertia for a body of mass M whose farthest point is at distance R from the axis. Why? Because every single gram of mass is at the maximum distance R . No mass is wasted near the axis where it would contribute less. A ring is the “most efficient” shape for resisting rotation: all its mass works at full r^2 leverage.*

This is also why the ring result serves as the reference against which all other shapes are compared. Every other shape of the same mass and outer radius will have $I < MR^2$, because some of its mass will be closer to the axis than R . The disc, the sphere, the cylinder: all have coefficients less than 1 in front of MR^2 . The ring is the champion, with a coefficient of exactly 1.

If every derivation were this friendly, physics students would never complain. Unfortunately, the disc is next, and it requires actual work. But not much.

Disc: Axis Through the Centre, Perpendicular to the Plane

A solid disc differs from a ring in one crucial way: its mass is not all at the rim. Mass is spread from the centre ($r = 0$) to the edge ($r = R$), and the mass near the centre contributes less to the moment of inertia than the mass near the edge. We therefore expect $I < MR^2$, but we need the integration to find out by how much.

Consider a uniform solid disc of mass M , radius R , and negligible thickness. The axis of rotation passes through the centre, perpendicular to the disc.

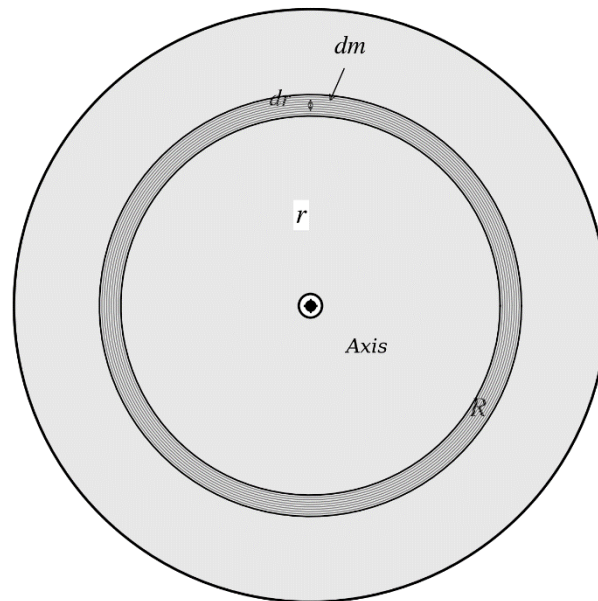


Figure: A uniform solid disc of mass M and radius R , with the rotation axis through the centre, perpendicular to the plane. The disc is divided into concentric thin ring elements. A typical ring at radius r with width dr has area $dA = 2\pi r dr$ and mass $dm = \sigma dA$.

Choosing the element

The key insight is to choose an element that is entirely at one distance from the axis. *What shape satisfies this?* A thin ring! If we slice the disc into concentric thin rings, each ring has all its mass at the same distance from the axis. Each ring can then use the result we just derived ($dI = r^2 dm$) without further thought.

Consider a thin ring of radius r and width dr within the disc. Its area is approximately the circumference times the width:

$$dA = 2\pi r dr$$

Since the disc is uniform, the surface-mass density (mass per unit area) is:

$$\sigma = \frac{M}{\pi R^2}$$

The mass of the thin ring element is:

$$dm = \sigma dA = \frac{M}{\pi R^2} \times 2\pi r dr = \frac{2Mr dr}{R^2}$$

Setting up the integral

The moment of inertia contribution from this thin ring is:

$$dI = r^2 dm = r^2 \times \frac{2Mr dr}{R^2} = \frac{2M}{R^2} r^3 dr$$

Performing the integration

Integrating from the centre ($r = 0$) to the edge ($r = R$):

$$\begin{aligned} I &= \frac{2M}{R^2} \int_0^R r^3 dr \\ &= \frac{2M}{R^2} \left[\frac{r^4}{4} \right]_0^R \\ &= \frac{2M}{R^2} \times \frac{R^4}{4} \\ &= \frac{2MR^2}{4} \\ I &= \frac{1}{2} MR^2 \end{aligned}$$

The moment of inertia of a uniform solid disc about an axis through its centre, perpendicular to its plane, is $\frac{1}{2} MR^2$.

Making Sense of the Result: *The disc has half the moment of inertia of a ring of the same mass and radius. This makes excellent physical sense: the ring has all its mass at distance R , while the disc has mass spread from $r = 0$ to $r = R$. The mass near the centre contributes very little (because of the r^2 factor), so the average contribution is less than R^2 . The factor of $\frac{1}{2}$ tells us that the “effective distance” of the disc’s mass from the axis is $\frac{R}{\sqrt{2}} \approx 0.707R$, which is a reasonable average between 0 and R , weighted by r^2 .*

Another way to see it: if you could magically redistribute the disc’s mass so that all of it sits at the rim (turning it into a ring), the moment of inertia would double. This is why a bicycle wheel (essentially a ring, with most mass in the rim and tyre) has a larger moment of inertia than a solid disc of the same total mass and radius. The wheel is harder to spin up but also harder to slow down, which is why heavy rims help maintain cycling speed on flat roads.

That is four shapes down and four to go. The pattern should feel familiar by now: choose a thin element whose mass is all at the same distance from the axis, write dm , write $dI = r^2 dm$, integrate. If you have been following the recipe, the next four are variations on the same theme, just with slightly different geometry. Let us step into three dimensions.

Solid Sphere: Axis Through the Centre

Welcome to three dimensions. The solid sphere is the most elegant derivation in this section, and the trick that makes it work is the same trick we used for the disc: choose an element whose moment of inertia you already know. For the disc, we sliced it into thin rings. For the sphere, we slice it into thin discs.

Consider a uniform solid sphere of mass M and radius R . The axis of rotation passes through the centre.

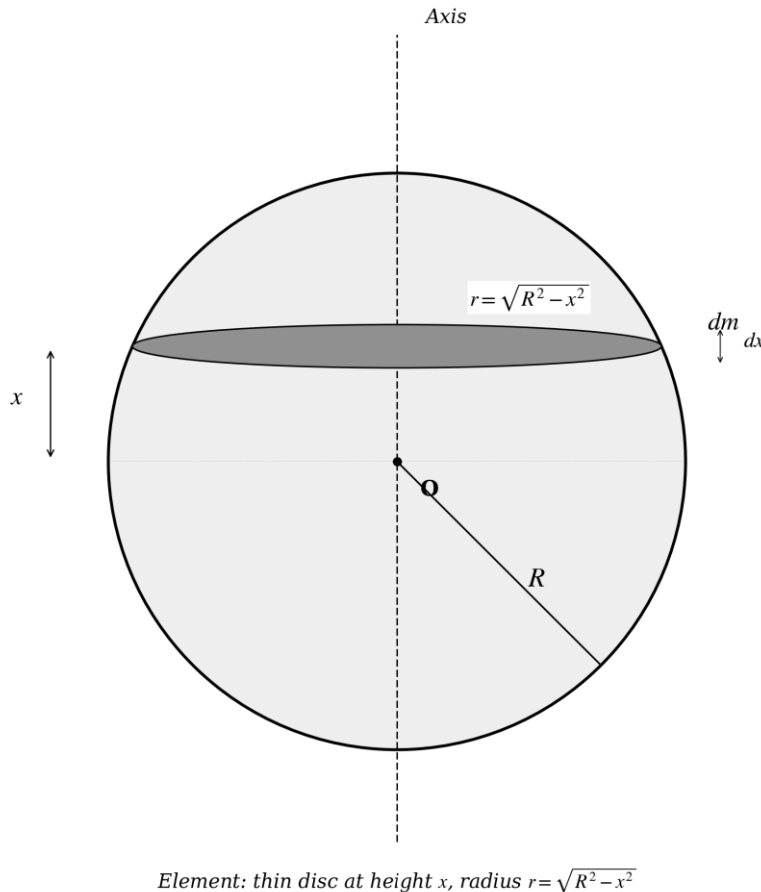


Figure: A uniform solid sphere of mass M and radius R , with the rotation axis passing vertically through the centre. The sphere is sliced into thin disc elements perpendicular to the axis. A typical disc at height x above the centre has radius $r = \sqrt{R^2 - x^2}$ and thickness dx .

Choosing the element

We slice the sphere into thin circular discs perpendicular to the rotation axis. Imagine the axis running vertically through the sphere. Each horizontal slice is a thin disc at height x above the centre, with thickness dx .

At height x , the cross-section of the sphere is a circle. Using the equation of the sphere ($x^2 + r^2 = R^2$), the radius of this disc is:

$$r = \sqrt{R^2 - x^2}$$

The volume of the thin disc element is:

$$dV = \pi r^2 dx = \pi(R^2 - x^2) dx$$

Since the sphere is uniform, the volume-mass density is:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

The mass of the thin disc element is:

$$dm = \rho dV = \frac{3M}{4\pi R^3} \times \pi(R^2 - x^2) dx = \frac{3M}{4R^3} (R^2 - x^2) dx$$

Moment of inertia of each disc element

Each thin disc has mass dm and radius $r = \sqrt{R^2 - x^2}$. From the previous subsection, the moment of inertia of a disc about its central axis is $\frac{1}{2}m_{\text{disc}}r_{\text{disc}}^2$. Therefore:

$$\begin{aligned} dI &= \frac{1}{2}dm \times r^2 = \frac{1}{2} \times \frac{3M}{4R^3} (R^2 - x^2) dx \times (R^2 - x^2) \\ &= \frac{3M}{8R^3} (R^2 - x^2)^2 dx \end{aligned}$$

Performing the integration

We integrate from $x = -R$ (bottom of sphere) to $x = +R$ (top of sphere):

$$I = \frac{3M}{8R^3} \int_{-R}^{+R} (R^2 - x^2)^2 dx$$

Expanding the bracket:

$$\begin{aligned} (R^2 - x^2)^2 &= R^4 - 2R^2x^2 + x^4 \\ I &= \frac{3M}{8R^3} \int_{-R}^{+R} (R^4 - 2R^2x^2 + x^4) dx \end{aligned}$$

Integrating term by term:

$$\begin{aligned} \int_{-R}^{+R} R^4 dx &= R^4 [x]_{-R}^{+R} = R^4 \times 2R = 2R^5 \\ \int_{-R}^{+R} 2R^2x^2 dx &= 2R^2 \left[\frac{x^3}{3} \right]_{-R}^{+R} = 2R^2 \times \frac{2R^3}{3} = \frac{4R^5}{3} \\ \int_{-R}^{+R} x^4 dx &= \left[\frac{x^5}{5} \right]_{-R}^{+R} = \frac{2R^5}{5} \end{aligned}$$

Combining:

$$I = \frac{3M}{8R^3} \left(2R^5 - \frac{4R^5}{3} + \frac{2R^5}{5} \right)$$

Finding a common denominator (15):

$$2R^5 - \frac{4R^5}{3} + \frac{2R^5}{5} = R^5 \left(\frac{30 - 20 + 6}{15} \right) = \frac{16R^5}{15}$$

Therefore:

$$\begin{aligned} I &= \frac{3M}{8R^3} \times \frac{16R^5}{15} = \frac{48MR^2}{120} = \frac{2MR^2}{5} \\ \mathbf{I} &= \mathbf{\frac{2}{5}MR^2} \end{aligned}$$

The moment of inertia of a uniform solid sphere about any axis through its centre is $\frac{2}{5}MR^2$.

Making Sense of the Result: The coefficient $\frac{2}{5}$ is less than $\frac{1}{2}$ (the disc) and much less than 1 (the ring). This is because the sphere has mass distributed in three dimensions, and a significant fraction of that mass is close to the axis. Think of it this way: in a sphere, only the “equatorial belt” of mass is at the full distance R from the axis. Mass near the poles is very close to the axis and contributes almost nothing to I . The three-dimensional spreading of mass reduces the effective r^2 contribution compared to a flat disc.

Notice that the result says “any axis through the centre.” Because a sphere is perfectly symmetric, every axis through the centre gives the same moment of inertia. A rod has different values for different axes; a sphere does not. Symmetry is a beautiful thing.

That derivation was the longest so far, but the strategy was clean: slice the sphere into discs, use the disc formula, and add them up. The hollow sphere is next, and it is actually shorter because a hollow sphere is just a surface with no interior to worry about.

Hollow Sphere (Thin Spherical Shell): Axis Through the Centre

A thin spherical shell is the three-dimensional cousin of the ring. Just as a ring has all its mass at one distance from the centre (in a plane), a spherical shell has all its mass at one distance from the centre (in space). However, unlike the ring, the mass on a spherical shell is not all at the same distance from the rotation axis, because the axis passes through only two points on the shell (the poles), while the equator is at the maximum distance R .

Consider a thin uniform spherical shell of mass M and radius R . The axis of rotation passes through the centre.

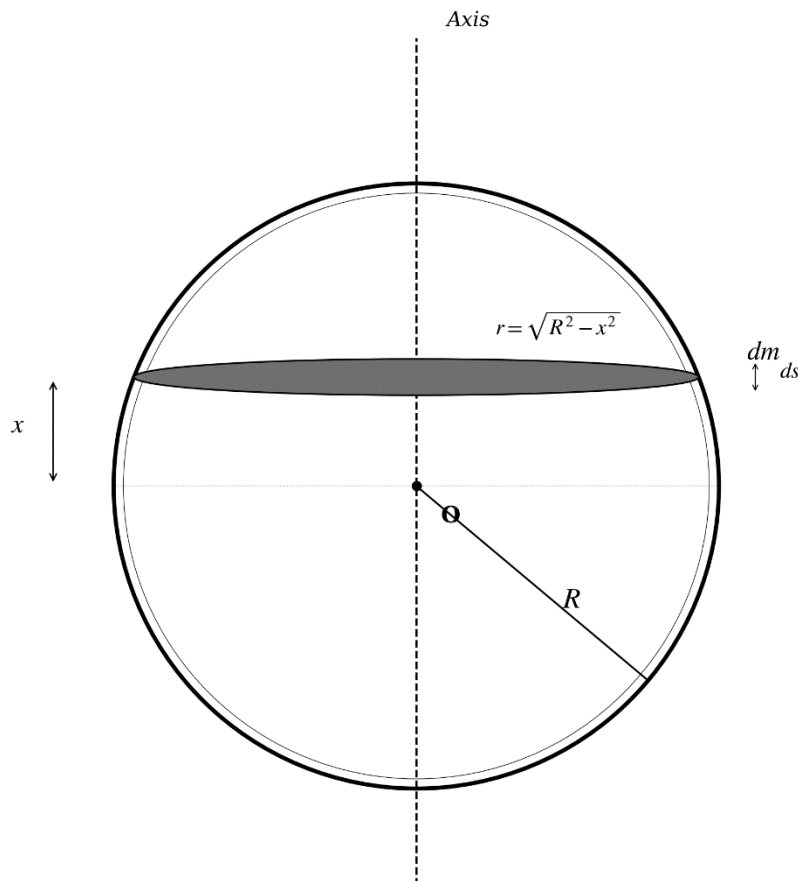


Figure: A thin uniform spherical shell of mass M and radius R , with the rotation axis passing vertically through the centre. The shell is sliced into thin ring elements at height x . Each ring has radius $r = \sqrt{R^2 - x^2}$ and surface width ds . The area of each ring element is $dA = 2\pi R dx$.

Choosing the element

We slice the shell into thin rings by cutting it with planes perpendicular to the rotation axis. Each ring is at height x above the centre, has radius $r = \sqrt{R^2 - x^2}$, and has width ds measured along the surface of the sphere.

The width along the surface is not dx but $ds = R d\theta$, where θ is the polar angle measured from the top of the sphere. However, it is simpler to work with x as the variable. From the geometry: $x = R \cos \theta$, so $dx = -R \sin \theta d\theta$. The circumference of the ring at height x is $2\pi r = 2\pi \sqrt{R^2 - x^2}$ and its width along the surface is $ds = \frac{R dx}{\sqrt{R^2 - x^2}}$.

The area of the ring element is:

$$dA = 2\pi\sqrt{R^2 - x^2} \times \frac{R dx}{\sqrt{R^2 - x^2}} = 2\pi R dx$$

This is a remarkably clean result: the area of each ring element depends only on dx , not on x . Every thin horizontal strip of a sphere has the same area!

The surface-mass density is:

$$\sigma = \frac{M}{4\pi R^2}$$

The mass of the ring element is:

$$dm = \sigma dA = \frac{M}{4\pi R^2} \times 2\pi R dx = \frac{M}{2R} dx$$

Setting up the integral

Each ring element has mass dm and is at distance $r = \sqrt{R^2 - x^2}$ from the rotation axis. Since the ring is thin (all its mass at the same distance from the axis), its moment of inertia is:

$$dI = r^2 dm = (R^2 - x^2) \times \frac{M}{2R} dx$$

Performing the integration

$$\begin{aligned} I &= \frac{M}{2R} \int_{-R}^{+R} (R^2 - x^2) dx \\ &= \frac{M}{2R} \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{+R} \\ &= \frac{M}{2R} \left(\left(R^3 - \frac{R^3}{3} \right) - \left(-R^3 + \frac{R^3}{3} \right) \right) \\ &= \frac{M}{2R} \left(\frac{2R^3}{3} + \frac{2R^3}{3} \right) \\ &= \frac{M}{2R} \times \frac{4R^3}{3} \\ &= \frac{4MR^2}{6} = \frac{2MR^2}{3} \\ \mathbf{I} &= \mathbf{\frac{2}{3}MR^2} \end{aligned}$$

The moment of inertia of a thin uniform spherical shell about any axis through its centre is $\frac{2}{3}MR^2$.

Making Sense of the Result: The hollow sphere $\left(\frac{2}{3}\right)$ has a larger coefficient than the solid sphere $\left(\frac{2}{5}\right)$. This should not be surprising: the hollow sphere has all its mass at the surface (distance R from the centre), while the solid sphere has mass distributed throughout the interior, including regions close to the axis. Mass near the axis contributes little to I , so the solid sphere's "wasted" interior mass pulls its coefficient down.

Comparing all the round shapes so far: ring (1), hollow sphere $\left(\frac{2}{3}\right)$, disc $\left(\frac{1}{2}\right)$ solid sphere $\left(\frac{2}{5}\right)$. The trend is clear: the more mass is pushed outward (toward the maximum distance R), the larger the coefficient. The ring is the extreme case (all mass at R), and the solid sphere has the most mass near the axis (filling the entire interior).

As with the solid sphere, the result is the same for any axis through the centre, because the spherical shell is symmetric about every such axis.

Six down, two to go. The cylinders are next, and if you have survived the spheres, the cylinders will feel like a gentle stroll. They are essentially stretched versions of the disc and the ring.

Solid Cylinder: Axis Along the Cylinder

If you survived the sphere, this one will make you smile. A solid cylinder rotating about its own axis is mathematically identical to a solid disc. Not similar. Not approximately the same. Identical.

Consider a uniform solid cylinder of mass M , radius R , and length L . The axis of rotation is the central axis of the cylinder (the axis running along its length through the centre of each circular face).

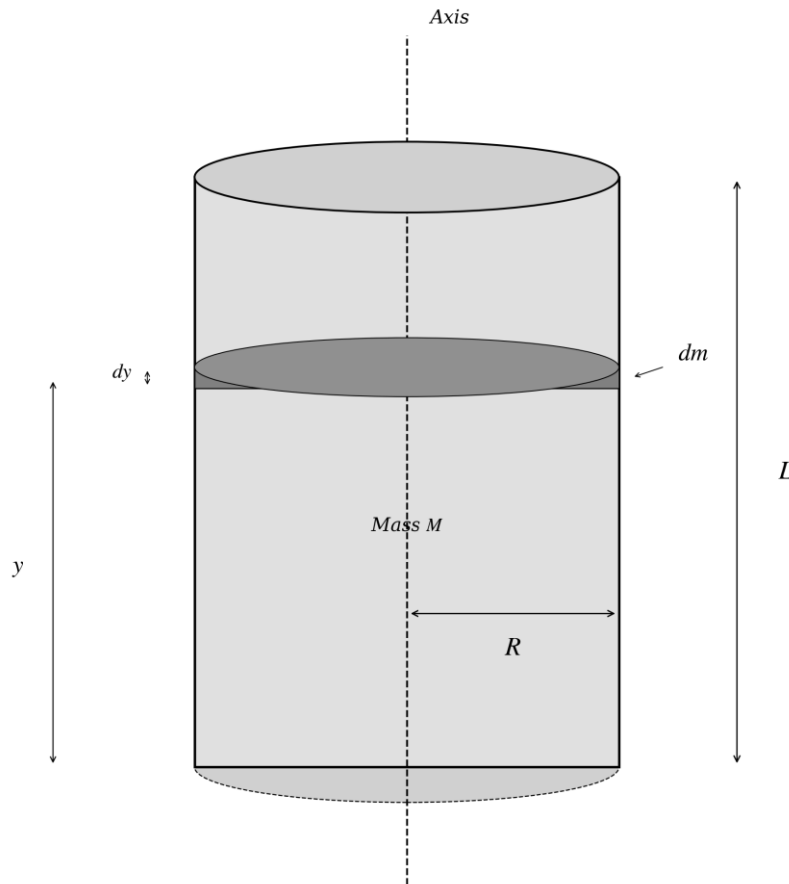


Figure: A uniform solid cylinder of mass M , radius R , and length L , with the rotation axis along the central axis. The cylinder is sliced into thin disc elements at height y . Each disc has the same radius R , so the moment of inertia is identical to that of a disc.

Why it is the same as a disc

Imagine slicing the cylinder into many thin circular discs, each perpendicular to the axis. Every disc has the same radius R . For each disc, the distance of any mass element from the rotation axis depends only on how far that element is from the central axis, which is the radial distance r . The position of the disc along the length of the cylinder (its height) does not affect this distance at all, because the axis runs along the length.

In other words, moving a disc up or down along the axis does not change its distance from the axis. Every disc, regardless of where it sits along the cylinder, contributes to the moment of inertia in exactly the same way as every other disc. Stacking identical discs on top of each other does not change the geometry relative to the axis.

The derivation

Each thin disc at position y along the cylinder has mass $dm = \frac{M}{L} dy$ and moment of inertia (as learned earlier in the ring subsection):

$$dI = \frac{1}{2} dm \times R^2 = \frac{1}{2} \times \frac{M}{L} dy \times R^2$$

Integrating over the full length of the cylinder from $y = 0$ to $y = L$:

$$I = \int_0^L \frac{1}{2} \frac{M}{L} R^2 dy = \frac{MR^2}{2L} \int_0^L dy = \frac{MR^2}{2L} \times [y]_0^L = \frac{MR^2}{2L} \times L$$

$$I = \frac{1}{2} MR^2$$

The moment of inertia of a uniform solid cylinder about its central axis is $\frac{1}{2} MR^2$, exactly the same as a disc.

Making Sense of the Answer: The length L has cancelled completely. A short, fat cylinder and a long, thin cylinder of the same mass and radius have exactly the same moment of inertia about their central axis. This makes perfect sense: stretching a cylinder along the axis does not move any mass farther from the axis or closer to it. Every particle stays at the same radial distance r , regardless of how long the cylinder is. The moment of inertia cares only about the perpendicular distance from the axis, and the length is parallel to the axis.

This is why a solid cylinder and a solid disc give the same formula. A disc is just a very short cylinder, and a cylinder is just a very tall disc. From the axis's point of view, they look identical.

Practical implication: a coin, a hockey puck, a tin can (solid), and a telegraph pole all have $I = \frac{1}{2} MR^2$ about their central axis, provided they are uniform. The only things that matter are M and R .

One more to go. The last derivation in this section. Let us finish strong.

Hollow Cylinder (Cylindrical Shell): Axis Along the Cylinder

Just as the solid cylinder is a stretched disc, the hollow cylinder is a stretched ring. And just as the ring was the simplest derivation in the flat world, the hollow cylinder will be almost as simple in the three-dimensional world.

Consider a thin uniform hollow cylinder (cylindrical shell) of mass M , radius R , and length L . The axis of rotation is the central axis of the cylinder.

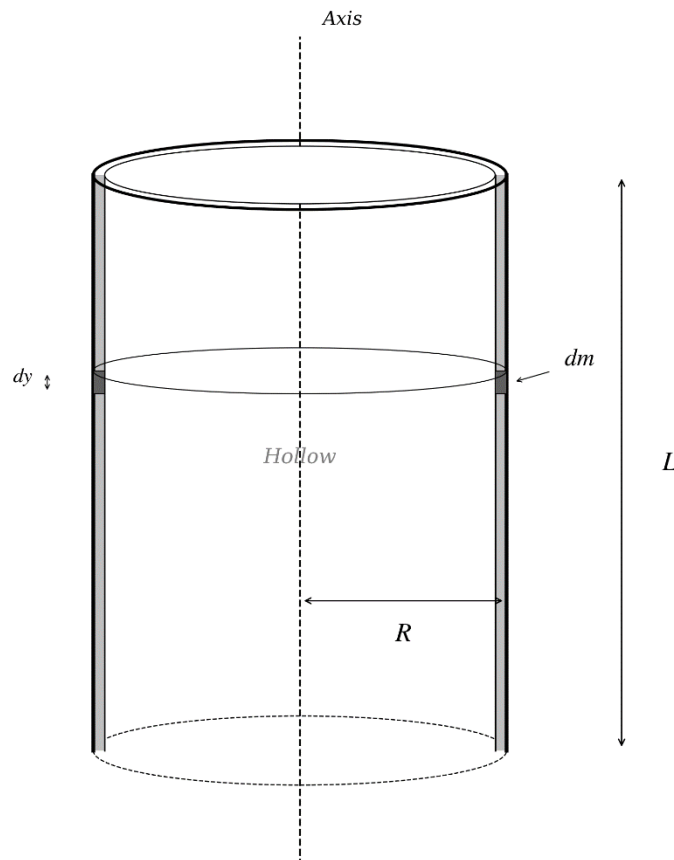


Figure: A thin uniform hollow cylinder (cylindrical shell) of mass M , radius R , and length L , with the rotation axis along the central axis. The cylinder is sliced into thin ring elements at height y . All mass is at distance R from the axis, so the moment of inertia is identical to that of a ring.

Why it is the same as a ring

Every element of mass in the cylindrical shell sits on the curved surface, at distance R from the central axis. Just as with the ring, the distance from the axis is the same for every element. And just as with the solid cylinder, the position along the length (the height y) does not affect this distance, because the axis runs parallel to the length.

The derivation

Each thin ring element at position y has mass $dm = \frac{M}{L} dy$ and moment of inertia (as learned earlier in the ring subsection):

$$dI = dm \times R^2 = \frac{M}{L} dy \times R^2$$

Integrating over the full length from $y = 0$ to $y = L$:

$$I = \int_0^L \frac{MR^2}{L} dy = \frac{MR^2}{L} \int_0^L dy = \frac{MR^2}{L} \times [y]_0^L = \frac{MR^2}{L} \times L$$

$$I = MR^2$$

The moment of inertia of a thin uniform hollow cylinder about its central axis is MR^2 , exactly the same as a ring.

Making Sense of the Answer: *Once again, the length L cancels. A short pipe and a long pipe of the same mass and radius have the same moment of inertia about their central axis. Every element of mass is at distance R from the axis, regardless of where it sits along the length. The hollow cylinder is to the ring what the solid cylinder is to the disc: the same shape, just taller.*

And just as the ring was the “champion” of flat shapes (coefficient = 1), the hollow cylinder is the champion of three-dimensional shapes rotating about their central axis. No other shape of the same mass and outer radius can have a larger moment of inertia, because no other shape has all its mass at the maximum distance from the axis.

Eight derivations complete. The integrals are done. If you are still reading, you have just accomplished something that most textbooks ask you to accept on faith. You did not accept it on faith. You derived every result from scratch, one careful integral at a time. That is not just learning physics; that is *doing* physics.

Now let us collect the fruits of our labour into one beautiful table.

Summary table: Moments of inertia of standard shapes

The table below collects all eight results, plus the discrete particle definition. This is one of the most frequently referenced tables in rotational mechanics. Commit the results to memory, but more importantly, understand *why* each coefficient is what it is.

Shape	Axis	Moment of inertia	Coefficient
Discrete particles	Any	$I = \sum m_i r_i^2$	depends on arrangement
Uniform rod	Through centre, $\perp$ to rod	$\frac{1}{12} ML^2$	$\frac{1}{12} = 0.083$
Uniform rod	Through one end, $\perp$ to rod	$\frac{1}{3} ML^2$	$\frac{1}{3} = 0.333$
Thin ring (hoop)	Through centre, $\perp$ to plane	MR^2	1
Uniform disc	Through centre, $\perp$ to plane	$\frac{1}{2} MR^2$	$\frac{1}{2} = 0.5$
Solid sphere	Any axis through centre	$\frac{2}{5} MR^2$	$\frac{2}{5} = 0.4$
Thin hollow sphere	Any axis through centre	$\frac{2}{3} MR^2$	$\frac{2}{3} = 0.667$
Solid cylinder	Central (long) axis	$\frac{1}{2} MR^2$	$\frac{1}{2} = 0.5$
Thin hollow cylinder	Central (long) axis	MR^2	1

Reading the table:

The coefficient tells you how efficiently the shape uses its mass to resist rotation. A coefficient of 1 (ring, hollow cylinder) means all mass is at the maximum distance from the axis. A coefficient of $\frac{2}{5}$ (solid sphere) means the mass is spread throughout the interior, with much of it close to the axis where it contributes less. The larger the coefficient, the harder the object is to spin for a given mass and size.

Notice the pairings: the solid cylinder has the same formula as the disc ($\frac{1}{2}MR^2$), and the hollow cylinder has the same formula as the ring (MR^2). This is because stretching a shape along the rotation axis does not change the radial distribution of mass.

Also notice: the hollow version of any shape always has a larger coefficient than the solid version, because the hollow version pushes all its mass to the outside. Hollow sphere ($\frac{2}{3}$) beats solid sphere ($\frac{2}{5}$). Hollow cylinder (1) beats solid cylinder ($\frac{1}{2}$).

This table will serve you for the rest of this chapter and beyond. Treat it as a trusted friend.

Now that the derivations are complete and the table is assembled, let us put these results to work. The following worked examples are where the formulas come alive: real objects, real numbers, real insight.

BINDER Example 14

A uniform solid sphere of mass 4kg and radius 0.1m spins about an axis through its centre at 20rad/s.

- Calculate the moment of inertia of the sphere.
- Calculate the rotational kinetic energy.
- A thin hollow sphere of the same mass and radius spins at the same angular velocity. Calculate its moment of inertia and explain why it is larger.

Solution

- From the summary table, for a solid sphere: $I = \frac{2}{5}MR^2$.

Substituting:

$$I = \frac{2}{5} \times 4\text{kg} \times (0.1\text{m})^2 = \frac{2}{5} \times 4 \times 0.01\text{kgm}^2 = 0.016\text{kgm}^2$$

The moment of inertia of the solid sphere is 0.016kgm².

- Using $\text{KE} = \frac{1}{2}I\omega^2$ and substituting:

$$\text{KE} = \frac{1}{2} \times 0.016\text{kgm}^2 \times (20\text{rad/s})^2 = 3.2\text{J}$$

The rotational kinetic energy is 3.2J.

- For a thin hollow sphere of the same mass and radius: $I = \frac{2}{3}MR^2$.

$$I = \frac{2}{3} \times 4\text{kg} \times (0.1\text{m})^2 = 0.0267\text{kgm}^2$$

The moment of inertia of the hollow sphere is 0.0267kgm².

This is larger than the solid sphere (0.016kgm²) because the hollow sphere has all its mass on the surface, at the maximum distance R from the axis of rotation. The solid sphere has mass distributed throughout the interior, including regions close to the axis that contribute little to I. This is justified by the ratio, $\frac{2/3}{2/5} = \frac{5}{3} \approx 1.67$ which implies that the hollow sphere has about 67% more rotational inertia.

Making Sense of the Answer: The solid sphere stores 3.2J of rotational kinetic energy at 20rad/s. The hollow sphere, with its larger moment of inertia, would store $\frac{1}{2} \times 0.0267 \times 400 = 5.33\text{J}$ at the same angular velocity. More inertia means more energy stored at the same spin rate, just as more mass means more kinetic energy at the same translational velocity.

Think Like a Physicist: Whenever a problem asks you to compare solid and hollow versions of the same shape, the answer is always the same: the hollow version has a larger moment of inertia because its mass is farther from the axis on average. This principle applies to spheres, cylinders, and any other shape.

REAL Example 15

A bicycle wheel can be modelled as a thin ring of mass 1.5kg and radius 0.33m (most of the mass is in the rim and tyre). A solid disc of the same mass and radius is attached to a motor for comparison.

- Calculate the moment of inertia of each.
- The motor applies the same constant torque to both objects, starting from rest. Using $\tau = I\alpha$ (which we will derive formally later, but the twin of $F = ma$ should feel natural by now), explain which object reaches 10rad/s first.
- Both objects are spinning at 10rad/s and the motor is switched off. If friction provides the same decelerating torque to both, which one takes longer to stop?

Solution

- Bicycle wheel (ring): $I_{\text{ring}} = MR^2$. Substituting:

$$I_{\text{ring}} = 1.5\text{kg} \times (0.33\text{m})^2 = 0.163\text{kgm}^2$$

The moment of inertia of the bicycle wheel is 0.163kgm².

- Disc: $I_{\text{disc}} = \frac{1}{2}MR^2$. Substituting:

$$I_{\text{disc}} = \frac{1}{2} \times 1.5\text{kg} \times (0.33\text{m})^2 = 0.0817\text{kgm}^2$$

The moment of inertia of the disc is 0.0817kgm².

- From $\tau = I\alpha$, making α the subject: $\alpha = \frac{\tau}{I}$.

Since the same torque τ is applied to both, the object with the smaller I gets the larger angular acceleration α . The disc ($I = 0.0817\text{kgm}^2$) has half the moment of inertia of the ring (0.163kgm²), so the disc has twice the angular acceleration and reaches 10rad/s in half the time.

Hence, the disc reaches 10rad/s first.

- When decelerating, $\alpha = \frac{\tau}{I}$ again. The ring has the larger I , so for the same decelerating torque, the ring has the smaller angular deceleration. With a smaller deceleration, it takes longer to slow from 10rad/s to zero.

Hence, the bicycle wheel takes longer to stop.

Making Sense of the Answer: The ring is harder to start spinning (larger I , smaller acceleration for the same torque) but also harder to stop. This is the essence of rotational inertia: it resists all changes in angular velocity, whether speeding up or slowing down. This is why heavy bicycle wheels maintain their speed well on flat roads but feel sluggish when accelerating from traffic lights, and why racing cyclists prefer lighter wheels (smaller I) for quick acceleration.

Think Like a Physicist: Parts (b) and (c) are the same physics seen from two directions. A large I means small α for a given torque, whether that torque is driving or braking. This is the rotational twin of Newton's second law: a heavy object (m large) accelerates slowly for a given force, and also decelerates slowly for a given braking force. The analogy is exact because $\tau = I\alpha$ has the same mathematical structure as $F = ma$.

BINDER Example 16

A uniform rod of mass 0.3kg and length 0.8m can be pivoted either at its centre or at one end.

- Calculate the moment of inertia about each axis.
- Calculate the ratio of the two results and explain why it is exactly 4.
- A torque of 0.06Nm is applied. Calculate the angular acceleration for each case.

Solution

(a) Axis through the centre: using $I = \frac{1}{12}ML^2$ and substituting:

$$I_{\text{centre}} = \frac{1}{12} \times 0.3\text{kg} \times (0.8\text{m})^2 = 0.016\text{kgm}^2$$

The moment of inertia about the centre is 0.016kgm^2 .

Axis through one end: using $I = \frac{1}{3}ML^2$ and substituting:

$$I_{\text{end}} = \frac{1}{3} \times 0.3\text{kg} \times (0.8\text{m})^2 = 0.064\text{kgm}^2$$

The moment of inertia about one end is 0.064kgm^2 .

(b) The ratio is:

$$\frac{I_{\text{end}}}{I_{\text{centre}}} = \frac{0.064\text{kgm}^2}{0.016\text{kgm}^2} = 4$$

The end-axis moment of inertia is exactly 4 times the centre-axis moment of inertia. This is because $\frac{1/3}{1/12} = \frac{12}{3} = 4$. Physically, when the axis moves from the centre to the end, all the mass that was previously distributed symmetrically on both sides is now entirely on one side, extending to a full length L from the axis instead of $\frac{L}{2}$. Because I depends on r^2 , the mass at the far end (distance L) contributes enormously compared to what it contributed when it was only at distance $\frac{L}{2}$.

(c) Using $\tau = I\alpha$ and making α the subject:

$$\text{Centre axis: } \alpha = \frac{\tau}{I} = \frac{0.06\text{Nm}}{0.016\text{kgm}^2} = 3.75\text{rad/s}^2$$

$$\text{End axis: } \alpha = \frac{\tau}{I} = \frac{0.06\text{Nm}}{0.064\text{kgm}^2} = 0.9375\text{rad/s}^2$$

The angular acceleration about the centre is 3.75rad/s^2 and about one end is 0.9375rad/s^2 .

The same torque produces four times the angular acceleration when the rod is pivoted at the centre than when pivoted at the end. This is exactly what Kipanga felt when he tried to spin his ruler from different points.

Making Sense of the Answer: The ratio of 4 is universal for any uniform rod: it does not depend on the mass or length. A toothpick, a metre ruler, and a telephone pole all show the same factor-of 4 as difference between centre-pivot and end-pivot rotation. The angular acceleration at the centre (3.75rad/s^2) is comfortably large, meaning the rod responds easily to the torque. At the end (0.94rad/s^2), the response is sluggish. Same rod, same torque, four times more stubborn.

Think Like a Physicist: This example is the numerical proof of what we derived earlier in different subsections. The Parallel Axis Theorem will give us a shortcut to get I_{end} from I_{centre} without re-deriving. But for now, the direct comparison confirms both derivations are correct and consistent.

REAL Example 17

Kipanga bets Kipute that he can spin a hard-boiled egg faster than she can spin a raw egg, both starting from rest with the same push. Kipute accepts the bet.

- Explain, using the concept of moment of inertia, why Kipanga wins the bet.
- A hard-boiled egg can be modelled as a uniform solid sphere of mass 60g and radius 2.5cm. Calculate its moment of inertia.
- After both eggs are spinning, Kipute briefly touches the raw egg to stop it, then releases it. She notices it starts spinning again on its own. Explain this observation, and why the hard-boiled egg would behave differently.

Solution

- (a) The hard-boiled egg is rigid: every particle inside rotates together as one body. Its moment of inertia is that of a solid sphere: $I = \frac{2}{5}MR^2$. On another hand, the raw egg contains liquid that does not rotate rigidly with the shell. When the egg is pushed, the shell begins to rotate but the liquid inside lags behind due to its own inertia. The effective moment of inertia of the system (shell + liquid) is larger because the internal liquid resists being dragged into rotation. Therefore, for the same applied torque and the same duration, the hard-boiled egg achieves a higher angular velocity.
- (b) Converting units: $M = 60\text{g} = 0.060\text{kg}$, $R = 2.5\text{cm} = 0.025\text{m}$.

Using $I = \frac{2}{5}MR^2$ and substituting:

$$I = \frac{2}{5} \times 0.060\text{kg} \times (0.025\text{m})^2 = 1.5 \times 10^{-5}\text{kgm}^2$$

The moment of inertia of the hard-boiled egg is $1.5 \times 10^{-5}\text{kgm}^2$.

- (c) When Kipute touches the raw egg, only the shell stops. The liquid inside is still swirling because it has angular momentum that was not removed by the brief touch. When she releases the egg, the still-moving liquid exerts forces on the inside of the shell, dragging it back into rotation. The egg appears to restart on its own. A hard-boiled egg, if stopped and released, stays still because the entire body (shell + solid interior) stops together.

This is actually a classic kitchen test for distinguishing raw eggs from boiled ones.

Making Sense of the Answer: The moment of inertia of the egg ($1.5 \times 10^{-5}\text{kgm}^2$) is tiny, which is why eggs spin so easily. The key physics is not the number but the comparison: rigid bodies spin predictably because I is well-defined, while non-rigid bodies (raw egg, liquid in a container) behave unpredictably because the internal mass does not follow the external rotation immediately.

Think Like a Physicist: This example connects back to the introduction of the chapter, where Mr. Akilikubwa demonstrated the difference between the hard-boiled egg and the raw egg. The moment of inertia formula works perfectly for the rigid egg, but the raw egg breaks the rigid body assumption. Recognising when a formula applies and when it does not is just as important as knowing the formula itself.

HOT Example 18

A compound pulley consists of a solid disc of mass 6kg and radius 0.20m, with a thin ring of mass 2kg and radius 0.20m attached around its rim (both share the same axis).

- (a) Calculate the total moment of inertia of the compound pulley about its central axis.
- (b) If a torque of 0.5Nm is applied to the pulley, calculate the angular acceleration.
- (c) Starting from rest, calculate the angular velocity after 4s and the number of revolutions completed.

Solution

- (a) For the disc: $I_{\text{disc}} = \frac{1}{2}MR^2 = \frac{1}{2} \times 6\text{kg} \times (0.20\text{m})^2 = 0.12\text{kgm}^2$

For the ring: $I_{\text{ring}} = MR^2 = 2\text{kg} \times (0.20\text{m})^2 = 0.08\text{kgm}^2$

Since both share the same axis, the total moment of inertia is the sum:

$$I_{\text{total}} = I_{\text{disc}} + I_{\text{ring}} = 0.12\text{kgm}^2 + 0.08\text{kgm}^2 = 0.20\text{kgm}^2$$

The total moment of inertia of the compound pulley is 0.20kgm^2 .

- (b) Using $\tau = I\alpha$ and making α the subject:

$$\alpha = \frac{\tau}{I} = \frac{0.5\text{Nm}}{0.20\text{kgm}^2} = 2.5\text{rad/s}^2$$

The angular acceleration is 2.5rad/s^2 .

- (c) Using $\omega = \omega_0 + \alpha t$ and substituting:

$$\omega = 0 + 2.5\text{rad/s}^2 \times 4\text{s} = 10\text{rad/s}$$

The angular velocity after 4s is 10rad/s.

Using $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$ and substituting:

$$\theta = 0 + \frac{1}{2} \times 2.5\text{rad/s}^2 \times (4\text{s})^2 = 20\text{rad}$$

$$\text{Number of revolutions: } \frac{20\text{rad}}{2\pi\text{rad/rev}} = 3.18\text{rev}$$

The pulley completes 3.18 revolutions in 4s.

Making Sense of the Answer: The ring (2kg) contributes 0.08kgm^2 to the moment of inertia, while the disc (6kg, three times heavier) contributes only 0.12kgm^2 . The ring contributes 40% of the total I despite being only 25% of the total mass. This is the r^2 effect: the ring has all its mass at the rim, while the disc spreads its mass from centre to rim. Position matters as much as mass.

Think Like a Physicist: Part (a) uses a crucial principle: when multiple bodies share the same rotation axis, their moments of inertia simply add. This is because $I = \int r^2 dm$ is an integral over all the mass, and splitting the mass into components and integrating separately gives the same total.

HOT Example 19

A flywheel is designed to store 10kJ of rotational kinetic energy while spinning at 3000rpm. The flywheel is a uniform solid disc.

- Calculate the required moment of inertia.
- If the flywheel is made of steel with mass 25kg, calculate the required radius.
- A designer proposes replacing the solid disc with a thin ring of the same mass to reduce the size. Calculate the radius of the ring that stores the same energy at the same angular velocity, and explain why it is smaller.

Solution

$$(a) \text{ Converting: } \omega = \frac{2\pi \times 3000}{60} = 100\pi\text{rad/s} = 314.2\text{rad/s}$$

$$\text{KE} = 10\text{kJ} = 10000\text{J}$$

Using $\text{KE} = \frac{1}{2} I \omega^2$ and making I the subject:

$$I = \frac{2\text{KE}}{\omega^2} = \frac{2 \times 10000\text{J}}{(314.2\text{rad/s})^2} = 0.2026\text{kgm}^2$$

The required moment of inertia is 0.203kgm^2 .

- For a solid disc: $I = \frac{1}{2} MR^2$. Making R the subject:

$$R = \sqrt{\frac{2I}{M}} = \sqrt{\frac{2 \times 0.2026\text{kgm}^2}{25\text{kg}}} = 0.127\text{m}$$

The required radius of the solid disc is 0.127m (about 12.7cm).

- For a thin ring: $I = MR^2$. Making R the subject:

$$R = \sqrt{\frac{I}{M}} = \sqrt{\frac{0.2026\text{kgm}^2}{25\text{kg}}} = 0.0900\text{m}$$

The required radius of the ring is 0.0900m (about 9.0cm).

The ring is smaller (9.0cm) than the disc (12.7cm) because the ring concentrates all its mass at radius R , while the disc wastes mass near the centre where it contributes less to I . Since every kilogram of the ring

works at full R^2 leverage, the ring achieves the same moment of inertia with a smaller radius. The ratio of radii is $\frac{0.090}{0.127} = \frac{1}{\sqrt{2}} \approx 0.707$, which follows from $MR_{\text{ring}}^2 = \frac{1}{2}MR_{\text{disc}}^2$, giving $R_{\text{ring}} = \frac{R_{\text{disc}}}{\sqrt{2}}$.

Making Sense of the Answer: A flywheel of radius 12.7cm is roughly the size of a dinner plate, which is a reasonable size for a 25kg steel disc storing 10kJ. The ring alternative is about 30% smaller in radius, which is a significant space saving. This is why real flywheel energy storage systems (used in some hybrid vehicles and power grid stabilisation) often use ring-like designs with heavy rims rather than solid discs.

Think Like a Physicist: This is a design problem: given the energy requirement and the angular velocity, work backward to find the physical dimensions. The key insight from part (c) is that the ring achieves the same I with a smaller radius because it distributes mass more efficiently. In engineering, this means ring-like flywheels are more compact for the same energy storage capacity. The factor of $\frac{1}{\sqrt{2}}$ is exact and universal: it applies to any solid disc versus ring comparison of the same mass and moment of inertia.

HOT Example 20

A uniform circular disc of mass 5kg and radius 0.30m has a circular hole of radius 0.10m cut from it. The centre of the hole is at the centre of the disc (a concentric hole, forming an annular disc).

- Calculate the mass of the removed piece.
- Calculate the moment of inertia of the remaining annular disc about an axis through the centre, perpendicular to the plane.
- Explain why the result is not simply $\frac{1}{2}MR^2$ with the remaining mass substituted for M .

Solution

- The disc is uniform, so mass is proportional to area.

$$\text{Area of full disc: } A_{\text{full}} = \pi(0.30)^2 = 0.09\pi\text{m}^2$$

$$\text{Area of removed piece: } A_{\text{hole}} = \pi(0.10)^2 = 0.01\pi\text{m}^2$$

$$\text{Fraction removed: } \frac{A_{\text{hole}}}{A_{\text{full}}} = \frac{0.01\pi}{0.09\pi} = \frac{1}{9}$$

$$\text{Mass of removed piece: } m_{\text{hole}} = \frac{1}{9} \times 5\text{kg} = 0.556\text{kg}$$

The mass of the removed piece is 0.556kg.

- We use the subtraction method: (full disc) = (remaining annular disc) + (removed small disc).

$$\text{Therefore: } I_{\text{remaining}} = I_{\text{full}} - I_{\text{hole}}$$

For the full disc: using $I = \frac{1}{2}MR^2$ and substituting:

$$I_{\text{full}} = \frac{1}{2} \times 5\text{kg} \times (0.30\text{m})^2 = 0.225\text{kgm}^2$$

For the removed small disc (mass 0.556kg, radius 0.10m): using $I = \frac{1}{2}mR^2$ and substituting:

$$I_{\text{hole}} = \frac{1}{2} \times 0.556\text{kg} \times (0.10\text{m})^2 = 0.00278\text{kgm}^2$$

Therefore:

$$I_{\text{remaining}} = 0.225\text{kgm}^2 - 0.00278\text{kgm}^2 = 0.222\text{kgm}^2$$

The moment of inertia of the annular disc is 0.222kgm^2 .

- If we wrongly used $\frac{1}{2}M_{\text{remaining}}R_{\text{outer}}^2$ with $M_{\text{remaining}} = 5\text{kg} - 0.556\text{kg} = 4.444\text{kg}$:

$$\frac{1}{2} \times 4.444\text{kg} \times (0.30\text{m})^2 = 0.200\text{kgm}^2$$

This is 0.200kgm^2 , which is less than the correct answer 0.222kgm^2 . The wrongly used formula underestimates because it treats the annular disc as if its mass were distributed from $r = 0$ to $r = 0.30\text{m}$, just

like a full disc. But the annular disc has no mass between $r = 0$ and $r = 0.10\text{m}$. Its mass is concentrated in the region $0.10\text{m} \leq r \leq 0.30\text{m}$, farther from the axis on average than a full disc's mass. Because the mass is farther out, the true moment of inertia is larger. The subtraction method correctly accounts for the missing central region; the wrongly used formula does not.

Making Sense of the Answer: The removed piece (0.556kg, radius 0.10m) contributes only 0.00278kgm^2 to the moment of inertia, which is barely 1% of the full disc's 0.225kgm^2 . This is because the removed piece is small and close to the axis, where the r^2 factor keeps contributions tiny. Removing the central region barely changes I , but it removes 11% of the mass. This is why annular (ring-like) shapes are more efficient than solid discs for rotational inertia: the central mass contributes almost nothing to I but adds to the weight.

Think Like a Physicist: The subtraction method ($I_{\text{remaining}} = I_{\text{full}} - I_{\text{removed}}$) is the rotational analogue of the subtraction method we used earlier for centre of mass in (Example 11, the disc with a hole). The principle is the same: the properties of the full body equal the sum of the properties of its parts. This technique appears frequently in examinations, and the most common error is exactly the one exposed in part (c): using the formula for a full disc with the reduced mass. Always use subtraction, never substitution.

And with that, our interesting moment of inertia subtopic is truly complete. The hard work of this chapter is behind us. What follows builds on these results but requires no more plenty of integrations. If you survived subtopic, the rest of the chapter will feel like coasting downhill, which, as we will soon discover in a subsequent subtopic, is something that rolling objects do with surprising elegance.

THE PARALLEL AND PERPENDICULAR AXIS THEOREMS

If the previous subtopic was the hard climb, this section is the view from the top. You have just derived eight moments of inertia the honest way, integral by integral, element by element. Now we introduce two theorems that will save you from ever repeating most of that work. These theorems are shortcuts, powerful ones, and they feel like a reward for surviving the derivations.

The first theorem tells you how to find the moment of inertia about any axis parallel to one you already know. The second tells you how to relate moments of inertia about perpendicular axes in a flat body. Between them, they extend your table of eight results to an almost unlimited number of axes, without a single new integral.

The Parallel Axis Theorem (Steiner's Theorem)

The problem: You know the moment of inertia of a body about an axis through its centre of mass. You now need the moment of inertia about a different axis, parallel to the first, but displaced by some distance d . *Must you redo the entire integration?*

The answer: No. The Parallel Axis Theorem gives you the result in one line.

Statement:

If I_{cm} is the moment of inertia of a body of mass M about an axis through its centre of mass, then the moment of inertia about any parallel axis a distance d away is:

$$I = I_{\text{cm}} + Md^2$$

The new moment of inertia equals the centre-of-mass value plus an extra term Md^2 that accounts for the shift. This extra term is exactly what you would get if the entire mass M were concentrated at the centre of mass and rotated about the new axis at distance d .

Derivation:

Consider a body of mass M . Let the centre of mass be at the origin, and let the CM-axis (passing through the centre of mass) be the original rotation axis. We want the moment of inertia about a new axis, parallel to the CM-axis, passing through a point at distance d from the centre of mass.

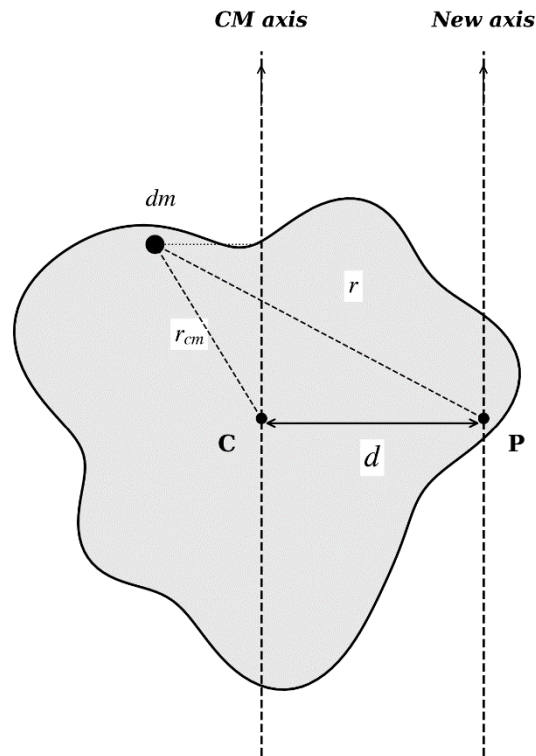


Figure: An irregular body of mass M with two parallel axes, both perpendicular to the page. The CM axis passes through the centre of mass C . The new axis passes through point P , at distance d from C . A mass element dm at position (x, y) is at distance r_{cm} from the CM axis and distance r from the new axis.

Set up coordinates so that the centre of mass is at the origin and the new axis is at position $(d, 0)$ in the x - y plane. For any mass element dm at position (x, y) :

Distance from the cm-axis: $r_{cm}^2 = x^2 + y^2$

Distance from the new axis: $r^2 = (x - d)^2 + y^2 = x^2 - 2xd + d^2 + y^2$

The moment of inertia about the new axis is:

$$I = \int_0^M r^2 dm = \int_0^M (x^2 - 2xd + d^2 + y^2) dm$$

Rearranging:

$$I = \int_0^M (x^2 + y^2) dm - 2d \int_0^M x dm + d^2 \int_0^M dm$$

The first integral is I_{cm} (moment of inertia about the cm-axis).

The second integral is $\int x dm = Mx_{cm} = M \times 0 = 0$, because the origin is at the centre of mass, so $x_{cm} = 0$.

The third integral is simply M .

Therefore:

$$I = I_{cm} + 0 + Md^2$$

$$I = I_{cm} + Md^2$$

The middle term vanishes because we measured from the centre of mass. This is precisely why the theorem works only when one of the two axes passes through the centre of mass. If neither axis passes through the centre of mass, you cannot use the theorem directly; you must first go back to the cm-axis, then shift to the desired axis.

Verification: the rod

Earlier, we derived $I_{cm} = \frac{1}{12}ML^2$ for a rod about its centre. Then we derived $I_{end} = \frac{1}{3}ML^2$ about one end.

The end is at distance $d = \frac{L}{2}$ from the centre. Applying the Parallel Axis Theorem:

$$I_{end} = I_{cm} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1+3}{12}ML^2 = \frac{4}{12}ML^2 = \frac{1}{3}ML^2$$

This matches the result we derived independently. The theorem works perfectly, and it took one line instead of a full integration. If this does not bring a smile to your face after the effort of the previous subtopic, nothing will!

Key points about the Parallel Axis Theorem:

1. One of the two axes **must** pass through the centre of mass. You cannot shift directly from one off-centre axis to another.
2. The shift term Md^2 is always **positive**, so the moment of inertia about any off-centre axis is always larger than I_{cm} . The centre-of-mass axis always gives the **minimum** moment of inertia for any set of parallel axes. This makes physical sense: the centre of mass is the point where mass is most “compactly arranged” relative to the axis.
3. The theorem works for **any** shape, not just symmetric ones. It applies to rods, discs, spheres, irregular blobs, and anything else.

The Perpendicular Axis Theorem

The problem: You have a flat (planar) body lying in the x-y plane. You know the moments of inertia about the x-axis (I_x) and the y-axis (I_y), both lying in the plane of the body. You need the moment of inertia about the z-axis, perpendicular to the body and passing through the intersection of x and y. *Must you set up a new integral?*

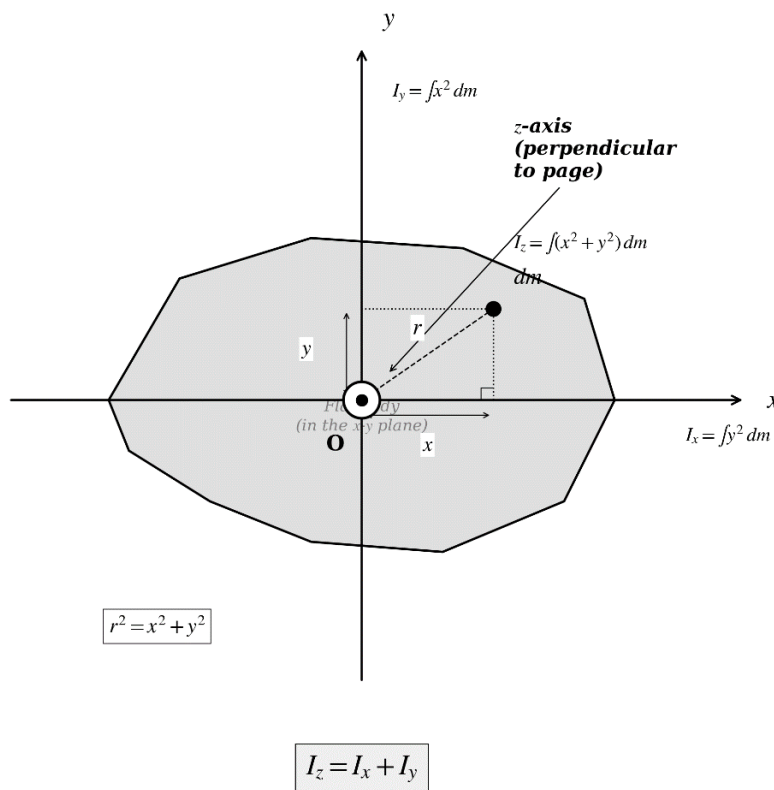
The answer: No. The Perpendicular Axis Theorem gives the result by simple addition.

Statement:

For a planar (flat) body lying in the x-y plane:

$$I_z = I_x + I_y$$

The moment of inertia about the perpendicular axis equals the sum of the moments of inertia about any two mutually perpendicular axes in the plane of the body, all three axes meeting at the same point.



(valid only for flat bodies in the x-y plane)

Figure: A flat body lying in the x-y plane. The x-axis and y-axis lie in the plane of the body; the z-axis is perpendicular to the plane, passing through point O. A mass element dm at position (x, y) is at distance y from the x-axis, distance x from the y-axis, and distance $r = \sqrt{x^2 + y^2}$ from the z-axis. Since $r^2 = x^2 + y^2$, the Perpendicular Axis Theorem gives $I_z = I_x + I_y$.

Derivation:

For any mass element dm at position (x, y) in the plane, the perpendicular distances from each axis are:

From the x-axis (which runs along x): the distance is $|y|$, so $I_x = \int_0^M y^2 dm$

From the y-axis (which runs along y): the distance is $|x|$, so $I_y = \int_0^M x^2 dm$

From the z-axis (perpendicular to the plane): the distance is $r = \sqrt{x^2 + y^2}$, so:

$$I_z = \int_0^M (x^2 + y^2) dm$$

Since $x^2 + y^2$ can be split:

$$I_z = \int_0^M (x^2 + y^2) dm = \int_0^M x^2 dm + \int_0^M y^2 dm = I_y + I_x$$

$$I_z = I_x + I_y$$

The derivation works because $r^2 = x^2 + y^2$ splits cleanly into two independent terms. This is a consequence of Pythagoras's theorem applied to the position of dm in the plane.

Critical restriction: This theorem applies **only** to flat (planar) bodies, such as discs, rings, flat plates, and thin sheets. It does **not** apply to three-dimensional objects like spheres, solid cylinders, or cubes. The reason is that the derivation requires every mass element to lie in the x-y plane (i.e., $z = 0$ for all dm). If the body has thickness in the z-direction, the theorem fails because the distance from the x-axis is not simply $|y|$ but $\sqrt{y^2 + z^2}$.

Applications: new results without integration

The Perpendicular Axis Theorem is most powerful when combined with the symmetry of the body. Here are three important results that fall out instantly:

1) Disc about a diameter:

We know $I_z = \frac{1}{2}MR^2$ for a disc (axis through centre, perpendicular to plane). By symmetry, every diameter of a disc is equivalent, so $I_x = I_y$. Applying the theorem:

$$\frac{1}{2}MR^2 = I_x + I_x = 2I_x$$

Making I_x the subject:

$$I_{\text{diameter}} = \frac{1}{4}MR^2$$

A disc is twice as easy to spin about a diameter as about the perpendicular central axis.

2) Ring about a diameter:

We know $I_z = MR^2$ for a ring. By symmetry, $I_x = I_y$:

$$MR^2 = 2I_x$$

$$I_{\text{diameter}} = \frac{1}{2}MR^2$$

3) Rectangular plate about a perpendicular axis:

A uniform rectangular plate of mass M and sides a and b , centred at the origin. Viewed along the x-axis, the plate looks like a rod of length b ; viewed along the y-axis, it looks like a rod of length a . Therefore:

$$I_x = \frac{1}{12}Mb^2 \quad I_y = \frac{1}{12}Ma^2$$

$$I_z = I_x + I_y = \frac{1}{12}M(a^2 + b^2)$$

For a square plate ($a = b$): $I_z = \frac{1}{6}Ma^2$.

None of these results required a single new integral. Each was obtained by combining a known result with the Perpendicular Axis Theorem and, where applicable, symmetry. This is the theorem's gift: it turns what would be difficult integrations into one-line algebra.

Summary of new results from the Perpendicular Axis Theorem:

Body	Axis	Moment of inertia
Disc	Through centre, along a diameter	$\frac{1}{4}MR^2$
Ring	Through centre, along a diameter	$\frac{1}{2}MR^2$
Rectangular plate	Through centre, $\perp$ to plate	$\frac{1}{12}M(a^2 + b^2)$
Square plate	Through centre, $\perp$ to plate	$\frac{1}{6}Ma^2$

With two theorems in hand, we can now handle almost any axis without returning to the integral. Time to put them to work.

BINDER Example 21

A uniform circular disc of mass 2kg and radius 0.20m lies flat on a table.

- Using the Perpendicular Axis Theorem, calculate the moment of inertia of the disc about an axis along one of its diameters.
- Using the Parallel Axis Theorem on the result from (a), calculate the moment of inertia about an axis tangent to the disc and lying in the plane of the disc.

Solution

- The moment of inertia about the perpendicular central axis is:

$$I_z = \frac{1}{2}MR^2 = \frac{1}{2} \times 2\text{kg} \times (0.20\text{m})^2 = 0.04\text{kgm}^2$$

By symmetry, every diameter of the disc is equivalent, so $I_x = I_y$. Applying the Perpendicular Axis Theorem $I_z = I_x + I_y = 2I_x$ and making I_x the subject:

$$I_{\text{diameter}} = \frac{I_z}{2} = \frac{0.04}{2} = 0.02\text{kgm}^2$$

The moment of inertia about a diameter is 0.02kgm^2 .

- A tangent axis in the plane of the disc is parallel to a diameter, at distance $d = R = 0.20\text{m}$ from the centre. Applying the Parallel Axis Theorem:

$$I_{\text{tangent}} = I_{\text{diameter}} + Md^2 = 0.02\text{kgm}^2 + 2\text{kg} \times (0.20\text{m})^2 = 0.10\text{kgm}^2$$

The moment of inertia about the tangent axis in the plane is 0.10kgm^2 .

Making Sense of the Answer: Starting from one known value ($I_z = 0.04\text{kgm}^2$), we obtained two new values ($I_{\text{diameter}} = 0.02$ and $I_{\text{tangent}} = 0.10$) without any integration. The tangent value is five times the diameter value ($0.10/0.02 = 5$) because moving the axis from the centre to the rim adds a substantial Md^2 term. Notice the progression: diameter (0.02) < perpendicular centre (0.04) < tangent (0.10). The perpendicular axis has a larger I than the diameter because it “sees” the full radial spread of the disc, while the diameter axis has mass distributed on both sides, some of it very close.

Think Like a Physicist: This example uses both theorems in sequence: Perpendicular first (to get the diameter result from the perpendicular result), then Parallel (to shift from the centre to the rim). This “Perpendicular then Parallel” combination is one of the most common examination techniques. Master it, and you can find the moment of inertia of any flat body about almost any axis from a single known result.

BINDER Example 22

A uniform disc of mass 3kg and radius 0.25m is mounted on an axle that passes through a point on its rim, perpendicular to the plane of the disc.

- Calculate the moment of inertia of the disc about an axis through its centre, perpendicular to its plane.
- Using the Parallel Axis Theorem, calculate the moment of inertia about the rim axis.

Solution

- From the summary table: $I_{\text{cm}} = \frac{1}{2}MR^2$.

$$\text{Substituting: } I_{\text{cm}} = \frac{1}{2} \times 3\text{kg} \times (0.25\text{m})^2 = \frac{1}{2} \times 3 \times 0.0625 = 0.09375\text{kgm}^2$$

The moment of inertia about the centre is 0.09375kgm^2 .

- The rim axis is parallel to the centre axis, at distance $d = R = 0.25\text{m}$ from the centre.

Using $I = I_{\text{cm}} + Md^2$ and substituting:

$$I = 0.09375\text{kgm}^2 + 3\text{kg} \times (0.25\text{m})^2 = 0.281\text{kgm}^2$$

The moment of inertia about the rim axis is 0.281kgm^2 .

Making Sense of the Answer: The rim-axis value (0.281) is exactly three times the centre-axis value (0.09375). This is not a coincidence: $I_{\text{rim}} = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$, which is indeed three times $\frac{1}{2}MR^2$.

Moving the axis from the centre to the rim always triples the moment of inertia for a disc. The Md^2 term (0.1875) is twice as large as I_{cm} (0.09375), showing that the shift contributes more than the original value.

Think Like a Physicist: The Parallel Axis Theorem always increases the moment of inertia when you move the axis away from the centre of mass. The increase is Md^2 , which can be substantial. For the disc, shifting by one radius doubles the shift term relative to the original. This is why doors have hinges at the edge, not at the centre: the larger moment of inertia about the hinge axis makes the door swing in a stable, controlled way rather than fluttering like a butterfly.

BINDER Example 23

A thin uniform ring of mass 0.8kg and radius 0.15m lies flat on a table. Calculate the moment of inertia about:

- An axis through the centre, perpendicular to the plane of the ring.
- An axis along a diameter of the ring (an axis in the plane of the ring through the centre).
- An axis tangent to the ring in its plane (touching the ring at one point, lying in the plane).

Solution

- From the summary table: $I_z = MR^2$.

$$\text{Substituting: } I_z = 0.8\text{kg} \times (0.15\text{m})^2 = 0.018\text{kgm}^2$$

The moment of inertia about the perpendicular axis through the centre is 0.018kgm^2 .

- Using the Perpendicular Axis Theorem: $I_z = I_x + I_y$.

By symmetry, $I_x = I_y$ (every diameter of a ring is equivalent). Therefore:

$$I_z = 2I_x$$

Making I_x the subject:

$$I_x = \frac{I_z}{2} = \frac{0.018\text{kgm}^2}{2} = 0.009\text{kgm}^2$$

The moment of inertia about a diameter is 0.009kgm^2 .

- The tangent axis is parallel to a diameter, at distance $d = R = 0.15\text{m}$ from the centre. Using the Parallel Axis Theorem (shifting from the diameter axis):

$$I_{\text{tangent}} = I_{\text{diameter}} + Md^2 = 0.009\text{kgm}^2 + 0.8\text{kg} \times (0.15\text{m})^2 = 0.027\text{kgm}^2$$

The moment of inertia about the tangent axis is 0.027kgm^2 .

Making Sense of the Answer: The three results form a satisfying progression: perpendicular centre (0.018), diameter (0.009), tangent (0.027). The diameter value is the smallest because the mass is distributed both above and below the axis, with much of it close to the axis. The tangent value is the largest because the axis is shifted to the rim, putting all the mass on one side. Notice how we used both theorems in sequence: Perpendicular Axis Theorem first (to get the diameter), then Parallel Axis Theorem (to shift from diameter to tangent). This combination is a common examination technique.

Think Like a Physicist: This example demonstrates the power of combining the two theorems. Starting from just one known result ($I_z = MR^2$), we obtained two more (I_{diameter} and I_{tangent}) without any integration. The Perpendicular Axis Theorem moved us from a perpendicular axis to an in-plane axis. The Parallel Axis Theorem then moved us from the centre to the rim. Together, these two theorems can reach almost any axis from any known starting point.

REAL Example 24

A 0.5kg uniform ruler is 1m long. A student pivots the ruler at a point 0.2m from one end.

- Explain why the Parallel Axis Theorem is needed here, rather than using a formula directly from the summary table.
- Calculate the moment of inertia about this pivot.
- If a torque of 0.1Nm is applied, calculate the angular acceleration.

Solution

- The summary table gives the moment of inertia for a rod about two specific axes: through the centre and through one end. The pivot at 0.2m from one end is neither of these. It is an off-centre axis that does not appear in the table. The Parallel Axis Theorem lets us start from the known centre-axis result and shift to any parallel axis, without deriving a new integral.
- The centre of the ruler is at 0.5m from either end. The pivot is at 0.2m from one end, which is $0.5\text{m} - 0.2\text{m} = 0.3\text{m}$ from the centre. So $d = 0.3\text{m}$.

From the summary table: $I_{\text{cm}} = \frac{1}{12}ML^2$.

Substituting: $I_{\text{cm}} = \frac{1}{12} \times 0.5\text{kg} \times (1\text{m})^2 = 0.04167\text{kgm}^2$

Using $I = I_{\text{cm}} + Md^2$ and substituting:

$$I = 0.04167\text{kgm}^2 + 0.5\text{kg} \times (0.3\text{m})^2 = 0.0867\text{kgm}^2$$

The moment of inertia about the pivot is 0.0867kgm^2 .

- Using $\tau = I\alpha$ and making α the subject:

$$\alpha = \frac{\tau}{I} = \frac{0.1\text{Nm}}{0.0867\text{kgm}^2} = 1.15\text{rad/s}^2$$

The angular acceleration is 1.15rad/s^2 .

Making Sense of the Answer: The moment of inertia (0.0867) lies between the centre value (0.0417) and the end value ($\frac{1}{3} \times 0.5 \times 1 = 0.1667$). This is correct: the pivot is between the centre and the end, so the moment of inertia should be between the two extreme values. If it had come out larger than the end value, we would know an error had occurred.

Think Like a Physicist: Part (a) highlights an important point: the summary table covers only specific axes. Any other axis requires the Parallel Axis Theorem. In examinations, the pivot is often placed at an arbitrary point precisely to test whether you know the theorem. The procedure is always the same: find I_{cm} from the table, identify d as the distance from the centre of mass to the new axis, and apply $I = I_{\text{cm}} + Md^2$.

HOT Example 25

A uniform square plate of mass M and side length a lies in the x-y plane, centred at the origin.

- Using the Perpendicular Axis Theorem and the known result for a rod, show that the moment of inertia about an axis through the centre, perpendicular to the plate, is $I = \frac{1}{6}Ma^2$.

- (b) Using the Parallel Axis Theorem, find the moment of inertia about an axis along one edge of the plate (perpendicular to the plate).

Solution

- (a) Consider the square plate lying in the x-y plane with its centre at the origin, sides parallel to the axes. The plate extends from $-\frac{a}{2}$ to $+\frac{a}{2}$ in both x and y.

The moment of inertia about the x-axis (I_x) can be found by noting that, viewed from the side, the plate looks like a uniform rod of length a in the y-direction. Every thin horizontal strip of the plate at height y is at distance $|y|$ from the x-axis. The result is the same as a rod of mass M and length a about its centre:

$$I_x = \frac{1}{12}Ma^2$$

By the same reasoning (the plate looks like a rod of length a in the x-direction when viewed along y):

$$I_y = \frac{1}{12}Ma^2$$

Applying the Perpendicular Axis Theorem:

$$I_z = I_x + I_y = \frac{1}{12}Ma^2 + \frac{1}{12}Ma^2 = \frac{2}{12}Ma^2 = \frac{1}{6}Ma^2$$

The moment of inertia about the perpendicular axis through the centre is $\frac{1}{6}Ma^2$.

- (b) Taking the midpoint of one edge, the distance from the centre is $d = \frac{a}{2}$.

Using $I = I_{\text{cm}} + Md^2$ and substituting:

$$I = \frac{1}{6}Ma^2 + M\left(\frac{a}{2}\right)^2 = \frac{1}{6}Ma^2 + \frac{1}{4}Ma^2 = Ma^2\left(\frac{1}{6} + \frac{1}{4}\right) = Ma^2\left(\frac{2+3}{12}\right) = \frac{5}{12}Ma^2$$

The moment of inertia about the axis through the midpoint of one edge is $\frac{5}{12}Ma^2$.

Making Sense of the Answer: In part (a), the coefficient $\frac{1}{6}$ for the square plate is larger than $\frac{1}{12}$ for a rod but smaller than $\frac{1}{2}$ for a disc. This is reasonable: a square plate has mass spread in two dimensions (x and y), so it contributes to I from both directions. The Perpendicular Axis Theorem captures this by adding the two in-plane contributions. In part (b), shifting the axis from the centre to the edge increases I by $\frac{1}{4}Ma^2$, nearly doubling the value. Rotation about the edge is significantly harder than rotation about the centre.

Think Like a Physicist: Part (a) is a beautiful example of how the Perpendicular Axis Theorem combines with an existing result (the rod formula) to produce a new result (the square plate formula) without any new integration. The argument that “the plate looks like a rod when viewed from the side” is the key insight. This reasoning generalises: the moment of inertia of any uniform rectangular plate of sides a and b about a perpendicular axis through the centre is $I = \frac{1}{12}M(a^2 + b^2)$. The square is the special case where $a = b$.

RADIUS OF GYRATION

This is a one-idea subtopic, and the idea is elegant. After deriving moments of inertia for eight shapes, you may have noticed that every result has the form:

$$I = (\text{some fraction}) \times M \times (\text{some length})^2$$

The fraction and the length depend on the shape and the axis. For a disc, $I = \frac{1}{2}MR^2$. For a rod about its centre, $I = \frac{1}{12}ML^2$. For a solid sphere, $I = \frac{2}{5}MR^2$.

In every case, we can write:

$$I = Mk^2$$

where k is a length that absorbs both the fraction and the characteristic dimension. This length k is called the **radius of gyration**.

By definition: the **radius of gyration k** of a body about a given axis is the distance from the axis at which the entire mass of the body would need to be concentrated (as a single point) to give the same moment of inertia as the actual distributed body.

$$I = Mk^2 \quad \text{therefore} \quad k = \sqrt{\frac{I}{M}}$$

The radius of gyration has units of metres. It is not the distance of any particular particle from the axis; it is the “effective distance” of the mass as a whole.

Physical meaning

Imagine you could take all the mass of a body and compress it into a thin ring of the same mass, centred on the rotation axis. The radius of that ring that would give the same moment of inertia as the original body is the radius of gyration.

For a thin ring: $I = MR^2 = Mk^2$, so $k = R$. The radius of gyration equals the actual radius, because all the mass is already at distance R .

For a solid disc: $I = \frac{1}{2}MR^2 = Mk^2$, so $k = R\sqrt{\frac{1}{2}} = \frac{R}{\sqrt{2}} \approx 0.707R$. The effective distance is about 71% of the actual radius, because some mass is close to the axis.

For a solid sphere: $I = \frac{2}{5}MR^2 = Mk^2$, so $k = R\sqrt{\frac{2}{5}} \approx 0.632R$. Even less, because the three-dimensional spreading puts more mass near the axis.

The radius of gyration compresses the shape information and the axis information into a single number. It tells you, at a glance, how “spread out” the mass is relative to the axis. The larger the radius of gyration, the more spread out the mass, and the harder the body is to rotate.

You have to understand that: shapes that push mass outward (ring, hollow sphere, hollow cylinder) have larger radii of gyration, closer to the actual radius R . Shapes that spread mass throughout the interior (disc, solid sphere, solid cylinder) have smaller radii of gyration.

Why radius of gyration is useful

1. **Quick comparisons:** The radius of gyration immediately tells you which of two bodies of the same mass is harder to rotate. The one with the larger k wins. You do not need to compare the full I values.
2. **Rolling motion:** Later, we will discover that the acceleration of a body rolling down an incline depends on $\frac{k^2}{R^2}$. The radius of gyration appears naturally in the rolling equation, making it the perfect tool for comparing how different shapes roll.
3. **Engineering specifications:** In structural and mechanical engineering, the radius of gyration is used to describe the stiffness of columns and the rotational properties of machine parts. It appears in the Euler buckling formula for columns and in the specifications of flywheels and rotors.

The radius of gyration is not a new physical quantity; it is a convenient repackaging of the moment of inertia. But like all good repackaging, it makes the contents easier to use.

The concept is simple enough. Let us make sure the calculations are equally comfortable.

BINDER Example 26

A uniform solid cylinder has mass 8kg and radius 0.12m. It rotates about its central axis.

- (a) Calculate the moment of inertia.
- (b) Calculate the radius of gyration.
- (c) A thin hollow cylinder of the same mass and radius rotates about the same axis. State its radius of gyration and explain why it is larger.

Solution

- (a) Using $I = \frac{1}{2}MR^2$ and substituting:

$$I = \frac{1}{2} \times 8\text{kg} \times (0.12\text{m})^2 = 0.0576\text{kgm}^2$$

The moment of inertia is 0.0576kgm^2 .

- (b) Using $k = \sqrt{\frac{I}{M}}$ and substituting:

$$k = \sqrt{\frac{0.0576 \text{ kgm}^2}{8 \text{ kg}}} = 0.0849 \text{ m}$$

The radius of gyration is 0.0849m (about 8.5cm).

Alternatively: $k = \frac{R}{\sqrt{2}} = \frac{0.12}{\sqrt{2}} = 0.0849 \text{ m}$. Same answer.

- (c) For the hollow cylinder: $I = MR^2$, so $k = R = 0.12 \text{ m}$.

The hollow cylinder's radius of gyration (0.12m) is larger than the solid cylinder's (0.085m) because all the mass of the hollow cylinder is at the maximum distance R from the axis. The solid cylinder has mass distributed from $r = 0$ to $r = R$, so its effective distance (radius of gyration) is smaller. This is supported by the ratio $\frac{k_{\text{hollow}}}{k_{\text{solid}}} = \frac{R}{R/\sqrt{2}} = \sqrt{2} \approx 1.41$, implying that the hollow cylinder's effective distance is about 41% larger.

Making Sense of the Answer: The solid cylinder's radius of gyration (8.5cm) is about 71% of the actual radius (12cm). This means the "average" mass in the solid cylinder behaves as if it were at 71% of the way from the axis to the rim. The hollow cylinder's k equals R exactly, because all mass is at the rim. These numbers will reappear later when we compare rolling speeds.

Think Like a Physicist: The radius of gyration is a single number that captures how a body's mass is distributed relative to the axis. Two bodies with the same k and the same mass have the same moment of inertia, even if they look completely different. A solid disc and a solid cylinder of the same mass and radius have identical radii of gyration ($R/\sqrt{2}$), which makes sense because they have the same moment of inertia formula.

REAL Example 27

A traditional Tanzanian wooden club (rungu) has a heavy head and a thin handle. The total mass is 0.6kg. When swung about the end of the handle (axis at the grip), the moment of inertia is measured to be 0.15 kgm^2 .

- Calculate the radius of gyration about this axis.
- Explain what the radius of gyration tells us about the mass distribution of the rung.
- If a uniform rod of the same mass and length had the same radius of gyration, what would its length be?

Solution

- (a) Using $k = \sqrt{\frac{I}{M}}$ and substituting:

$$k = \sqrt{\frac{0.15 \text{ kgm}^2}{0.6 \text{ kg}}} = 0.50 \text{ m}$$

The radius of gyration is 0.50m.

- (b) The radius of gyration of 0.50m tells us that the mass of the rung behaves as though it were all concentrated at 0.50m from the grip. Since the rung has a heavy head at the far end and a light handle near the grip, most of the mass is far from the axis. This pushes the effective distance (radius of gyration) toward the far end. If the mass were evenly distributed (like a uniform rod), the radius of gyration would be smaller because more mass would be near the grip.

- (c) For a uniform rod pivoted at one end: $I = \frac{1}{3} ML^2$, so $k = \frac{L}{\sqrt{3}}$.

Setting $k = 0.50 \text{ m}$ and making L the subject:

$$L = k\sqrt{3} = 0.50 \text{ m} \times \sqrt{3} = 0.866 \text{ m}$$

A uniform rod of the same mass and radius of gyration would be 0.866m long. If the actual rung is shorter than this (say 0.7m), it confirms that the mass concentration at the head gives it a larger radius of gyration than a uniform rod of the same length, which is exactly the design intent: a heavy head maximizes the striking force.

Making Sense of the Answer: A radius of gyration of 0.50m for a weapon designed for impact is sensible. The heavy head, placed far from the grip, maximizes the moment of inertia and therefore the angular momentum and KE at impact. This is the same principle behind a hammer, a golf club, and a baseball bat: mass at the far end makes the tool more effective.

Think Like a Physicist: The radius of gyration allows us to compare objects of very different shapes on equal footing. The rungu is nothing like a rod, a disc, or a sphere, yet its radius of gyration tells us exactly how its mass is distributed relative to the axis, in a single number. This is especially useful for irregular bodies where the moment of inertia cannot be calculated from a simple formula but can be measured experimentally. From the measured I and the known M , the radius of gyration follows immediately.

HOT Example 28

Two solid spheres, A and B, have the same mass M but different radii: $R_A = 0.10\text{m}$ and $R_B = 0.20\text{m}$.

- Calculate the ratio of their radii of gyration.
- Both spheres are released from rest at the top of the same inclined plane and roll without slipping. Using the fact that the acceleration of a rolling body is $a = \frac{g \sin \theta}{1 + k^2/R^2}$, determine which sphere reaches the bottom first.
- Explain why the result of (b) is surprising at first glance, and why it makes perfect sense when examined carefully.

Solution

- For a solid sphere: $k = R\sqrt{\frac{2}{5}}$.

$$k_A = 0.10\text{m} \sqrt{\frac{2}{5}} = 0.0632\text{m}$$

$$k_B = 0.20\text{m} \sqrt{\frac{2}{5}} = 0.1265\text{m}$$

$$\frac{k_A}{k_B} = \frac{0.0632}{0.1265} = \frac{1}{2}$$

The ratio of their radii of gyration equals the ratio of their radii, which is $\frac{1}{2}$.

- The acceleration of a rolling body is $a = \frac{g \sin \theta}{1 + k^2/R^2}$.

For any solid sphere: $k^2 = \frac{2}{5}R^2$, so $\frac{k^2}{R^2} = \frac{2}{5}$.

This ratio is the same for sphere A and sphere B, regardless of their radii! Therefore:

$$a = \frac{g \sin \theta}{1 + \frac{2}{5}} = \frac{g \sin \theta}{\frac{7}{5}} = \frac{5g \sin \theta}{7}$$

Both spheres have exactly the same acceleration. They reach the bottom at the same time.

- At first glance, it seems like the larger sphere should reach the bottom first (or latter) because it has a larger moment of inertia (or larger radius). But the rolling acceleration neither depends on I nor on R alone but on $\frac{k^2}{R^2}$, which is a pure number determined by the shape, not the size. For all solid spheres, regardless of radius, this ratio is $\frac{2}{5}$.

Understand: The physical reason is that a larger sphere has more rotational inertia (which slows it down) but also has a larger radius (which means its linear velocity is greater for the same angular velocity). These two effects cancel exactly. The result is that all solid spheres roll down an incline at the same rate, just as all objects fall at the same rate in free fall (regardless of mass). The rolling race depends on shape, not on size or mass.

Making Sense of the Answer: The ratio $\frac{k^2}{R^2} = \frac{2}{5}$ is a property of the shape “solid sphere,” not of any particular sphere. A marble, a bowling ball, and the Earth (if it could roll) would all have the same ratio

and the same rolling acceleration. This is one of the most beautiful results in rotational mechanics, and the radius of gyration is what makes it transparent.

Think Like a Physicist: The ratio $\frac{k^2}{R^2}$ is the key quantity for rolling motion. It is a dimensionless number that depends only on the shape and the axis, not on M or R . Shapes with larger $\frac{k^2}{R^2}$ roll more slowly because a greater fraction of their energy goes into rotation rather than translation. This is why a hollow cylinder ($\frac{k^2}{R^2} = 1$) always loses the rolling race to a solid sphere ($\frac{k^2}{R^2} = \frac{2}{5}$). We will explore this fully later.

That wraps up the radius of gyration: one concept, one formula, and a glimpse of the rolling race that awaits. Short section, big payoff. Now let us move to the real engine of rotational dynamics: torque.

TORQUE: THE ROTATIONAL FORCE

Up to this point, we have described rotation without asking what causes it. We know *how* a body rotates, *where* it rotates, and *how stubbornly* it resists rotation. But we have said nothing about what makes a body start rotating, or stop rotating, or rotate faster. In Chapters 3, we answered the equivalent question for translation: force is the cause of acceleration ($F = ma$). Now we answer it for rotation.

The rotational twin of force is **torque**.

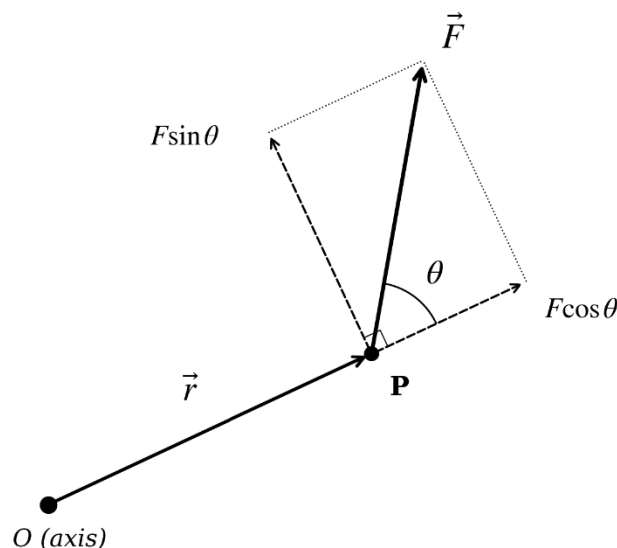
If force is a push or pull that tends to change an object's linear motion, torque *is a twist that tends to change an object's rotational motion*. You apply torque every time you open a door, turn a tap, tighten a bolt with a spanner, or stir ugali with a big wooden spoon (mwiko). In each case, a force is applied at some distance from the rotation axis, and it is this combination of force and distance that produces the twisting effect.

Definition of torque

Consider a force F applied at a point that is at position vector r from the rotation axis. The angle between r and F is θ . The **torque τ** (also called the **moment of force**) is defined as:

$$\tau = rF\sin\theta$$

where: - τ (Greek letter tau) is the torque, measured in newton-metres (Nm) - r is the distance from the axis to the point where the force is applied (m) - F is the magnitude of the applied force (N) - θ is the angle between r and F



$$\tau = rF\sin\theta = r \times (F\sin\theta) = F \times (r\sin\theta)$$

Figure: A force $\vec{F}$ is applied at point P, located at position $\vec{r}$ from the axis at O. The angle between $\vec{r}$ and $\vec{F}$ is θ . The force is resolved into two components: $F\sin\theta$ perpendicular to $\vec{r}$ (which produces torque) and $F\cos\theta$ along $\vec{r}$ (which produces no torque). The torque is $\tau = rF\sin\theta$.

The product $r\sin\theta$ is called the **perpendicular distance** (or **lever arm**) from the axis to the line of action of the force. Equivalently, $F\sin\theta$ is the component of force perpendicular to r . Either interpretation gives the same result.

Why $\sin\theta$?

Only the component of force perpendicular to the radius produces rotation. A force directed straight toward the axis (along r) produces no rotation at all, no matter how large it is. Think of pushing a door by pressing directly toward the hinge: the door does not move. But pushing perpendicular to the door (at right angles to r) produces maximum rotation. The $\sin\theta$ factor captures this: when $\theta = 90^\circ$, $\sin\theta = 1$ and the torque is maximum. When $\theta = 0^\circ$ or 180° , $\sin\theta = 0$ and the torque is zero.

Maximum torque occurs when the force is perpendicular to the radius ($\theta = 90^\circ$):

$$\tau_{\max} = rF$$

This is why door handles are placed as far as possible from the hinges (large r) and why we push perpendicular to the door (maximizing $\sin\theta$). Both choices maximise the torque.

Sign convention: We take anticlockwise torque as positive and clockwise torque as negative, consistent with our angular convention we learned earlier.

Torque as a vector

For advanced work, torque is defined as the cross product of the position vector and the force:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The magnitude is $|\vec{\tau}| = rF\sin\theta$, which is our scalar definition. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$, given by the right-hand rule: curl the fingers of the right hand from $\vec{r}$ toward $\vec{F}$, and the thumb points in the direction of $\vec{\tau}$. For rotation in a plane, this direction is simply “into the page” (clockwise) or “out of the page” (anticlockwise).

For most problems in this chapter, the scalar form $\tau = rF\sin\theta$ with the sign convention is sufficient.

Net torque

When several forces act on a body, each produces its own torque about the axis. The **net torque** is the algebraic sum of all individual torques:

$$\tau_{\text{net}} = \sum \tau_i = \sum r_i F_i \sin\theta_i$$

taking signs into account (positive for anticlockwise, negative for clockwise).

A body is in **rotational equilibrium** when the net torque about any point is zero:

$$\sum \tau = 0 \quad (\text{condition for rotational equilibrium})$$

This is the rotational counterpart of $\sum F = 0$ for translational equilibrium.

Newton's second law for rotation

This is the master equation of rotational dynamics, the twin of $F = ma$.

Derivation:

Consider a single particle of mass m at distance r from a fixed rotation axis. A tangential force F_t acts on it. By Newton's second law for the particle:

$$F_t = ma_t$$

where $a_t = \alpha r$ is the tangential acceleration. Substituting:

$$F_t = m\alpha r$$

The torque due to this force about the axis is $\tau = F_t \times r$ (the force is tangential, so $\theta = 90^\circ$):

$$\tau = m\alpha r \times r = \mathbf{mr^2\alpha} \quad (\text{for one particle})$$

For a rigid body consisting of many particles, the net torque is the sum over all particles:

$$\tau_{\text{net}} = \sum m_i r_i^2 \alpha = \alpha \sum m_i r_i^2$$

We can factor out α because it is the same for all particles in a rigid body. The remaining sum is the moment of inertia $I = \sum m_i r_i^2$:

$$\tau_{\text{net}} = I\alpha$$

This is **Newton's second law for rotation**. It says: *the net torque on a body equals the moment of inertia times the angular acceleration*. Compare with $F_{\text{net}} = ma$: force is replaced by torque, mass by moment of inertia, linear acceleration by angular acceleration. The twin language is complete.

Linear	Rotational
$F_{\text{net}} = ma$	$\tau_{\text{net}} = I\alpha$
Force causes linear acceleration	Torque causes angular acceleration
Greater mass $\rightarrow$ less acceleration for same force	Greater $I \rightarrow$ less angular acceleration for same torque

Couples

A **couple** consists of two forces that are equal in magnitude, opposite in direction, and parallel but not collinear (they act along different lines). A couple produces pure rotation with no translation, because the net force is zero but the net torque is not.

Torque due to a couple:

If two forces, each of magnitude F , are separated by a perpendicular distance d , the torque of the couple is:

$$\tau_{\text{couple}} = Fd$$

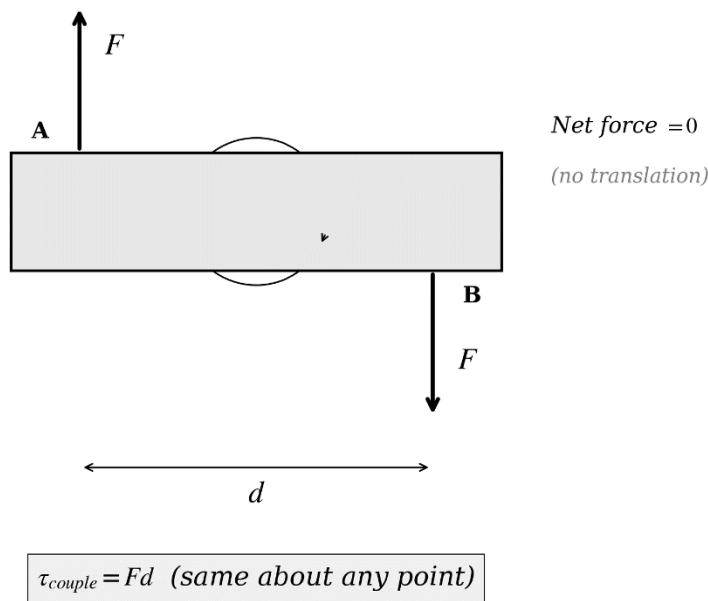


Figure: A couple consists of two equal and opposite forces F , applied at points A and B, separated by perpendicular distance d . The net force is zero (no translation), but both forces produce torques in the same rotational direction, giving a net torque $\tau = Fd$. This torque is the same about any point.

This torque is the same about **any** point. This is a special property of couples: unlike a single force (whose torque depends on where you measure it from), a couple produces the same torque regardless of the chosen pivot. This is because the couple produces pure rotation and no translation.

Proof: Let the two forces $+F$ and $-F$ act at points A and B, separated by distance d . Take torques about any point O at distance x from A. Then the distance from O to B is $(d - x)$ if O is between A and B, (or $(d + x)$ if O is outside).

For O between A and B: $\tau = F \times x + F \times (d - x) = Fx + Fd - Fx = Fd$

The distance x cancels, confirming that the torque of a couple is independent of the reference point.

Examples of couples in everyday life:

- Turning a steering wheel with both hands (one pushes up, the other pushes down).
- Turning a tap with thumb and finger.
- Winding a clock with a key.
- Opening a jar lid with two hands.

In each case, two opposite forces produce pure rotation without moving the axis sideways.

Difference between torque and couple:

Students often confuse these. A **torque** is produced by a single force about a chosen axis; its value depends on which axis you choose. A **couple** is produced by two equal and opposite forces; its torque is the same about any axis. A single force can produce both translation and rotation. A couple produces only rotation.

Now that we have the full machinery of rotational dynamics, from torque to $\tau = I\alpha$, and from single forces to couples, let us see it all in action.

BINDER Example 29

A force of 15N is applied at the end of a 0.4m spanner, at an angle of 60° to the handle.

- Calculate the torque about the bolt at the other end.
- Calculate the maximum torque that could be produced by the same force at the same point on the spanner.
- Explain why a longer spanner makes it easier to loosen a tight bolt.

Solution

- Using $\tau = rF\sin\theta$ and substituting:

$$\tau = 0.4\text{m} \times 15\text{N} \times \sin 60^\circ = 5.20\text{Nm}$$

The torque is 5.20Nm.

- Maximum torque occurs when $\theta = 90^\circ$ ($\sin 90^\circ = 1$):

$$\tau_{\max} = rF = 0.4\text{m} \times 15\text{N} = 6.0\text{Nm}$$

The maximum torque is 6.0Nm.

- Torque is $\tau = rF\sin\theta$. A longer spanner means a larger r . For the same applied force and angle, a larger r gives a larger torque. Thus, the extra length gives the force more leverage.

Making Sense of the Answer: At 60° , the torque (5.20Nm) is 87% of the maximum (6.0Nm). This means applying the force at 60° instead of 90° costs only 13% of the torque. The $\sin\theta$ factor is quite forgiving near 90° : even at 45° , you still get 71% of the maximum. It is only at very small angles (pushing nearly along the handle) that the torque drops dramatically.

Think Like a Physicist: The torque equation has three “knobs” you can turn: r , F , and θ . To maximise torque, make all three as large as possible: use a long lever (large r), push hard (large F), and push perpendicular to the lever (large $\sin\theta$). In practice, the angle is the easiest to optimize (just push at 90°), while the force is limited by human strength, and the lever length is limited by the tool available.

BINDER Example 30

A uniform disc of mass 4kg and radius 0.25m is free to rotate about its central axis. A constant tangential force of 8N is applied at the rim.

- Calculate the torque about the central axis.
- Calculate the angular acceleration.
- Starting from rest, calculate the angular velocity after 3s.

Solution

- The force is tangential (perpendicular to the radius), so $\theta = 90^\circ$ and $\sin\theta = 1$:

$$\tau = rF\sin 90^\circ = 0.25\text{m} \times 8\text{N} \times 1 = 2.0\text{Nm}$$

The torque is 2.0Nm.

(b) First, the moment of inertia of the disc: $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 4 \times (0.25)^2 = 0.125\text{kgm}^2$

Using $\tau = I\alpha$ and making α the subject:

$$\alpha = \frac{\tau}{I} = \frac{2.0\text{Nm}}{0.125\text{kgm}^2} = 16\text{rad/s}^2$$

The angular acceleration is 16rad/s^2 .

(c) Using $\omega = \omega_0 + \alpha t$ and substituting:

$$\omega = 0 + 16\text{rad/s}^2 \times 3\text{s} = 48\text{rad/s}$$

The angular velocity after 3s is 48rad/s .

Making Sense of the Answer: An angular acceleration of 16rad/s^2 is large. In 3 seconds, the disc reaches 48rad/s , which is about 7.6 revolutions per second. This rapid spin-up happens because the disc has a small moment of inertia (0.125kgm^2) and the applied torque (2Nm) is substantial relative to it. A heavier disc or one with a larger radius would accelerate more slowly for the same torque.

Think Like a Physicist: This example combines three subtopics: torque, moment of inertia, and angular kinematics. This is the typical structure of a rotational dynamics problem: calculate the torque, find I , use $\tau = I\alpha$ to get α , then use the kinematic equations to find ω or θ . Master this chain, and you can solve any constant-torque rotation problem.

REAL Example 31

Mr. Akilikubwa demonstrates couples by asking Kipanga and Kipute to stand on opposite sides of a laboratory turntable. Kipanga pushes the rim with a force of 20N to the left, and Kipute pushes the opposite rim with a force of 20N to the right. The turntable has a diameter of 0.6m .

- Explain why the turntable rotates but does not translate sideways.
- Calculate the torque of the couple.
- If the turntable has mass 5kg , calculate the angular acceleration.

Solution

- (a) The two forces are equal in magnitude (20N each) and opposite in direction (left and right). Therefore the net force on the turntable is $20\text{N} - 20\text{N} = 0$. With zero net force, there is no translational acceleration, so the centre of the turntable does not move sideways.

However, both forces produce torques in the same rotational direction. Kipanga's force pushes the rim to the left at the top, creating a clockwise torque. Kipute's force pushes the rim to the right at the bottom, also creating a clockwise torque. The torques add rather than cancel, producing net rotation. This is the defining property of a couple: zero net force but non-zero net torque.

- (b) The perpendicular distance between the two forces is the diameter: $d = 0.6\text{m}$.

Using $\tau_{\text{couple}} = Fd$ and substituting:

$$\tau = 20\text{N} \times 0.6\text{m} = 12\text{Nm}$$

The torque of the couple is 12Nm .

- (c) The turntable is a uniform disc: $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 5 \times (0.3)^2 = 0.225\text{kgm}^2$

Using $\tau = I\alpha$ and making α the subject:

$$\alpha = \frac{\tau}{I} = \frac{12\text{Nm}}{0.225\text{kgm}^2} = 53.3\text{rad/s}^2$$

The angular acceleration is 53.3rad/s^2 .

Making Sense of the Answer: The angular acceleration is very large (53.3rad/s^2) because the couple torque (12Nm) is substantial and the disc's moment of inertia (0.225kgm^2) is modest. In practice, friction would reduce this acceleration significantly. But the demonstration makes the point: a couple is an efficient way to spin something, because both forces contribute to rotation and neither is "wasted" on translation.

Think Like a Physicist: Note that we could have calculated the torque by taking moments about the centre: $\tau = 20\text{N} \times 0.3\text{m} + 20\text{N} \times 0.3\text{m} = 12\text{Nm}$. We could also have taken moments about any other point and obtained the same answer, because the torque of a couple is independent of the pivot.

REAL Example 32

A merry-go-round at a Tanzanian school playground can be modelled as a uniform disc of mass 120kg and radius 1.5m. A student applies a constant tangential force of 60N at the rim.

- Calculate the torque and the angular acceleration.
- How long does it take to reach an angular velocity of 2rad/s (a gentle spin), starting from rest?
- Once spinning at 2rad/s, the student lets go. A friction torque of 15Nm opposes the motion. Calculate the time for the merry-go-round to stop.

Solution

- Torque: $\tau = rF = 1.5\text{m} \times 60\text{N} = 90\text{Nm}$ (force is tangential, $\theta = 90^\circ$)

Moment of inertia: $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 120\text{kg} \times (1.5\text{m})^2 = 135\text{kgm}^2$

Using $\tau = I\alpha$ and making α the subject:

$$\alpha = \frac{90\text{Nm}}{135\text{kgm}^2} = 0.667\text{rad/s}^2$$

The torque is 90Nm and the angular acceleration is 0.667rad/s².

- Using $\omega = \omega_0 + \alpha t$ and making t the subject:

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{2\text{rad/s} - 0}{0.667\text{rad/s}^2} = 3.0\text{s}$$

It takes 3.0 seconds to reach 2rad/s.

- The friction torque decelerates the merry-go-round. Using $\tau = I\alpha$ and making α the subject:

$$\alpha_{\text{friction}} = \frac{15\text{Nm}}{135\text{kgm}^2} = 0.111\text{rad/s}^2$$

Using $\omega = \omega_0 + \alpha t$ with $\omega = 0$ (stops) and $\alpha = -0.111\text{rad/s}^2$ (decelerating):

$$0 = 2\text{rad/s} + (-0.111\text{rad/s}^2) \times t$$

$$t = \frac{2\text{rad/s}}{0.111\text{rad/s}^2} = 18\text{s}$$

The merry-go-round takes 18 seconds to stop after the student lets go.

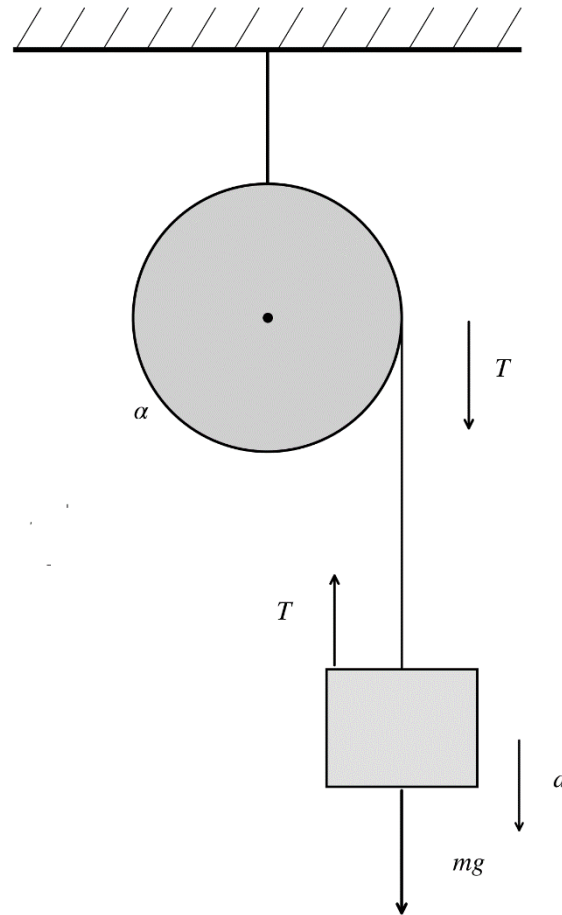
Making Sense of the Answer: It takes only 3 seconds to spin up but 18 seconds to stop. The asymmetry comes from the torques: the student applies 90Nm (strong push), but friction provides only 15Nm (gentle resistance). The moment of inertia of 135kgm² is large (the merry-go-round is heavy and wide), which is why even the strong push gives a modest angular acceleration and why the weak friction takes a long time to bring it to rest. This is rotational inertia in action: a massive spinning object resists changes in both directions.

Think Like a Physicist: This problem follows the same chain as Example 29 but with a more realistic scenario and a two-phase motion (spinning up, then coasting down). In part (c), the angular deceleration is 6 times smaller than the acceleration in part (a), because the friction torque is 6 times smaller than the student's torque. The ratio of stopping time to spin-up time ($18/3 = 6$) is exactly this torque ratio. This makes the problem easy to check without recalculating everything.

HOT Example 33

A pulley of mass 2kg and radius 0.10m is modelled as a uniform disc and is free to rotate about its central axis. A light string is wound around the pulley, and a block of mass 3kg hangs from the free end. The system is released from rest. Take $g = 9.8\text{m/s}^2$.

- Draw a free body diagrams for the pulley-block system.
- If the string does not slip on the pulley, solve for the acceleration of the block and the tension in the string.



Solution

(a)

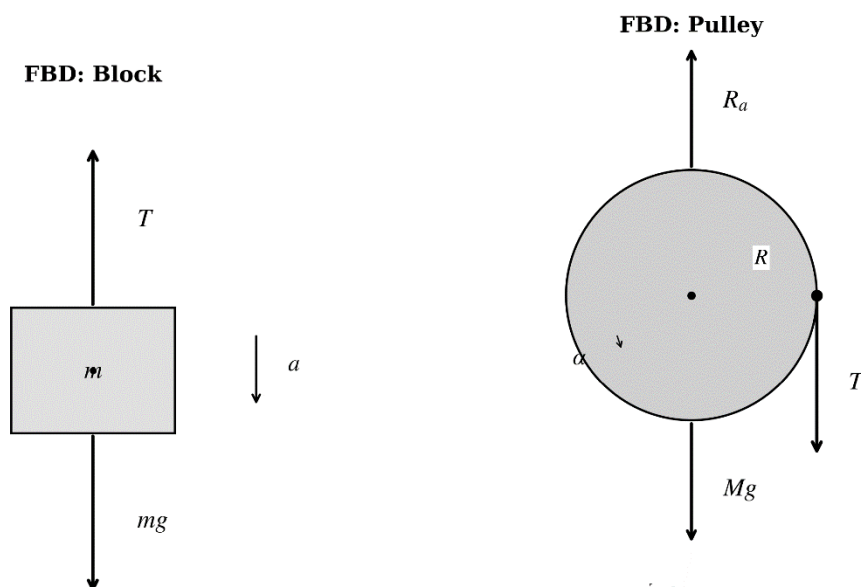


Figure: Free body diagrams for the pulley-block system. Left: the block of mass m , with tension T upward and weight mg downward, accelerating at a downward. Right: the pulley of mass M and radius R , with tension T at the rim, axle reaction R_a , weight Mg , and angular acceleration α

- (b) **For block:** The block has two forces: weight $mg = 3\text{kg} \times 9.8\text{m/s}^2 = 29.4\text{N}$ downward, and tension T upward. Taking downward as positive (the block accelerates downward):

$$\begin{aligned} mg - T &= ma \\ 29.4 - T &= 3a \quad \dots (i) \end{aligned}$$

For pulley: R_a balances with mg and both act at axis, and thus there is zero torque about the axis. The torque on the pulley is produced by the tension T in the string, acting at the rim (radius $R = 0.10\text{ m}$). The torque is $\tau = TR$.

The pulley's moment of inertia is $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 2 \times (0.10)^2 = 0.01\text{kgm}^2$.

Using $\tau = I\alpha$:

$$\begin{aligned} TR &= I\alpha \\ T \times 0.10 &= 0.01\alpha \quad \dots (ii) \end{aligned}$$

If the string does not slip on the pulley, then $a = \alpha R$, which gives $\alpha = \frac{a}{R} = \frac{a}{0.10} = 10a$.

Substituting into (ii):

$$\begin{aligned} 0.10T &= 0.01 \times 10a = 0.1a \\ T &= a \quad \dots (iii) \end{aligned}$$

Substituting (iii) into (i):

$$\begin{aligned} 29.4 - a &= 3a \\ 29.4 &= 4a \\ a &= 7.35\text{m/s}^2 \end{aligned}$$

From (iii): $T = 7.35\text{N}$

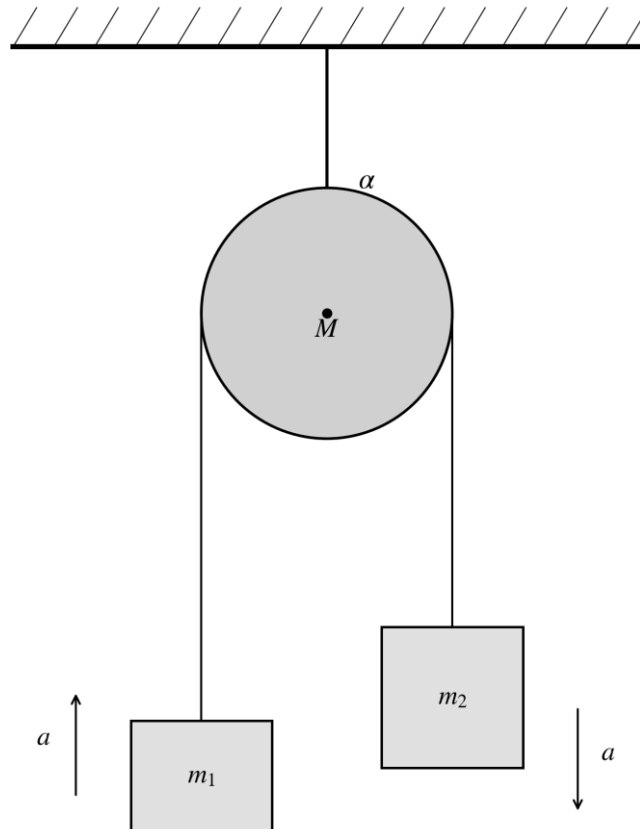
The acceleration of the block is 7.35m/s^2 and the tension in the string is 7.35N .

Making Sense of the Answer: If the pulley were massless ($I = 0$), the block would fall freely at 9.8m/s^2 with $T = 0$. With the massive pulley, the acceleration is 7.35m/s^2 , which is 75% of g . The pulley's rotational inertia slows the block down because some of the gravitational potential energy goes into spinning the pulley rather than accelerating the block. The tension (7.35N) is much less than the weight (29.4N), confirming that the block is accelerating downward. If T equalled mg , the block would not accelerate at all.

Think Like a Physicist: This is the classic "Atwood machine with a massive pulley" problem. The key idea is that the pulley is not just a frictionless redirector of the string; it has its own moment of inertia and must be accelerated rotationally by the string tension. The constraint $a = \alpha R$ connects the linear motion of the block to the rotational motion of the pulley, giving us the third equation needed to solve for three unknowns (a , α , T). This type of coupled linear-rotational problem is one of the most important in this chapter and appears frequently in examinations.

HOT Example 34

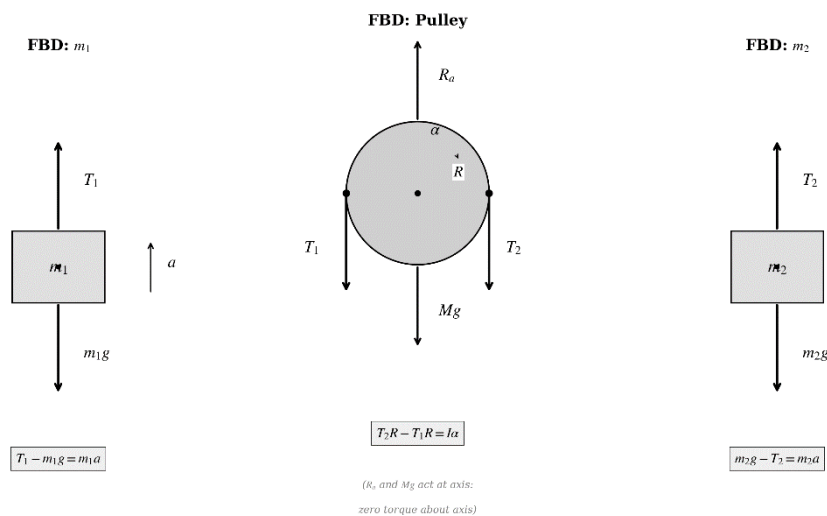
Two blocks of masses $m_1 = 4\text{kg}$ and $m_2 = 6\text{kg}$ are connected by a light string that passes over a uniform disc pulley of mass $M = 2\text{kg}$ and radius $R = 0.15\text{m}$. The string does not slip on the pulley. The system is released from rest. Take $g = 9.8\text{m/s}^2$.



- (a) Explain why the tensions on the two sides of the string are not equal.
 (b) Calculate the acceleration of the system, the angular acceleration of the pulley, and both tensions.

Solution

- (a) If the pulley were massless and frictionless, the string would simply transmit force without change, and the tensions would be equal on both sides. But this pulley has mass, which means it has a moment of inertia ($I = \frac{1}{2}MR^2$). To accelerate the pulley rotationally, there must be a net torque on it. The net torque is $(T_2 - T_1)R$. If $T_1 = T_2$, the net torque would be zero and the pulley would not accelerate. Therefore the tensions must be different: $T_2 > T_1$ (the tension on the heavier side is larger) to provide the torque that spins the pulley.
- (b) Forces acting in each object are clearly shown in the following free body diagrams:



Since $m_2 > m_1$, the heavier block m_2 accelerates downward and the lighter block m_1 accelerates upward, both with the same magnitude a .

For m_1 (accelerating upward, taking upward as positive):

$$T_1 - m_1 g = m_1 a$$

$$T_1 - 4 \times 9.8 = 4a$$

$$T_1 - 39.2 = 4a \quad \dots (i)$$

For m_2 (accelerating downward, taking downward as positive):

$$m_2 g - T_2 = m_2 a$$

$$6 \times 9.8 - T_2 = 6a$$

$$58.8 - T_2 = 6a \quad \dots (ii)$$

For the pulley (net torque produces angular acceleration):

$$T_2 R - T_1 R = (T_2 - T_1) R = I \alpha$$

The moment of inertia: $I = \frac{1}{2} M R^2 = \frac{1}{2} \times 2 \times (0.15)^2 = 0.0225 \text{ kgm}^2$

$$(T_2 - T_1) \times 0.15 = 0.0225 \alpha \quad \dots (iii)$$

The no-slip constraint gives: $a = \alpha R$, so $\alpha = \frac{a}{R} = \frac{a}{0.15}$

Substituting into (iii):

$$(T_2 - T_1) \times 0.15 = 0.0225 \times \frac{a}{0.15}$$

$$(T_2 - T_1) \times 0.15 = 0.15a$$

$$T_2 - T_1 = a \quad \dots (iv)$$

Adding equations (i) and (ii):

$$(T_1 - 39.2) + (58.8 - T_2) = 4a + 6a$$

$$19.6 + T_1 - T_2 = 10a$$

From (iv): $T_1 - T_2 = -a$

$$19.6 - a = 10a$$

$$a = \frac{19.6}{11} = 1.78 \text{ m/s}^2$$

The acceleration of the system is 1.78 m/s^2 .

Angular acceleration: $\alpha = \frac{a}{R} = \frac{1.78}{0.15} = 11.9 \text{ rad/s}^2$

The angular acceleration of the pulley is 11.9 rad/s^2 .

From (i): $T_1 = 4a + 39.2 = 4 \times 1.78 + 39.2 = 46.3 \text{ N}$

From (ii): $T_2 = 58.8 - 6a = 58.8 - 6 \times 1.78 = 48.1 \text{ N}$

The tension on the m_1 side is 46.3 N and the tension on the m_2 side is 48.1 N .

Making Sense of the Answer: If the pulley were massless, the acceleration would be $\frac{(m_2 - m_1)g}{m_1 + m_2} = \frac{2 \times 9.8}{10} = 1.96 \text{ m/s}^2$ and both tensions would be equal. With the massive pulley, the acceleration drops to 1.78 m/s^2 (about 9% less) because the pulley's rotational inertia acts as extra "effective mass" in the system. The total effective mass is $m_1 + m_2 + \frac{1}{2} M = 4 + 6 + 1 = 11 \text{ kg}$, and indeed $\frac{(m_2 - m_1)g}{11} = \frac{19.6}{11} = 1.78 \text{ m/s}^2$. The two tensions differ by only 1.8 N , which is the small torque needed to rotationally accelerate the relatively light pulley.

Think Like a Physicist: This is the full Atwood machine with a massive pulley, one of the most classic problems in rotational dynamics. The key differences from the massless-pulley version (Chapter 4) are: (1) the tensions on the two sides are different, (2) the pulley's rotational inertia adds to the system's effective mass, and (3) you need three equations (two for the blocks, one for the pulley) plus one constraint ($a = \alpha R$) to solve for four unknowns (a, α, T_1, T_2). The effective mass trick $(m_1 + m_2 + \frac{1}{2}M)$ is a powerful shortcut for checking your answer.

ROTATIONAL WORK, ENERGY, AND POWER

We are already familiar with the energy toolkit for translational motion: work done by a force ($W = F \cos \theta$), the kinetic energy of a moving body ($KE = \frac{1}{2}mv^2$), and the power delivered by a force ($P = Fv$). These tools allow us to solve problems that would have been difficult or impossible using forces alone.

We now build the identical toolkit for rotation. And by “identical,” we mean it: every formula is the translational formula with each quantity replaced by its rotational twin. If you understood the energy methods studied earlier, this section will feel like meeting an old friend in new clothes.

Work Done by a Torque

When a force F moves an object through a displacement s , the work done is $W = Fs$ (for a constant force in the direction of motion). The rotational twin replaces force with torque and linear displacement with angular displacement.

Consider a torque τ acting on a body that rotates through an angular displacement θ (in radians). The **work done by the torque** is:

$$W = \tau \theta$$

This is easily derived from the linear definition. If a tangential force F_t acts at radius r , the torque is $\tau = F_t r$ and the arc length travelled is $s = r\theta$. Therefore:

$$W = F_t \times s = F_t \times r\theta = (F_t r)\theta = \tau \theta$$

For a variable torque, the work is:

$$W = \int_{\theta_1}^{\theta_2} \tau \, d\theta$$

The unit of rotational work is the joule (J), just as for translational work. Work is work, whether the motion is linear or rotational.

The Work-Energy Theorem for Rotation

The work-energy theorem states that the net work done on a body equals the change in its translational KE:

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

The rotational twin is:

$$W_{\text{net}} = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

The net work done by all torques on a rotating body equals the change in its rotational KE.

Derivation:

Starting from $\tau = I\alpha$ and using $\alpha = \frac{d\omega}{dt}$:

$$W = \int \tau \, d\theta = \int I\alpha \, d\theta = \int I \frac{d\omega}{dt} \, d\theta = \int I \frac{d\omega}{dt} \times \frac{d\theta}{dt} \, dt$$

But $\frac{d\theta}{dt} = \omega$, so:

$$W = \int I\omega \, d\omega = I \left[\frac{\omega^2}{2} \right]_{\omega_0}^{\omega} = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

This is the rotational work-energy theorem. It is the rotational twin of $W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$, with m replaced by I and v replaced by ω .

Rotational Kinetic Energy

We already met this earlier when we derived the moment of inertia. The **rotational KE** of a body spinning with angular velocity ω about a fixed axis is:

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2$$

This is the twin of $KE_{\text{trans}} = \frac{1}{2}mv^2$.

When a body both translates and rotates (as we will explore later on rolling motion), the total KE is the sum:

$$KE_{\text{total}} = \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$$

The first term is the translational KE of the centre of mass. The second term is the rotational KE about the centre of mass. The two are independent and simply add.

Rotational Power

Power is the rate of doing work. In translation, $P = \frac{dW}{dt} = Fv$. In rotation:

$$P = \frac{dW}{dt} = \frac{d(\tau\theta)}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

$$\mathbf{P = \tau\omega}$$

The rotational power equals the torque times the angular velocity. This is the twin of $P = Fv$.

The unit is the watt (W), the same as for translational power.

This formula is enormously useful in engineering. The power output of an engine, a turbine, or a motor is specified in terms of the torque it delivers and the angular velocity at which it operates.

The complete twin table:

Translational	Rotational
$W = Fs$	$W = \tau\theta$
$KE = \frac{1}{2}mv^2$	$KE = \frac{1}{2}I\omega^2$
$W_{\text{net}} = \Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$	$W_{\text{net}} = \Delta KE = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$
$P = Fv$	$P = \tau\omega$

The twin language is now complete for energy. Every formula is the same structure with the rotational substitutions. If you know one column, you know both.

With the energy toolkit assembled, let us put it to work. These examples will feel familiar from what you have already know, just dressed in rotational clothes.

BINDER Example 35

A constant torque of 8Nm is applied to a wheel that is initially at rest. The wheel rotates through 20rad.

- Calculate the work done by the torque.
- If the wheel has a moment of inertia of 0.5kgm^2 , calculate the angular velocity after 20rad using the work-energy theorem.
- Verify your answer using $\omega^2 = \omega_0^2 + 2\alpha\theta$.

Solution

(a) Using $W = \tau\theta$ and substituting:

$$W = 8\text{Nm} \times 20\text{rad} = 160\text{J}$$

The work done is 160J.

(b) Using the work-energy theorem $W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$. Since $\omega_0 = 0$:

$$160\text{J} = \frac{1}{2} \times 0.5\text{kgm}^2 \times \omega^2$$

$$\omega = \sqrt{\frac{2 \times 160\text{J}}{0.5\text{kgm}^2}} = 25.3\text{rad/s}$$

The angular velocity after 20rad is 25.3rad/s.

(c) First, find α from $\tau = I\alpha$:

$$\alpha = \frac{\tau}{I} = \frac{8\text{Nm}}{0.5\text{kgm}^2} = 16\text{rad/s}^2$$

Using $\omega^2 = \omega_0^2 + 2\alpha\theta$ and substituting:

$$\omega^2 = 0 + 2 \times 16\text{rad/s}^2 \times 20\text{rad} = 640\text{rad}^2/\text{s}^2$$

$$\omega = \sqrt{640\text{rad}^2/\text{s}^2} = 25.3\text{rad/s}$$

Both methods give the same answer. ✓

Making Sense of the Answer: The work-energy theorem and the kinematic approach must always agree, because they describe the same physics from different perspectives. The energy method is often quicker because it bypasses the intermediate step of finding α , going directly from work to final velocity. This advantage becomes even greater when the torque varies with angle, making the kinematic approach impractical.

Think Like a Physicist: Compare this with a more familiar problem: a force of 8N pushes a 0.5kg box from rest through 20m. Work = 160J, final velocity = $\sqrt{640} = 25.3\text{m/s}$. The numbers are identical because the mathematical structure is identical. The twin language is not just a metaphor; it is an exact correspondence.

REAL Example 36

A traditional Tanzanian grain mill uses a heavy circular grinding stone (modelled as a uniform disc) of mass 15kg and radius 0.25m, spinning at 4rev/s.

- Calculate the rotational KE stored in the grinding stone.
- A friction torque of 3Nm acts on the stone when grain is fed in. Using the work-energy theorem, calculate how many revolutions the stone makes before stopping (if not pushed again).
- Calculate the average power dissipated during the stopping process.

Solution

(a) Moment of inertia: $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 15\text{kg} \times (0.25\text{m})^2 = 0.469\text{kgm}^2$

Angular velocity: $\omega = 2\pi \times 4 = 8\pi\text{rad/s} = 25.13\text{rad/s}$

Using $\text{KE} = \frac{1}{2}I\omega^2$ and substituting:

$$\text{KE} = \frac{1}{2} \times 0.469\text{kgm}^2 \times (25.13\text{rad/s})^2 = 148\text{J}$$

The rotational KE is 148J.

(b) The friction torque does negative work (it opposes the motion), converting KE into heat. By the work-energy theorem:

$$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = 0 - 148\text{J} = -148\text{J}$$

Since $W = -\tau\theta$ (negative because friction opposes motion), making θ the subject:

$$\theta = \frac{148\text{J}}{3\text{Nm}} = 49.3\text{rad}$$

Converting: $\frac{49.3}{2\pi} = 7.85\text{rev}$

The stone makes about 7.9 revolutions before stopping.

(c) First, find the time to stop. Using $\omega = \omega_0 + \alpha t$ with $\alpha = -\frac{\tau}{I} = -\frac{3}{0.469} = -6.40\text{rad/s}^2$:

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 25.13}{-6.40} = 3.93\text{s}$$

Average power: $P_{\text{avg}} = \frac{W}{t} = \frac{148\text{J}}{3.93\text{s}} = 37.7\text{W}$

The average power dissipated by friction is 37.7W.

Making Sense of the Answer: The grinding stone stores 148J at 4rev/s. This is the energy available for grinding before the stone needs another push. The stone makes about 8 revolutions while this energy is used up by the friction of grinding. The average power (38W) is modest, roughly equivalent to a dim light bulb. This explains why traditional grinding is slow and requires frequent pushing: the stone does not store enough energy for prolonged grinding at high power.

Think Like a Physicist: Part (b) uses the work-energy theorem instead of kinematics. Compare: using kinematics would require finding α , then using $\omega^2 = \omega_0^2 + 2\alpha\theta$, then converting to revolutions. The energy method goes directly from initial KE to the target, bypassing α . Both give the same answer, but the energy route is shorter. Energy methods are especially powerful when forces (or torques) vary, because the work-energy theorem still works while $\tau = I\alpha$ requires constant torque.

REAL Example 37

A car engine delivers a torque of 180Nm at 4000rpm.

- Calculate the power output in horsepower (1hp = 746W).
- If the engine speed drops to 2000rpm but the torque remains the same, calculate the new power output and explain why it is halved.
- Explain why car manufacturers quote “peak torque” and “peak power” at different engine speeds.

Solution

(a) Converting: $\omega = \frac{2\pi \times 4000}{60} = \frac{8000\pi}{60} = 418.9\text{rad/s}$

Using $P = \tau\omega$ and substituting:

$$P = 180\text{Nm} \times 418.9\text{rad/s} = 75402\text{W} = 75.4\text{kW}$$

Converting to horsepower: $\frac{75400}{746} = 101\text{hp}$

The power output is 101hp.

(b) At 2000rpm: $\omega = \frac{2\pi \times 2000}{60} = 209.4\text{rad/s}$

$$P = 180 \times 209.4 = 37700\text{W} = 37.7\text{kW}$$

The power is halved because $P = \tau\omega$. If τ stays the same but ω halves, then P halves. Power depends on both torque and angular velocity, not torque alone. An engine producing high torque at low rpm delivers less power than the same engine producing the same torque at high rpm.

- (c) Torque measures the engine’s turning effect at any instant. Power measures the rate at which the engine delivers energy. Because $P = \tau\omega$, peak power occurs not at peak torque but at the product $\tau\omega$ being maximum. In most engines, torque peaks at a moderate rpm (where combustion is most efficient), while power peaks at a higher rpm (where the higher ω compensates for the slightly reduced torque). Car manufacturers quote both because they measure different things: peak torque determines how much load the engine can handle (towing, hill climbing), while peak power determines the maximum speed and acceleration.

Making Sense of the Answer: A 101hp engine is a typical mid-range car engine. The numbers are realistic. The key insight is that torque alone does not determine power: a tractor engine may have more torque than a sports car engine, but the sports car produces more power because it operates at much higher rpm.

Think Like a Physicist: The formula $P = \tau\omega$ is one of the most practically useful in all of rotational mechanics. Engineers use it daily to specify motors, turbines, and engines. If you ever need to choose a motor for a machine, you will use this formula: determine the torque required by the load and the angular velocity at which it must operate, multiply, and you have the power rating of the motor you need.

HOT Example 38

A flywheel (uniform disc, mass 6kg, radius 0.50m) is spinning at 80rad/s. A braking torque is applied that decreases linearly with angle: $\tau = 50 - 0.5\theta$, where τ is in Nm and θ is in radians (measured from the instant the brake is applied). The brake weakens as the flywheel turns because the brake pad wears down.

- Calculate the initial rotational KE.
- Using the work-energy theorem with the variable torque, calculate the angle through which the flywheel turns before stopping.
- Explain why the kinematic equation $\omega^2 = \omega_0^2 + 2\alpha\theta$ cannot be used for this problem.

Solution

$$(a) \quad I = \frac{1}{2}MR^2 = \frac{1}{2} \times 6\text{kg} \times (0.50\text{m})^2 = 0.75\text{kgm}^2$$

$$KE_0 = \frac{1}{2}I\omega_0^2 = \frac{1}{2} \times 0.75\text{kgm}^2 \times (80\text{rad/s})^2 = 2400\text{J}$$

The initial rotational KE is 2400J.

- The braking torque does negative work as it opposes the rotation. The work done by the variable braking torque as the flywheel rotates from $\theta = 0$ to $\theta = \theta_{\text{stop}}$ is:

$$W = - \int_0^{\theta_{\text{stop}}} (50 - 0.5\theta) d\theta = - \left[50\theta - \frac{0.5\theta^2}{2} \right]_0^{\theta_{\text{stop}}} = -(50\theta_{\text{stop}} - 0.25\theta_{\text{stop}}^2)$$

By the work-energy theorem, when the flywheel stops ($\omega = 0$):

$$W = 0 - KE_0 = -2400\text{J}$$

Therefore:

$$50\theta_{\text{stop}} - 0.25\theta_{\text{stop}}^2 = 2400$$

Rearranging:

$$0.25\theta_{\text{stop}}^2 - 50\theta_{\text{stop}} + 2400 = 0$$

Solving the quadratic equation in θ_{stop} :

$$\theta_{\text{stop}} = 80\text{rad} \quad \text{or} \quad \theta_{\text{stop}} = 120\text{rad}$$

The braking torque vanishes when $50 - 0.5\theta = 0$, i.e., at $\theta = 100\text{rad}$. The solution $\theta = 120\text{rad}$ is beyond this point (it gives the negative value of $50 - 0.5\theta$, which implies that the brake has already released), so it is physically invalid.

The flywheel stops after turning through $\theta = 80\text{rad}$ (12.7 revolutions).

You may check: at $\theta = 80\text{rad}$, the braking torque is $\tau = 50 - 0.5 \times 80 = 10\text{Nm}$, which is still positive. The brake is still active when the flywheel stops. ✓

- The kinematic equation $\omega^2 = \omega_0^2 + 2\alpha\theta$ requires **constant** angular acceleration. Since the braking torque varies with angle ($\tau = 50 - 0.5\theta$), the angular acceleration $\alpha = \frac{\tau}{I}$ also varies with angle. The kinematic equations are therefore invalid for this problem. Only the work-energy theorem, which handles variable torques through integration, can solve it correctly.

Making Sense of the Answer: The flywheel stores 2400J and the brake removes this energy over 80rad (about 13 revolutions). At the start, the brake exerts 50Nm, but by the time the flywheel stops, the brake has weakened to 10Nm. The two solutions of the quadratic (80 and 120) correspond to two angles where the cumulative work equals 2400J: the flywheel reaches zero speed at 80rad (real stopping point), while 120rad is a mathematical solution that has no physical meaning because the brake has already disappeared at 100rad. Rejecting the unphysical root is an important skill in examination problems.

Think Like a Physicist: This is the kind of problem that makes energy methods indispensable. The variable torque makes $\tau = I\alpha$ useless for finding the total angle directly, but the work-energy theorem handles it

elegantly through integration. The quadratic with two roots is a bonus lesson: always check whether both roots are physically valid. The brake's limited range (0 to 100rad) eliminates one root, and only the student who checks the physics, not just the mathematics, gets the correct answer.

Experimental Determination of the Moment of Inertia of a Flywheel

The derivations of moments of inertia studied earlier, give us the moment of inertia for ideal, perfectly uniform shapes. But real flywheels are not perfect: they have spokes, hubs, keyways, and manufacturing imperfections. For a real flywheel, the moment of inertia is best determined experimentally, and the method uses exactly the energy principles we have just developed.

The method

A flywheel is mounted on a horizontal axle so that it is free to rotate. A known mass m is attached to a string wound around the axle (radius r). The mass is released from rest and falls through a measured height h , unwinding the string. At the instant the string detaches (all the string has unwound), the flywheel is spinning at its maximum angular velocity ω . The flywheel then continues to spin, gradually slowing due to friction, and eventually comes to rest after n additional revolutions.

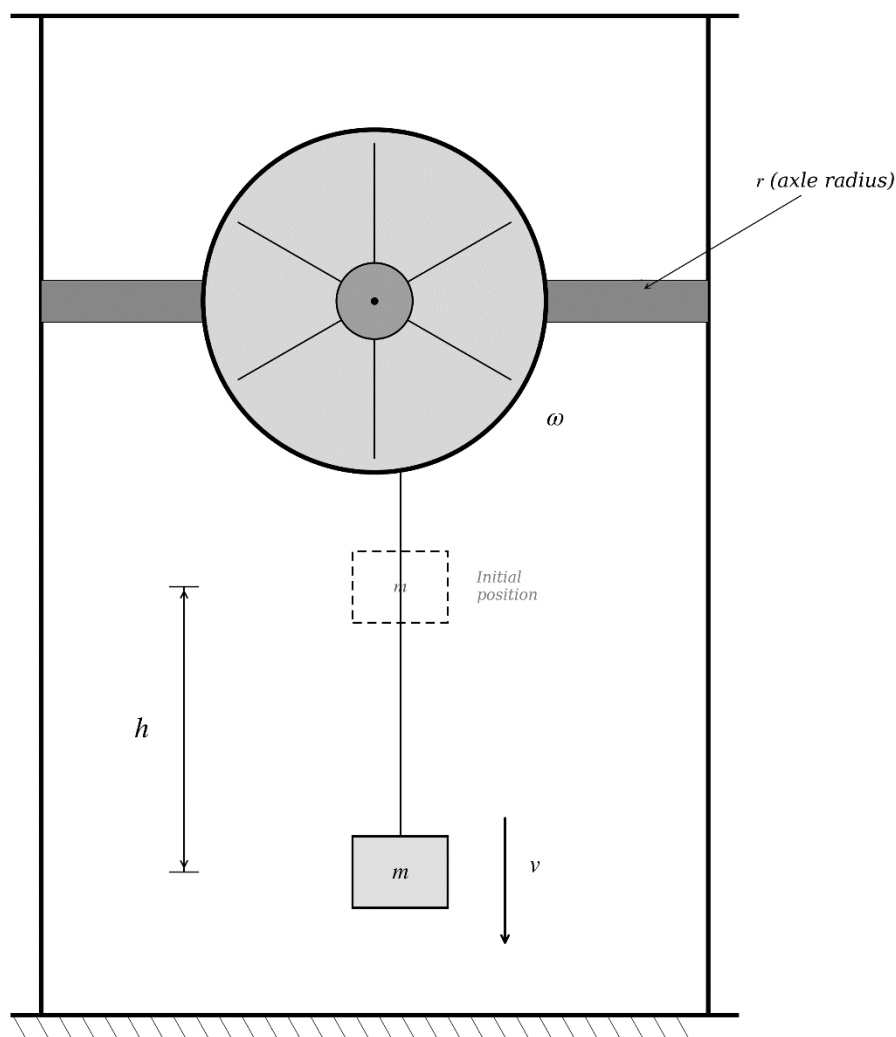


Figure: Experimental setup for determining the moment of inertia of a flywheel. The flywheel is mounted on a horizontal axle (radius r) supported by a rigid frame. A mass m is attached to a string wound around the axle. Released from its initial position (dashed), the mass falls through height h , unwinding the string and spinning the flywheel to angular velocity ω . At the final position, the mass has downward velocity $v = \omega r$. After the string detaches, the flywheel coasts for n_2 additional revolutions before friction brings it to rest.

The energy equation

As the mass falls through height h , gravitational potential energy is converted into three forms:

- 1) Translational KE of the falling mass: $\frac{1}{2}mv^2$
- 2) Rotational KE of the flywheel: $\frac{1}{2}I\omega^2$
- 3) Energy lost to friction during the winding phase

By conservation of energy:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + W_{\text{friction (winding)}}$$

Dealing with friction

The friction torque τ_f acts throughout both the winding phase (while the mass falls) and the unwinding phase (after the string detaches). If the flywheel makes n_1 revolutions during the fall and n_2 revolutions after the string detaches before stopping:

Work done by friction during the fall: $W_{f1} = \tau_f \times 2\pi n_1$

Work done by friction after detachment: $W_{f2} = \tau_f \times 2\pi n_2$

After the string detaches, the flywheel has $KE = \frac{1}{2}I\omega^2$ and this is entirely used up by friction over n_2 revolutions:

$$\frac{1}{2}I\omega^2 = \tau_f \times 2\pi n_2$$

This gives us the friction torque:

$$\tau_f = \frac{I\omega^2}{4\pi n_2}$$

The number of revolutions during the fall, n_1 , is found from the geometry: the string length unwound equals the distance fallen, so $h = 2\pi n_1 r$, giving $n_1 = \frac{h}{2\pi r}$.

The no-slip condition relates v and ω at the moment of detachment: $v = \omega r$.

Substituting everything into the energy equation:

$$mgh = \frac{1}{2}m(\omega r)^2 + \frac{1}{2}I\omega^2 + \frac{I\omega^2}{4\pi n_2} \times 2\pi n_1$$

$$mgh = \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}I\omega^2 + \frac{I\omega^2 n_1}{2n_2}$$

$$mgh = \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}I\omega^2 \left(1 + \frac{n_1}{n_2}\right)$$

Making I the subject:

$$\frac{1}{2}I\omega^2 \left(1 + \frac{n_1}{n_2}\right) = mgh - \frac{1}{2}m\omega^2 r^2$$

$$I = \frac{2mgh - m\omega^2 r^2}{\omega^2 \left(1 + \frac{n_1}{n_2}\right)}$$

Finding ω :

The angular velocity at the moment of detachment can be found by measuring the time for a few revolutions immediately after detachment (using a stopwatch and counting), or from the average angular velocity during the fall: $\omega = \frac{2 \times 2\pi n_1}{t}$ where t is the time of fall (since the flywheel accelerates from 0 to ω , the average is $\omega/2$).

REAL Example 39

In a flywheel experiment, a mass of 2kg is attached to a string wound around the axle of radius 0.03m. The mass is released from rest and falls through 1.5m in 10s, making 8 complete turns of the flywheel. After the string detaches, the flywheel makes 40 more revolutions before stopping. Take $g = 9.8\text{m/s}^2$.

- Calculate the angular velocity of the flywheel, and linear velocity of the falling mass at the instant the string detaches
- Calculate the moment of inertia of the flywheel.
- Calculate the friction torque.

Solution

- During the fall, the flywheel accelerates uniformly from rest. The number of revolutions is $n_1 = 8$, so the total angle is $\theta = 2\pi \times 8 = 16\pi\text{rad}$.

Using $\theta = \frac{1}{2}(\omega_0 + \omega)t$ with $\omega_0 = 0$ and making ω the subject:

$$\omega = \frac{2\theta}{t} = \frac{2 \times 16\pi \text{ rad}}{10\text{s}} = \frac{32\pi \text{ rad}}{10\text{s}} = 10.05\text{rad/s}$$

The angular velocity at detachment is 10.05rad/s.

Alternative solution

The angular velocity is directly given by the following formula:

$$\omega = \frac{2 \times 2\pi n_1}{t} = \frac{2 \times 2\pi \times 8\text{rad}}{10\text{s}} = \frac{32\pi \text{ rad}}{10\text{s}} = 10.05\text{rad/s}$$

The linear velocity of the falling mass: $v = \omega r = 10.05 \times 0.03 = 0.302\text{m/s}$

- Using the formula derived above and substituting:

$$I = \frac{2mgh - m\omega^2 r^2}{\omega^2 \left(1 + \frac{n_1}{n_2}\right)}$$

Substituting:

$$I = \frac{2 \times 2\text{kg} \times 9.8\text{m/s}^2 \times 1.5\text{m} - 2 \times (10.05\text{rad/s})^2 \times (0.03\text{m})^2}{(10.05\text{rad/s})^2 \times \left(1 + \frac{8}{40}\right)} = 0.484\text{kgm}^2$$

The moment of inertia of the flywheel is 0.484kgm^2 .

- Using $\frac{1}{2}I\omega^2 = \tau_f \times 2\pi n_2$ and making τ_f the subject:

$$\tau_f = \frac{I\omega^2}{4\pi n_2} = \frac{0.484\text{kgm}^2 \times (10.05\text{rad/s})^2}{4\pi \times 40} = 0.0973\text{Nm}$$

The friction torque is 0.097Nm.

Making Sense of the Answer: The moment of inertia of 0.484kgm^2 is a reasonable value for a laboratory flywheel. The friction torque (0.097Nm) is small, which is expected for a well-lubricated bearing, and it allows the flywheel to spin for 40 revolutions after detachment, consistent with low friction. The translational KE of the falling mass $\left(\frac{1}{2} \times 2 \times 0.302^2 = 0.091\text{J}\right)$ is tiny compared to the potential energy released (58.8J), which tells us that almost all the energy went into spinning the flywheel and overcoming friction. This is typical: the falling mass moves slowly because the axle radius is small.

Think Like a Physicist: This experiment is a beautiful application of conservation of energy. Three energy terms (translational KE, rotational KE, friction work) compete for the gravitational PE released by the falling mass. The clever part is using the coasting phase (after detachment) to independently determine the friction torque, which then feeds back into the main energy equation. Without this trick, we would have two unknowns (I and τ_f) in one equation. By splitting the motion into two phases (winding and coasting), we get two equations for two unknowns. This is experimental physics at its best.

So far, every object in this chapter has either rotated in place (spinning about a fixed axis) or translated in a straight line. Real objects often do both at the same time. A ball rolling down a hill translates forward while spinning about its own centre. A wheel on a car translates along the road while rotating about its axle. In the next subtopic, we bring translation and rotation together for the first time, and discover that the shape of a rolling object determines how fast it goes. The famous rolling race is about to begin.

ROLLING MOTION

A ball rolls down a hill. A wheel rolls along a road. A cylinder rolls off a table. In each case, the object does two things at once: it translates (the centre moves forward) and it rotates (the body spins about its centre). Neither description alone captures the full motion. Translation without rotation is sliding. Rotation without translation is spinning in place. **Rolling** is the beautiful combination of both.

This subtopic brings together everything we have built in this chapter. The moment of inertia decides how stubborn the body is about spinning. The radius of gyration decides who wins the rolling race. The energy methods of the previous subtopic decide how fast the body moves. And Newton's second law, applied simultaneously to translation and rotation, reveals the role of friction in making rolling possible in the first place.

Rolling motion is where rotational dynamics meets the real world. Let us begin.

Pure Rolling: The No-slip Condition

When a wheel rolls along a road without skidding, the bottom of the wheel is momentarily at rest relative to the road. There is no sliding, no skidding, no slipping. This is called **pure rolling** (or rolling without slipping), and it imposes a precise mathematical relationship between the translational velocity of the centre and the angular velocity of the body.

The constraint:

Consider a circular body (sphere, cylinder, disc, wheel) of radius R rolling without slipping on a flat surface. In one complete revolution, the body rotates through 2π radians. During this revolution, the centre of mass advances by exactly one circumference, $2\pi R$, because each point on the rim touches the ground exactly once.

In general, if the body rotates through angle θ , the centre of mass advances by:

$$s = R\theta$$

Differentiating with respect to time:

$$\frac{ds}{dt} = R \frac{d\theta}{dt}$$

$$\mathbf{v}_{\text{cm}} = \boldsymbol{\omega} R$$

This is the **no-slip condition** for pure rolling. It connects the translational speed of the centre of mass to the angular velocity. If you know one, you know the other.

Differentiating once more:

$$\frac{dv_{\text{cm}}}{dt} = R \frac{d\omega}{dt}$$

$$\mathbf{a}_{\text{cm}} = \boldsymbol{\alpha} R$$

The translational acceleration of the centre of mass equals the angular acceleration times the radius.

What pure rolling means at the contact point:

At the contact point (where the body touches the surface), two velocities combine:

- 1) The translational velocity of the centre: v_{cm} (forward)
- 2) The rotational velocity of the contact point relative to the centre: ωR (backward, because the bottom of the body moves backward relative to the centre)

The net velocity of the contact point is:

$$v_{\text{contact}} = v_{\text{cm}} - \omega R$$

For pure rolling, $v_{\text{cm}} = \omega R$, so:

$$v_{\text{contact}} = \omega R - \omega R = 0$$

The contact point has **zero velocity**. It is instantaneously at rest. This is the defining feature of pure rolling: no sliding at the contact point.

What happens at other points on the rim:

The topmost point has the translational velocity v_{cm} (forward) plus the rotational velocity ωR (also forward at the top), giving:

$$v_{\text{top}} = v_{\text{cm}} + \omega R = 2v_{\text{cm}}$$

The top of the rolling body moves at twice the speed of the centre. This is why mud flies backward off the top of a spinning car tyre, not off the bottom.

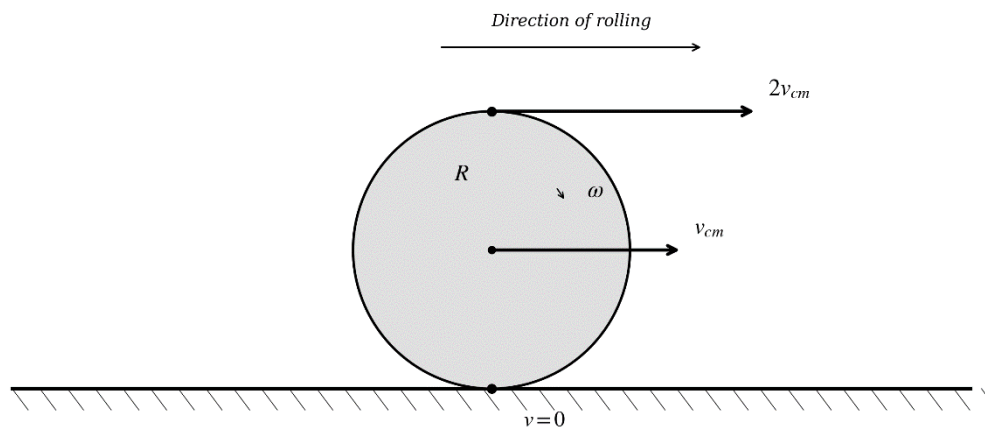


Figure: A circular body of radius R rolling without slipping on a flat surface. At the centre of mass, the velocity is v_{cm} (forward). At the contact point (bottom), the translational velocity v_{cm} and the rotational velocity ωR cancel exactly, giving zero net velocity. At the topmost point, the two velocities add, giving $2v_{\text{cm}}$. The angular velocity ω satisfies the no-slip condition $v_{\text{cm}} = \omega R$.

Three types of motion:

1. **Pure rolling** ($v_{\text{cm}} = \omega R$): no slipping at the contact point.
2. **Skidding** ($v_{\text{cm}} > \omega R$): the body slides forward faster than it spins. The contact point moves forward. Kinetic friction acts backward on the body, slowing the translation and speeding up the rotation until pure rolling is established.
3. **Spinning** ($v_{\text{cm}} < \omega R$): the body spins faster than it translates. The contact point moves backward. Kinetic friction acts forward on the body, speeding up the translation and slowing the rotation until pure rolling is established.

In all three cases, friction acts to bring v_{cm} and ωR into agreement. Once pure rolling is achieved, the friction at the contact point becomes **static friction** (no sliding), and it does **no work** (because the contact point is at rest). This is a crucial point that we will return to later.

Total Kinetic Energy of a Rolling Body

A rolling body has both translational and rotational KE. The total KE is their sum:

$$\text{KE}_{\text{total}} = \text{KE}_{\text{trans}} + \text{KE}_{\text{rot}} = \frac{1}{2} M v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega^2$$

Using the no-slip condition $v_{\text{cm}} = \omega R$ and $I_{\text{cm}} = M k^2$:

$$\text{KE}_{\text{total}} = \frac{1}{2} M v_{\text{cm}}^2 + \frac{1}{2} M k^2 \left(\frac{v_{\text{cm}}}{R} \right)^2$$

$$= \frac{1}{2} M v_{cm}^2 \left(1 + \frac{k^2}{R^2} \right)$$

$$KE_{total} = \frac{1}{2} M v_{cm}^2 \left(1 + \frac{k^2}{R^2} \right)$$

This is a powerful result. The factor $\left(1 + \frac{k^2}{R^2} \right)$ tells us how much extra KE the body has because of its rotation. For a solid sphere ($\frac{k^2}{R^2} = \frac{2}{5}$), the total KE is $\frac{7}{5}$ times the translational KE alone. For a hollow cylinder ($\frac{k^2}{R^2} = 1$), it is exactly double.

Energy partition:

The fraction of total KE that is translational:

$$\frac{KE_{trans}}{KE_{total}} = \frac{\frac{1}{2} M v_{cm}^2}{\frac{1}{2} M v_{cm}^2 \left(1 + \frac{k^2}{R^2} \right)} = \frac{1}{1 + \frac{k^2}{R^2}}$$

The fraction that is rotational:

$$\frac{KE_{rot}}{KE_{total}} = \frac{\frac{1}{2} M k^2 \left(\frac{v_{cm}}{R} \right)^2}{\frac{1}{2} M v_{cm}^2 \left(1 + \frac{k^2}{R^2} \right)} = \frac{\frac{k^2}{R^2}}{1 + \frac{k^2}{R^2}}$$

Rolling Down an Inclined Plane: Energy Method

A body of mass M and radius R starts from rest at the top of an inclined plane of height h and rolls without slipping to the bottom.

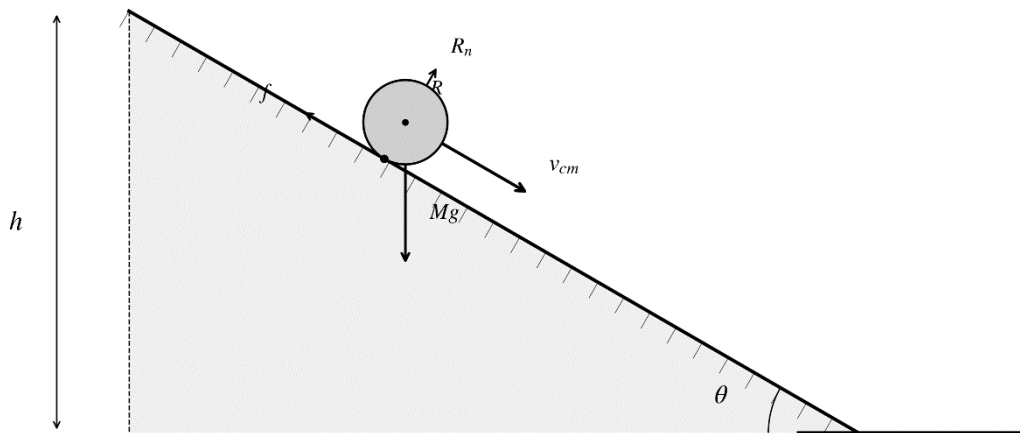


Figure: A body of mass M and radius R rolling without slipping down an inclined plane of height h and angle θ . The forces acting on the body are: weight Mg (vertically downward from the centre of mass), normal reaction R_n (perpendicular to the surface at the contact point), and static friction f (up the slope at the contact point). The velocity of the centre of mass v_{cm} is directed down the slope. The contact point, where friction acts, is instantaneously at rest during pure rolling.

Speed at the bottom:

By conservation of energy, the gravitational PE at the top converts entirely into KE at the bottom (no work done by friction, since static friction at the contact point does no work during pure rolling):

$$Mgh = \frac{1}{2} M v_{cm}^2 \left(1 + \frac{k^2}{R^2} \right)$$

Making v_{cm} the subject:

$$v_{\text{cm}}^2 = \frac{2gh}{1 + \frac{k^2}{R^2}}$$

$$v_{\text{cm}} = \sqrt{\frac{2gh}{1 + \frac{k^2}{R^2}}}$$

Compare this with a sliding (non-rolling) body, which would have $v = \sqrt{2gh}$. The rolling body is always slower, by the factor $\frac{1}{\sqrt{1+k^2/R^2}}$, because some energy goes into rotation.

Acceleration down the incline:

Using $v^2 = u^2 + 2as$ with $u = 0$ and $v^2 = \frac{2gh}{1+k^2/R^2}$, and noting that $h = s\sin\theta$ where s is the distance along the incline and θ is the angle of inclination:

$$\frac{2gssin\theta}{1 + k^2/R^2} = 2as$$

$$a = \frac{gsin\theta}{1 + \frac{k^2}{R^2}}$$

This is a constant acceleration (independent of position), which means the body rolls down with uniform acceleration, just like a block sliding down a frictionless incline but with the effective gravitational pull reduced by the factor $\frac{1}{1+k^2/R^2}$.

Key observations:

1. The acceleration depends on the **shape** (through $\frac{k^2}{R^2}$) but not on the **mass** or the **size**. A marble and a bowling ball (both solid spheres) have the same acceleration and reach the bottom at the same time. This is the rotational equivalent of Galileo's discovery that all objects fall at the same rate.
2. The acceleration is always less than $gsin\theta$ (the acceleration of a sliding body on a frictionless incline). The "missing" acceleration goes into spinning the body up.
3. The smaller the $\frac{k^2}{R^2}$, the larger the acceleration and the faster the body reaches the bottom.

Now let us put numbers to these results.

BINDER Example 40

A solid sphere of mass 2kg and radius 0.08m rolls without slipping down an inclined plane of height 1.5m and angle 30° . Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the speed of the centre of mass at the bottom of the incline.
- (b) Calculate the translational and rotational KE at the bottom.
- (c) Calculate the acceleration down the incline.
- (d) Calculate the time to reach the bottom.

Solution

- (a) For a solid sphere: $\frac{k^2}{R^2} = \frac{2}{5}$.

Using $v_{\text{cm}} = \sqrt{\frac{2gh}{1+k^2/R^2}}$ and substituting:

$$v_{\text{cm}} = \sqrt{\frac{2 \times 9.8 \text{ m/s}^2 \times 1.5 \text{ m}}{1 + \frac{2}{5}}} = 4.58 \text{ m/s}$$

The speed at the bottom is 4.58 m/s.

(b) Translational KE:

$$\text{KE}_{\text{trans}} = \frac{1}{2} M v_{\text{cm}}^2 = \frac{1}{2} \times 2 \text{ kg} \times (4.58 \text{ m/s})^2 = 21.0 \text{ J}$$

$$\text{Total KE} = Mgh = 2 \times 9.8 \text{ m/s}^2 \times 1.5 \text{ m} = 29.4 \text{ J}$$

$$\text{Rotational KE} = \text{Total} - \text{Translational} = (29.4 - 21.0) \text{ J} = 8.4 \text{ J}$$

The translational KE is 21.0 J and the rotational KE is 8.4 J.

(c) Using $a = \frac{g \sin \theta}{1 + k^2/R^2}$ and substituting:

$$a = \frac{9.8 \text{ m/s}^2 \times \sin 30^\circ}{1 + \frac{2}{5}} = 3.50 \text{ m/s}^2$$

The acceleration is 3.50 m/s^2 .

$$(d) \text{ The length of the incline: } s = \frac{h}{\sin \theta} = \frac{1.5 \text{ m}}{\sin 30^\circ} = \frac{1.5}{0.5} = 3.0 \text{ m}$$

Using $s = \frac{1}{2} at^2$ (starting from rest) and making t the subject:

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 3.0 \text{ m}}{3.50 \text{ m/s}^2}} = 1.31 \text{ s}$$

The time to reach the bottom is 1.31 s.

Making Sense of the Answer: If the sphere were sliding (not rolling) on a frictionless incline, it would reach $v = \sqrt{2 \times 9.8 \times 1.5} = 5.42 \text{ m/s}$. The rolling sphere is slower (4.58 m/s, which is 84% of the sliding speed) because 29% of the energy goes into rotation. The acceleration (3.50 m/s^2) is 71% of the sliding value ($g \sin 30^\circ = 4.9 \text{ m/s}^2$). Notice that neither the mass (2 kg) nor the radius (0.08 m) appeared in the final answers for speed and acceleration. Only the shape matters.

Think Like a Physicist: This example uses the energy method: conservation of energy gives the speed, kinematics gives the acceleration and time. No forces or torques are needed. The energy method is elegant but has one limitation: it does not tell us the friction force. For that, we need the force method. But for finding speed, acceleration, and time, the energy method is unbeatable.

REAL Example 41

After school, Kipanga and Kipute are cycling home. Kipanga notices that when his bicycle rides through a muddy puddle, the mud flies off the **top** of the wheel, not the bottom. He expected the mud to fly off from the bottom, where the wheel touches the mud.

Explain why the mud flies off from the top of the spinning wheel.

Solution

At the contact point (bottom of the wheel), the velocity is zero because the translational velocity v_{cm} (forward) and the rotational velocity ωR (backward) cancel exactly. Since the contact point is momentarily at rest, mud at the bottom has no tendency to fly off.

At the topmost point, the two velocities add: the translational velocity v_{cm} (forward) and the rotational velocity ωR (also forward at the top) give a total velocity of $2v_{\text{cm}}$. This is the fastest-moving point on the wheel. Mud at the top is moving at twice the bicycle's speed and is flung off tangentially when the adhesion to the tyre can no longer provide the centripetal force needed to keep it on the curved path.

Making Sense of the Answer: The bottom of the wheel is the safest place for mud: it is momentarily at rest. The top is the most dangerous: it moves at $2v_{cm}$, the fastest point on the entire wheel. This is also why mudguards on bicycles are placed above the wheel, not below it. The mud threat comes from the top, not the bottom.

Think Like a Physicist: This observation is a direct consequence of the no-slip condition $v_{cm} = \omega R$. The velocity at any point on the wheel is the vector sum of the translational velocity (same for all points) and the rotational velocity (different direction at each point). At the bottom they cancel. At the top they reinforce. At the sides they are perpendicular, giving $v = \sqrt{2} v_{cm}$ at 45° to the horizontal.

HOT Example 42

A solid cylinder and a solid sphere, both of mass M and radius R , are released from rest at the top of an inclined plane of angle θ and length L . They roll without slipping.

- Show that the cylinder arrives at the bottom later than the sphere, and find the ratio of their arrival times.
- If $\theta = 20^\circ$ and $L = 2.5\text{m}$, calculate the arrival time for each. Take $g = 9.8\text{m/s}^2$.
- How much later does the cylinder arrive?

Solution

- The acceleration of a rolling body is $a = \frac{g \sin \theta}{1 + k^2/R^2}$.

For the solid sphere: $\frac{k^2}{R^2} = \frac{2}{5}$, so $a_{\text{sphere}} = \frac{g \sin \theta}{\frac{7}{5}} = \frac{5g \sin \theta}{7}$

For the solid cylinder: $\frac{k^2}{R^2} = \frac{1}{2}$, so $a_{\text{cylinder}} = \frac{g \sin \theta}{\frac{3}{2}} = \frac{2g \sin \theta}{3}$

Since $\frac{5}{7} > \frac{2}{3}$, the sphere has the larger acceleration and arrives first.

Using $L = \frac{1}{2}at^2$ and making t the subject: $t = \sqrt{\frac{2L}{a}}$

$$\frac{t_{\text{cylinder}}}{t_{\text{sphere}}} = \sqrt{\frac{a_{\text{sphere}}}{a_{\text{cylinder}}}} = \sqrt{\frac{5/7}{2/3}} = \mathbf{1.0351}$$

(The cylinder takes 3.5% longer than the sphere. The ratio does not depend on θ , L , M , R , or g : it is a universal constant for the sphere versus cylinder race.)

- For the sphere:

$$a_{\text{sphere}} = \frac{5 \times 9.8\text{m/s}^2 \times \sin 20^\circ}{7} = 2.394\text{m/s}^2$$

$$t_{\text{sphere}} = \sqrt{\frac{2 \times 2.5\text{m}}{2.394\text{m/s}^2}} = 1.445\text{s}$$

For the cylinder:

$$a_{\text{cylinder}} = \frac{2 \times 9.8\text{m/s}^2 \times \sin 20^\circ}{3} = 2.235\text{m/s}^2$$

$$t_{\text{cylinder}} = \sqrt{\frac{2 \times 2.5\text{m}}{2.235\text{m/s}^2}} = 1.496\text{s}$$

The sphere arrives in 1.445s and the cylinder in 1.496s.

- Time difference: $t_{\text{cylinder}} - t_{\text{sphere}} = 1.496\text{s} - 1.445\text{s} = 0.051\text{s}$

The cylinder arrives 0.051 seconds (51 milliseconds) after the sphere.

Check: $\text{ratio} = \frac{1.496}{1.445} = 1.035$. Matches part (a). ✓

Making Sense of the Answer: The difference is only 51 milliseconds on a 2.5m ramp. This is small but measurable with a stopwatch if you know what to look for. On a longer ramp, the gap grows (it scales as $\sqrt{L}$). On a 10m ramp, the gap would be about 0.1s, easily visible. The fact that the ratio $\sqrt{15/14}$ is independent of everything except the shapes is remarkable: the same 3.5% time advantage holds whether the ramp is steep or gentle, short or long, on Earth or on the Moon.

Think Like a Physicist: Part (a) is the kind of algebraic proof that examiners love: show that the ratio depends only on shape, not on any other variable. The trick is to express both times in terms of $L/g\sin\theta$ and take the ratio, which cancels everything except the shape factors. Part (b) puts concrete numbers to the algebra. Part (c) shows that while the sphere always wins, the margin is slim: the cylinder is a respectable runner-up, only 3.5% behind.

Rolling Down an Inclined Plane: Force and Torque method

The energy method gives us the speed and acceleration of a rolling body but tells us nothing about the forces involved. In particular, it cannot tell us the friction force. To find friction, we must apply Newton's second law for translation and Newton's second law for rotation simultaneously.

Setup: A body of mass M and radius R rolls without slipping down an inclined plane of angle θ (refer to the diagram provided in the previous subsection: energy method).

Three forces act on the body:

- 1) Weight Mg vertically downward (through the centre of mass)
- 2) Normal reaction R_n perpendicular to the surface (at the contact point)
- 3) Static friction f up the slope (at the contact point)

Translational equation (along the incline, taking down the slope as positive):

$$Mg\sin\theta - f = Ma_{\text{cm}} \quad \dots (i)$$

Rotational equation (about the centre of mass, taking clockwise as positive since the body rolls down):

The weight acts through the centre of mass, so its torque about the centre of mass is zero. The normal reaction acts at the point of contact and is directed along the radius of the body, passing through the centre of mass; hence, it also produces zero torque about the centre. Only friction acts at the contact point in a direction that produces a nonzero torque.

$$fR = I_{\text{cm}}\alpha \quad \dots (ii)$$

Constraint: $a_{\text{cm}} = \alpha R$, so $\alpha = \frac{a_{\text{cm}}}{R}$

Substituting into (ii):

$$fR = I_{\text{cm}} \times \frac{a_{\text{cm}}}{R}$$

$$f = \frac{I_{\text{cm}} a_{\text{cm}}}{R^2} \quad \dots (iii)$$

Substituting (iii) into (i):

$$Mg\sin\theta - \frac{I_{\text{cm}} a_{\text{cm}}}{R^2} = Ma_{\text{cm}}$$

$$Mg\sin\theta = Ma_{\text{cm}} + \frac{I_{\text{cm}} a_{\text{cm}}}{R^2} = a_{\text{cm}} \left(M + \frac{I_{\text{cm}}}{R^2} \right)$$

Using $I_{\text{cm}} = Mk^2$:

$$Mg\sin\theta = a_{\text{cm}} \times M \left(1 + \frac{k^2}{R^2} \right)$$

Making a_{cm} the subject:

$$a_{\text{cm}} = \frac{g \sin \theta}{1 + \frac{k^2}{R^2}}$$

This is exactly the same result as the energy method. Good. The two methods must agree.

Finding the friction force:

Substituting a_{cm} back into (iii) where $I_{\text{cm}} = Mk^2$:

$$f = \frac{Mk^2}{R^2} \times \frac{g \sin \theta}{1 + \frac{k^2}{R^2}} = \frac{Mg \sin \theta \times \frac{k^2}{R^2}}{1 + \frac{k^2}{R^2}} = \frac{Mg \sin \theta \times \frac{k^2}{R^2} \left(\frac{R^2}{k^2} \right)}{1 + \frac{k^2}{R^2}} = \frac{Mg \sin \theta}{\frac{R^2}{k^2} + 1}$$

$$f = \frac{Mg \sin \theta}{1 + \frac{R^2}{k^2}}$$

This is the result the energy method could not give us. The friction force depends on the shape (through k^2/R^2), the mass, and the angle, but not on the radius. Let us see what it gives for specific shapes:

Body	$\frac{k^2}{R^2}$	Friction f
Solid sphere	$\frac{2}{5}$	$\frac{2}{7} Mg \sin \theta$
Solid cylinder	$\frac{1}{2}$	$\frac{1}{3} Mg \sin \theta$
Hollow sphere	$\frac{2}{3}$	$\frac{2}{5} Mg \sin \theta$
Hollow cylinder	1	$\frac{1}{2} Mg \sin \theta$

The hollow cylinder needs the most friction ($\frac{1}{2} Mg \sin \theta$) and the solid sphere the least ($\frac{2}{7} Mg \sin \theta$). This has practical consequences: on a surface with limited friction, the hollow cylinder is the first to start slipping, while the solid sphere is the last.

The Condition for Pure Rolling

For pure rolling to be maintained, the required friction must not exceed the maximum available static friction:

$$f \leq \mu_s R_n$$

Since $R_n = Mg \cos \theta$ on an incline:

$$\frac{Mg \sin \theta}{1 + \frac{R^2}{k^2}} \leq \mu_s Mg \cos \theta$$

Dividing $\cos \theta$ throughout:

$$\frac{\tan \theta}{1 + \frac{R^2}{k^2}} \leq \mu_s$$

$$\tan \theta \leq \mu_s \left(1 + \frac{R^2}{k^2} \right)$$

If the incline is too steep (large θ) or the surface too smooth (small μ_s), the body cannot roll without slipping. It will slide and spin simultaneously.

For a solid sphere: $\tan\theta \leq \mu_s \left(1 + \frac{5}{2}\right) = \frac{7}{2}\mu_s$, so pure rolling holds for $\tan\theta \leq 3.5\mu_s$.

For a hollow cylinder: $\tan\theta \leq \mu_s(1 + 1) = 2\mu_s$, so pure rolling holds for $\tan\theta \leq 2\mu_s$.

The solid sphere can roll on steeper inclines than the hollow cylinder before slipping begins, because it demands less friction.

The Role of Friction in Rolling

Students often find friction in rolling confusing. Here is a clear summary:

1. Static friction, not kinetic: During pure rolling, the contact point is at rest. Therefore, the friction is **static**, not kinetic. This is the same type of friction that prevents a book from sliding on a tilted table.

2. Friction does no work: Because the contact point (where friction acts) is instantaneously at rest, the friction force does no work. Work requires force and displacement at the same point, and the displacement of the contact point is zero. This is why the energy method works without including friction: it contributes zero to the energy equation.

3. Friction is essential for rolling: Without friction, a body on an incline would slide, not roll. Friction provides the torque about the centre of mass that spins the body up. Remove friction (a perfectly smooth surface), and the body slides without rotating, like a block on a frictionless incline. Friction is the agent that converts some of the translational acceleration into rotational acceleration.

4. Friction acts up the slope: When a body rolls down an incline, friction acts up the slope. This may seem strange (friction opposing the translational motion), but it makes sense: friction must produce a torque that spins the body in the rolling direction. The only way to produce a clockwise torque (for a body rolling down to the right) is for friction to act up the slope at the contact point.

5. Friction does not always act up the slope: If a body is rolling up an incline (decelerating), friction acts **down** the slope. It must now decelerate the spin as well as the translation. The direction of friction depends on the situation, not on a fixed rule.

Now that we understand both the energy and force methods, and the role of friction, let us tackle some challenging problems.

BINDER Example 43

A solid cylinder of mass 4kg and radius 0.10m rolls without slipping down an incline of angle 25° . Take $g = 9.8\text{m/s}^2$.

- Calculate the acceleration of the centre of mass.
- Calculate the friction force.
- Calculate the minimum coefficient of static friction required for pure rolling.

Solution

- (a) For a solid cylinder: $I_{\text{cm}} = Mk^2 = \frac{1}{2}MR^2$. So:

$$\frac{k^2}{R^2} = \frac{1}{2}.$$

Using $a = \frac{g\sin\theta}{1+k^2/R^2}$ and substituting:

$$a = \frac{9.8\text{m/s}^2 \times \sin 25^\circ}{1 + \frac{1}{2}} = 2.76\text{m/s}^2$$

The acceleration is 2.76m/s^2 .

- (b) Using $f = \frac{Mg\sin\theta}{1+R^2/k^2}$ and substituting:

$$f = \frac{4\text{kg} \times 9.8\text{m/s}^2 \times \sin 25^\circ}{1 + \frac{1}{1/2}} = 5.52\text{N}$$

The friction force is 5.52N, acting up the slope.

(c) For pure rolling: $f \leq \mu_s Mg \cos \theta$

Making μ_s the subject:

$$\mu_s \geq \frac{f}{Mg \cos \theta} = \frac{5.52\text{N}}{4\text{kg} \times 9.8\text{m/s}^2 \times \cos 25^\circ} = 0.155$$

The minimum coefficient of static friction is 0.155.

Making Sense of the Answer: The friction force (5.52N) is about one-third of the gravitational component along the slope ($Mg \sin 25^\circ = 16.56\text{N}$). This is exactly the fraction $\frac{1}{3}$ from the friction table for a solid cylinder. The required μ_s of 0.155 is moderate, is easily achieved by most surfaces (rubber on concrete has $\mu_s \approx 0.6$), so the cylinder rolls comfortably on most real surfaces. Only very slippery surfaces (ice, oiled metal) would cause it to slide.

Think Like a Physicist: The force method gives three things the energy method cannot: the friction force, the normal reaction, and the condition for pure rolling. Whenever a problem asks for “the friction force” or “the minimum coefficient of friction,” you must use the force method. The energy method is faster for speed and acceleration but blind to forces.

REAL Example 44

On a rainy day, Kipute observes a car wheel spinning in place on a muddy patch. The car is stuck: the engine roars, the wheel spins, but the car does not move forward.

- (i) Explain, using the physics of rolling, why the car cannot move.
- (ii) Suggest any practical solution that will enable the car to move forward.

Solution

- (i) For a wheel to propel a car forward, it must roll without slipping. During pure rolling, static friction at the contact point pushes the car forward (this is the reaction to the wheel pushing backward on the road). The wheel’s rotation is converted into forward translation through friction. On a muddy surface, the coefficient of static friction is very low. The torque from the engine spins the wheel, but the mud cannot provide enough static friction to prevent slipping. The contact point slides backward (the wheel spins faster than v_{cm}/R), and kinetic friction replaces static friction. Kinetic friction is weaker than static friction and cannot accelerate the car effectively. The wheel spins in place, the car barely moves, and the mud flies everywhere.
- (ii) The solution is to increase the available friction: place sand, gravel, or a mat under the wheel. This raises μ_s , restores the no-slip condition, and allows the wheel to grip the road again.

Making Sense of the Answer: A spinning wheel that cannot grip the road is the rolling condition in reverse: the wheel violates $v_{\text{cm}} = \omega R$ because ωR far exceeds v_{cm} . Friction tries to bring them into agreement by accelerating the car forward and decelerating the spin, but on mud the friction is too weak to do this effectively. The car is stuck because rolling requires friction, and the mud cannot provide enough.

Think Like a Physicist: The condition for pure rolling requires μ_s to be above a minimum value. On mud, μ_s drops below this threshold, and the wheel transitions from rolling to spinning. The physics of a stuck car is the same physics as a cylinder failing to roll on a steep, smooth incline: insufficient friction to maintain the no-slip condition.

HOT Example 45

A solid sphere and a hollow cylinder, both of mass M and radius R , are released from rest at the top of a rough inclined plane of angle θ and height h . Both roll without slipping.

- (a) Find the friction force on each body as a fraction of $Mg \sin \theta$.
- (b) Show that the hollow cylinder requires 75% more friction than the solid sphere.
- (c) If $\mu_s = 0.25$ and $\theta = 35^\circ$, determine which body (if any) can roll without slipping.

Solution

(a) From the friction formula $f = \frac{Mg \sin \theta}{1 + R^2/k^2}$:

Solid sphere ($k^2/R^2 = 2/5$): $\frac{R^2}{k^2} = \frac{5}{2}$

$$f_{\text{sphere}} = \frac{Mg \sin \theta}{1 + \frac{5}{2}} = \frac{Mg \sin \theta}{\frac{7}{2}} = \frac{2}{7} Mg \sin \theta$$

Hollow cylinder ($k^2/R^2 = 1$): $\frac{R^2}{k^2} = 1$

$$f_{\text{cylinder}} = \frac{Mg \sin \theta}{1 + 1} = \frac{1}{2} Mg \sin \theta$$

The solid sphere needs $\frac{2}{7}$ of $Mg \sin \theta$ and the hollow cylinder needs $\frac{1}{2}$.

(b) The ratio:

$$\frac{f_{\text{cylinder}}}{f_{\text{sphere}}} = \frac{1/2}{2/7} = 1.75$$

$$\% \text{ increase} = \left(\frac{1.75 - 1}{1} \right) \times 100 = 75\%$$

Hence, the hollow cylinder requires **75% more friction** than the solid sphere.

(c) For pure rolling, we need $f \leq \mu_s Mg \cos \theta$.

For the solid sphere:

$$\frac{2}{7} Mg \sin \theta \leq \mu_s Mg \cos \theta$$

$$\frac{2}{7} \sin \theta \leq \mu_s \cos \theta$$

$$\frac{2}{7} \tan \theta \leq \mu_s$$

$$\frac{2}{7} \tan 35^\circ = 0.2 < \mu_s (0.25)$$

This is satisfied. The solid sphere rolls without slipping.

Performing the same for the hollow cylinder:

$$\frac{1}{2} \tan 35^\circ = 0.35 > \mu_s (0.25)$$

This is **not** satisfied. The hollow cylinder cannot roll without slipping at 35° with $\mu_s = 0.25$. It will slide and spin.

The solid sphere rolls without slipping, but the hollow cylinder slides.

Hence, the body which can roll without slipping is solid sphere.

Making Sense of the Answer: On the same incline with the same friction, the solid sphere rolls cleanly while the hollow cylinder fails. This is because the hollow cylinder demands more friction (it has more rotational inertia to spin up) but the surface cannot provide it. The solid sphere is the most “friction-efficient” roller: it demands the least friction of all standard shapes and can therefore roll on the widest range of surfaces and angles. The hollow cylinder is the least efficient, requiring the most friction and failing first on slippery surfaces.

Think Like a Physicist: Part (c) is exactly the kind of question that appears in examinations: “determine whether the body rolls or slides.” The method is always the same: calculate the required friction from the rolling formula, compare with $\mu_s Mg \cos \theta$, and check the inequality. If satisfied, pure rolling. If not, the body slides.

That concludes our study of rolling motion: the no-slip condition, the energy partition, the acceleration down an incline (by both energy and force methods), the friction requirement, and the condition for pure rolling. Rolling is where every idea in this chapter comes together: moment of inertia, radius of gyration, torque, energy, and Newton's laws for both translation and rotation, all working in concert.

In the next subtopic, we turn to the last great conservation law of mechanics: angular momentum, the rotational twin of linear momentum. It will complete the twin language and open the door to some of the most spectacular phenomena in physics, from pirouetting ice skaters to collapsing stars.

ANGULAR MOMENTUM AND ITS CONSERVATION

In Chapter 5, we discovered that when no external force acts on a system, its total linear momentum is conserved. This principle explained collisions, explosions, recoil, and rocket propulsion. Now we meet the rotational twin: when no external torque acts on a system, its total **angular momentum** is conserved. This principle explains why an ice skater spins faster when she pulls her arms in, why a collapsing star spins up to incredible speeds, why a cat always lands on its feet, and why a bicycle stays upright while moving.

Angular momentum completes the twin language. With it, every translational concept from Chapters 2 through 5 has a rotational counterpart:

Translational	Rotational
Force F	Torque τ
Mass m	Moment of inertia I
Velocity v	Angular velocity ω
Acceleration a	Angular acceleration α
Momentum $p = mv$	Angular momentum $L = I\omega$
$F = ma$	$\tau = I\alpha$
$F = dp/dt$	$\tau = dL/dt$
If $F = 0$, $p = \text{constant}$	If $\tau = 0$, $L = \text{constant}$

Definition of angular momentum

The **angular momentum** L of a rigid body rotating about a fixed axis is defined as *the product of moment of inertia and its angular velocity*:

$$L = I\omega$$

where I is the moment of inertia about the rotation axis and ω is the angular velocity.

The unit of angular momentum is $\text{kgm}^2\text{s}^{-1}$ (or equivalently, $\text{kgm}^2\text{rad/s}$, but since radians are dimensionless, these are the same).

Angular momentum is a vector quantity. Its direction is along the rotation axis, given by the right-hand rule: curl the fingers of the right hand in the direction of rotation, and the thumb points in the direction of L .

For a **single particle** of mass m moving with velocity v at perpendicular distance r from the axis:

$$L = mvr$$

This is consistent with $L = I\omega$ because for a single particle, $I = mr^2$ and $v = \omega r$, giving $L = mr^2 \times \frac{v}{r} = mvr$.

Torque and Angular Momentum

Just as force equals the rate of change of linear momentum ($F = \frac{dp}{dt}$), torque equals the rate of change of angular momentum:

$$\tau = \frac{dL}{dt}$$

Derivation:

Starting from $\tau = I\alpha$ and using $\alpha = \frac{d\omega}{dt}$:

$$\tau = I \frac{d\omega}{dt}$$

If I is constant (rigid body, fixed axis): $I \frac{d\omega}{dt} = \frac{d(I\omega)}{dt}$

$$\tau = \frac{d(I\omega)}{dt} = \frac{dL}{dt}$$

This is the rotational form of Newton's second law in its most general form. It is more general than $\tau = I\alpha$ because it also applies when I changes (as in the spinning skater problem).

Conservation of Angular Momentum

Angular momentum is conserved when no external torque acts on a system. If the net external torque on a system is zero:

$$\tau_{\text{ext}} = 0 \Rightarrow \frac{dL}{dt} = 0 \Rightarrow L = \text{constant, and hence:}$$

$$I_1\omega_1 = I_2\omega_2$$

This is the **law of conservation of angular momentum**: if no external torque acts on a system, the total angular momentum remains constant, even if the moment of inertia changes.

The key insight: When I decreases, ω must increase to keep L constant. When I increases, ω must decrease. This is the physics behind every spinning-and-pulling-in phenomenon in nature.

Applications of Conservation of Angular Momentum**1. Ice skater pulling arms in:**

A skater begins a spin with arms outstretched (large I , small ω). She then pulls her arms close to her body, reducing I . Since no external torque acts about her vertical axis (friction at the ice is negligible), angular momentum is conserved: $I_1\omega_1 = I_2\omega_2$. With $I_2 < I_1$, we get $\omega_2 > \omega_1$. She spins faster. To slow down, she extends her arms again.

2. Diver performing somersaults:

A diver leaves the board with some angular momentum. In the air, no external torque acts about the centre of mass (gravity acts through the CM, producing zero torque about it). To spin faster (for somersaults), the diver pulls into a tight tuck (small I , large ω). To stop spinning before entry, the diver straightens out (large I , small ω). The angular momentum is constant throughout.

3. Collapsing star (neutron star):

A large, slowly rotating star collapses under gravity. Its radius shrinks enormously (from millions of km to about 10 km), reducing I by a factor of 10^{10} or more. Conservation of angular momentum demands a corresponding increase in ω . The resulting neutron star spins at hundreds of revolutions per second. This is the most dramatic example of angular momentum conservation in nature.

4. Rotating chair with dumbbells:

Kipute sits on a freely rotating chair holding two heavy dumbbells with arms outstretched and is given a gentle spin. She rotates slowly (large I , small ω). She pulls the dumbbells to her chest, reducing I . With no external torque (the chair rotates freely), angular momentum is conserved and she spins faster. Extending the arms again restores the original slow spin. This is the classroom version of the ice skater.

5. Planet in elliptical orbit:

A planet moves closer to the Sun at perihelion and farther at aphelion. Gravity acts along the radius, producing zero torque about the Sun. Angular momentum $L = mvr$ is conserved. At perihelion (small r), the

planet must move faster (large v). At aphelion (large r), it moves slower. This is Kepler's second law, and it is nothing more than conservation of angular momentum.

6. Spinning raw egg restarting:

In the chapter introduction, Kipute stopped a spinning raw egg and released it. The egg restarted on its own. When she touched the shell, only the shell stopped; the liquid inside kept swirling (its angular momentum was not removed). When released, the still-spinning liquid dragged the shell back into rotation. Angular momentum cannot simply disappear from the liquid without an external torque, and a brief touch on the shell is not enough.

These six examples share a single story: when no external torque acts, angular momentum is locked in. The only freedom the system has is to trade moment of inertia for angular velocity, or vice versa. The product $I\omega$ never changes.

With the principle established and the examples fresh in mind, let us see how the numbers work. And as you will discover, conservation of angular momentum produces some of the most satisfying calculations in this entire chapter.

BINDER Example 46

A disc of moment of inertia 0.4kgm^2 rotates at 10rad/s about its central axis.

- Calculate its angular momentum.
- A ring of moment of inertia 0.1kgm^2 is dropped coaxially onto the spinning disc. The ring is initially not rotating. After contact, they rotate together. Calculate the final angular velocity.
- Show that KE is lost in this process and explain where it goes.

Solution

- Using $L = I\omega$ and substituting:

$$L = 0.4\text{kgm}^2 \times 10\text{rad/s} = 4.0\text{kgm}^2\text{s}^{-1}$$

The angular momentum is $4.0\text{kgm}^2\text{s}^{-1}$.

- No external torque acts on the disc-ring system (the axis is fixed and frictionless at the pivot). Angular momentum is conserved:

$$I_1\omega_1 = (I_1 + I_2)\omega_f$$

Making ω_f the subject:

$$\omega_f = \frac{I_1\omega_1}{I_1 + I_2} = \frac{0.4\text{kgm}^2 \times 10\text{rad/s}}{0.4\text{kgm}^2 + 0.1\text{kgm}^2} = 8.0\text{rad/s}$$

The final angular velocity is 8.0rad/s .

- Initial KE:

$$\text{KE}_i = \frac{1}{2}I_1\omega_1^2 = \frac{1}{2} \times 0.4\text{kgm}^2 \times (10\text{rad/s})^2 = 20\text{J}$$

Final KE:

$$\text{KE}_f = \frac{1}{2}(I_1 + I_2)\omega_f^2 = \frac{1}{2} \times 0.5\text{kgm}^2 \times (8.0\text{rad/s})^2 = 16\text{J} < \text{KE}_i (20\text{J})$$

KE lost: $20 - 16 = 4\text{J}$

The KE decreases by 4J .

This energy is converted into heat and sound at the contact surface as the ring accelerates to match the disc's rotation through friction between the surfaces.

Relate: This is analogous to an inelastic collision in linear momentum: momentum is conserved but KE is not.

Making Sense of the Answer: The ring was dropped onto a spinning disc, so friction between the ring and disc had to bring the ring up to speed. This friction dissipated energy as heat, just as kinetic friction always does. The angular momentum ($4.0\text{kgm}^2\text{s}^{-1}$) is the same before and after, but the KE dropped from 20J to 16J. The fraction lost is $\frac{4}{20} = 20\%$, which equals $\frac{I_2}{I_1+I_2} = \frac{0.1}{0.5} = 20\%$. This fraction depends only on the ratio of moments of inertia, just as in linear inelastic collisions.

Think Like a Physicist: This problem is the rotational twin of a perfectly inelastic collision. A stationary object (ring) is struck by a moving object (disc), and they stick together. Angular momentum is conserved; KE is not. The mathematics is identical to $m_1 v_1 = (m_1 + m_2) v_f$, with I replacing m and ω replacing v . If you understood inelastic collisions in Chapter 5, you already understand this problem.

REAL Example 47

A student sits on a freely rotating stool holding a spinning bicycle wheel horizontally above her head. The wheel spins clockwise when viewed from above. The student then flips the wheel upside down so that it now spins anticlockwise when viewed from above.

- Describe what happens to the student and the stool when the wheel is flipped.
- Explain why this happens using conservation of angular momentum.

Solution

- When the student flips the wheel, the stool and student begin to rotate clockwise (when viewed from above), which is the same direction as the wheel was originally spinning.
- Before the flip, the system (student + stool + wheel) has a total angular momentum equal to that of the spinning wheel. Call this L , directed clockwise from above. The student and stool are initially stationary, contributing zero.

Total before: $L_{\text{total}} = L_{\text{wheel}} + 0 = L$

The stool is free to rotate, so no external torque acts on the system. Therefore L_{total} must remain equal to L after the flip.

When the student flips the wheel upside down, its spin direction reverses. What was clockwise becomes anticlockwise. The wheel now carries angular momentum $-L$ (opposite to the original direction).

Total after: $L_{\text{total}} = L_{\text{student+stool}} + (-L)$

Setting total before equal to total after: $L = L_{\text{student+stool}} - L$

Making $L_{\text{student+stool}}$ the subject: $L_{\text{student+stool}} = 2L$

So the student and stool must acquire angular momentum $2L$ in the original (clockwise) direction. The student and stool therefore spin clockwise (from above) with angular momentum equal to twice the wheel's original angular momentum.

Making Sense of the Answer: Before the flip, only the wheel had angular momentum. After the flip, the wheel has the same magnitude of angular momentum but in the opposite direction, and the student has twice the original angular momentum in the original direction. The total angular momentum is $2L - L = L$, which equals the original total. Nothing was created or destroyed; the angular momentum was merely redistributed within the system.

Think Like a Physicist: This demonstration can be performed in physics class. The surprise is that flipping a wheel makes a person spin on a stool without touching anything. But there is no mystery: the system's total angular momentum is constant. The student "borrows" angular momentum from the wheel by reversing it. If she flips the wheel back, she stops spinning. The angular momentum flows back and forth between wheel and person, always conserving the total.

HOT Example 48

A uniform disc of mass 3kg and radius 0.20m rotates freely at 15rad/s about its central axis. A piece of putty of mass 0.5kg is dropped vertically onto the edge of the disc and sticks there.

- Calculate the angular momentum before the putty lands.
- Calculate the moment of inertia of the system after the putty lands.

- (c) Calculate the final angular velocity.
 (d) Calculate the percentage of KE lost.

Solution

(a) Moment of inertia of the disc: $I_{\text{disc}} = \frac{1}{2}MR^2 = \frac{1}{2} \times 3\text{kg} \times (0.20\text{m})^2 = 0.06\text{kgm}^2$

$$L_i = I_{\text{disc}} \times \omega_i = 0.06\text{kgm}^2 \times 15\text{rad/s} = 0.9\text{kgm}^2\text{s}^{-1}$$

The angular momentum before impact is $0.9\text{kgm}^2\text{s}^{-1}$.

(b) After landing, the putty sits at the rim ($r = R = 0.20\text{m}$). Its moment of inertia about the axis is $I_{\text{putty}} = mR^2 = 0.5\text{kg} \times (0.20\text{m})^2 = 0.02\text{kgm}^2$.

$$I_{\text{total}} = I_{\text{disc}} + I_{\text{putty}} = (0.06 + 0.02)\text{kgm}^2 = 0.08\text{kgm}^2$$

The total moment of inertia is 0.08kgm^2 .

- (c) No external torque acts on the system (the axis is fixed). Angular momentum is conserved:

$$L_i = I_{\text{total}} \times \omega_f$$

Making ω_f the subject:

$$\omega_f = \frac{L_i}{I_{\text{total}}} = \frac{0.9\text{kgm}^2\text{s}^{-1}}{0.08\text{kgm}^2} = 11.25\text{rad/s}$$

The final angular velocity is 11.25rad/s .

(d) Initial KE: $\text{KE}_i = \frac{1}{2} \times 0.06\text{kgm}^2 \times (15\text{rad/s})^2 = 6.75\text{J}$

Final KE: $\text{KE}_f = \frac{1}{2} \times 0.08\text{kgm}^2 \times (11.25\text{rad/s})^2 = 5.06\text{J}$

KE lost: $(6.75 - 5.06)\text{J} = 1.69\text{J}$

Percentage lost: $\frac{1.69\text{J}}{6.75\text{J}} \times 100\% = 25\%$

The system loses 25% of its KE.

Making Sense of the Answer: The disc slows from 15 to 11.25rad/s (a 25% decrease in angular velocity). The moment of inertia increased by 33% (from 0.06 to 0.08), and the angular velocity decreased by 25% to compensate and keep L constant. Note that ω did not decrease by 33% because L depends on $I\omega$, not on I alone. The 25% KE loss is significant and is converted deforming the putty and generating heat at the moment of impact.

Think Like a Physicist: This is a rotational inelastic collision. The putty “collides” with the disc and sticks (perfectly inelastic). Angular momentum is conserved, but KE is lost. The procedure is identical to a linear perfectly inelastic collision: find L before, find I after (with the new mass included), then find ω_f from conservation. The KE calculation is always a separate step, never assumed to be conserved.

HOT Example 49

A merry-go-round (uniform disc, mass 100kg, radius 2.0m) is spinning freely at 1.5rad/s . A child of mass 30kg runs tangentially at 4m/s in the same direction as the merry-go-round rotates and jumps onto the edge.

- (a) Calculate the angular momentum of the merry-go-round before the child jumps on.
 (b) Calculate the angular momentum of the running child about the axis of the merry-go-round.
 (c) Determine the final angular velocity after the child is on the merry-go-round.
 (d) State whether the merry-go-round speeds up or slows down, and explain why.

Solution

- (a) Moment of inertia of the merry-go-round:

$$I_m = \frac{1}{2}MR^2 = \frac{1}{2} \times 100\text{kg} \times (2.0\text{m})^2 = 200\text{kgm}^2$$

Angular momentum:

$$L_m = I_m \omega = 200 \text{kgm}^2 \times 1.5 \text{rad/s} = 300 \text{kgm}^2 \text{s}^{-1}$$

The angular momentum of the merry-go-round is $300 \text{kgm}^2 \text{s}^{-1}$.

- (b) The child runs tangentially at speed $v = 4 \text{m/s}$ at distance $r = R = 2.0 \text{m}$ from the axis. The angular momentum of the child about the axis is:

$$L_{\text{child}} = mvr = 30 \text{kg} \times 4 \text{m/s} \times 2.0 \text{m} = 240 \text{kgm}^2 \text{s}^{-1}$$

The angular momentum of the child is $240 \text{kgm}^2 \text{s}^{-1}$.

- (c) After the child lands, the system (merry-go-round + child) rotates together. The child, sitting at the edge, contributes $I_{\text{child}} = mR^2 = 30 \text{kg} \times (2.0 \text{m})^2 = 120 \text{kgm}^2$ to the moment of inertia.

$$I_{\text{total}} = I_m + I_{\text{child}} = (200 + 120) \text{kgm}^2 = 320 \text{kgm}^2$$

Total angular momentum before: $L_{\text{total}} = (300 + 240) \text{kgm}^2 \text{s}^{-1} = 540 \text{kgm}^2 \text{s}^{-1}$

Conservation of angular momentum:

$$L_{\text{total}} = I_{\text{total}} \times \omega_f$$

Making ω_f the subject:

$$\omega_f = \frac{540 \text{kgm}^2 \text{s}^{-1}}{320 \text{kgm}^2} = 1.6875 \text{rad/s}$$

The final angular velocity is 1.69rad/s .

- (d) The merry-go-round speeds up slightly from 1.5rad/s to 1.69rad/s . This happens because the child brings additional angular momentum ($240 \text{kgm}^2 \text{s}^{-1}$) into the system by running in the same direction as the rotation. Although the child also adds moment of inertia (which tends to slow things down), the extra angular momentum outweighs the extra moment of inertia, producing a net speed increase.

Making Sense of the Answer: The child contributes $240 \text{kgm}^2 \text{s}^{-1}$ of angular momentum, which is 80% of the merry-go-round's own $300 \text{kgm}^2 \text{s}^{-1}$. The combined 540 is then shared across 320kgm^2 of inertia, giving a modest increase in ω . If the child had run faster (say, 6m/s), the speed-up would have been more dramatic. If the child had run in the opposite direction, the merry-go-round would have slowed down.

Think Like a Physicist: This problem combines two sources of angular momentum: a rotating body ($L = I\omega$) and a moving particle ($L = mvr$). Both must be included in the total. The particle's angular momentum depends on how it moves relative to the axis (tangentially, radially, or at an angle). For a tangential approach, $L = mvr$ with full v . For a radial approach (running straight toward the centre), the angular momentum would be zero. The direction and speed of approach both matter.

Angular momentum subtopic is complete. Angular momentum is the final piece of the twin language. With it, every concept from translational mechanics has a rotational counterpart: displacement and angle, velocity and angular velocity, acceleration and angular acceleration, mass and moment of inertia, force and torque, momentum and angular momentum, and the conservation laws that govern them all. We are now going to bring the twins face to face in a final comparison table.

COMPARISON TABLE: TRANSLATIONAL AND ROTATIONAL QUANTITIES

Throughout this chapter, we have called rotational quantities the “twins” of translational quantities. The table below places all the twins side by side. Every formula in the left column has an exact counterpart in the right column, obtained by replacing each translational quantity with its rotational twin.

This table is not decoration. It is a thinking tool. Whenever you face a rotational problem and feel lost, find the translational version you already know and translate it using this table.

Translational quantity	Symbol	Rotational quantity	Symbol
Displacement	s	Angular displacement	θ
Velocity	v	Angular velocity	ω
Acceleration	a	Angular acceleration	α
Mass (inertia)	m	Moment of inertia	I
Force	F	Torque	τ
Momentum	$p = mv$	Angular momentum	$L = I\omega$

Translational law/formula	Rotational twin
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s = \frac{1}{2}(u + v)t$	$\theta = \frac{1}{2}(\omega_0 + \omega)t$
$s_{nth} = u + \frac{1}{2}a(2n - 1)$	$\theta_{nth} = \omega_0 + \frac{1}{2}\alpha(2n - 1)$
$F = ma$	$\tau = I\alpha$
$F = \frac{dp}{dt}$	$\tau = \frac{dL}{dt}$
$W = Fs$	$W = \tau\theta$
$KE = \frac{1}{2}mv^2$	$KE = \frac{1}{2}I\omega^2$
$P = Fv$	$P = \tau\omega$
If $F = 0$, $p = \text{constant}$	If $\tau = 0$, $L = \text{constant}$

Translational connection	Rotational connection
$s = r\theta$	Links displacement and angle
$v = r\omega$	Links linear and angular velocity
$a_t = r\alpha$	Links tangential and angular acceleration
$v_{cm} = \omega R$	No-slip condition for rolling

How to use the table:

If a translational problem says “a force of 10N accelerates a 2kg mass,” the rotational twin says “a torque of 10Nm angularly accelerates a body of moment of inertia 2kgm².” The mathematics is identical: $a = F/m = 5\text{m/s}^2$ becomes $\alpha = \tau/I = 5\text{rad/s}^2$. If you can solve one, you can solve the other.

This twin correspondence is not a coincidence. It arises because the mathematical structure of Newton's laws is the same for both translation and rotation. The physics is different (forces push, torques twist), but the equations have the same form. Learning one set of equations teaches you both.

APPLICATIONS OF ROTATIONAL MECHANICS

Rotational mechanics is not confined to textbooks and examination halls. It is at work in every machine, every vehicle, every spinning object in the universe. This section highlights some of the most important real-world applications.

1. Flywheels

A flywheel is a heavy rotating disc (or wheel) designed to store rotational KE. Because $KE = \frac{1}{2}I\omega^2$, a flywheel with a large moment of inertia spinning at high angular velocity stores substantial energy.

In an engine, the power strokes are intermittent (they happen only during part of each cycle). Between power strokes, the engine would slow down and jerk. A flywheel attached to the crankshaft stores energy during the power stroke and releases it between strokes, smoothing out the rotation. This is why all internal combustion engines have flywheels.

To maximize energy storage for a given mass, the mass should be placed as far from the axis as possible (maximizing I). This is why flywheels are designed with heavy rims and light centres, approaching the ring shape ($I = MR^2$) rather than the solid disc shape ($I = \frac{1}{2}MR^2$).

Modern flywheel energy storage is used in some hybrid buses, uninterruptible power supplies, and spacecraft attitude control systems. Advanced flywheels spin at tens of thousands of rpm in vacuum chambers on magnetic bearings to minimize friction losses.

2. Gyroscopes and gyroscopic stability

A gyroscope is a spinning body mounted so that its axis of rotation can point in any direction. Once spinning, the axis resists changes in direction. This resistance comes from the large angular momentum $L = I\omega$ along the spin axis. Changing the direction of L requires an external torque ($\tau = \frac{dL}{dt}$), and without such a torque, the axis stays fixed in space.

Gyroscopes are used in aircraft, ships, submarines, and spacecraft to maintain a reference direction independent of the vehicle's motion. The gyrocompass always points in the same direction regardless of how the vehicle turns.

Gyroscopic effects also keep a moving bicycle upright. The spinning wheels have angular momentum along a horizontal axis. Tilting the bicycle requires changing the direction of this angular momentum, which requires torque. The larger the angular momentum (faster wheel spin), the harder it is to tilt. This is why a bicycle is easier to balance at speed than at rest.

3. The spinning top and precession

A spinning top does not fall over immediately because its angular momentum resists the change in direction that falling would require. Instead, gravity applies a torque that gradually changes the direction of L , causing the spin axis to sweep out a cone. This slow sweeping motion is called **precession**. The precession rate is slower when the top spins faster (larger L means more resistance to direction change), which is why a fast-spinning top appears almost perfectly stable.

The Earth itself **precesses** (slowly change the direction of its axis of rotation). Its axis traces a cone with a period of about 26000 years, caused by the gravitational torques of the Sun and Moon on the Earth's equatorial bulge. This precession slowly shifts the position of the North Celestial Pole among the stars.

4. Rifling of gun barrels

Gun barrels have spiral grooves (rifling) that make the bullet spin as it leaves the barrel. The spinning bullet has angular momentum along its direction of travel. This angular momentum stabilizes the bullet's orientation, preventing it from tumbling in the air. A tumbling bullet has poor accuracy and less penetrating power. Rifling exploits gyroscopic stability to keep the bullet's nose pointing forward throughout its flight.

5. Moment of inertia in sports

Athletes instinctively use moment of inertia to control their rotation. A gymnast performing a somersault tucks tightly (small I) to spin fast, then opens out (large I) to slow the rotation before landing. A long jumper swings the arms forward during landing to transfer angular momentum from the body to the arms, preventing a backward fall. A discus thrower extends the arm fully (large r) to maximize the release speed ($v = \omega r$). In every case, the athlete is exploiting the relationship between moment of inertia, angular velocity, and angular momentum.

6. Tanzanian applications

Traditional grain mills use heavy circular grinding stones that store rotational KE and maintain steady grinding between pushes (the flywheel principle). The rungu (wooden club) concentrates mass at the far end to maximize the moment of inertia about the grip, giving more angular momentum and KE at impact. Potters' wheels use the flywheel effect to maintain smooth rotation while the potter shapes clay. Bicycle wheels on Tanzanian roads demonstrate rolling motion, gyroscopic stability, and conservation of angular momentum every day, whether the rider knows the physics or not.

This chapter has taken us from Kipanga's fumbled ruler to flywheels, from spinning eggs to collapsing stars, and from a single formula $\tau = I\alpha$ to a complete twin language that mirrors everything we ever learned about translational motion. The twin language has been spoken. Now it must be tested. In the next section, the concepts of this entire chapter: moment of inertia, torque, energy, rolling motion, and angular momentum will stop introducing themselves politely and start arriving together, uninvited, in miscellaneous worked examples. Bring your calculator, and make sure your pen does not slip, like a like a car's wheel skidding helplessly down a muddy slope!

MISCELLANEOUS WORKED EXAMPLES ON ROTATION OF RIGID BODIES

Example 50

- (a) Explain why a hollow cylinder has a larger moment of inertia than a solid cylinder of the same mass and radius.
- (b) A uniform solid disc of mass 5kg and radius 0.30m is mounted on a frictionless horizontal axle. A constant force of 12N is applied tangentially at the rim. Take $g = 9.8\text{m/s}^2$.
 - (i) Calculate the moment of inertia of the disc.
 - (ii) Calculate the angular acceleration.
 - (iii) Calculate the angular velocity after 4s, starting from rest.
 - (iv) Calculate the angle turned through in the 4th second.

Solution

- (a) In a hollow cylinder, all the mass is concentrated at the outer radius, far from the axis of rotation. In a solid cylinder, some mass is close to the axis, where it contributes less to the moment of inertia. Since moment of inertia depends on mr^2 , mass far from the axis contributes more. Therefore, the hollow cylinder, with all its mass at the maximum distance, has the larger moment of inertia.
- (b) The solution of each part is as follows:

$$(i) \quad I = \frac{1}{2}MR^2 = \frac{1}{2} \times 5\text{kg} \times (0.30\text{m})^2 = 0.225\text{kgm}^2$$

The moment of inertia is 0.225kgm^2 .

- (ii) The tangential force at the rim produces torque $\tau = FR = 12\text{N} \times 0.30\text{m} = 3.6\text{Nm}$

$$\alpha = \frac{\tau}{I} = \frac{3.6\text{Nm}}{0.225\text{kgm}^2} = 16\text{rad/s}^2$$

The angular acceleration is 16rad/s^2 .

$$(iii) \quad \omega = \omega_0 + \alpha t = 0 + 16\text{rad/s}^2 \times 4\text{s} = 64\text{rad/s}$$

The angular velocity after 4s is 64rad/s .

$$(iv) \theta_{nth} = \omega_0 + \frac{1}{2}\alpha(2n - 1) = 0 + \frac{1}{2} \times 16 \times (2 \times 4 - 1) = 56\text{rad}$$

The angle turned through in the 4th second is 56rad.

Example 51

- (a) Explain why the parallel axis theorem always gives a moment of inertia that is larger than the moment of inertia about the centre of mass.
- (b) A uniform rod of mass 1.2kg and length 0.80m is pivoted at a point 0.20m from one end.
 - (i) Calculate the moment of inertia about the centre of mass.
 - (ii) Calculate the moment of inertia about the pivot.
- (iii) A couple of magnitude 2.4Nm is applied to the rod. Calculate the angular acceleration.
- (iv) Calculate the rotational KE after 3s from rest.

Solution

- (a) The parallel axis theorem states $I = I_{cm} + Md^2$. Since Md^2 is always positive (mass is positive, d^2 is positive), I about any other axis is always greater than I_{cm} . The centre of mass is the axis of minimum moment of inertia.

Physically, the moment of inertia measures how far the mass is spread from the rotation axis. When the axis passes through the centre of mass, the mass is distributed as closely as possible around the axis (some mass is near, some is far, but on average the distribution is the most compact it can be). When the axis is shifted to a distance d away from the centre of mass, every piece of mass in the body is, on average, farther from the new axis than it was from the centre of mass. Farther mass means more resistance to rotation. Therefore, the moment of inertia about any axis away from the centre of mass is always larger. The centre of mass is the "most balanced" point of the body, and rotating about it requires the least effort.

- (b) The answer of each part is as follows:

$$(i) I_{cm} = \frac{1}{12}ML^2 = \frac{1}{12} \times 1.2\text{kg} \times (0.80\text{m})^2 = \frac{1}{12} \times 1.2 \times 0.64 = 0.064\text{kgm}^2$$

The moment of inertia about the centre is 0.064kgm^2 .

- (ii) The centre of mass is at the midpoint, 0.40m from each end. The pivot is 0.20m from one end, so the distance from the pivot to the centre of mass is $d = 0.40\text{m} - 0.20\text{m} = 0.20\text{m}$.

$$I_{pivot} = I_{cm} + Md^2 = 0.064\text{kgm}^2 + 1.2\text{kg} \times (0.20\text{m})^2 = (0.064 + 0.048)\text{kgm}^2 = 0.112\text{kgm}^2$$

The moment of inertia about the pivot is 0.112kgm^2 .

$$(iii) \alpha = \frac{\tau}{I} = \frac{2.4\text{Nm}}{0.112\text{kgm}^2} = 21.4\text{rad/s}^2$$

The angular acceleration is 21.4rad/s^2 .

- (iv) After 3s from rest: $\omega = \alpha t = 21.4\text{rad/s}^2 \times 3\text{s} = 64.3\text{rad/s}$

$$KE = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.112\text{kgm}^2 \times (64.3\text{rad/s})^2 = 231.5\text{J}$$

The rotational KE is 232J.

Example 52

- (a) **Kipanga** claims: "A body with a larger moment of inertia always has more rotational KE." Explain why this claim is not necessarily true.
- (b) Two discs, A and B, rotate about their central axes. Disc A has $I_A = 0.5\text{kgm}^2$ and $\omega_A = 10\text{rad/s}$. Disc B has $I_B = 2.0\text{kgm}^2$ and $\omega_B = 4\text{rad/s}$.
 - (i) Calculate the rotational KE of each disc.
 - (ii) Calculate the angular momentum of each disc.
 - (iii) State which disc has more KE and which has more angular momentum.

Solution

- (a) Rotational KE is $\frac{1}{2}I\omega^2$. It depends on both I and ω^2 . A body with larger I but much smaller ω can have less KE than a body with smaller I but larger ω . The ω^2 factor means angular velocity has a greater influence on KE than moment of inertia does.
- (b) The solution of each part is as follows:
- (i) $KE_A = \frac{1}{2} \times 0.5 \text{ kg m}^2 \times (10 \text{ rad/s})^2 = 25 \text{ J}$
 $KE_B = \frac{1}{2} \times 2.0 \text{ kg m}^2 \times (4 \text{ rad/s})^2 = 16 \text{ J}$
- (ii) $L_A = I_A \omega_A = 0.5 \times 10 = 5 \text{ kg m}^2 \text{ s}^{-1}$
 $L_B = I_B \omega_B = 2.0 \times 4 = 8 \text{ kg m}^2 \text{ s}^{-1}$
- (iii) Disc A has more KE (25J versus 16J), but disc B has more angular momentum ($8 \text{ kg m}^2 \text{ s}^{-1}$ versus $5 \text{ kg m}^2 \text{ s}^{-1}$).

Interesting! A body can have more angular momentum but less KE than another body. This is because KE depends on ω^2 (favouring fast-spinning bodies) while L depends on ω (favouring high-inertia bodies). In this example, disc A spins faster so it wins on KE, but disc B has more inertia so it wins on angular momentum.

Example 53

- (a) Explain why static friction does no work on a body that rolls without slipping on a flat surface.
- (b) A solid sphere and a hollow cylinder, each of mass 2kg, are released from rest at the top of a rough incline of height 3m and angle 30° . Both roll without slipping. Take $g = 9.8 \text{ m/s}^2$.
- (i) Calculate the speed of each body at the bottom.
- (ii) Calculate the acceleration of each body.
- (iii) Calculate the friction force on each body.
- (iv) Determine which body arrives at the bottom first and by how much time.

Solution

- (a) Static friction acts at the contact point. During pure rolling, the contact point is instantaneously at rest (zero velocity). Since work requires both force and displacement at the point of application, and the displacement of the contact point is zero, the work done by static friction is zero.
- (b) The solution of each part is as follows:
- (i) For the solid sphere ($k^2/R^2 = 2/5$):

$$v = \sqrt{\frac{2gh}{1 + k^2/R^2}} = \sqrt{\frac{2 \times 9.8 \text{ m/s}^2 \times 3 \text{ m}}{1 + 2/5}} = 6.48 \text{ m/s}$$

For the hollow cylinder ($k^2/R^2 = 1$):

$$v = \sqrt{\frac{2 \times 9.8 \times 3}{1 + 1}} \text{ m/s}^2 = 5.42 \text{ m/s}$$

- (ii) For the solid sphere:

$$a = \frac{g \sin 30^\circ}{1 + 2/5} = \frac{9.8 \times 0.5}{1.4} \text{ m/s}^2 = 3.50 \text{ m/s}^2$$

For the hollow cylinder:

$$a = \frac{9.8 \times 0.5}{2} \text{ m/s}^2 = 2.45 \text{ m/s}^2$$

- (iii) For the solid sphere:

$$f = \frac{2}{7} Mg \sin 30^\circ = \frac{2}{7} \times 2\text{kg} \times 9.8\text{m/s}^2 \times 0.5 = 2.80\text{N}$$

For the hollow cylinder:

$$f = \frac{1}{2} Mg \sin 30^\circ = \frac{1}{2} \times 2\text{kg} \times 9.8\text{m/s}^2 \times 0.5 = 4.90\text{N}$$

(iv) Length of incline: $s = h/\sin 30^\circ = 3/0.5 = 6\text{m}$

For the solid sphere: $t = \sqrt{2s/a} = \sqrt{2 \times 6\text{m}/3.50\text{m/s}^2} = 1.852\text{s}$

For the hollow cylinder: $t = \sqrt{2 \times 6\text{m}/2.45\text{m/s}^2} = 2.213\text{s}$

Time difference: $2.213 - 1.852 = 0.361\text{s}$

The solid sphere arrives first, beating the hollow cylinder by 0.36s.

Example 54

- (a) Explain why a torque is needed to produce angular acceleration, even if the body is already rotating.
- (b) A flywheel of moment of inertia 0.8kgm^2 is rotating at 300rpm. A braking torque is applied and the flywheel comes to rest in 20s.
 - (i) Calculate the braking torque.
 - (ii) Calculate the number of revolutions before stopping.
 - (iii) Calculate the KE dissipated as heat.

Solution

- (a) A rotating body with no net torque continues at constant angular velocity (rotational analogue of Newton's first law). To change the angular velocity (i.e. to produce angular acceleration), a net torque is required (rotational analogue of Newton's second law, $\tau = I\alpha$).
- (b) $\omega_0 = 300\text{rpm} \times \frac{2\pi\text{rad}}{60\text{s}} = \frac{300 \times 2\pi}{60} = 10\pi = 31.4\text{rad/s}$

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 31.4\text{rad/s}}{20\text{s}} = -1.57\text{rad/s}^2$$

The angular deceleration is 1.57rad/s^2 .

- (i) $\tau = I|\alpha| = 0.8\text{kgm}^2 \times 1.57\text{rad/s}^2 = 1.26\text{Nm}$

The braking torque is 1.26Nm .

- (ii) $\theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{0 - (31.4)^2}{2 \times (-1.57)} = \frac{-986}{-3.14} = 314\text{rad}$

Number of revolutions: $N = \frac{\theta}{2\pi} = \frac{314}{2\pi} = 50$ revolutions

The flywheel makes 50 revolutions before stopping.

- (iii) $\text{KE} = \frac{1}{2} I \omega_0^2 = \frac{1}{2} \times 0.8\text{kgm}^2 \times (31.4\text{rad/s})^2 = 394\text{J}$

The KE dissipated as heat is 394J .

Be careful! Always convert rpm to rad/s before using any rotational formula. Many students forget this and substitute 300 directly, which gives wrong answers throughout the problem.

Example 55

- (a) Two objects of different shapes but the same mass and radius roll down the same incline. Explain why the acceleration does not depend on mass or radius, only on shape.
- (b) A uniform solid sphere of mass M and radius R rolls without slipping up an inclined plane with initial speed v_0 at the bottom.

- (i) Show that the height reached is $h = \frac{7v_0^2}{10g}$.
- (ii) If $v_0 = 5\text{m/s}$ and $g = 9.8\text{m/s}^2$, calculate h .
- (iii) Compare this with the height a sliding block would reach with the same initial speed on a frictionless incline.

Solution

- (a) The acceleration is $a = \frac{g\sin\theta}{1+k^2/R^2}$. The numerator ($g\sin\theta$) contains no mass or radius. The denominator ($1 + k^2/R^2$) depends only on the ratio k^2/R^2 , which is a pure number determined by the shape of the body.

Physically, two competing effects cancel mass and radius simultaneously. On the driving side, a heavier object has a larger gravitational pull down the incline ($Mg\sin\theta$), which would accelerate it faster. But on the resisting side, a heavier object also has a larger moment of inertia ($I = Mk^2$, proportional to M), which makes it harder to spin up. The larger driving force and the larger rotational resistance both increase proportionally with M , so mass cancels. Similarly, a larger radius increases the moment of inertia (more mass farther from the axis), but it also increases the lever arm through which friction acts to produce torque. These two radius effects cancel each other exactly. What remains is the shape factor k^2/R^2 , which describes how the mass is distributed relative to the outer radius.

- (b) The solution of each part is as follows:

(i) At the bottom: $KE_{\text{total}} = \frac{1}{2}Mv_0^2 \left(1 + \frac{k^2}{R^2}\right) = \frac{1}{2}Mv_0^2 \left(1 + \frac{2}{5}\right) = \frac{1}{2}Mv_0^2 \times \frac{7}{5} = \frac{7Mv_0^2}{10}$

At the top: $PE = Mgh$, and $KE = 0$ (the sphere stops momentarily).

By conservation of mechanical energy: $\frac{7Mv_0^2}{10} = Mgh$

$$h = \frac{7v_0^2}{10g}$$

(ii) $h = \frac{7 \times (5\text{m/s})^2}{10 \times 9.8\text{m/s}^2} = 1.79\text{m}$

The height reached is 1.79m.

- (ii) A sliding block:

$$\frac{1}{2}Mv_0^2 = Mgh_{\text{slide}}$$

$$h_{\text{slide}} = \frac{v_0^2}{2g} = \frac{25}{19.6} = 1.28\text{m}$$

Interesting! The rolling sphere reaches 1.79m, which is higher than the sliding block (1.28m). This seems paradoxical. The explanation is that the rolling sphere started with more total KE than the block (it had both translational and rotational KE at the bottom), so it has more energy to convert into PE. The extra height is $\frac{7}{10g} \div \frac{1}{2g} = \frac{7}{5}$ times the sliding height, exactly the factor $(1 + k^2/R^2)$.

Example 56

- (a) What is the difference between torque and couple? State one similarity and one difference.
- (b) A grinding wheel (solid disc, mass 8kg, radius 0.25m) rotates at 120rad/s. A brake pad presses against the rim with a normal force of 40N. The coefficient of kinetic friction between the pad and the wheel is 0.6.
- (i) Calculate the time to stop the wheel.
- (ii) Calculate the total energy dissipated as heat.

Solution

- (a) **Similarity:** Both torque and couple produce rotation.

Difference: A single torque requires a pivot point and the body may translate as well as rotate while a couple consists of two equal, opposite, parallel forces separated by a distance, and it produces pure rotation without translation.

- (b) The solution of each part is as follows:

(i) Finding the friction force

$$f = \mu_k F_n = 0.6 \times 40\text{N} = 24\text{N}$$

Finding the braking torque

$$\tau = fR = 24\text{N} \times 0.25\text{m} = 6\text{Nm}$$

Finding moment of inertia and hence deceleration

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 8\text{kg} \times (0.25\text{m})^2 = 0.25\text{kgm}^2$$

$$\alpha = \frac{\tau}{I} = \frac{6\text{Nm}}{0.25\text{kgm}^2} = 24\text{rad/s}^2$$

Using $\omega = \omega_0 + \alpha t$ with $\omega = 0$

$$t = \frac{-\omega_0}{\alpha} = \frac{-120\text{rad/s}}{-24\text{rad/s}^2} = 5\text{s}$$

The wheel stops in 5s.

$$(ii) \quad KE = \frac{1}{2}I\omega_0^2 = \frac{1}{2} \times 0.25\text{kgm}^2 \times (120\text{rad/s})^2 = 1800\text{J}$$

The energy dissipated as heat is 1800J.

Example 57

- (a) Explain why a person spinning on a freely rotating chair speeds up when she pulls her arms inward.
- (b) A disc of moment of inertia 0.6kgm^2 rotates at 8rad/s . A second disc of moment of inertia 0.2kgm^2 , initially at rest, is dropped coaxially onto it. They rotate together.
- (i) Calculate the final angular velocity.
- (ii) Show that the fraction of KE lost equals $\frac{I_2}{I_1 + I_2}$ where I_2 is the moment of inertia of the dropped disc.

Solution

- (a) No external torque acts on the person (the chair rotates freely), so angular momentum $L = I\omega$ is conserved. When she pulls her arms inward, her moment of inertia I decreases. Since L must remain constant, ω must increase and therefore, she spins faster.
- (b) The solution of each part is as follows:
- (i) By conservation of angular momentum:

$$I_1\omega_1 = (I_1 + I_2)\omega_f$$

$$\omega_f = \frac{I_1\omega_1}{I_1 + I_2} = \frac{0.6\text{kgm}^2 \times 8\text{rad/s}}{(0.6 + 0.2)\text{kgm}^2} = 6\text{rad/s}$$

The final angular velocity is 6rad/s .

$$(ii) \quad KE_i = \frac{1}{2}I_1\omega_1^2$$

$$KE_f = \frac{1}{2}(I_1 + I_2)\omega_f^2 = \frac{1}{2}(I_1 + I_2)\left(\frac{I_1\omega_1}{I_1 + I_2}\right)^2 = \frac{I_1^2\omega_1^2}{2(I_1 + I_2)}$$

Fraction lost:

$$\Delta KE = \frac{KE_i - KE_f}{KE_i} = 1 - \frac{KE_f}{KE_i} = 1 - \frac{\frac{I_1^2\omega_1^2}{2(I_1 + I_2)}}{\frac{1}{2}I_1\omega_1^2} = 1 - \frac{I_1}{I_1 + I_2} = \frac{I_2}{I_1 + I_2}$$

Hence:

$$\text{Fraction lost} = \frac{I_2}{I_1 + I_2}$$

Example 58

- (a) A student says: “Friction always opposes motion, so during rolling it must act in the direction opposite to the velocity of the centre of mass.” Explain why this statement is misleading.
- (b) A solid cylinder of mass 3kg and radius 0.10m is given a translational velocity of 6m/s along a rough horizontal surface without any initial spin. The coefficient of kinetic friction is $\mu_k = 0.4$. Take $g = 9.8\text{m/s}^2$.
- (i) Explain why the cylinder initially slides rather than rolls.
- (ii) Calculate the friction force.
- (iii) Calculate the translational deceleration and the angular acceleration.
- (iv) Calculate the time at which pure rolling begins.
- (v) Calculate the velocity when pure rolling begins.

Solution

- (a) The statement is misleading because friction opposes **sliding at the contact point**, not motion of the centre of mass. During rolling down an incline, friction acts up the slope (opposing the sliding tendency at the contact point), but the centre of mass moves down the slope. Friction and the centre-of-mass velocity are in the same direction when a body accelerates rotationally.
- (b) The solution of each part is as follows:
- (i) Initially $v_{\text{cm}} = 6\text{m/s}$ but $\omega = 0$, so $v_{\text{cm}} \neq \omega R$. The no-slip condition is violated ($v_{\text{cm}} > \omega R$), and hence the contact point slides forward while kinetic friction acts backward.

(ii) $f = \mu_k Mg = 0.4 \times 3\text{kg} \times 9.8\text{m/s}^2 = 11.76\text{N}$

(iii) Translational: $a = -\frac{f}{M} = -\frac{11.76\text{N}}{3\text{kg}} = -3.92\text{m/s}^2$ (deceleration)

Rotational: $\alpha = \frac{fR}{I} = \frac{fR}{\frac{1}{2}MR^2} = \frac{11.76\text{N} \times 0.10\text{m}}{\frac{1}{2} \times 3\text{kg} \times (0.1\text{m})^2} = 78.4\text{rad/s}^2$ (spin-up)

- (iv) Pure rolling begins when $v_{\text{cm}} = \omega R$:

Using $v = u + at$ and $\omega = \omega_0 + \alpha t$ with $\omega_0 = 0$:

$$v_{\text{cm}} = 6 - 3.92t \text{ and } \omega = 78.4t$$

Setting $v_{\text{cm}} = \omega R$ (when rolling begins):

$$6 - 3.92t = 78.4t \times 0.10$$

$$t = 0.51\text{s}$$

Pure rolling begins at $t = 0.51\text{s}$.

(v) $v = 6\text{m/s} - 3.92\text{m/s}^2 \times 0.51\text{s} = 4.0\text{m/s}$

The velocity when pure rolling begins is 4.0m/s.

Example 59

- (a) Explain why the total angular momentum of a system is conserved even when internal torques act between parts of the system.
- (b) A merry-go-round (mass 80kg, radius 1.5m) rotates at 2.0rad/s. A child of mass 25kg standing at the edge walks 1m inward toward the centre.
- (i) Calculate the initial moment of inertia of the system.
- (ii) Calculate the final moment of inertia of the system.
- (iii) Calculate the final angular velocity.
- (iv) Calculate the factor by which the KE changes, and explain the source of the extra KE.

Solution

- (a) Internal torques come in action-reaction pairs (Newton’s third law). When one part of the system exerts a torque on another, the second part exerts an equal and opposite torque back. These internal torques

cancel in the total, leaving the total angular momentum unchanged. Only external torques can change the total angular momentum.

(b) The solution of each part is as follows:

$$(i) \quad I_{\text{disc}} = \frac{1}{2}MR^2 = \frac{1}{2} \times 80\text{kg} \times (1.5\text{m})^2 = 90\text{kgm}^2$$

$$I_{\text{child},i} = mR^2 = 25\text{kg} \times (1.5\text{m})^2 = 56.25\text{kgm}^2$$

$$I_i = (90 + 56.25)\text{kgm}^2 = 146.25\text{kgm}^2$$

The initial moment of inertia of the system is 146.25kgm^2 .

(ii) New position of a child from the centre: $r = 1.5\text{m} - 1\text{m} = 0.5\text{m}$

$$I_{\text{child},f} = mr^2 = 25\text{kg} \times (0.5\text{m})^2 = 6.25\text{kgm}^2$$

$$I_f = (90 + 6.25)\text{kgm}^2 = 96.25\text{kgm}^2$$

The final moment of inertia of the system is 96.25kgm^2 ,

(iii) By conservation of angular momentum: $I_i\omega_i = I_f\omega_f$

$$\omega_f = \frac{I_i\omega_i}{I_f} = \frac{146.25\text{kgm}^2 \times 2.0\text{rad/s}}{96.25\text{kgm}^2} = 3.04\text{rad/s}$$

The final angular velocity is 3.04rad/s .

$$(iv) \quad KE_i = \frac{1}{2} \times 146.25\text{kgm}^2 \times (2.0\text{rad/s})^2 = 292.5\text{J}$$

$$KE_f = \frac{1}{2} \times 96.25\text{kgm}^2 \times (3.04\text{rad/s})^2 = 444.6\text{J}$$

$$\text{Factor: } \frac{KE_f}{KE_i} = \frac{444.6\text{J}}{292.5\text{J}} = 1.52$$

The KE increased by a factor of 1.52 (a 52% increase).

The extra KE comes from the **work done by the child** in walking inward against the centripetal acceleration. The child must exert a radial force to move toward the centre while the system is spinning; this force does positive work, increasing the total KE of the system.

Interesting! Unlike the disc-drop problem (Example 57) where KE decreased, here the KE increased. The difference is that in Example 57 the second disc was passive (dropped on), while here the child actively does work by walking inward. Conservation of angular momentum does not require conservation of KE.

Example 60

(a) A uniform metre stick is balanced on a finger at the 50cm mark. A 200g lump of clay is then stuck to one end. Explain, without calculation, in which direction the balance point must shift and why.

(b) Three particles are positioned in space as follows:

$$\text{Particle 1: mass } 2\text{kg} \text{ at position } \vec{r}_1 = (3\hat{i} + 2\hat{j} + 4\hat{k})\text{m}$$

$$\text{Particle 2: mass } 3\text{kg} \text{ at position } \vec{r}_2 = (-1\hat{i} + 4\hat{j} + 2\hat{k})\text{m}$$

$$\text{Particle 3: mass } 5\text{kg} \text{ at position } \vec{r}_3 = (2\hat{i} - 2\hat{j} + 1\hat{k})\text{m}$$

- Explain why the centre of mass lies closer to particle 3 than to particle 1, even without performing any calculation.
- Calculate the position vector of the centre of mass.
- Calculate the distance of the centre of mass from the origin.

Solution

(a) The balance point of a uniform metre stick is at its centre of mass, which is at the 50cm mark (the geometric centre). When the clay is added to one end, the system's centre of mass shifts toward the added mass, because the centre of mass is the mass-weighted average position. The balance point must

therefore shift toward the end where the clay is attached. The heavier the clay relative to the stick, the further the shift.

(b) The solution of each part is as follows:

(i) The centre of mass is the mass-weighted average position. Particle 3 has the largest mass (5kg out of a total of 10kg), contributing 50% of the weighting. The centre of mass is pulled most strongly toward the heaviest particle. Therefore, without calculating, we can predict that the centre of mass lies closer to particle 3 than to either of the other two.

(ii) The position vector of the centre of mass is:

$$\vec{r}_{\text{cm}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3}$$

The total mass is:

$$M = 2\text{kg} + 3\text{kg} + 5\text{kg} = 10\text{kg}$$

Calculating each component separately:

$\hat{i}$ component:

$$x_{\text{cm}} = \frac{2\text{kg} \times 3\text{m} + 3\text{kg} \times (-1)\text{m} + 5\text{kg} \times 2\text{m}}{10\text{kg}} = 1.3\text{m}$$

$\hat{j}$ component:

$$y_{\text{cm}} = \frac{2\text{kg} \times 2\text{m} + 3\text{kg} \times 4\text{m} + 5\text{kg} \times (-2)\text{m}}{10\text{kg}} = 0.6\text{m}$$

$\hat{k}$ component:

$$z_{\text{cm}} = \frac{2\text{kg} \times 4\text{m} + 3\text{kg} \times 2\text{m} + 5\text{kg} \times 1\text{m}}{10\text{kg}} = 1.9\text{m}$$

Therefore:

$$\vec{r}_{\text{cm}} = (1.3\hat{i} + 0.6\hat{j} + 1.9\hat{k})\text{m}$$

Be careful! Each term in the numerator must carry the correct sign from the position vector. Particle 2 has $x = -1\text{m}$ and particle 3 has $y = -2\text{m}$. Missing a negative sign here changes the answer completely.

(iv) The distance of the centre of mass from the origin is the magnitude of $\vec{r}_{\text{cm}}$:

$$r_{\text{cm}} = |\vec{r}_{\text{cm}}| = \sqrt{x_{\text{cm}}^2 + y_{\text{cm}}^2 + z_{\text{cm}}^2}$$

Substituting:

$$r_{\text{cm}} = \sqrt{(1.3\text{m})^2 + (0.6\text{m})^2 + (1.9\text{m})^2} = \sqrt{1.69 + 0.36 + 3.61}\text{m} = 2.38\text{m}$$

The centre of mass is 2.38m from the origin.

Example 61

- (a) A body rotating at constant angular velocity has zero angular acceleration ($\alpha = 0$). Does this mean a point on the rim of the body has zero acceleration? Explain your answer.
- (b) A grinding wheel of radius 0.25m starts from rest and accelerates uniformly at 4rad/s^2 .
- (i) Calculate the tangential acceleration of a point on the rim.
- (ii) Calculate the centripetal acceleration of a point on the rim at $t = 3\text{s}$.
- (iii) Calculate the magnitude and direction of the total linear acceleration at $t = 3\text{s}$.

Solution

- (a) No. Zero angular acceleration means the angular velocity is not changing, so the speed of the rim point is constant. However, the direction of the velocity changes continuously as the point moves around the circle. This continuous change in direction means the point has centripetal acceleration directed toward the centre, given by $a_c = \omega^2 R$. This acceleration is not zero as long as the body is rotating ($\omega \neq 0$). What is zero is the tangential acceleration ($a_t = \alpha R = 0$), which is responsible for changing the speed.

So a point on the rim of a body rotating at constant angular velocity has centripetal acceleration but no tangential acceleration. The total acceleration is purely centripetal.

(b) The solution of each part is as follows:

(i) The tangential acceleration is:

$$a_t = \alpha R = 4 \text{ rad/s}^2 \times 0.25 \text{ m} = 1.0 \text{ m/s}^2$$

The tangential acceleration is 1.0 m/s^2 .

Be careful! The tangential acceleration is constant (because α is constant), so it does not depend on time. It is the same at $t = 1 \text{ s}$ as at $t = 3 \text{ s}$.

(iii) First, finding ω at $t = 3 \text{ s}$:

$$\omega = \omega_0 + \alpha t = 0 + 4 \text{ rad/s}^2 \times 3 \text{ s} = 12 \text{ rad/s}$$

The centripetal acceleration is:

$$a_c = \omega^2 R = (12 \text{ rad/s})^2 \times 0.25 \text{ m} = 36 \text{ m/s}^2$$

The centripetal acceleration at $t = 3 \text{ s}$ is 36 m/s^2 .

Another point to note! Unlike the tangential acceleration, the centripetal acceleration is not constant. It grows with time because ω increases. At $t = 0$, $a_c = 0$ (the wheel is not yet turning). At $t = 3 \text{ s}$, $a_c = 36 \text{ m/s}^2$. This is already 36 times larger than the tangential acceleration!

(iv) Since a_t and a_c are perpendicular, the magnitude of the total linear acceleration is:

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{(1.0 \text{ m/s}^2)^2 + (36 \text{ m/s}^2)^2} = 36.01 \text{ m/s}^2$$

The angle ϕ between the total acceleration and the radius (centripetal direction) is:

$$\tan \phi = \frac{a_t}{a_c} = \frac{1.0 \text{ m/s}^2}{36 \text{ m/s}^2} = 0.0278$$

$$\phi = \tan^{-1}(0.0278) = 1.59^\circ$$

The total acceleration has magnitude 36.0 m/s^2 and is directed almost radially inward, tilted just 1.6° from the radius toward the tangent direction.

Interesting! At $t = 3 \text{ s}$, the centripetal acceleration (36 m/s^2) completely dominates the tangential acceleration (1.0 m/s^2). The total acceleration is practically centripetal. This is typical: once a body has been spinning for a few seconds, the centripetal component grows as ω^2 and quickly dwarfs the constant tangential component. Early in the motion (e.g. at $t = 0.25 \text{ s}$, where $\omega = 1 \text{ rad/s}$ and $a_c = 0.25 \text{ m/s}^2$), the tangential component actually dominates. The balance shifts as ω builds up.

Example 62

- (a) In a simple Atwood machine with a massless pulley, the tension is the same on both sides of the string. Explain why this is no longer true when the pulley has mass.
- (b) Two masses $m_1 = 5 \text{ kg}$ and $m_2 = 3 \text{ kg}$ are connected by a light, inextensible string that passes over a solid cylindrical pulley of mass $M = 2 \text{ kg}$ and radius $R = 0.10 \text{ m}$. The pulley rotates about a fixed frictionless axle. Mass m_1 hangs on the left and m_2 hangs on the right. The system is released from rest with m_1 at a height of 1.5 m above m_2 . Take $g = 9.8 \text{ m/s}^2$.
- (i) Calculate the speed of the masses when m_1 has descended 0.8 m .
- (ii) Calculate the acceleration of the system.
- (iii) Calculate the tensions T_1 and T_2 .

Solution

- (a) A massless pulley has zero moment of inertia ($I = 0$). By Newton's second law for rotation, $\tau = I\alpha = 0$, so the net torque on the pulley is zero. The only torques come from the tensions on each side: $(T_1 - T_2)R = 0$, which gives $T_1 = T_2$. So equal tensions follow automatically from zero mass.

When the pulley has mass, $I > 0$. The pulley must angularly accelerate as the system moves, requiring a net torque: $(T_1 - T_2)R = I\alpha \neq 0$. This demands $T_1 \neq T_2$. The tension on the heavier side must be larger than on the lighter side, and the difference provides the torque that spins the pulley.

- (b) The moment of inertia of the solid cylindrical pulley:

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 2\text{kg} \times (0.10\text{m})^2 = 0.01\text{kgm}^2$$

- (i) When m_1 descends by $h = 0.8\text{m}$, m_2 rises by 0.8m (the string is inextensible). The pulley rotates, gaining rotational KE.

By conservation of energy, the PE lost by m_1 equals the PE gained by m_2 plus the KE gained by both masses plus the rotational KE gained by the pulley:

$$m_1gh = m_2gh + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}I\omega^2$$

Using the no-slip condition: $v = \omega R$, so $\omega = \frac{v}{R}$. Substituting:

$$m_1gh = m_2gh + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}I\frac{v^2}{R^2}$$

Rearranging (moving m_2gh to the left side):

$$\begin{aligned} m_1gh - m_2gh &= \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}I\frac{v^2}{R^2} \\ (m_1 - m_2)gh &= \frac{1}{2}v^2 \left(m_1 + m_2 + \frac{I}{R^2} \right) \end{aligned}$$

Substituting all values:

$$(5\text{kg} - 3\text{kg}) \times 9.8\text{m/s}^2 \times 0.8\text{m} = \frac{1}{2}v^2 \times \left(5\text{kg} + 3\text{kg} + \frac{0.01\text{kgm}^2}{(0.10\text{m})^2} \right)$$

$$v^2 = 3.484\text{m}^2/\text{s}^2$$

$$v = \sqrt{3.484\text{m}^2/\text{s}^2} = 1.87\text{m/s}$$

The speed of the masses when m_1 has descended 0.8m is 1.87m/s .

- (ii) Since the acceleration is constant, using $v^2 = u^2 + 2as$ with $u = 0$:

$$a = \frac{v^2}{2s} = \frac{3.484\text{m}^2/\text{s}^2}{2 \times 0.8\text{m}} = 2.178\text{m/s}^2$$

The acceleration is 2.178m/s^2 .

- (iii) For m_1 (descending), applying Newton's second law:

$$m_1g - T_1 = m_1a$$

Making T_1 the subject:

$$T_1 = m_1(g - a) = 5\text{kg} \times (9.8\text{m/s}^2 - 2.178\text{m/s}^2) = 38.1\text{N}$$

For m_2 (ascending), applying Newton's second law:

$$T_2 - m_2g = m_2a$$

Making T_2 the subject:

$$T_2 = m_2(g + a) = 3\text{kg} \times (9.8\text{m/s}^2 + 2.178\text{m/s}^2) = 35.9\text{N}$$

The tensions are $T_1 = 38.1\text{N}$ and $T_2 = 35.9\text{N}$.

The plates are cleared, the table is wiped, and every dish has been tasted twice. If your brain feels like a flywheel that has been spinning for too long, congratulations! That is rotational KE, and it means you have been working hard. Now let us see if the Digging Deeper exercise can provide enough torque to slow you down!

DIGGING DEEPER EXERCISE 9

EXERCISE 9A: BINDER QUESTIONS

Question 1

A student claims: “If I know how to solve $F = ma$, I already know how to solve $\tau = I\alpha$, because the mathematics is identical.” Is this claim correct? Justify your answer by identifying the translational twin of each rotational quantity in $\tau = I\alpha$.

Question 2

A solid sphere and a hollow sphere have the same mass and the same outer radius.

- (a) State which one has the larger moment of inertia about an axis through the centre. Explain why.
- (b) If both are acted upon by the same torque, state which one has the larger angular acceleration. Explain.

Question 3

A uniform rod can be rotated about its centre or about one end.

- (a) Without using the parallel axis theorem, explain physically why the moment of inertia about the end is larger than about the centre.
- (b) Using the parallel axis theorem, show that $I_{\text{end}} = 4I_{\text{centre}}$ for a uniform rod.

Question 4

By using parallel axis theorem, explain why the moment of inertia about the centre of mass is always the minimum moment of inertia for any rigid body.

Question 5

A solid disc and a ring have the same mass and the same outer radius R .

- (a) Which one has the larger radius of gyration about the central axis? Explain in terms of how mass is distributed.
- (b) A student says: “If I double the mass of the disc without changing its radius, the radius of gyration doubles.” Is the student correct? Explain.

Question 6

A student tries to open a heavy door by pushing at the centre of the door instead of at the handle (far from the hinges).

- (a) Explain why pushing at the centre requires more force than pushing at the handle to achieve the same angular acceleration.
- (b) Explain why a couple is more effective at producing pure rotation than a single force applied at one point.

Question 7

A wheel starts from rest and undergoes uniform angular acceleration. A student measures the angular displacements in four consecutive seconds and obtains: 2rad, 6rad, 10rad, 14rad. Explain how these values confirm that the angular acceleration is uniform.

Question 8

- (a) Distinguish between rotational KE and translational KE of a rolling body.
- (b) Explain why a rolling body always has more KE than a sliding body of the same mass and speed.

Question 9

A solid cylinder and a hollow cylinder of the same mass and radius are released from rest at the top of an identical rough incline.

- (a) Explain why the solid cylinder reaches the bottom first, even though both have the same mass and start with the same potential energy.

- (b) If the incline were made perfectly frictionless, explain what would happen to both cylinders.

Question 10

During pure rolling on a flat surface, the contact point has zero velocity.

- (a) Explain how the contact point can have zero velocity when the body is clearly moving forward.
- (b) A motorcycle wheel picks up a small stone from the road. At what point on the wheel's rotation does the stone is more likely to fly off? Explain why.

Question 11

Friction acts up the slope when a body rolls down an incline, yet it does no work.

- (a) Explain how friction can exert a force without doing any work.
- (b) If friction does no work, explain where the rotational KE of the rolling body comes from.

Question 12

A disc spinning freely on a frictionless axle has a lump of clay dropped onto it.

- (a) Explain why the angular velocity decreases after the clay lands.
- (b) Explain why the KE of the system decreases, even though angular momentum is conserved.

Question 13

A body can have zero angular velocity but nonzero angular acceleration.

- (a) Give a physical example of this situation.
- (b) Can a body have nonzero angular velocity but zero angular acceleration? Give an example and explain.

Question 14

Two objects of different shapes but the same mass and radius roll without slipping down the same incline from rest.

- (a) Explain why the acceleration of each object depends only on its shape and not on its mass or radius.
- (b) Rank the following in order of arrival at the bottom (fastest first): solid sphere, solid cylinder, hollow sphere, hollow cylinder. Justify your ranking.

Question 15

Two students argue about a spinning raw egg that is briefly stopped and released.

Student A: *"The egg restarts because of Newton's third law."*

Student B: *"The egg restarts because of conservation of angular momentum."*

Which student is correct? Explain the physics carefully.

Question 16

The moment of inertia of a body depends on the axis of rotation.

- (a) Explain, with an example, why the same body can have different moments of inertia about different axes.
- (b) State the theorem that allows you to relate the moment of inertia about one axis to that about a perpendicular axis. To which shapes does this theorem apply?

Question 17

The perpendicular axis theorem applies to flat (planar) bodies but not to three-dimensional bodies like spheres.

- (a) Explain physically why this restriction exists.
- (b) A student tries to use the perpendicular axis theorem on a solid sphere to relate the moment of inertia about a diameter to the moment of inertia about a perpendicular diameter. Explain why this is unnecessary and what the correct relationship is.

Question 18

A flywheel and a solid ball have the same mass and the same outer radius. The flywheel is designed with a heavy rim (like a ring), while the ball has mass distributed throughout its volume.

- (a) If both rotate at the same angular velocity, which stores more rotational KE? Explain in terms of moment of inertia.
- (b) An engineer needs to design a rotating body that reaches high angular velocity quickly from rest under a given torque. Should the engineer choose the flywheel shape or the ball shape? Explain.

Question 19

A flywheel is designed to store rotational KE. Explain why flywheels are made with heavy rims and light centres rather than being solid discs of uniform thickness.

Question 20

A body rolling without slipping on a rough incline may begin to slip if the incline angle is increased beyond a certain value.

- (a) Explain why increasing the angle can cause the body to slip.
- (b) For the same coefficient of static friction and the same incline angle, which is more likely to slip: a solid sphere or a hollow cylinder? Explain.

EXERCISE 9B: REAL QUESTIONS**Question 21**

A door has its handle placed at the edge farthest from the hinges, not at the centre or near the hinges. Explain, using the concept of torque, why this placement makes the door easier to open.

Question 22

A traditional Tanzanian potter shapes clay on a heavy stone wheel. The potter gives the wheel a strong push and then works the clay while the wheel gradually slows down.

- (a) Explain why a heavier wheel is preferred over a lighter one for pottery.
- (b) Explain why the wheel continues to spin for a long time after the push, even though no one is pushing it.

Question 23

Kipanga is helping his mother grind maize using a traditional hand-powered grinding stone. He pushes the handle at the edge of the stone. His younger brother tries to help by pushing near the centre.

Kipanga says: *"Push at the edge! You are wasting your effort near the centre."*

- (a) Explain why Kipanga is correct.
- (b) The grinding stone continues to rotate smoothly between pushes. Explain this observation in terms of moment of inertia and rotational KE.

Question 24

A figure skater begins a spin with both arms stretched outward. She then pulls her arms tight against her body, and her spin rate increases dramatically. No one pushes her, and the ice is nearly frictionless.

- (a) Explain why pulling her arms in causes her to spin faster.
- (b) Explain the change in kinetic energy as she pulls her arms tight against her body.

Question 25

A car is stuck in mud. The driver presses the accelerator hard, and the wheels spin rapidly but the car does not move.

- (a) Explain, using the physics of rolling, why the car cannot move forward.
- (b) A passenger places wooden planks under the spinning wheels. Explain why this helps the car to move.

Question 26

Kipute and Kipanga are watching a circus performer balance a spinning plate on a stick. The performer keeps the plate spinning fast, and it stays balanced. When the spin slows down, the plate wobbles and eventually falls.

Kipanga: *“The spinning must create some upward force that holds the plate.”*

Kipute: *“No, the weight of the plate has not changed. Spinning does not reduce gravity. Something else is keeping it stable.”*

Who is correct? Explain.

Question 27

A long metal rod and a short metal rod, both uniform and of the same material, are held vertically and released from the same height to fall and rotate about their lower ends (which are pinned to the ground).

- (a) Explain why the longer rod takes more time to fall to the ground than the shorter rod.
- (b) Explain why this result seems to contradict free fall (where all objects fall at the same rate).

Question 28

When opening a tight bottle cap, people instinctively wrap a piece of cloth around the cap before twisting.

- (a) Explain how the piece of cloth helps.
- (b) Some people use pliers with long handles to open very tight caps. Explain the advantage of long handles.

Question 29

In competitive diving, a diver performs three somersaults in the tuck position but straightens out before entering the water with almost no rotation.

- (a) Explain how the diver controls the rate of somersaulting without any external force or torque while in the air.
- (b) Explain why straightening out before entry reduces the rotation rate.

Question 30

In many Tanzanian villages, water is drawn from a well using a rope wound around a cylindrical drum. The operator turns a handle attached to the drum.

- (a) Explain why a larger drum radius makes it harder to turn the handle when pulling up a full bucket, even though the bucket weight has not changed.
- (b) Explain why the drum continues to rotate slightly after the operator stops turning the handle, especially when the drum is made of heavy wood.

Question 31

A rungu has most of its mass concentrated at the far end (the knob) and a thin handle.

- (a) Explain why this design makes the rungu more effective as a striking weapon than a uniform stick of the same total mass and length.
- (b) When thrown, the rungu rotates end over end. Explain why concentrating mass at one end affects the rotation of the thrown rungu.

Question 32

A child sits at the edge of a stationary merry-go-round. An adult gives the merry-go-round a push and it begins to spin. The child then moves to the centre.

- (a) Explain what happens to the angular velocity of the merry-go-round when the child moves to the centre.
- (b) The child now moves back to the edge. Explain what happens to the angular velocity.

Question 33

Mr. Akilikubwa demonstrates two identical tin cans in class. One is filled with water, the other with frozen ice (same mass of water in both). He releases both simultaneously from the top of a tilted plank.

“The ice-filled can reaches the bottom first.”

Kipanga: *“But they weigh the same! How can one be faster?”*

Mr. Akilikubwa: *“The weight is the same, yes. But think about what happens inside each can as it rolls.”*

Explain why the ice-filled can reaches the bottom first.

EXERCISE 9C: HOT QUESTIONS**Question 34**

A wheel starts from rest and accelerates uniformly. It completes 20 revolutions in the first 5s.

- Calculate the angular acceleration.
- Calculate the angular velocity at $t = 5\text{s}$.
- Calculate the angle turned through during the 5th second only.
- If the wheel has radius 0.30m, calculate the total linear acceleration of a point on the rim at $t = 5\text{s}$.

Question 35

Three masses are arranged in the x-y plane as follows:

$$m_1 = 4\text{kg at } (0,0)$$

$$m_2 = 2\text{kg at } (3\text{m}, 0)$$

$$m_3 = 6\text{kg at } (1\text{m}, 4\text{m})$$

- Calculate the position of the centre of mass.
- Calculate the distance of the centre of mass from m_3 .
- If m_3 is removed, calculate the new position of the centre of mass and the direction in which it shifts.

Question 36

A uniform rectangular plate of mass M , length a , and width b lies in the x-y plane.

- Using the result $I = \frac{1}{12}ML^2$ for a rod about its centre, and the perpendicular axis theorem, show that the moment of inertia of the plate about an axis perpendicular to its surface through its centre is $I = \frac{1}{12}M(a^2 + b^2)$.
- Calculate the moment of inertia of a rectangular plate of mass 2.4kg, length 0.60m, and width 0.40m about an axis perpendicular to the plate through its centre.
- Using the parallel axis theorem, calculate the moment of inertia about an axis perpendicular to the plate through one corner.

Question 37

A uniform disc of mass 5kg and radius 0.20m is mounted on a frictionless horizontal axle. A light cord is wound around the rim, and a constant force of 15N is applied to the free end of the cord.

- Calculate the moment of inertia of the disc about the axle.
- Calculate the torque applied to the disc.
- Calculate the angular acceleration.
- Calculate the angular velocity after 3s, starting from rest.
- Calculate the number of revolutions in the first 3s.

Question 38

A grinding stone (solid disc) of mass 8kg and radius 0.25m rotates at 600rpm. A brake pad is pressed against the rim with a normal force of 40N. The coefficient of kinetic friction between the pad and the stone is 0.5.

- (a) Calculate the initial rotational KE of the stone.
- (b) Calculate the friction force and the braking torque.
- (c) Calculate the angular deceleration.
- (d) Calculate the time to bring the stone to rest.
- (e) Calculate the number of revolutions before stopping.

Question 39

A solid sphere of mass 3kg and radius 0.10m and a hollow cylinder of mass 3kg and radius 0.10m are both released from rest at the top of a rough incline of height 2.0m and angle 30° .

- (a) Calculate the speed of each at the bottom of the incline.
- (b) Calculate the time for each to reach the bottom.
- (c) Calculate the time difference between their arrivals.

Question 40

A solid cylinder of mass 6kg and radius 0.15m rolls without slipping down a rough incline of angle 40° . Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the acceleration of the centre of mass.
- (b) Calculate the friction force required for pure rolling.
- (c) Calculate the minimum coefficient of static friction needed.
- (d) If the surface is oiled so that $\mu_s = 0.10$, show that the cylinder cannot roll without slipping and describe what happens instead.

Question 41

A merry-go-round (uniform disc, mass 120kg, radius 1.8m) rotates freely at 2.0rad/s . A boy of mass 40kg stands at the rim. He then walks inward to a distance of 0.6m from the centre.

- (a) Calculate the moment of inertia of the system before the boy moves.
- (b) Calculate the moment of inertia of the system after the boy moves.
- (c) Calculate the new angular velocity.
- (d) Calculate the ratio of the final KE to the initial KE, and explain where the extra energy comes from.

Question 42

Two masses $m_1 = 8\text{kg}$ and $m_2 = 5\text{kg}$ are connected by a light string passing over a solid cylindrical pulley of mass $M = 4\text{kg}$ and radius $R = 0.12\text{m}$. The system is released from rest. Take $g = 9.8\text{m/s}^2$.

- (a) Calculate the acceleration of the system.
- (b) Calculate the tensions T_1 and T_2 .
- (c) Calculate the speed of the masses after m_1 has descended 1.2m.

Question 43

A uniform rod of mass 1.5kg and length 0.80m is pivoted at one end and held horizontally. It is released from rest and swings downward.

- (a) Calculate the moment of inertia of the rod about the pivot.
- (b) Using conservation of energy, calculate the angular velocity of the rod as it passes through the vertical position.
- (c) Calculate the linear speed of the free end as the rod passes through the vertical.

- (d) Calculate the angular acceleration of the rod at the instant of release (when it is horizontal).
- (e) Calculate the angular acceleration of the rod as it passes through the vertical.

Question 44

A flywheel (solid disc, mass 10kg, radius 0.30m) is driven by a motor that delivers a constant torque of 4.5Nm. A friction torque of 0.5Nm opposes the rotation.

- (a) Calculate the net torque on the flywheel.
- (b) Calculate the angular acceleration.
- (c) Calculate the time to reach 120rad/s from rest.
- (d) Calculate the power delivered by the motor at 120rad/s.
- (e) Calculate the power dissipated by friction at 120rad/s.

Question 45

The moment of inertia of a flywheel is determined experimentally. A mass $m = 1.5\text{kg}$ is attached to a string wound around the axle (radius $r = 0.025\text{m}$). The mass is released from rest and falls $h = 0.90\text{m}$, reaching a speed of 0.75m/s. After the string detaches, the flywheel makes $n_2 = 30$ additional revolutions before stopping.

- (a) Calculate the angular velocity of the flywheel at the moment the string detaches.
- (b) Using conservation of energy for the falling phase, derive an expression for the moment of inertia I of the flywheel in terms of m , g , h , v , r , and n_2 .
- (c) Calculate I .
- (d) Calculate the friction torque.

Question 46

A solid sphere of mass 2kg and radius 0.05m rolls without slipping along a horizontal surface at 3m/s. It then encounters a rough incline of angle 25° .

- (a) Calculate the total KE of the sphere on the horizontal surface.
- (b) Calculate the maximum height the sphere reaches on the incline before momentarily stopping.
- (c) Calculate the acceleration of the sphere on the incline.
- (d) Calculate the friction force acting on the sphere while it rolls up the incline.
- (e) In which direction does friction act on the sphere as it rolls up the incline? Explain why this is different from rolling down.

Question 47

An engineer needs to design a flywheel that stores at least 5000J of rotational KE at a maximum angular velocity of 300rad/s. The flywheel is to be a solid steel disc (density of steel = 7800kg/m^3).

- (a) Calculate the minimum moment of inertia required.
- (b) If the disc has thickness $t = 0.05\text{m}$, calculate the minimum radius of the disc.
- (c) Calculate the mass of this disc.
- (d) If the engineer instead uses a ring (same mass, same outer radius, inner radius = $0.8R$), calculate the moment of inertia and the energy stored at the same angular velocity. Is the ring design better or worse for energy storage?

Question 48

A factory conveyor belt is driven by a solid cylindrical roller of mass 50kg and radius 0.20m. The belt must accelerate from rest to a linear speed of 2.0m/s in 4s. Assume the belt does not slip on the roller.

- (a) Calculate the angular velocity of the roller at full speed.
- (b) Calculate the angular acceleration during startup.

- (c) Calculate the torque required from the motor to accelerate the roller (ignoring friction losses).
- (d) If friction losses add a constant opposing torque of 3Nm, calculate the total motor torque needed and the motor power at full speed.

Question 49

A disc of moment of inertia 0.5kgm^2 rotates at 20rad/s . A second disc of moment of inertia 0.3kgm^2 , initially rotating at 8rad/s in the **opposite direction**, is dropped coaxially onto the first.

- (a) Taking the direction of the first disc as positive, calculate the total angular momentum before contact.
- (b) Calculate the final angular velocity after they lock together.
- (c) In which direction does the combined system rotate?
- (d) Calculate the KE lost and express it as a percentage of the initial total KE.

Question 50

A uniform rod of mass 2kg and length 1.0m is free to rotate about a horizontal axis through one end. A bullet of mass 10g travelling horizontally at 400m/s embeds itself in the free end of the rod, which is initially hanging vertically at rest.

- (a) Calculate the angular momentum of the bullet about the pivot just before impact.
- (b) Calculate the moment of inertia of the system (rod + embedded bullet) about the pivot.
- (c) Calculate the angular velocity of the system immediately after impact.
- (d) Calculate the maximum angle through which the rod swings after impact.

Question 51

A student is asked: *“A solid sphere rolls without slipping down an incline. What provides the centripetal acceleration for the circular motion of the sphere?”*

The student answers: *“Friction provides the centripetal acceleration, because friction is the only horizontal force.”*

- (a) Identify the error in the student’s reasoning.
- (b) Explain what friction actually does in rolling on an incline.
- (c) If the sphere rolls on a flat horizontal surface at constant velocity, is friction needed? Explain.

Question 52

A solid cylinder of mass 4kg and radius 0.08m rolls without slipping at 5m/s along a horizontal surface. It then rolls up a rough incline of angle 35° .

- (a) Calculate the total KE on the horizontal surface.
- (b) Calculate the maximum distance along the incline the cylinder travels before stopping.
- (c) Calculate the friction force while the cylinder rolls up.
- (d) After momentarily stopping, the cylinder rolls back down. Calculate its speed when it returns to the horizontal surface.

Question 53

A solid sphere of mass M and radius R rolls without slipping at speed v_0 on a horizontal surface. It encounters a step of height h ($h < R$). The sphere collides with the edge of the step.

- (a) Show that the angular momentum of the sphere about the edge of the step, just before the collision, is $L = Mv_0(R - h) + \frac{2}{5}MRv_0$.
- (b) Assuming the sphere does not slip or bounce at the edge, and angular momentum about the edge is conserved during the very brief collision, find the angular velocity of the sphere about the edge immediately after the collision.

- (c) Using energy conservation after the collision, find the minimum v_0 for the sphere to climb over the step.

Question 54

A uniform disc of mass 3kg and radius 0.20m can rotate about a fixed horizontal axis through its centre. A constant tangential force of 12N is applied at the rim for 5s, after which the force is removed.

- (a) Calculate the angular velocity at the end of 5s.
- (b) Calculate the rotational KE at the end of 5s.
- (c) After the force is removed, a brake applies a constant friction torque of 0.8Nm. Calculate the time for the disc to stop.
- (d) Calculate the total number of revolutions from start to stop.

Question 55

A solid ball of mass 0.5kg and radius 0.04m rolls without slipping at 6m/s along a flat surface toward a loop-the-loop track of radius $R_{\text{loop}} = 0.50\text{m}$.

- (a) Calculate the total KE of the ball before it enters the loop.
- (b) Calculate the speed of the ball at the top of the loop. Assume the ball continues to roll without slipping.
- (c) For the ball to maintain contact with the track at the top of the loop, the centripetal acceleration must be at least g . Determine whether the ball completes the loop.

Question 56

A torque of $\tau = (6t - 2)\text{Nm}$ acts on a disc of moment of inertia 0.4kgm^2 , where t is in seconds. The disc starts from rest at $t = 0$.

- (a) At what time is the angular acceleration zero?
- (b) Calculate the angular velocity at $t = 3\text{s}$.
- (c) Calculate the angular momentum at $t = 3\text{s}$.

Question 57

A hollow sphere ($I = \frac{2}{3}MR^2$) of mass M and radius R rolls without slipping at speed v on a horizontal surface toward a ramp of angle θ .

- (a) Show that the acceleration of the sphere on the ramp is $a = \frac{3g\sin\theta}{5}$.
- (b) Show that the friction force required is $f = \frac{2}{5}Mg\sin\theta$.
- (c) Show that the minimum coefficient of static friction for pure rolling is $\mu_{s,\text{min}} = \frac{2}{5}\tan\theta$.
- (d) For $\theta = 45^\circ$ and $\mu_s = 0.35$, determine whether the sphere rolls or slips. If it slips, describe qualitatively what happens.

Chapter 10

FLUID MECHANICS

INTRODUCTION

Picture this scene: It is a hot Saturday afternoon in Sinza, Dar es Salaam. A group of boys are washing a car in the yard using a garden hose connected to a tap. The water flows out gently, splashing harmlessly onto the dusty bonnet. Then one of the boys, feeling clever, presses his thumb over the end of the hose.

The water explodes outward in a thin, powerful jet. It shoots across the yard, misses the car entirely, and hits a neighbouring *mama lishe* square in the back while she is frying *sambusa*. Chaos. Shouting. A plate of *sambusa* hits the ground. The boy releases his thumb immediately, the jet dies back to a gentle flow, and he spends the rest of the afternoon apologising and buying replacement *sambusa* with money he had been saving for airtime.

But here is the physics question nobody at the scene thought to ask: why did the water speed up? The tap pressure did not change. Nobody pumped harder. The boy simply narrowed the opening. Yet the water, passing through the smaller gap, shot out with dramatically more speed. Where did the extra velocity come from?

Now picture a second scene. After the apology, the boy notices a sewing needle on the ground. He drops it into a bucket of water. It sinks, as steel should. But then his younger sister places an identical needle *flat* on the water surface, with the care of a person laying a ruler on a sleeping cat. The needle floats. Steel, denser than water, sitting on the surface as though resting on an invisible mattress. How?

And one more. That evening, the boy watches his mother pour honey into hot chai. She tilts the jar. The honey responds like a government office at lunchtime: slowly, reluctantly, as though every millimetre of progress required a committee meeting, a signed form, and three cups of chai. Meanwhile the chai swirls freely in the cup. Same gravity pulls both. One flows eagerly. The other resists every step. Why?

Three ordinary moments. Three mysteries. And behind all three lies a single, beautiful truth: **fluids have three hidden personalities.**

The first personality is **surface tension**: an invisible elastic skin on every liquid surface, pulling inward, resisting anything that tries to break through. The flat needle spreads its weight across this skin; a needle dropped point-first punctures it and sinks. Surface tension is why raindrops are spherical, why insects walk on ponds, and why water climbs up narrow tubes against gravity.

The second personality is **Bernoulli's equation**: when a fluid speeds up, its pressure drops. This single principle explains how aeroplanes generate lift, why roofs blow *off* during storms instead of being crushed *in*, and why a boy's thumb turned a harmless garden hose into a *sambusa*-destroying weapon.

The third personality is **viscosity**: internal friction between fluid layers that resists flow. It is why honey ignores gravity's polite requests, why engines in Moshi need different oil from engines in Dar es Salaam, and how Stokes' law predicts the speed at which a sphere sinks through any fluid you care to drop it into.

Welcome to fluid mechanics, the final chapter of this volume and the gateway from rigid bodies to the physics of things that flow, deform, and reshape themselves. By the end, you will understand why that boy's thumb turned a gentle stream into a jet, why his sister's needle refused to sink, and why his mother's honey took its own sweet time. You will even understand why the *sambusa* vendor was perfectly right to be angry, but wrong to blame the boy. The real culprit was Bernoulli.

PRESSURE IN FLUIDS

Before we explore the three personalities of fluids, we need to make sure we speak the same language. Pressure is the foundation on which surface tension, Bernoulli's equation, and viscosity are all built. You met these ideas at O-level; this section sharpens them for the work ahead. Think of it as tuning your instrument before the concert begins.

What is Pressure?

When you stand on soft mattress, your feet sink in. When you lie flat on the same mattress, you do not sink as much, even though your weight has not changed. The difference is not in the force, but in how that force is *spread*. Standing concentrates your entire weight on the small area of your soles. Lying down distributes the same weight over a much larger area. The quantity that captures this idea is called **pressure**.

Pressure is defined as the *force acting per unit area perpendicular to the surface*:

$$P = \frac{F}{A}$$

Where:

P = pressure (measured in pascals, Pa)

F = force perpendicular to the surface (N)

A = area over which the force acts (m^2)

Since $1 \text{ Pa} = 1 \text{ N/m}^2$, the pascal is a remarkably small unit. Atmospheric pressure, for instance, is approximately 101,325 Pa, which is why we often use kilopascals (kPa) or the atmosphere ($1 \text{ atm} = 101,325 \text{ Pa}$) as practical units.

One crucial property of pressure in fluids deserves emphasis: *pressure at a point in a fluid acts equally in all directions*. This is fundamentally different from a solid surface, where force acts in a specific direction. A fluid pushes upward, downward, and sideways with equal intensity at any given depth. This is why a balloon submerged in water is squeezed equally from all sides, not just from above.

Pressure at a Depth in a Fluid

Consider a fluid of density ρ at rest in an open container. We want to find the pressure at a depth h below the surface. Imagine a thin horizontal disc of area A at this depth. The fluid above the disc forms a column of height h and cross-sectional area A .

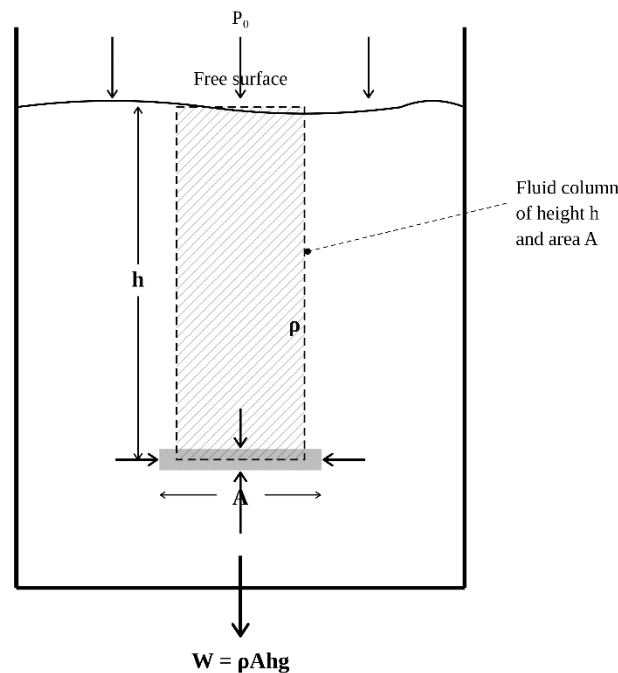


Figure: Pressure at a depth h in a fluid of density ρ . The hatched column (height h , cross-sectional area A) has weight $W = \rho Ahg$. Arrows at the disc show that pressure acts equally in all directions. P_0 is the atmospheric pressure at the free surface.

The volume of this column is:

$$V = Ah$$

The mass of the fluid in the column is:

$$m = \rho V = \rho Ah$$

The weight of the fluid column, which acts downward on the disc, is:

$$W = mg = \rho Ahg$$

This weight is spread over the area A of the disc, so the pressure due to the fluid column alone is:

$$P = \frac{W}{A} = \frac{\rho Ahg}{A}$$

$$P = \rho hg$$

This is the pressure due to the fluid itself at depth h . It depends only on the depth, the density of the fluid, and the acceleration due to gravity. It does not depend on the shape of the container or the total amount of fluid. A narrow tube and a wide lake, filled with the same fluid to the same depth, produce the same pressure at the bottom.

The **total pressure** (also called **absolute pressure**) at depth h in an open container includes the atmospheric pressure P_0 acting on the surface:

$$P_{\text{abs}} = P_0 + \rho hg$$

Pascal's Principle

Pascal's Principle states that *a change in pressure applied to an enclosed fluid at rest is transmitted undiminished to every point in the fluid and to the walls of the container*. In simpler terms: if you push harder at one point in a closed fluid system, every other point feels the same increase in pressure.

This principle is the foundation of all hydraulic machines. Consider a hydraulic press with a small piston of area A_1 and a large piston of area A_2 , connected by an enclosed fluid. When a force F_1 is applied to the small piston, the pressure increase in the fluid is $P = \frac{F_1}{A_1}$. By Pascal's Principle, this same pressure acts on the large piston, producing an output force:

$$F_2 = P \times A_2 = F_1 \times \frac{A_2}{A_1}$$

Since $A_2 > A_1$, the output force F_2 is greater than the input force F_1 . The hydraulic press is a force multiplier. This is exactly how the braking system of a daladala works: a small force applied by the driver's foot on the brake pedal is transmitted through brake fluid and amplified at the brake pads, producing enough force to stop a vehicle carrying passengers and their luggage.

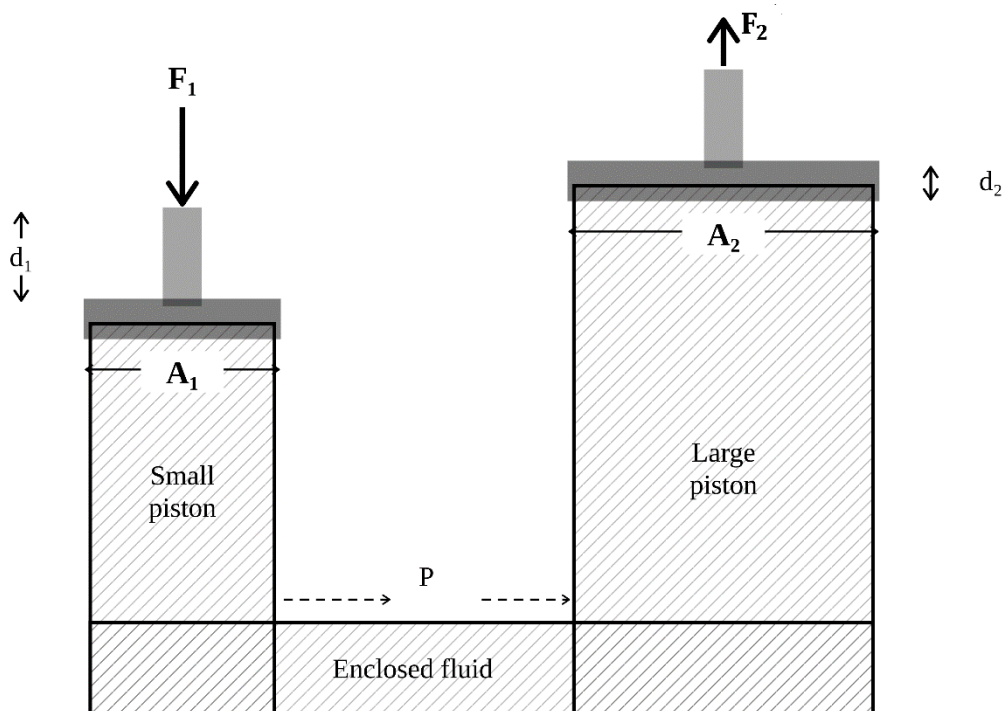


Figure: A hydraulic press. A force F_1 applied to the small piston (area A_1) is transmitted through the enclosed fluid to the large piston (area A_2), producing a larger output force F_2 . The small piston moves a distance d_1 while the large piston moves a smaller distance d_2 .

An important consequence: while the force is multiplied, **energy is conserved**. The small piston moves a larger distance d_1 while the large piston moves a smaller distance d_2 , such that the work done on both sides is equal:

$$F_1 \times d_1 = F_2 \times d_2$$

A hydraulic press gives you more force, but not more work. There is no free lunch in physics.

Atmospheric Pressure and Gauge Pressure

Atmospheric pressure is the pressure exerted by the weight of the atmosphere above a given point. At sea level, its standard value is approximately:

$$P_0 = 1.013 \times 10^5 \text{ Pa} = 101.3 \text{ kPa} = 1 \text{ atm}$$

This is the pressure that acts on every surface exposed to the atmosphere, including the surface of a liquid in an open container. It is the reason the equation of absolute pressure P_{abs} includes P_0 .

Gauge pressure is the pressure measured above atmospheric pressure. When a tyre pressure gauge reads 200 kPa, it means the air inside the tyre is at 200 kPa above atmospheric pressure. The absolute pressure inside the tyre is therefore:

$$P_{\text{abs}} = P_0 + P_{\text{gauge}} = 101.3 \text{ kPa} + 200 \text{ kPa} = 301.3 \text{ kPa}$$

In most problems in this chapter, when we write $P = \rho gh$, we are calculating gauge pressure. Always read the question carefully to determine whether atmospheric pressure should be included.

With the language of pressure now freshly sharpened, let us put it to work. The following examples are not meant to challenge you; they are meant to remind your hands how to hold the tools before we use those tools to build something extraordinary in the subtopics ahead.

BINDER Example 1

A cylindrical water storage tank in a village in Dodoma is filled to a depth of 4.0 m. Calculate the pressure exerted by the water at the base of the tank. Take the density of water as 1000 kg/m^3 and $g = 9.8 \text{ m/s}^2$.

Solution

Using:

$$P = \rho gh$$

$$P = 4.0 \text{ m} \times 1000 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 = 39,200 \text{ Pa} = 39.2 \text{ kPa}$$

Therefore, the pressure due to the water at the base of the tank is 39.2 kPa.

Making Sense of the Answer: This is about 39% of atmospheric pressure. Even a modest 4-metre column of water produces significant pressure, which is why water tanks are placed on elevated platforms: the height creates the pressure that pushes water through the pipes to your tap.

Think Like a Physicist: The pressure at the base depends only on the depth of the water, not on the width or shape of the tank. A narrow pipe and a wide tank, both filled to 4.0 m, give exactly the same pressure at the bottom. This is called the hydrostatic paradox, and it surprises students every time.

REAL Example 2

While on a school trip to Mbudya Island, **Kipanga** decides to dive into the sea. As he goes deeper, he notices a sharp, uncomfortable pain in his ears. Back on the surface, he tells Mr. Akilikubwa:

“Sir, the deeper I went, the more my ears hurt. But the water was not hitting my ears from above. The pain felt like it was coming from all sides. How can water push sideways?”

Explain why Kipanga’s ears hurt more as he dived deeper, and why the pressure felt as though it came from all directions.

Solution

The pressure at a depth h in a fluid is given by $P = \rho gh$. As Kipanga dives deeper, h increases, so the pressure on his body increases.

The pain comes from all directions because pressure in a fluid at rest acts equally in all directions at a given depth. The water does not just push downward like a weight sitting on a table. It pushes sideways, upward,

and at every angle with equal intensity. Kipanga's eardrums, which are vertical membranes, are pushed inward by horizontal water pressure. This is why the pain feels like it comes from everywhere, not just from above.

Making Sense of the Answer: Pressure in a fluid increases significantly with depth. Even at just 3 metres, the additional pressure is about 30% of atmospheric pressure. Professional divers descend to 30m or more, where the water pressure alone exceeds 3 atmospheres. Without equalising the pressure in their ears (by pinching the nose and blowing gently), the eardrums can be seriously damaged.

Think Like a Physicist: Whenever you encounter pressure in a fluid, remember that it is not directional like force. Pressure at a point acts equally in every direction. Drawing a single downward arrow to represent fluid pressure is a common mistake. Instead, imagine arrows pointing inward from all sides.

HOT Example 3

A hydraulic press has a small piston of radius 2.0cm and a large piston of radius 15cm. A force of 50N is applied to the small piston.

- Find the force exerted by the large piston.
- If the small piston is pushed down by 20cm, find the distance the large piston moves upward.
- Calculate the work done by the input force and the work done by the output force. Comment on your result.

Solution

(a) Finding the output force:

Area of small piston:

$$A_1 = \pi r_1^2 = \pi \times (0.020 \text{ m})^2 = 1.257 \times 10^{-3} \text{ m}^2$$

Area of large piston:

$$A_2 = \pi r_2^2 = \pi \times (0.15 \text{ m})^2 = 7.069 \times 10^{-2} \text{ m}^2$$

By Pascal's Principle, pressure is transmitted equally:

$$\begin{aligned} \frac{F_2}{A_2} &= \frac{F_1}{A_1} \\ F_2 &= F_1 \times \frac{A_2}{A_1} \\ F_2 &= 50 \text{ N} \times \frac{7.069 \times 10^{-2} \text{ m}^2}{1.257 \times 10^{-3} \text{ m}^2} = 2812.5 \text{ N} \end{aligned}$$

The force exerted by the large piston is 2812.5N.

(b) Since the fluid is incompressible, the volume of fluid pushed down by the small piston equals the volume displaced upward at the large piston:

$$\begin{aligned} A_1 d_1 &= A_2 d_2 \\ d_2 &= d_1 \times \frac{A_1}{A_2} \\ d_2 &= 0.20 \text{ m} \times \frac{1.257 \times 10^{-3} \text{ m}^2}{7.069 \times 10^{-2} \text{ m}^2} = 3.56 \times 10^{-3} \text{ m} = 3.56 \text{ mm} \end{aligned}$$

The distance moved by large piston is $3.56 \times 10^{-3} \text{ m}$ or 3.56mm.

(c) Comparing work done:

Work done by the input force:

$$W_1 = F_1 \times d_1 = 50 \text{ N} \times 0.20 \text{ m} = 10.0 \text{ J}$$

Work done by the output force:

$$W_2 = F_2 \times d_2 = 2812.5 \text{ N} \times 3.56 \times 10^{-3} \text{ m} = 10.0 \text{ J}$$

Comment: The work done is the same on both sides. The hydraulic press multiplies force, but it does not multiply energy. Energy is conserved.

Making Sense of the Answer: A small 50N force (about the weight of 5kg) produced an output force of over 2800N (about the weight of 280kg). That is a force multiplication factor of 56.25, which is simply the ratio of the piston areas. But the large piston moved only 3.56mm while the small piston moved 200mm. Enormous force, tiny movement. This is the trade-off at the heart of every hydraulic machine, from car jacks to the brakes on a daladala.

Think Like a Physicist: Whenever a machine multiplies force, always ask: what is the cost? In levers, the cost is distance. In hydraulic systems, the cost is also distance. In gears, the cost is angular speed. Nature never gives you more force for free. Conservation of energy is the referee that keeps all machines honest.

That completes our brief review of pressure. The tools are sharpened: $P = \frac{F}{A}$ tells us what pressure is, $P = \rho gh$ tells us how pressure builds with depth, and Pascal's Principle tells us how pressure is transmitted through enclosed fluids. These three ideas will reappear throughout the chapter, sometimes quietly in the background, sometimes at the centre of a derivation.

In the next section, we meet the first hidden personality of fluids: **surface tension**. We begin at the molecular level, where the physics of why a needle can float on water and why raindrops are spherical has been hiding all along.

SURFACE TENSION: MOLECULAR THEORY AND SURFACE ENERGY

We now meet the first hidden personality of fluids. In the introduction, we watched a steel needle float on water and asked: *how can something denser than water refuse to sink?* The answer lives at the surface of the liquid, where molecules behave differently from those buried deep inside. This subtopic builds the explanation from the molecular level upward, and by the end of it, the floating needle will make perfect sense.

The Concept of Surface Tension

You have seen this personality before, even if you did not know its name. Water droplets on a freshly waxed car form nearly perfect spheres instead of spreading flat. A steel sewing needle, placed carefully on a still water surface, floats even though steel is nearly eight times denser than water. Insects called pond skaters walk across ponds without breaking the surface. In every case, the liquid surface behaves as though it is covered by a thin, invisible elastic skin that resists being stretched or broken.

This behaviour is called **surface tension**. It is not caused by an actual membrane. There is no physical skin on the surface of water. The effect arises entirely from the forces between molecules, as we will see later. But the elastic-skin analogy is useful because it captures the key observation: the surface of a liquid resists any attempt to increase its area, just as a stretched rubber sheet resists being pulled further.

To define surface tension precisely, imagine a straight line drawn on the surface of a liquid, dividing it into two halves. The molecules on one side of the line pull on the molecules on the other side with a force that acts along the surface and perpendicular to the line. Surface tension is defined as this *force per unit length*:

$$\gamma = \frac{F}{l}$$

where γ is the coefficient of surface tension, F is the force acting along the surface perpendicular to the line, and l is the length of the line.

The SI unit of surface tension is N/m (newtons per metre), which is equivalent to Nm^{-1} . Its dimensions are:

$$\gamma = \frac{\text{MLT}^{-2}}{\text{L}} = \text{MT}^{-2}$$

An important distinction: **surface tension** is the physical phenomenon (the tendency of a liquid surface to contract), while the **coefficient of surface tension (γ)** is the numerical value that measures how strong this tendency is.

Molecular Theory of Surface Tension

Why does the surface behave like a stretched skin? The answer lies in the forces between molecules.

Every molecule in a liquid attracts its neighbours. This *attractive force between molecules of the same substance* is called a **cohesive force**. The range of this force is very short, typically a few molecular diameters (about 10^{-9}m). Beyond this range, the force is effectively zero.

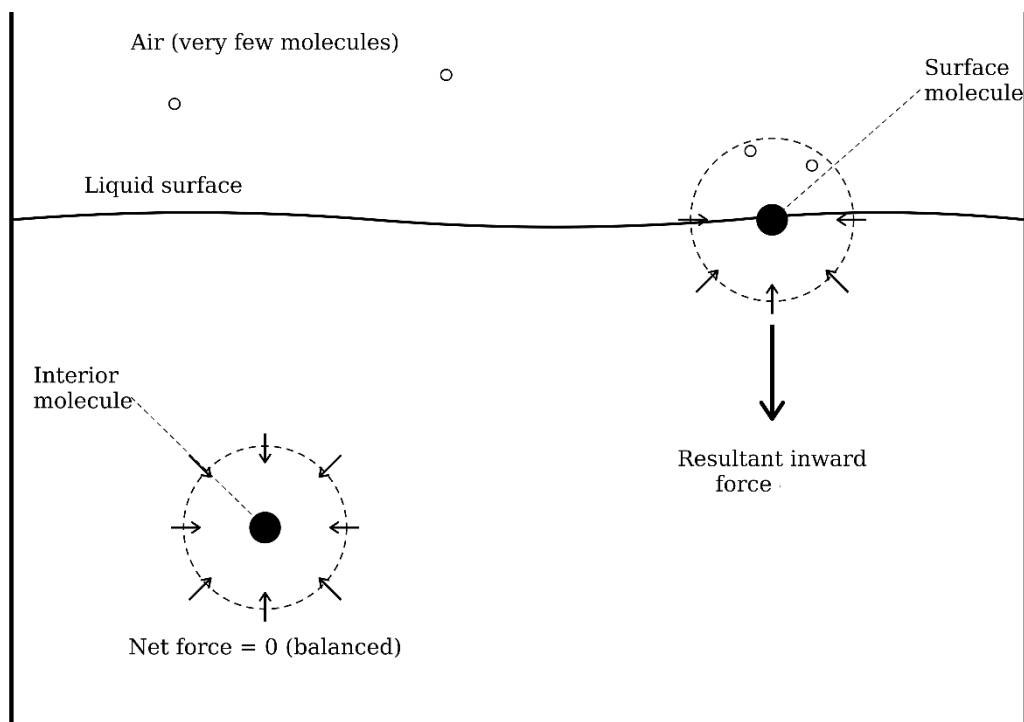


Figure: Molecular forces in a liquid. Interior molecule (deep inside) is pulled equally in all directions by its neighbours; the net force is zero. Surface molecule has very few neighbours above; the unbalanced cohesive forces produce a net inward force pulling it into the bulk of the liquid.

Now consider what happens to molecules at different positions in the liquid.

Interior molecule (deep inside the liquid): This molecule is surrounded by other molecules on all sides. The cohesive forces pull it equally in every direction. The net force on the molecule is zero. It is in equilibrium and can move freely within the liquid.

Surface molecule: This molecule has neighbours below it and to its sides, but very few above it (only the sparse molecules of air or vapour). The cohesive forces from below and from the sides are not balanced by forces from above. The result is a **net inward force** pulling the molecule downward, into the bulk of the liquid.

This net inward force on every surface molecule has two important consequences:

First consequence: Surface molecules are pulled inward, which means the liquid surface tends to **contract to the smallest possible area**. For a given volume of liquid, the shape with the smallest surface area is a sphere. This is why free-falling liquid drops are spherical (in the absence of air resistance distorting them). It is not that something pushes the drop into a sphere; rather, the surface contracts until no further contraction is possible, and that minimum-area shape is a sphere.

Second consequence: Because surface molecules are pulled inward, they must have been **pulled away from their neighbours** compared to molecules in the bulk. Work was done against cohesive forces to bring them to the surface. This means surface molecules have **higher potential energy** than interior molecules. The liquid surface is therefore a region of stored energy, which is why it resists being stretched: stretching the surface means bringing more molecules from the interior to the surface, which requires work.

This is the molecular origin of surface tension. The “elastic skin” (surface tension) is *really the collective pull of many (billions) surface molecules trying to minimise the surface area and reduce their excess potential energy*.

A common misconception is that surface tension is merely a description, not a real force. This is wrong. Surface tension is measurable, it does work, and it can support the weight of objects (like a needle or an insect). It is as real as the tension in a stretched rubber band. The difference is that a rubber band has material connecting its parts, while the “membrane” on a liquid surface is made of nothing but molecular forces.

Surface Energy

Since surface molecules have higher potential energy than interior molecules, any increase in surface area requires energy. This energy is called **surface energy**.

Consider stretching a film of liquid on a horizontal rectangular frame with a sliding wire of length l , as shown in the following figure.

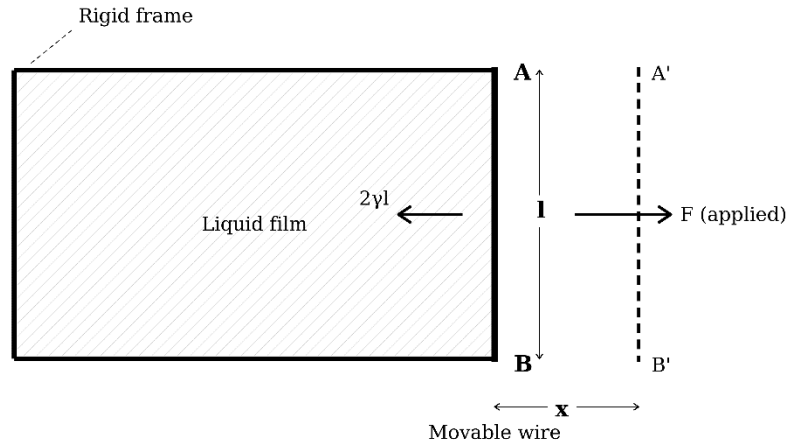


Figure: Stretching a liquid film on a rectangular frame. The movable wire AB of length l is held in place against the surface tension force $2\gamma l$ by an applied force F . When pulled to position $A'B'$ through a distance x , the work done is $W=2\gamma lx$.

The liquid film has two free surfaces (upper and lower), both in contact with the wire. Each surface exerts a force of γl on the wire, pulling it inward (trying to contract the film). The total surface tension force on the wire is therefore:

$$F = 2\gamma l$$

To hold the wire in place, an equal and opposite external force must be applied. If the wire is now pulled outward through a distance x (slowly, so kinetic energy is negligible), the work done against the surface tension force is:

$$W = F \times x = 2\gamma l \times x$$

The increase in surface area is $\Delta A = 2lx$ (because both the upper and lower surfaces increase by lx each). The work done per unit increase in surface area is:

$$\delta = \frac{W}{\Delta A} = \frac{2\gamma lx}{2lx}$$

$$\delta = \gamma$$

This is an elegant result: the **surface energy** (work done per unit area in extending the liquid surface) is **numerically equal to the coefficient of surface tension**. This means surface tension can be interpreted in two equivalent ways: as a force per unit length (Nm^{-1}), or as energy per unit area (Jm^{-2}). Both descriptions are correct and both have the same numerical value.

For a **single liquid surface** (such as the surface of a liquid drop or an air bubble in liquid), the work done to increase the surface area by ΔA is:

$$W = \gamma \Delta A$$

For a soap film or any thin film with **two free surfaces**, the work done is:

$$W = 2\gamma \Delta A$$

The factor of 2 appears because a soap film always has an inner surface and an outer surface, and both must be stretched.

Factors Affecting Surface Tension

Surface tension is not a fixed property. It depends on the conditions:

1. Nature of the liquid (strength of intermolecular forces)

This is the most fundamental factor. Surface tension exists because cohesive forces pull surface molecules inward. The stronger these forces, the stronger the inward pull, and the higher the surface tension. Mercury, whose atoms are held together by strong metallic bonds, has a very high surface tension ($\gamma \approx 0.47\text{Nm}^{-1}$). Water, with its strong hydrogen bonds, has $\gamma \approx 0.073\text{Nm}^{-1}$. Ethanol, with weaker hydrogen bonds, has $\gamma \approx 0.022\text{Nm}^{-1}$. Oils and organic solvents, held together only by weak van der Waals forces, have even lower values (typically 0.020 to 0.030Nm^{-1}).

2. Temperature

The surface tension of most liquids **decreases** with increasing temperature. As temperature rises, molecules move faster and the average distance between them increases. The cohesive forces weaken, and the surface contracts less strongly. This is why hot water wets surfaces more easily than cold water, and why washing with warm water is more effective than washing with cold water.

3. Impurities and dissolved substances

Adding substances to a liquid can either increase or decrease its surface tension. Soap and detergents **decrease** the surface tension of water dramatically (from 0.073Nm^{-1} to about 0.025Nm^{-1}), which is why soapy water spreads more easily and is better at wetting surfaces and removing grease. Detergent molecules have one end that attracts water and one end that repels it; they wedge themselves into the surface layer and weaken the cohesive forces between water molecules, reducing the surface tension. On the other hand, dissolving salt in water **increases** its surface tension slightly, because the dissolved ions strengthen the attractive forces between water molecules near the surface. This is why saltwater forms slightly larger droplets than freshwater.

4. Nature of the medium above the surface

Surface tension depends not only on the liquid itself but also on what is above it. The surface tension of water in contact with air is 0.073Nm^{-1} , but the surface tension of water in contact with oil is different, because oil molecules above the surface exert some attractive force on the water molecules, partially compensating the inward pull. In general, the greater the difference in intermolecular forces between the liquid and the medium above it, the higher the surface tension at the interface.

5. Electrification

When an electric charge is placed on the surface of a liquid, the surface tension **decreases**. The charged surface molecules repel each other, which opposes the inward pull of cohesive forces and makes the surface easier to stretch. This effect is used in electrospraying, where a high voltage causes a liquid to break into a fine mist of tiny droplets.

With these factors, the theory is now in place. You now understand what surface tension is, why it exists at the molecular level, and how it connects to surface energy. Before the next section arrives, let us anchor these ideas with worked examples that put the formulas to work.

BINDER Example 4

A rectangular wire frame has a sliding wire of length 15cm that can move freely along the frame. The frame is dipped in a soap solution and a thin film forms across it. Calculate the force required to hold the sliding wire in equilibrium. Take the surface tension of the soap solution as 0.025Nm^{-1} .

Solution

The soap film has two free surfaces (upper and lower), both in contact with the sliding wire. The total surface tension force pulling the wire inward is:

$$F = 2\gamma l$$

$$F = 2 \times 0.025\text{Nm}^{-1} \times 0.15\text{m} = 7.5 \times 10^{-3}\text{N}$$

Therefore, the force required to hold the sliding wire in equilibrium is $7.5 \times 10^{-3}\text{N}$ (or 7.5mN).

Making Sense of the Answer: This is a tiny force, less than the weight of a one-gram mass. Yet it is enough to visibly pull the wire inward when released. Surface tension forces are small by everyday standards, but they dominate at small scales, which is why they matter for drops, bubbles, needles, and insects, but not for swimming pools or oceans.

Think Like a Physicist: The factor of 2 appears because the film has two surfaces. Forgetting this factor is one of the most common errors in surface tension problems. Whenever you see the word “**film**,” immediately think: **two surfaces**. Whenever you see “**drop**” or “**bubble in liquid**,” think: **one surface**. Whenever you see “**soap bubble in air**,” think: **two surfaces** (inner and outer). Get this right, and half the battle is won!

BINDER Example 5

A soap film is formed on a rectangular frame measuring 8.0cm by 5.0cm. The surface tension of the soap solution is 0.030Nm^{-1} . Calculate the surface energy of the film.

Solution

The area of one surface of the film is:

$$A = 8.0 \times 10^{-2}\text{m} \times 5.0 \times 10^{-2}\text{m} = 4.0 \times 10^{-3}\text{m}^2$$

Since a soap film has two free surfaces, the total surface area is:

$$\Delta A = 2A = 2 \times 4.0 \times 10^{-3}\text{m}^2 = 8.0 \times 10^{-3}\text{m}^2$$

The surface energy (work stored in the film) is:

$$W = \gamma \Delta A$$

$$W = 0.030\text{Nm}^{-1} \times 8.0 \times 10^{-3}\text{m}^2 = 2.4 \times 10^{-4}\text{J}$$

Therefore, the surface energy of the soap film is $2.4 \times 10^{-4}\text{J}$ (or 0.24mJ).

Making Sense of the Answer: This is an extremely small amount of energy, far less than the energy needed to lift a grain of rice. But remember: this energy is spread over a very thin film. Per unit area, the energy density is significant enough to hold the film together and resist stretching. Scale matters in physics.

Think Like a Physicist: Notice that we used $W = \gamma \Delta A$ (not $W = 2\gamma \Delta A$) because we already accounted for the two surfaces when we calculated $\Delta A = 2A$. You can use either approach, but never count the factor of 2 twice. Be consistent.

REAL Example 6

While walking past a garage in Dar es Salaam, **Kipute** notices that a small amount of engine oil has leaked onto a puddle of rainwater. Instead of staying as a drop, the oil spreads out into a thin, shimmering film that covers a large area of the puddle.

“That is strange,” she says to **Mr. Akilikubwa**, who happens to be nearby. “On the ground, oil stays as a thick puddle. But on water, it spreads out thin. Why does oil behave differently on water than on a solid surface?”

Explain why oil spreads on water but not on a dry concrete surface.

Solution

Oil spreads on water because doing so **reduces the total surface energy** of the system.

When oil sits on water, there are three surfaces to consider: the oil-air surface, the oil-water interface, and the water-air surface. The surface tension of water is much higher than the surface tension of oil. When oil spreads, it replaces the high-energy water-air surface with two lower-energy surfaces (oil-air and oil-water). The total surface energy is thus decreases, so spreading is energetically favourable. The system moves spontaneously toward the lower-energy state.

On a dry concrete surface, the situation is different. Concrete has a relatively low surface energy, and there is no high-energy water surface to be replaced. Spreading the oil would increase the total surface area without a corresponding decrease in surface energy. So the oil stays as a thick puddle, minimising its own surface area.

Making Sense of the Answer: The oil is not “attracted” to water. It spreads because spreading lowers the total energy. Nature always moves toward lower energy when it can. This is the same principle that makes a ball roll downhill or a compressed spring expand.

Think Like a Physicist: Whenever you see a liquid spreading or contracting on a surface, ask: **does spreading reduce or increase the total surface energy?** If it reduces the total energy, the liquid spreads. If it increases the total energy, the liquid contracts into a drop. This single question explains an enormous range of wetting and non-wetting phenomena.

HOT Example 7

A rectangular wire frame of length 10cm and width 6.0cm is dipped in a soap solution of surface tension 0.030Nm^{-1} and then removed. A thin soap film forms across the frame. The sliding wire along one of the shorter sides is now pulled outward by a distance of 4.0cm.

- Calculate the force exerted by the film on the sliding wire.
- Calculate the work done in pulling the wire through the distance of 4.0cm.
- Calculate the increase in surface area and verify that the work done equals $\gamma \times$ (increase in total surface area).

Solution

(a) The sliding wire has length $l = 6.0 \times 10^{-2}\text{m}$. The soap film has two surfaces, so the total surface tension force on the wire is:

$$F = 2\gamma l$$

$$F = 2 \times 0.030\text{Nm}^{-1} \times 6.0 \times 10^{-2}\text{m} = 3.6 \times 10^{-3}\text{N}$$

Therefore, the force exerted by the soap film on the sliding wire is $3.6 \times 10^{-3}\text{N}$ (or 3.6mN).

(b) The wire is pulled through a distance $x = 4.0 \times 10^{-2}\text{m}$. The work done against the surface tension force is:

$$W = F \times x$$

$$W = 3.6 \times 10^{-3}\text{N} \times 4.0 \times 10^{-2}\text{m} = 1.44 \times 10^{-4}\text{J}$$

Therefore, the work done in pulling the wire is $1.44 \times 10^{-4}\text{J}$.

(c) The increase in area of one surface is:

$$\Delta A_{\text{one}} = l \times x = 6.0 \times 10^{-2}\text{m} \times 4.0 \times 10^{-2}\text{m} = 2.4 \times 10^{-3}\text{m}^2$$

The total increase in surface area (two surfaces) is:

$$\Delta A_{\text{total}} = 2 \times 2.4 \times 10^{-3}\text{m}^2 = 4.8 \times 10^{-3}\text{m}^2 = \text{Increase in total surface area}$$

The work done calculated from surface energy is:

$$W = \gamma \times \Delta A_{\text{total}}$$

$$W = 0.030\text{Nm}^{-1} \times 4.8 \times 10^{-3}\text{m}^2$$

$$W = 1.44 \times 10^{-4}\text{J}$$

This matches the work calculated in part (b) exactly. The two approaches are consistent, confirming that surface tension (force per unit length) and surface energy (energy per unit area) are two faces of the same quantity.

Making Sense of the Answer: The work done ($1.44 \times 10^{-4}\text{J}$) is extremely small. You could do this much work by lifting a grain of rice about 1.5cm. Yet this tiny amount of energy is what holds soap films together and allows them to form the beautiful, fragile structures that children blow into bubbles. At this scale, surface tension is the dominant force.

Think Like a Physicist: Part (c) is the heart of this problem. It shows that the force approach ($W = Fx$) and the energy approach ($W = \gamma\Delta A$) give identical results. This is not a coincidence. It is a consequence of the fact that γ is simultaneously a force per unit length and an energy per unit area. Whenever you can verify a result by two independent methods, do it. Agreement between methods is one of the strongest checks in physics.

That brings us to the end of this subtopic. We have built surface tension from the ground up: from the observation of floating needles and spherical drops, through the molecular theory that explains why surface molecules are special, to the mathematics of surface energy that will power everything ahead.

But surface tension does more than make needles float and drops spherical. It creates excess pressure inside bubbles and curved surfaces, and it pulls liquids up narrow tubes against gravity. These consequences are the subjects of the next two subtopics. We will derive the excess pressure inside bubbles and drops. And later, we will explore capillarity and the angle of contact. Both are built directly on the ideas of this section.

EXCESS PRESSURE IN CURVED SURFACES AND BUBBLES

In the previous subtopic, we learned that surface tension causes a liquid surface to behave like a stretched elastic membrane that tries to contract to the smallest possible area. But *what happens when a surface is forced to be curved, as in a bubble or a drop?* The stretched membrane pulls inward, and for the bubble to hold its shape, the pressure on the concave side (pressure inside) must be greater than the pressure on the convex side (pressure outside). This difference is called **excess pressure**, and it is the mathematical heart of surface tension.

Why Curved Surfaces Have Excess Pressure

Think about inflating a balloon. When the balloon is deflated, the rubber is relaxed and there is no extra pressure inside. The moment you start blowing, the rubber stretches and curves outward. The stretched rubber pulls inward (toward the centre), trying to contract. For the balloon to remain inflated, the air pressure inside must be greater than the atmospheric pressure outside. The difference between the inside and outside pressures is what keeps the balloon expanded against the inward pull of the rubber.

A curved liquid surface behaves in exactly the same way. The surface tension acts along the surface, pulling it inward, trying to flatten it. For the surface to remain curved, there must be a pressure difference across it: the pressure on the **concave side** (the inside of the curve) is always greater than the pressure on the **convex side** (the outside of the curve).

The smaller the radius of curvature, the greater this excess pressure. This is why it is harder to start blowing a balloon than to continue inflating it once it is already large: the small initial radius requires a large excess pressure, while the larger radius of the inflated balloon requires less.

We will now derive the excess pressure for three geometries: an air bubble inside a liquid, a soap bubble in air, and a liquid drop.

Excess Pressure Inside an Air Bubble in a Liquid (One Surface)

Consider a small air bubble of radius r formed inside a liquid of surface tension γ . The bubble has only **one** liquid surface (the boundary between the air inside and the liquid outside).

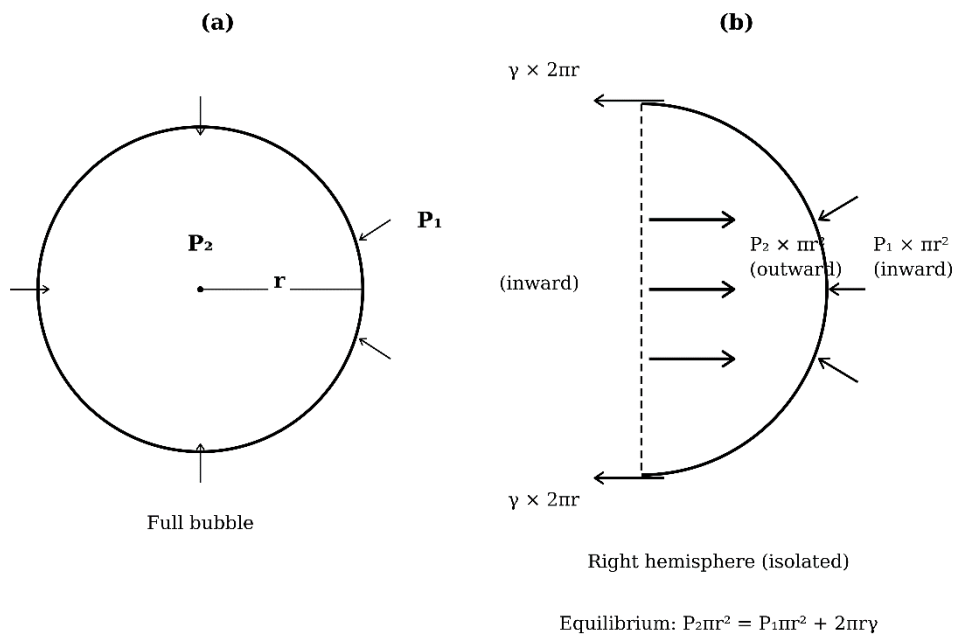


Figure: (a) An air bubble of radius r inside a liquid. The internal pressure P_2 is greater than the external pressure P_1 . (b) The right hemisphere isolated for force analysis. The outward force $P_2 \pi r^2$ on the flat face is balanced by the inward force $P_1 \pi r^2$ on the curved surface and the surface tension force $2\pi r \gamma$ along the equatorial rim.

To derive the excess pressure, consider the equilibrium of one hemisphere of the bubble. Imagine slicing the bubble in half along its equator. The flat circular face exposed by the cut has area πr^2 .

The forces acting on this hemisphere are:

1. The pressure force from outside (pushing the hemisphere inward):

$$F_{\text{outside}} = P_1 \times \pi r^2$$

2. The pressure force from inside (pushing the hemisphere outward):

$$F_{\text{inside}} = P_2 \times \pi r^2$$

3. The surface tension force along the circumference of the cut (pulling the hemisphere inward, trying to contract the bubble). The circumference is $2\pi r$, and the surface tension acts along this entire length:

$$F_\gamma = \gamma \times 2\pi r$$

For the hemisphere to be in equilibrium, the outward force must equal the total inward forces:

$$P_2 \times \pi r^2 = P_1 \times \pi r^2 + \gamma \times 2\pi r$$

Dividing both sides by πr^2 :

$$P_2 = P_1 + \frac{2\gamma}{r} \text{ or } P_2 - P_1 = \frac{2\gamma}{r}$$

Therefore, the excess pressure inside an air bubble in a liquid is:

$$\Delta P = \frac{2\gamma}{r}$$

where $\Delta P = P_2 - P_1$ is the difference between the pressure inside and outside the bubble.

This result tells us two important things. First, the pressure inside the bubble is always greater than the pressure outside ($P_2 > P_1$). Second, the excess pressure is inversely proportional to the radius: the smaller the bubble, the greater the excess pressure. A tiny bubble has extremely high internal pressure relative to its surroundings.

Excess Pressure Inside a Soap Bubble in Air (Two Surfaces)

A soap bubble floating in air is different from an air bubble in liquid. A soap bubble is a thin film of soapy water with air on both sides. This means it has **two** free surfaces: an inner surface and an outer surface. Each surface contributes a surface tension force.

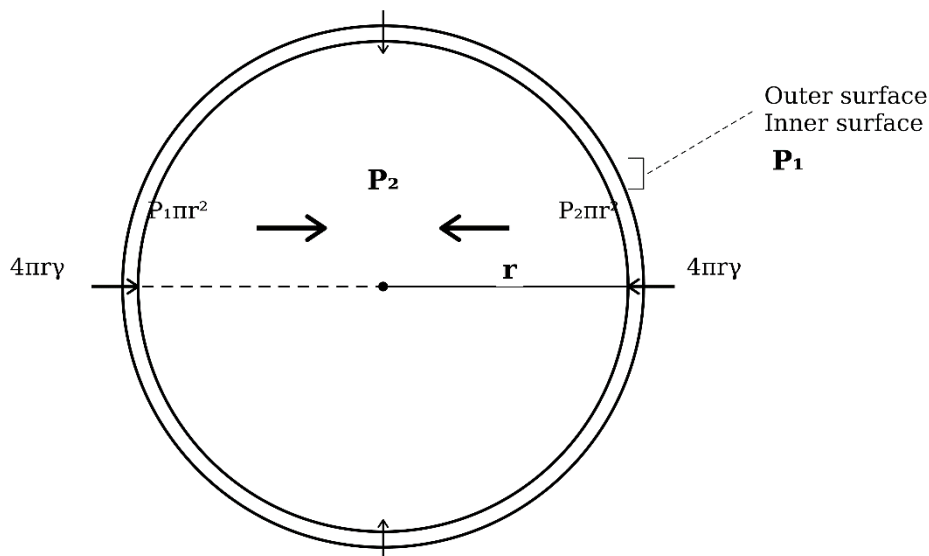


Figure: Equilibrium of one hemisphere of a soap bubble in air. The bubble has two surfaces (inner and outer), so the total surface tension force along the equator is $4\pi r\gamma$, double that of a single-surface bubble.

The derivation follows the same approach as in the previous derivation, except the surface tension force is doubled:

$$F_\gamma = 2 \times \gamma \times 2\pi r = 4\pi r\gamma$$

For equilibrium of one hemisphere:

$$P_2 \times \pi r^2 = P_1 \times \pi r^2 + 4\pi r\gamma$$

Dividing both sides by πr^2 :

$$P_2 = P_1 + \frac{4\gamma}{r}$$

$$P_2 - P_1 = \frac{4\gamma}{r}$$

Therefore, the excess pressure inside a soap bubble in air is:

$$\Delta P = \frac{4\gamma}{r}$$

The factor of 4 (instead of 2) is entirely due to the two surfaces. This is one of the most frequently tested distinctions in examinations: an air bubble in liquid has excess pressure $2\gamma/r$ (one surface), while a soap bubble in air has excess pressure $4\gamma/r$ (two surfaces). Confusing the two is a common and costly mistake.

Excess Pressure Inside a Liquid Drop (One Surface)

A liquid drop (such as a raindrop falling through air) has only **one** surface: the outer boundary between the liquid inside and the air outside. There is no air cavity inside a drop; it is solid liquid throughout.

The derivation is identical to that of an air bubble in a liquid, and the result is the same:

$$\Delta P = \frac{2\gamma}{r}$$

where P_2 is the pressure inside the drop and P_1 is the atmospheric pressure outside.

It is important to be clear about the distinction between the three cases:

- An **air bubble in liquid** has one liquid surface. Excess pressure: $\Delta P = \frac{2\gamma}{r}$.
- A **soap bubble in air** has two surfaces (inner and outer). Excess pressure: $\Delta P = \frac{4\gamma}{r}$.
- A **liquid drop** has one surface. Excess pressure: $\Delta P = \frac{2\gamma}{r}$.

Coalescing and Breaking of Drops and Bubbles

When drops or bubbles combine or break apart, the total surface area changes. Since surface energy equals $\gamma \times (\text{surface area})$, any change in surface area involves a transfer of energy. Understanding this energy change is essential for problems involving coalescing drops, breaking drops, and the common interface between two bubbles.

Coalescing of liquid drops in vacuum

Consider two liquid drops of radii r_1 and r_2 that coalesce to form a single drop of radius R under isothermal conditions in vacuum. Since the total volume of liquid is conserved:

$$\frac{4}{3}\pi r_1^3 + \frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi R^3$$

Simplifying:

$$r_1^3 + r_2^3 = R^3$$

If the drops are identical ($r_1 = r_2 = r$), then $2r^3 = R^3$, giving $R = r\sqrt[3]{2}$.

For the surface energy analysis, the initial total surface area is $4\pi r_1^2 + 4\pi r_2^2$ and the final surface area is $4\pi R^2$. Since a sphere has the minimum surface area for a given volume, the combined drop has less total surface area than the two separate drops. The process releases surface energy. By conservation of energy, this released energy appears as heat, slightly raising the temperature of the drop.

For the special case where we need R in terms of r_1 and r_2 using surface energy conservation (isothermal, in vacuum), the surface energies must balance:

$$4\pi r_1^2\gamma + 4\pi r_2^2\gamma = 4\pi R^2\gamma$$

$$R = \sqrt{r_1^2 + r_2^2}$$

For n identical drops of radius r coalescing into one drop of radius R :

$$n \times 4\pi r^2 \gamma = 4\pi R^2 \gamma$$

giving $R^2 = nr^2$, so $R = r\sqrt{n}$.

Be careful! The volume conservation formula ($R^3 = r_1^3 + r_2^3$) and the surface energy formula ($R^2 = r_1^2 + r_2^2$) give different results for R . Which one applies depends on the problem. If the coalescing happens in vacuum under isothermal conditions with no other energy input, the surface energy formula applies. If the problem simply states that two drops merge into one and asks for the new radius, use volume conservation. Always read the question carefully.

Breaking of a large liquid drop into smaller drops

When a large drop of radius R breaks into n identical smaller drops, each of radius r , the total volume is conserved:

$$\frac{4}{3}\pi R^3 = n \times \frac{4}{3}\pi r^3$$

This gives $R^3 = nr^3$, so $n = R^3/r^3$.

The total surface area increases (many small drops have more combined surface area than one large drop), so work must be done. The work done in breaking the drop is:

$$W = \gamma \times \Delta A = \gamma \times (n \times 4\pi r^2 - 4\pi R^2)$$

$$W = 4\pi\gamma(nr^2 - R^2)$$

Since $n = R^3/r^3$, we can substitute:

$$W = 4\pi\gamma\left(\frac{R^3}{r^3} \times r^2 - R^2\right)$$

$$W = 4\pi\gamma R^2 \left(\frac{R}{r} - 1\right)$$

Since $R > r$, the factor $(R/r - 1)$ is positive, confirming that work must be done to break a large drop into smaller ones. This energy goes into creating the additional surface area.

Interesting! If all this energy is absorbed by the liquid itself (no heat loss to the surroundings), it causes a slight temperature rise. If the mass of the liquid is m , the density is ρ , and the specific heat capacity is c , then the temperature rise ΔT can be found from:

$$W = mc\Delta T$$

$$4\pi\gamma R^2 \left(\frac{R}{r} - 1\right) = \frac{4}{3}\pi R^3 \rho c \Delta T$$

$$\Delta T = \frac{3\gamma}{\rho c} \left(\frac{1}{r} - \frac{1}{R}\right)$$

Coalescing of two soap bubbles (radius of common interface)

When two soap bubbles of different radii r_1 and r_2 (with $r_1 < r_2$) come into contact, a common interface forms between them. The smaller bubble has a higher internal pressure (since excess pressure is inversely proportional to radius), so the common interface bulges into the larger bubble.

Let P_A be the atmospheric pressure outside both bubbles, and let P_1 and P_2 be the pressures inside the small and large bubbles respectively.

$$\text{For the small bubble: } P_1 - P_A = \frac{4\gamma}{r_1}$$

$$\text{For the large bubble: } P_2 - P_A = \frac{4\gamma}{r_2}$$

The pressure difference across the common interface is:

$$P_1 - P_2 = \frac{4\gamma}{r_1} - \frac{4\gamma}{r_2} = 4\gamma \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

If R is the radius of curvature of the common interface, and the interface is itself a soap film with two surfaces:

$$P_1 - P_2 = \frac{4\gamma}{R}$$

Equating:

$$\begin{aligned}\frac{4\gamma}{R} &= 4\gamma \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \\ \frac{1}{R} &= \frac{1}{r_1} - \frac{1}{r_2} = \frac{r_2 - r_1}{r_1 r_2} \\ R &= \frac{r_1 r_2}{r_2 - r_1}\end{aligned}$$

Since $r_2 > r_1$, R is positive. The interface curves into the larger bubble, as expected.

The theory of this section is now complete. We have derived the excess pressure for all three geometries and the key results for coalescing and breaking of drops and bubbles. Let us now test these ideas with worked examples.

BINDER Example 8

An air bubble of radius 2.5cm is formed inside a liquid whose surface tension is $25 \times 10^{-3} \text{Nm}^{-1}$. Calculate the excess pressure inside the bubble.

Solution

An air bubble in liquid has one surface, so:

$$\Delta P = \frac{2\gamma}{r} = \frac{2 \times 25 \times 10^{-3} \text{Nm}^{-1}}{2.5 \times 10^{-2} \text{m}} = 2.0 \text{Nm}^{-2}$$

Therefore, the excess pressure inside the air bubble is 2.0Nm^{-2} (or 2.0Pa).

Making Sense of the Answer: This is a very small pressure, about 0.002% of atmospheric pressure. The bubble is relatively large (2.5cm radius), which is why the excess pressure is small. A bubble one thousand times smaller (radius 0.025mm) would have an excess pressure one thousand times larger: 2000Pa. This is the inverse relationship at work.

Think Like a Physicist: Before calculating, always identify the geometry. Is it an air bubble in liquid (one surface, $\Delta P = 2\gamma/r$), a soap bubble in air (two surfaces, $\Delta P = 4\gamma/r$), or a liquid drop (one surface, $\Delta P = 2\gamma/r$)? This single decision determines whether you use 2 or 4 in the numerator. Get it wrong, and the entire answer is wrong.

BINDER Example 9

A soap bubble has a diameter of 4.0mm. Calculate the pressure inside it if the atmospheric pressure is $1.0 \times 10^5 \text{Pa}$ and the surface tension of the soap solution is $2.8 \times 10^{-2} \text{Nm}^{-1}$.

Solution

A soap bubble in air has two surfaces, so the excess pressure is:

$$\Delta P = \frac{4\gamma}{r} = \frac{4 \times 2.8 \times 10^{-2} \text{Nm}^{-1}}{2.0 \times 10^{-3} \text{m}} = 56 \text{Pa}$$

The total pressure inside the bubble is:

$$\begin{aligned}P_{\text{inside}} &= P_A + \Delta P \\ P_{\text{inside}} &= 1.0 \times 10^5 \text{Pa} + 56 \text{Pa} = 1.00056 \times 10^5 \text{Pa}\end{aligned}$$

Therefore, the total pressure inside the soap bubble is $1.00056 \times 10^5 \text{Pa}$.

Making Sense of the Answer: The excess pressure (56Pa) is tiny compared to atmospheric pressure (100,000Pa). Yet this small difference is what keeps the bubble inflated. If the excess pressure were zero, the bubble would have no reason to maintain its spherical shape and would collapse.

Think Like a Physicist: The question asks for “the pressure inside,” not “the excess pressure.” These are different quantities. The excess pressure is $\Delta P = 56\text{Pa}$. The total pressure inside is $P_A + \Delta P = 1.00056 \times 10^5\text{Pa}$. Always check whether the question asks for the excess or the total.

BINDER Example 10

Two soap bubbles of radii 3.0cm and 4.0cm are made from the same soap solution. They coalesce to form a single bubble. Assuming the process is isothermal, find the radius of the new bubble.

Solution

For soap bubbles coalescing isothermally, the total surface energy is conserved. Since each bubble has two surfaces:

$$2 \times 4\pi r_1^2 \gamma + 2 \times 4\pi r_2^2 \gamma = 2 \times 4\pi R^2 \gamma$$

The factor $2 \times 4\pi\gamma$ cancels from all terms:

$$r_1^2 + r_2^2 = R^2$$

$$R = \sqrt{r_1^2 + r_2^2} = \sqrt{(3.0\text{cm})^2 + (4.0\text{cm})^2} = 5.0\text{cm}$$

Therefore, the radius of the new bubble formed after coalescence is 5.0cm.

Making Sense of the Answer: The new bubble ($R = 5.0\text{cm}$) is larger than either original bubble, as expected.

Think Like a Physicist: The assumption “isothermal” is critical. It means no heat is gained or lost, so all the surface energy of the original bubbles is transferred to the new bubble. Without this assumption, some energy could be lost as heat, and the calculation would be different.

REAL Example 11

At a birthday party, **Kipanga** tries to blow up a balloon. He notices something strange: the balloon is very hard to blow at the beginning, but once it starts expanding, it becomes much easier.

He complains to Mr. Akilikubwa: “Sir, I almost passed out trying to start this balloon. But now it inflates easily. Why is the first breath so hard?”

Explain why it is harder to begin inflating a balloon than to continue inflating it once it has already expanded.

Solution

The excess pressure required to inflate a curved surface is given by $\Delta P = 4\gamma/r$ (since the rubber membrane, like a soap bubble, has two surfaces). When the balloon is small (just starting to inflate), the radius r is very small. Since the excess pressure is inversely proportional to r , a small radius means a large excess pressure is needed.

As Kipanga continues blowing and the balloon expands, the radius r increases. The required excess pressure $\Delta P = 4\gamma/r$ decreases. This is why the balloon feels easier to inflate once it is already partially expanded: less pressure difference is needed to maintain and increase the larger radius.

In addition, the rubber of the balloon is initially stiff and resists stretching. Once it begins to stretch, the elastic restoring force per unit extension decreases (rubber does not obey Hooke’s law at large extensions), making further inflation easier.

Therefore, the first breath is hardest because both the small radius (requiring high excess pressure) and the initial stiffness of the rubber work against Kipanga.

Making Sense of the Answer: This is why balloon vendors at Kariakoo market always pre-stretch their balloons before selling them. Pre-stretching increases the initial radius and softens the rubber, making the balloon much easier for a child to inflate. Physics at the marketplace.

Think Like a Physicist: The formula $\Delta P = 4\gamma/r$ predicts that as r approaches zero, the excess pressure approaches infinity. In practice, this does not happen because the balloon material has a finite thickness and the formula breaks down at very small radii. But the trend is real: smaller radius means harder to inflate.

HOT Example 12

A large spherical drop of mercury of radius 2.0mm breaks into 1000 identical smaller drops. The surface tension of mercury is 0.472Nm^{-1} and its specific heat capacity is $140\text{Jkg}^{-1}\text{K}^{-1}$. The density of mercury is 13600kgm^{-3} .

- Find the radius of each small drop.
- Calculate the work done in breaking the large drop.
- If all the work done appears as heat in the mercury, find the rise in temperature.

Solution

(a) By conservation of volume:

$$\begin{aligned}\frac{4}{3}\pi R^3 &= n \times \frac{4}{3}\pi r^3 \\ R^3 &= nr^3 \\ r &= \frac{R}{\sqrt[3]{n}} = \frac{2.0 \times 10^{-3}\text{m}}{\sqrt[3]{1000}} = 2.0 \times 10^{-4}\text{m}\end{aligned}$$

Therefore, the radius of each small drop is $2.0 \times 10^{-4}\text{m}$ (or 0.20mm).

(b) The change in surface area is:

$$\begin{aligned}\Delta A &= n \times 4\pi r^2 - 4\pi R^2 \\ \Delta A &= 1000 \times 4\pi \times (2.0 \times 10^{-4}\text{m})^2 - 4\pi \times (2.0 \times 10^{-3}\text{m})^2 = 4.524 \times 10^{-4}\text{m}^2\end{aligned}$$

The work done is:

$$W = \gamma \times \Delta A = 0.472\text{Nm}^{-1} \times 4.524 \times 10^{-4}\text{m}^2 = 2.14 \times 10^{-4}\text{J}$$

Therefore, the work done in breaking the large drop is $2.14 \times 10^{-4}\text{J}$.

(c) The mass of mercury is:

$$m = \rho \times \frac{4}{3}\pi R^3 = 13,600\text{kgm}^{-3} \times \frac{4}{3}\pi \times (2.0 \times 10^{-3}\text{m})^3 = 4.557 \times 10^{-4}\text{kg}$$

Using $W = mc\Delta T$:

$$\Delta T = \frac{W}{mc} = \frac{2.14 \times 10^{-4}\text{J}}{4.557 \times 10^{-4}\text{kg} \times 140\text{Jkg}^{-1}\text{K}^{-1}} = 3.35 \times 10^{-3}\text{K}$$

Therefore, the rise in temperature of the mercury is $3.35 \times 10^{-3}\text{K}$ (or approximately 0.003K).

Making Sense of the Answer: The temperature rise is extraordinarily small, about three thousandths of a degree. This makes physical sense: the surface energy involved in breaking a 2mm drop is tiny, and mercury has significant thermal mass even in small quantities. In practice, this temperature rise would be undetectable by ordinary thermometers. But the calculation demonstrates that surface energy is real and can, in principle, be converted to thermal energy.

Think Like a Physicist: This problem combines three topics: surface tension (surface area change), energy conservation (work equals surface energy gained), and thermal physics (energy equals $mc\Delta T$). Multi-topic problems like this are frequently tested in examinations. The key strategy is to solve each part as a separate sub-problem and feed results forward: part (a) gives r , which is needed in part (b); part (b) gives W , which is needed in part (c). Never try to do everything in one equation.

HOT Example 13

Two soap bubbles of radii 10mm and 30mm are made from the same soap solution of surface tension 0.030Nm^{-1} . They coalesce so that they share a common interface.

- Find the radius of curvature of the common interface.
- Determine whether the interface curves into the larger or smaller bubble.

Solution

- Finding the radius of the common interface:

Interpreting data:

$$r_1 = 10\text{mm} = 10 \times 10^{-3}\text{m} \text{ (smaller bubble)}, r_2 = 30\text{mm} = 30 \times 10^{-3}\text{m} \text{ (larger bubble)}$$

The radius of curvature of the common interface is:

$$R = \frac{r_1 r_2}{r_2 - r_1} = \frac{10 \times 10^{-3}\text{m} \times 30 \times 10^{-3}\text{m}}{30 \times 10^{-3}\text{m} - 10 \times 10^{-3}\text{m}} = 15 \times 10^{-3}\text{m} = 15\text{mm}$$

Therefore, the radius of curvature of the common interface is 15mm.

- The smaller bubble ($r_1 = 10\text{mm}$) has a higher internal pressure than the larger bubble ($r_2 = 30\text{mm}$), because excess pressure is inversely proportional to radius:

$$P_1 - P_A = \frac{4\gamma}{r_1} = \frac{4 \times 0.030}{10 \times 10^{-3}} = 12\text{Pa}$$

$$P_2 - P_A = \frac{4\gamma}{r_2} = \frac{4 \times 0.030}{30 \times 10^{-3}} = 4.0\text{Pa}$$

Since $P_1 > P_2$, the higher pressure inside the smaller bubble pushes the common interface outward into the larger bubble. Therefore, the common interface **curves into the larger bubble**, bulging toward its centre.

Making Sense of the Answer: The common interface radius (15mm) is larger than the smaller bubble but smaller than the larger one. This makes physical sense: the interface is gently curved, not tightly bent, because the pressure difference between the two bubbles ($12 - 4 = 8\text{Pa}$) is relatively small compared to the excess pressure in either individual bubble.

Think Like a Physicist: The formula $R = r_1 r_2 / (r_2 - r_1)$ only works when $r_2 \neq r_1$. If both bubbles have the same radius, they have the same internal pressure, so there is no pressure difference across the interface. The common surface is flat (infinite radius of curvature). Mathematically, as $r_1 \rightarrow r_2$, the denominator approaches zero and $R \rightarrow \infty$, which correctly describes a flat surface.

That completes this beautiful subtopic. We have derived the excess pressure for all three standard geometries, explored the energy changes when drops and bubbles coalesce or break apart, and derived the radius of the common interface between two coalescing soap bubbles.

In the next subtopic, we apply surface tension to one of its most visible consequences: the rise (or depression) of liquids in narrow tubes. This phenomenon is called **capillarity**, and it connects surface tension to something you can see with your own eyes every time water climbs up a paper towel or mercury dips below the level in a glass tube.

CAPILLARITY AND ANGLE OF CONTACT

Surface tension does not only make needles float and drops spherical. It also pulls liquids up narrow tubes against gravity, and pushes other liquids down. If you have ever watched water climb up a paper towel, or noticed that mercury in a glass thermometer sits slightly below the level you would expect, you have already seen this phenomenon at work. It is called **capillarity**, and it connects the invisible molecular forces we discussed earlier to something you can measure with a ruler and see with your own eyes. By the end of this subtopic, you will understand exactly why water rises, why mercury falls, and how a simple glass tube can be used to measure the surface tension of any liquid.

Angle of Contact

When a liquid meets a solid surface, the liquid surface near the contact point is not flat. It curves, either upward or downward, depending on how strongly the liquid molecules are attracted to the solid compared to how strongly they are attracted to each other. The angle that describes this curvature is called the **angle of contact** (or **contact angle**), and it is denoted by θ .

Precisely, the angle of contact is *the angle between the solid surface and the tangent to the liquid surface at the point of contact, measured through the liquid*.

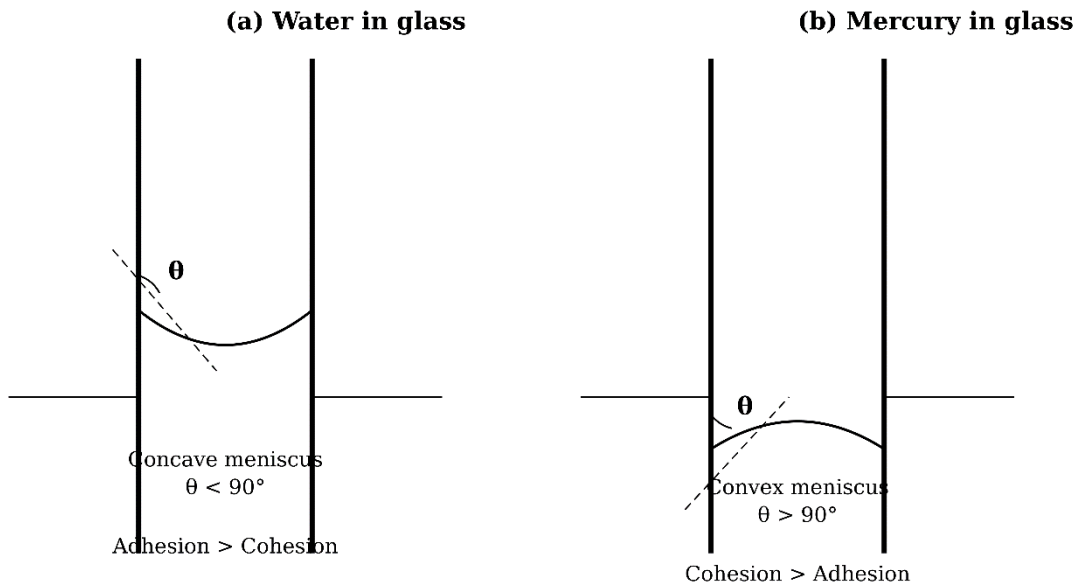


Figure: (a) *Water in a glass tube:* the meniscus is concave because adhesive forces between water and glass are stronger than cohesive forces between water molecules. The angle of contact $\theta < 90^\circ$. (b) *Mercury in a glass tube:* the meniscus is convex because cohesive forces between mercury atoms are stronger than adhesive forces between mercury and glass. The angle of contact $\theta > 90^\circ$.

The shape of the meniscus depends on the competition between two types of intermolecular forces:

- 1) **Adhesive forces** are the attractive forces between molecules of different substances (for example, between water molecules and glass molecules).
- 2) **Cohesive forces** are the attractive forces between molecules of the same substance (for example, between water molecules and other water molecules).

When **adhesion is stronger than cohesion** (as with water and glass), the liquid molecules are attracted to the solid surface more strongly than to each other. The liquid climbs up the wall, wetting it, and the meniscus is **concave** (curves upward). The angle of contact is $\theta < 90^\circ$. For water on clean glass, $\theta \approx 0^\circ$.

When **cohesion is stronger than adhesion** (as with mercury and glass), the liquid molecules are attracted to each other more strongly than to the solid surface. The liquid pulls away from the wall, and the meniscus is **convex** (curves downward). The angle of contact is $\theta > 90^\circ$. For mercury on glass, $\theta \approx 135^\circ$.

Factors Affecting the Angle of Contact

The angle of contact is not a fixed number. Like surface tension itself, it depends on several conditions. Understanding these factors will help you interpret experimental results and predict how different liquids behave on different surfaces.

1. Nature of the liquid and the solid surface

This is the most fundamental factor. The angle of contact depends on the relative strengths of adhesive and cohesive forces, which are determined by the chemical nature of both the liquid and the solid. Water on clean glass gives $\theta \approx 0^\circ$ (strong adhesion), but water on a waxed surface gives $\theta > 90^\circ$ (weak adhesion). Mercury on glass gives $\theta \approx 135^\circ$, but mercury on clean copper gives a much smaller angle because mercury adheres more strongly to copper than to glass.

2. Nature of the medium above the liquid

The angle of contact is not a property of the liquid alone. It also depends on what sits above the liquid surface. Water in contact with air has a different angle of contact from water in contact with oil vapour, because the molecules of the medium above the surface exert their own forces on the surface molecules of the liquid, partially altering the balance between adhesion and cohesion.

3. Impurities in the liquid

Adding impurities to the liquid changes the intermolecular forces at the surface, which alters the angle of contact. In general, adding a detergent or surfactant decreases the angle of contact of water on most surfaces, which is precisely why detergent helps water spread and wet surfaces more effectively.

4. Temperature

The angle of contact generally increases with increasing temperature. As temperature rises, the increased molecular motion weakens the adhesive forces at the liquid-solid interface, making the liquid less willing to cling to the solid surface.

Capillary Rise

Here is one of those moments in physics where something very simple leads to something very beautiful. Take a narrow glass tube, dip it vertically into a beaker of water, and watch. The water inside the tube quietly rises above the level of the water outside, and stays there, motionless, as though gravity has forgotten about it. It has not. The water is being held up by surface tension, and the balance between these two forces produces a formula of remarkable elegance.

This phenomenon is called **capillary rise**, and the narrow tube is called a **capillary tube**.

The rise happens because the adhesive forces between the liquid and the glass pull the liquid upward along the inner wall of the tube. Surface tension provides the upward force. The liquid continues to rise until the upward surface tension force is exactly balanced by the weight of the liquid column that has been lifted. At that point, the liquid stops rising and the system reaches equilibrium.

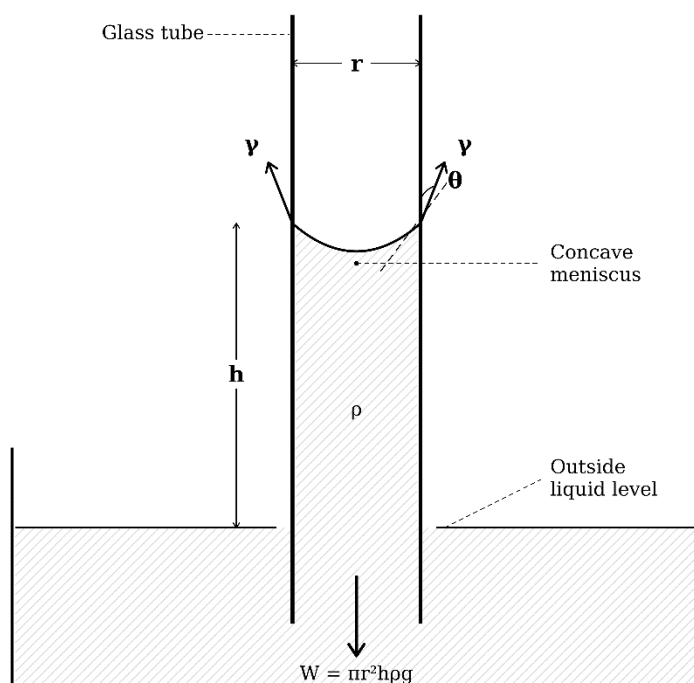


Figure: Capillary rise of a liquid in a narrow tube of internal radius r . The surface tension γ acts along the contact perimeter $2\pi r$ at an angle θ to the vertical. The vertical component of the total surface tension force, $\gamma \times 2\pi r \times \cos\theta$, supports the weight $W = \pi r^2 h \rho g$ of the liquid column of height h .

Derivation of the capillary rise formula:

The surface tension γ acts along the entire circumference of the contact line between the liquid and the inner wall of the tube. The length of this contact line is $2\pi r$, where r is the internal radius of the tube.

The surface tension force acts at an angle θ to the vertical (along the tangent to the meniscus at the contact point). Only the vertical component of this force supports the weight of the liquid column. The total upward force is:

$$F_{\text{up}} = \gamma \times 2\pi r \times \cos\theta$$

The weight of the liquid column of height h , radius r , and density ρ is:

$$W = \pi r^2 h \rho g$$

At equilibrium, the upward surface tension force equals the weight of the liquid column:

$$\gamma \times 2\pi r \times \cos\theta = \pi r^2 h \rho g$$

Solving for h :

$$h = \frac{2\gamma \cos\theta}{\rho g r}$$

This is the **capillary rise formula**, and it deserves a moment of appreciation. Look at what it tells us:

1. The height h is **inversely proportional to the radius** r of the tube. The narrower the tube, the higher the liquid rises. This is why capillary effects are dramatic in very thin tubes but invisible in wide containers.
2. The height depends on $\cos\theta$. For liquids that wet the surface ($\theta < 90^\circ$, so $\cos\theta > 0$), h is positive: the liquid rises. For liquids that do not wet the surface ($\theta > 90^\circ$, so $\cos\theta < 0$), h is negative: the liquid is depressed below the outside level. One formula handles both rise and depression. The sign of $\cos\theta$ does all the work.
3. The height does not depend on the shape of the tube, only on its internal radius. A tube with a circular cross-section and a tube with a square cross-section of the same effective radius give the same capillary rise.

Correction for meniscus volume: The derivation above assumes the liquid column is a perfect cylinder, but the meniscus at the top is curved, adding a small extra volume of liquid. For very narrow tubes where this extra volume matters, the corrected formula for the height of rise is:

$$h = \frac{2\gamma \cos\theta}{\rho g r} - \frac{r}{3}$$

The correction term $\frac{r}{3}$ is only significant when the tube radius is comparable to the height of rise. However, for most A-level problems, the simple formula is sufficient.

Practical Applications of Capillary Rise

Capillary action is not just a laboratory curiosity. It operates quietly and powerfully in everyday life. Here are the most important applications you should know:

1. Water transport in soil

Water rises from the water table to the roots of plants through capillary action in the tiny spaces between soil particles. Without this, crops would only grow where roots reach the water table directly.

2. Oil in a lamp wick

In a kerosene lamp, the liquid fuel rises from the reservoir up through the narrow fibres of the wick to the flame at the top. Capillary rise does the lifting; gravity would keep the kerosene at the bottom.

3. Ink in blotting paper and paper towels

When you touch blotting paper to a drop of ink, the ink spreads rapidly through the narrow spaces between the fibres. This is capillary action pulling the liquid into every available gap.

4. Ink in a fountain pen

The narrow channel inside a fountain pen nib draws ink from the reservoir to the writing tip by capillary action.

5. Construction

Damp-proof courses in buildings prevent water from rising through brick walls by capillary action. Without them, moisture would climb up from the foundation and damage the walls.

Capillary Depression

When a capillary tube is dipped into a liquid that does not wet the glass ($\theta > 90^\circ$), the liquid inside the tube is **depressed** below the outside level. Mercury in a glass tube is the classic example.

The physics is exactly the same as capillary rise, but the sign of $\cos\theta$ reverses everything. Since $\theta > 90^\circ$, $\cos\theta$ is negative, and the formula gives a negative value of h :

$$h = \frac{2\gamma\cos\theta}{\rho g r}$$

The magnitude of h gives the depth of depression below the outside level.

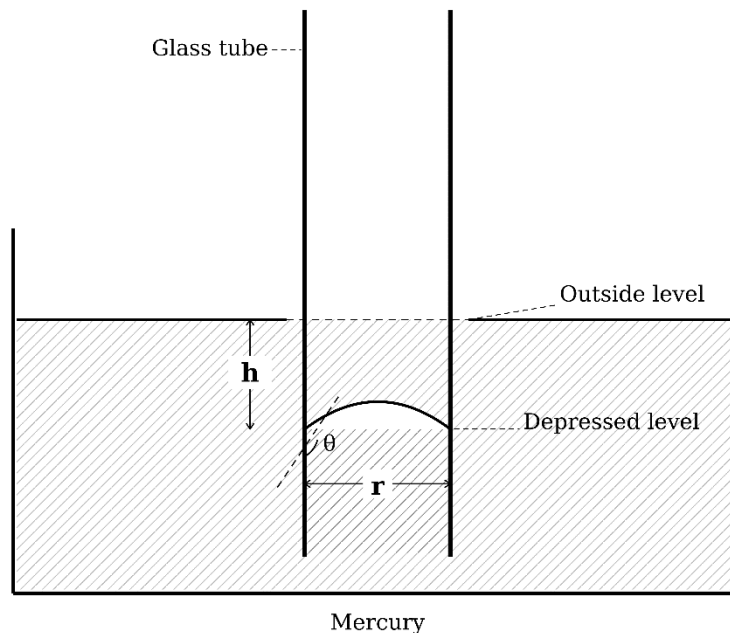


Figure: Capillary depression of mercury in a glass tube. The mercury level inside the tube is depressed by a height h below the outside level. The meniscus is convex with an angle of contact $\theta > 90^\circ$. The same formula $h = 2\gamma\cos\theta/(\rho g r)$ applies, with $\cos\theta$ negative, giving a negative h (depression).

The convex meniscus means the pressure just below the curved surface inside the tube is greater than the atmospheric pressure. This higher pressure pushes the mercury down until equilibrium is reached. There is no need to memorise a separate formula for depression. The capillary rise formula handles everything; the sign of $\cos\theta$ automatically tells you whether the liquid rises or falls.

Measuring Surface Tension by the Capillary Method

The capillary rise formula can be rearranged to find the surface tension of a liquid. If you can measure the height of rise, you can calculate γ :

$$\gamma = \frac{\rho g r h}{2\cos\theta}$$

This is the basis of the **capillary tube method** for measuring surface tension, and it is one of the simplest and most elegant experiments in fluid mechanics. Here is how it works:

A clean capillary tube of known internal radius r is clamped vertically and dipped into the liquid whose surface tension is to be measured. The liquid rises inside the tube. Once the liquid has settled, the height h

of the liquid above the outside level is measured using a travelling microscope. The density ρ of the liquid is measured separately (or looked up). If the angle of contact is known (for water on clean glass, $\theta \approx 0^\circ$, so $\cos\theta \approx 1$), then γ is calculated directly.

Sources of error and how to minimise them

Every experiment has enemies. In the capillary method, the main ones are:

1. Tube not perfectly vertical

If the tube is tilted, the measured height h is less than the true vertical rise, giving a surface tension value that is too low. To avoid this, clamp the tube carefully with a retort stand and use a spirit level to check.

2. Tube not perfectly clean

Grease, dust, or chemical residue on the inner wall changes the angle of contact, sometimes dramatically. A greasy glass tube will not give $\theta = 0^\circ$ for water. To avoid this, clean the tube thoroughly with chromic acid or distilled water before the experiment, and handle it only by the outer surface.

3. Non-uniform internal radius

If the tube is wider at one point and narrower at another, the rise depends on where the meniscus sits. To avoid this, use a high-quality capillary tube with a uniform bore, and measure the internal radius at several points.

4. Temperature change during the experiment

Surface tension decreases with temperature. If the room temperature changes during the experiment, γ changes too. To avoid this, take all measurements quickly, or work in a temperature-controlled environment.

5. Difficulty reading the meniscus

The exact position of the meniscus can be hard to judge with the naked eye. To avoid this, use a travelling microscope to read the height, and always read from the bottom of the meniscus (for concave menisci) or the top (for convex menisci).

Now let us put these ideas to work with some carefully chosen examples.

BINDER Example 14

A capillary tube of internal radius 0.25mm is dipped vertically into water. Calculate the height to which water rises in the tube. Take the surface tension of water as $7.2 \times 10^{-2} \text{Nm}^{-1}$, the density of water as 1000kgm^{-3} , the angle of contact as 0° , and $g = 9.8 \text{ms}^{-2}$.

Solution

Using the capillary rise formula:

$$h = \frac{2\gamma \cos\theta}{\rho g r}$$

$$h = \frac{2 \times 7.2 \times 10^{-2} \text{Nm}^{-1} \times \cos 0^\circ}{1000 \text{kgm}^{-3} \times 9.8 \text{ms}^{-2} \times 2.5 \times 10^{-4} \text{m}} = 5.88 \times 10^{-2} \text{m}$$

Therefore, water rises to a height of $5.88 \times 10^{-2} \text{m}$ (or approximately 5.9cm) in the capillary tube.

Making Sense of the Answer: Nearly 6cm of rise in a tube only 0.5mm wide. This is a substantial height, easily visible to the naked eye. In an even narrower tube (say, 0.05mm radius), the water would rise five times higher: about 29cm. This is why capillary effects are dramatic in very thin tubes but invisible in ordinary drinking glasses.

Think Like a Physicist: The angle of contact for water on clean glass is essentially 0° , which means $\cos\theta = 1$ and the formula simplifies to $h = 2\gamma/(\rho g r)$. Many problems give $\theta = 0^\circ$ for water on glass. If the angle of contact is not given in a problem, assume 0° unless stated otherwise.

BINDER Example 15

Water rises in a capillary tube to a height of 2.0cm. What will be the height of rise in a capillary tube of one-third the radius?

Solution

From the capillary rise formula, $h = \frac{2\gamma\cos\theta}{\rho g r}$.

For the same liquid, same temperature, and same solid surface, γ , θ , ρ , and g are all constant. Therefore:

$$h \propto \frac{1}{r}$$

This means:

$$h_1 r_1 = h_2 r_2$$

If $r_2 = \frac{r_1}{3}$. Then:

$$h_2 = h_1 \times \frac{r_1}{r_2} = 2.0\text{cm} \times \frac{r_1}{\frac{r_1}{3}} = 2.0\text{cm} \times 3 = 6.0\text{cm}$$

Therefore, water rises to a height of 6.0cm in the narrower tube.

Making Sense of the Answer: One-third the radius gives three times the height. This is the inverse proportionality at work: $h \propto 1/r$. If you halve the radius, the height doubles. If you reduce the radius to one-tenth, the height increases tenfold. This scaling law is why capillary action is powerful in the microscopic channels of plant xylem but unnoticeable in a wide bucket.

Think Like a Physicist: The relation $h_1 r_1 = h_2 r_2$ is one of the most useful shortcuts in capillary problems. It avoids the need to know γ , ρ , θ , or g . Whenever a problem compares two capillary tubes with the same liquid, reach for this relation first.

BINDER Example 16

A glass capillary tube of internal radius 0.75mm is dipped vertically into mercury. Given that the surface tension of mercury is 0.54Nm^{-1} , the density of mercury is 13600kgm^{-3} , the angle of contact of mercury with glass is 135° , and $g = 9.8\text{ms}^{-2}$, calculate the depression of mercury in the tube.

Solution

Using the capillary formula:

$$h = \frac{2\gamma\cos\theta}{\rho g r}$$

$$h = \frac{2 \times 0.54\text{Nm}^{-1} \times \cos 135^\circ}{13,600\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 7.5 \times 10^{-4}\text{m}} \quad h = -7.64 \times 10^{-3}\text{m}$$

The negative sign confirms that mercury is depressed below the outside level. Therefore, the depression of mercury in the glass tube is $7.64 \times 10^{-3}\text{m}$ (or approximately 7.6mm).

Making Sense of the Answer: A depression of about 7.6mm in a tube of radius 0.75mm. This is small but significant in precision instruments. Mercury thermometers and barometers use glass tubes, and the capillary depression introduces a systematic error in the readings. Manufacturers correct for this by calibrating the instruments against known standards.

Think Like a Physicist: The formula is identical for rise and depression. The sign of $\cos\theta$ does all the work: positive for $\theta < 90^\circ$ (rise), negative for $\theta > 90^\circ$ (depression). There is no need to memorise a separate formula for mercury. One formula, two behaviours. That is good physics.

REAL Example 17

After a rainy afternoon, Kipanga spills water on the classroom table. He grabs a paper towel to wipe it up and watches the water rush upward through the towel the moment it touches the puddle. Impressed, he tries the same thing with a sheet of waxed paper from his lunch wrapper. The water sits on the waxed surface and refuses to climb at all.

Explain why water rises through a paper towel but not through waxed paper.

Solution

A paper towel is made of loosely woven plant fibres with tiny air gaps between them. These gaps act as capillary tubes. Since paper fibres are hydrophilic (water-attracting), the adhesive forces between water and the fibres are stronger than the cohesive forces within the water. The angle of contact is very small ($\theta \approx 0^\circ$), and capillary rise draws the water upward through the narrow gaps against gravity.

Waxed paper, on the other hand, is coated with a hydrophobic (water-repelling) layer. The cohesive forces within the water are stronger than the adhesive forces between water and the waxy surface. As result, the angle of contact is large ($\theta > 90^\circ$), which means $\cos\theta$ is negative. The capillary formula gives a negative height which means that the water would be depressed, not raised. Since the water is not attracted to the waxed surface, it does not enter the tiny gaps and instead remains on top as a puddle.

Making Sense of the Answer: *The same liquid, the same gravity, the same physics. The only difference is the surface. Paper fibres attract water (adhesion wins), so capillary rise pulls the water in. Wax repels water (cohesion wins), so the water stays out. This is why waterproof jackets are waxed or coated: they turn every fibre gap from a capillary channel into a barrier.*

Think Like a Physicist: *The angle of contact is the gatekeeper. If $\theta < 90^\circ$, the liquid enters and rises. If $\theta > 90^\circ$, the liquid stays out and may even be depressed. Whenever you see a liquid being absorbed or repelled by a material, ask: is the contact angle acute or obtuse? That single question answers everything.*

HOT Example 18

The same capillary tube is used to measure the rise of water and the depression of mercury. Water rises to a height of 9.0cm, while mercury is depressed by 3.4cm in the same tube.

Given that the angle of contact for water with glass is 0° and for mercury with glass is 135° , and the density of mercury is 13600kgm^{-3} and the density of water is 1000kgm^{-3} , find the ratio of the surface tension of mercury to that of water.

Solution

For water: $h_w = 9.0\text{cm}$, $\theta_w = 0^\circ$, $\rho_w = 1000\text{kgm}^{-3}$

For mercury: $h_m = 3.4\text{cm}$ (depression), $\theta_m = 135^\circ$, $\rho_m = 13600\text{kgm}^{-3}$

From the capillary formula, for each liquid:

$$\gamma = \frac{\rho g r h}{2 \cos \theta}$$

Since the same tube is used, r and g are the same for both. Taking the ratio:

$$\frac{\gamma_m}{\gamma_w} = \frac{\rho_m \times h_m \times \cos \theta_w}{\rho_w \times h_w \times \cos \theta_m}$$

$$\frac{\gamma_m}{\gamma_w} = \frac{46,240}{-6364}$$

Taking the magnitude (since we are comparing strengths of surface tension):

$$\frac{\gamma_m}{\gamma_w} = 7.27$$

Therefore, the ratio of the surface tension of mercury to that of water is approximately 7.3: 1. (Mercury has about 7 times the surface tension of water.)

Making Sense of the Answer: *This ratio is consistent with known values: $\gamma_{\text{mercury}} \approx 0.47\text{Nm}^{-1}$ and $\gamma_{\text{water}} \approx 0.073\text{Nm}^{-1}$, giving a ratio of about 6.4. Our experimental result of 7.3 is in the right ballpark. The discrepancy arises from experimental uncertainties: imperfect angles of contact, non-uniform tube radius, and temperature variations.*

Think Like a Physicist: *This is a ratio problem. Whenever the same apparatus is used for two different liquids, set up the ratio to cancel common factors (r , g). This eliminates the need to know the tube radius, which may be difficult to measure precisely. Ratio methods are a powerful experimental strategy: they turn absolute measurements (which are hard) into relative measurements (which are easier and more reliable).*

That completes this subtopic. Up to this point, the first hidden personality of fluids is now fully developed: from the molecular theory of why surface tension exists, through the excess pressure it creates inside bubbles and drops, to the capillary rise and depression it produces in narrow tubes.

In the next subtopic, we leave static fluids behind and step into the world of moving fluids. We are going to introduce the types of fluid flow, the Reynolds number, and the equation of continuity. So the next subtopic is the gateway to the crown jewel of this chapter: **Bernoulli's equation**.

TYPES OF FLUID FLOW AND EQUATION OF CONTINUITY

Up to this point, every fluid we have studied has been sitting still. Surface tension, excess pressure, and capillary rise are all properties of fluids at rest. But fluids in the real world move. Water flows through pipes, blood flows through arteries, and air flows over aircraft wings. The moment a fluid begins to move, new physics enters the picture, and we need new language to describe it.

This section introduces that language. We will classify fluids, distinguish between orderly and chaotic flow, and derive a powerful equation that tells us exactly what happens to the speed of a fluid when it is forced through a narrow opening. By the end, you will have the tools to explain why a garden hose shoots faster when you cover the end with your thumb, and you will be ready for the crown jewel of this chapter: Bernoulli's equation.

Compressible and Incompressible Fluids

When you squeeze a balloon, the air inside is forced into a smaller space. The same number of molecules now occupy a smaller volume, so the density of the air increases. Air, like all gases, is a **compressible fluid**: its *density changes when pressure is applied*.

Now try squeezing a sealed syringe full of water. Push as hard as you like. The plunger barely moves. The water refuses to be compressed. Liquids are very nearly **incompressible**: their *density remains essentially constant no matter how much pressure you apply*.

This distinction matters enormously for flowing fluids. If a fluid is compressible, its density varies from place to place inside the pipe, and the mathematics becomes considerably more complex. If the fluid is incompressible, its density is the same everywhere, and the equations simplify beautifully.

In this chapter, unless stated otherwise, we deal exclusively with **incompressible fluids**. This is a reasonable assumption for all liquids and for gases moving at low speeds (below the speed of sound).

Viscous and Non-Viscous Fluids

Imagine two tall glasses on a table. One is filled with water, the other with honey. You tilt both glasses at the same angle. The water rushes to the rim instantly, eager to flow. The honey crawls forward reluctantly, as though every millimetre of progress is a personal struggle.

Both liquids are being pulled by the same gravity. Both are free to flow. Yet honey resists flowing in a way that water does not. Why?

The answer is that when honey flows, its layers do not all move at the same speed. The layer touching the glass wall does not move at all (it sticks to the surface). The layer just above it moves a little. The next layer moves a little faster. And so on, until the layer at the centre moves fastest. Each layer drags on the layer next to it, slowing it down, like a stack of heavy blankets being pulled sideways: the bottom blanket is pinned to the bed, and each blanket above it must overcome the friction of the one below before it can move.

This dragging between adjacent layers is called **internal friction**, and the property of a fluid that measures how strongly its layers resist sliding over one another is called **viscosity**. A fluid with high viscosity (honey, engine oil, glycerine) resists flowing strongly because its layers drag heavily on each other. A fluid with low viscosity (water, air, petrol) flows easily because the dragging between layers is weak.

A **viscous fluid** is one in which this internal friction is significant. A **non-viscous fluid** (also called an **inviscid fluid**) is one in which the internal friction is so small that it can be ignored. No real fluid is perfectly non-viscous, but water and air come close enough that, for many problems, the assumption of zero viscosity gives excellent results.

Why does this matter for flowing fluids? Because viscosity drains energy. In a viscous fluid, some of the fluid's kinetic energy is continuously converted into heat by the friction between layers. Consequently, the fluid gradually loses speed unless an external force (such as a pump or gravity) keeps pushing it. In a non-

viscous fluid, no energy is lost to internal friction. Hence, the total mechanical energy of the fluid is conserved as it flows, which is the key assumption behind Bernoulli's equation.

An **ideal fluid** is one that is both incompressible and non-viscous, and whose flow is irrotational (meaning the fluid elements do not spin about their own axes as they move along). Ideal fluids do not exist in nature, but the concept is extraordinarily useful. It allows us to derive clean, exact equations that describe real fluid behaviour to a very good approximation under many practical conditions. Think of the ideal fluid as the physicist's perfectly smooth surface or frictionless pulley: a simplification that captures the essential physics while removing the complications that would otherwise make the mathematics unmanageable.

Steady (Laminar) Flow and Turbulent Flow

Open a tap very slightly and let water trickle out. The stream is smooth, glassy, and predictable. Every drop follows the same path as the drop before it. Now open the tap fully. The stream becomes rough, noisy, and chaotic, splashing in all directions.

These are the two faces of fluid flow, and understanding the difference between them is essential.

Steady (laminar) flow

When a fluid flows slowly and smoothly, every particle that passes through a given point follows exactly the same path as every particle that passed through that point before it. The flow is orderly, layered, and predictable. This type of flow is called **steady flow** or **laminar flow** (from the Latin *lamina*, meaning a thin layer).

To picture laminar flow, imagine thin sheets of fluid sliding smoothly over one another, like pages in a book being pushed sideways. The path followed by any individual fluid particle is called a **streamline**. In laminar flow, streamlines are smooth, continuous curves, and they never cross one another. If two streamlines crossed at a point, a fluid particle arriving at that point would face two possible velocities at the same instant, which is physically impossible. Thus, the non-crossing of streamlines is not just a convention; it is a requirement of physical consistency.

Furthermore, in steady flow, the velocity of the fluid at any given point does not change with time. Different points may have different velocities (for example, the fluid moves faster in a narrow section of pipe than in a wide section), but at each fixed point, the velocity remains constant as long as the flow is steady.

Turbulent flow

When the flow speed increases beyond a certain limit, the orderly laminar pattern collapses. The fluid particles begin to move chaotically, with rapidly fluctuating velocities and unpredictable paths. Eddies and vortices form, fluid layers mix violently, and the streamline picture breaks down entirely. This is **turbulent flow**.

You have seen turbulence many times: the swirling water behind a rock in a fast river, the chaotic smoke rising from a campfire after it breaks into curls and twists, the bumpy ride on an aircraft passing through rough air. In every case, the flow has become too fast or too disturbed to remain orderly.

Turbulent flow dissipates far more energy than laminar flow at the same average speed. Consequently, turbulence is undesirable in many engineering applications: pipeline designers, aircraft engineers, and even blood vessel surgeons all work hard to keep flow laminar wherever possible.

The Reynolds Number

A natural question arises: *how fast is too fast? At what point does laminar flow give way to turbulence?* The answer depends not on the speed alone, but on a combination of the fluid's speed, density, viscosity, and the size of the pipe. The quantity that captures all of these factors in a single number is called the **Reynolds number**, denoted by R_e :

$$R_e = \frac{\rho v r}{\eta}$$

where ρ is the density of the fluid (kgm^{-3}), v is the speed of flow (ms^{-1}), r is the radius of the tube (m), and η is the coefficient of viscosity of the fluid (Nsm^{-2}).

The Reynolds number is a **dimensionless quantity** (it has no units), and it predicts the flow regime:

$R_e < 2000$: the flow is **laminar** (smooth and orderly).

$R_e > 3000$: the flow is **turbulent** (chaotic and irregular).

$2000 < R_e < 3000$: the flow is **unstable** (it may switch unpredictably between laminar and turbulent).

What does this number actually tell you? Physically, the Reynolds number is the ratio of **inertial forces** to **viscous forces** in the fluid.

$$R_e = \frac{\text{Inertial forces}}{\text{Viscous forces}}$$

Inertial forces are the forces associated with the fluid's momentum; they tend to keep the fluid moving and, if disturbed, to amplify the disturbance into turbulence. Viscous forces are the internal friction forces; they tend to damp out disturbances and restore order. When viscous forces dominate (small R_e), disturbances are suppressed and the flow stays laminar. When inertial forces dominate (large R_e), disturbances grow and the flow becomes turbulent.

This explains several everyday observations. A thick, viscous fluid like honey almost always flows in a laminar manner, because its high viscosity gives it a small Reynolds number even at moderate speeds. Water in a narrow pipe at low speed is also laminar, but the same water in a wide pipe at high speed becomes turbulent, because the increased speed and radius push the Reynolds number above 3000. Blood flow in healthy arteries is normally laminar ($R_e \approx 1000$), but if the artery narrows due to plaque buildup, the speed increases at the constriction, the Reynolds number rises, and the flow may become turbulent, producing the characteristic “murmur” that a doctor hears through a stethoscope.

Critical velocity

The speed at which the flow regime changes from laminar to turbulent is called the **critical velocity**, denoted by v_c . It is the maximum speed at which laminar flow can be maintained in a given system.

We can find its value directly from the Reynolds number. At the transition point, R_e equals the critical value (approximately 2000 for flow through a pipe). Setting $R_e = 2000$ and solving for v :

$$R_e = \frac{\rho v_c r}{\eta}$$

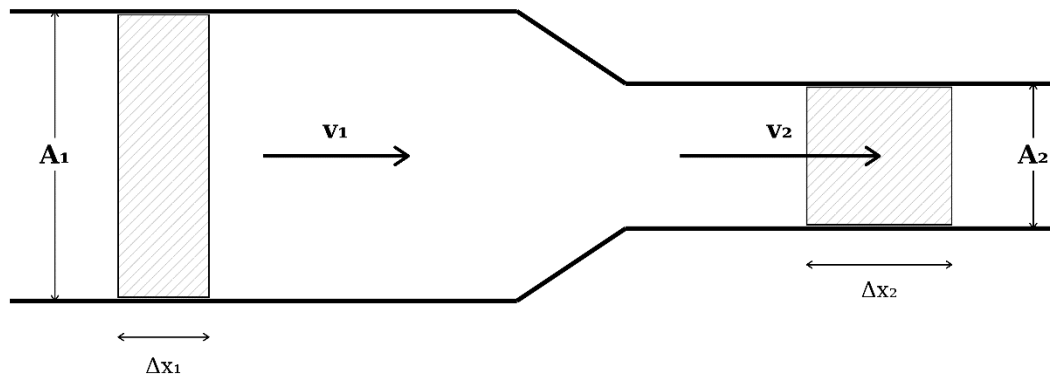
$$v_c = \frac{R_e \times \eta}{\rho r} = \frac{2000 \times \eta}{\rho r}$$

This formula reveals three important facts. First, a more viscous fluid (larger η) has a higher critical velocity, which is why honey can flow quite fast and still remain laminar. Second, a denser fluid (larger ρ) has a lower critical velocity, because its inertial forces overpower its viscous forces more easily. Third, a wider pipe (larger r) has a lower critical velocity, which is why turbulence appears more readily in large rivers than in narrow tubes.

The Equation of Continuity

We now derive one of the most important and most useful results in fluid mechanics. It is called the **equation of continuity**, and it is nothing more than the **law of conservation of mass** applied to a flowing fluid. The question it answers is deceptively simple: *if a fluid flows through a pipe that changes width, what happens to the speed?*

You already know the answer intuitively. When you partially cover the end of a garden hose with your thumb, the water shoots out faster. When a river narrows between two rocks, the current speeds up. When traffic on a highway is forced from three lanes into one, each car must move faster to maintain the overall flow. The equation of continuity makes this intuition precise and mathematical.



Fluid flows from left to right

Figure: Steady flow through a pipe of varying cross-section. At the wider section (area A_1), the fluid moves with velocity v_1 . At the narrower section (area A_2), the fluid moves with velocity v_2 . The shaded elements represent equal volumes of fluid passing each section in time Δt . By conservation of mass, $A_1 v_1 = A_2 v_2$.

Derivation

Consider an incompressible fluid flowing steadily through a pipe that narrows from cross-sectional area A_1 to cross-sectional area A_2 . Since the fluid is incompressible, its density ρ is the same everywhere in the pipe.

In a small time interval Δt , the fluid at the wide section moves a distance $\Delta x_1 = v_1 \Delta t$, sweeping out a volume:

$$\Delta V_1 = A_1 \times \Delta x_1 = A_1 v_1 \Delta t$$

The mass of fluid entering the wide section in time Δt is therefore:

$$\Delta m_1 = \rho A_1 v_1 \Delta t$$

Similarly, at the narrow section, the fluid moves a distance $\Delta x_2 = v_2 \Delta t$, and the mass leaving in time Δt is:

$$\Delta m_2 = \rho A_2 v_2 \Delta t$$

Now here is the key physical argument. The fluid is not leaking, not being created, and not being destroyed. Every drop that enters the wide end must eventually leave through the narrow end. Therefore, the mass entering per unit time must equal the mass leaving per unit time:

$$\Delta m_1 = \Delta m_2$$

$$\rho A_1 v_1 \Delta t = \rho A_2 v_2 \Delta t$$

Since the fluid is incompressible, ρ is the same on both sides. The time interval Δt is also the same. Both cancel, leaving:

$$A_1 v_1 = A_2 v_2$$

This is the **equation of continuity** for an incompressible fluid. It states that *the product of cross-sectional area and fluid velocity is the same at every point along the pipe.*

The quantity $Q = Av$ is called the **volume flow rate or volume flux**: *the volume of fluid passing a given cross-section per unit time.* Its SI unit is $\text{m}^3 \text{s}^{-1}$. The equation of continuity therefore says:

$$Q = A_1 v_1 = A_2 v_2 = \text{constant}$$

Take a moment to appreciate what this equation is telling you. Where the pipe narrows (A decreases), the speed must increase (v goes up) to keep the product Av constant. Where the pipe widens (A increases), the speed must decrease. No extra force is needed. No pump is required. The speed change is an automatic consequence of mass conservation: the same volume of fluid must pass through every cross-section in the same time, so where the area is smaller, the fluid must move faster to compensate.

This is precisely why the boy in the introduction turned a gentle hose into a powerful jet by pressing his thumb over the end. His thumb reduced the exit area. Consequently, the continuity equation demanded that the exit velocity increase. The tap pressure did not change. Conservation of mass did all the work.

Note for compressible fluids: If the fluid is compressible (such as a gas at high speed), the density differs at each cross-section, and the general form of the continuity equation becomes:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

For all problems in this chapter, we use the incompressible form $A_1 v_1 = A_2 v_2$.

With the language of fluid flow now in place, let us put it to work.

BINDER Example 19

Water flows steadily through a horizontal pipe that narrows from a cross-sectional area of 40cm^2 to 10cm^2 . If the velocity of water in the wider section is 2.0ms^{-1} , find the velocity in the narrower section.

Solution

Given: $A_1 = 40\text{cm}^2 = 40 \times 10^{-4}\text{m}^2$, $A_2 = 10\text{cm}^2 = 10 \times 10^{-4}\text{m}^2$, $v_1 = 2.0\text{ms}^{-1}$

Applying the equation of continuity for an incompressible fluid:

$$A_1 v_1 = A_2 v_2$$

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{40 \times 10^{-4}\text{m}^2 \times 2.0\text{ms}^{-1}}{10 \times 10^{-4}\text{m}^2} = 8.0\text{ms}^{-1}$$

Therefore, the velocity of water in the narrower section is 8.0ms^{-1} .

Making Sense of the Answer: The area decreased by a factor of 4, and consequently the velocity increased by the same factor. This is the continuity equation in its purest form: area down, speed up, in exact inverse proportion. The volume flow rate ($Q = 8.0 \times 10^{-3}\text{m}^3\text{s}^{-1}$) is the same in both sections, as conservation of mass requires.

Think Like a Physicist: The equation of continuity is a ratio equation. If the area halves, the speed doubles. If the area drops to one-tenth, the speed increases tenfold. Always check your answer against this proportionality before moving on.

REAL Example 20

Kipanga is helping his mother water the garden using a hose connected to the outdoor tap. The water flows out gently from the open end of the hose. Then Kipanga places his thumb over the end, covering most of the opening and leaving only a small gap. The water immediately shoots out as a fast, thin jet that reaches much further across the garden.

Explain why partially covering the end of the hose causes the water to exit at a higher speed.

Solution

When the hose end is fully open, the water exits through the full cross-sectional area of the hose opening. When Kipanga places his thumb over the end, the effective exit area is reduced to the small gap between his thumb and the hose rim.

The tap delivers a fixed volume flow rate Q to the hose. By the equation of continuity, $Q = Av$, where A is the exit area and v is the exit velocity. Since the tap setting has not changed, Q remains constant. Consequently, reducing A causes v to increase in the same proportion.

As a result, the water exits as a fast, narrow jet that travels a greater horizontal distance before gravity brings it to the ground.

Making Sense of the Answer: No extra force is applied. The tap pressure has not changed. The speed increase is entirely a consequence of conservation of mass: the same volume of water must pass through a smaller opening in the same time, and therefore it must move faster.

Think Like a Physicist: This example connects directly to the introduction of the chapter, where a boy's thumb turned a gentle hose into a sambusa-destroying weapon. The continuity equation explains what happens to the speed. Bernoulli's equation, which we meet next, will explain what happens to the pressure.

HOT Example 21

The cylindrical tube of a spray pump has a cross-sectional area of 8.0cm^2 . At one end, the tube is fitted with a nozzle containing 40 fine holes, each of diameter 0.10mm . If the liquid flows inside the main tube at 1.5m per minute, find the speed of ejection of the liquid through the holes.

Solution

Given: Area of main tube $A_1 = 8.0\text{cm}^2 = 8.0 \times 10^{-4}\text{m}^2$

Velocity in main tube: $v_1 = 1.5\text{m}/\text{min} = \frac{1.5}{60}\text{ms}^{-1} = 0.025\text{ms}^{-1}$

Each hole has diameter $d = 0.10\text{mm} = 1.0 \times 10^{-4}\text{m}$, hence radius $r = 0.50 \times 10^{-4}\text{m}$.

Area of one hole:

$$a = \pi r^2 = \pi \times (0.50 \times 10^{-4}\text{m})^2 = \pi \times 2.5 \times 10^{-9}\text{m}^2 = 7.854 \times 10^{-9}\text{m}^2$$

Total exit area through all 40 holes:

$$A_2 = 40 \times a = 40 \times 7.854 \times 10^{-9}\text{m}^2 = 3.142 \times 10^{-7}\text{m}^2$$

Applying the equation of continuity:

$$A_1 v_1 = A_2 v_2$$

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{8.0 \times 10^{-4}\text{m}^2 \times 0.025\text{ms}^{-1}}{3.142 \times 10^{-7}\text{m}^2} = 63.7\text{ms}^{-1}$$

Therefore, the speed of ejection of the liquid through the holes is approximately 64ms^{-1} .

Making Sense of the Answer: The liquid enters the tube at a leisurely 0.025ms^{-1} and exits the holes at approximately 64ms^{-1} (about 230kmh^{-1}). This represents a speed amplification factor of over 2500, which is consistent with the area ratio: the total exit area is roughly 2500 times smaller than the tube area. This enormous amplification is what makes spray nozzles effective: a slow, steady push produces a fine, high-speed mist.

Think Like a Physicist: The most common error in problems involving multiple holes is forgetting to multiply the area of one hole by the total number of holes. The exit area is the combined area of all 40 holes, not the area of a single hole. Always read the geometry carefully before applying the continuity equation.

That completes the foundations of fluid flow. We have classified fluids, distinguished laminar from turbulent flow, and derived the equation of continuity from conservation of mass. We now know that when an incompressible fluid speeds up through a constriction, it must obey $A_1 v_1 = A_2 v_2$. But a deeper question remains unanswered: *what happens to the pressure when the speed changes?* The answer is one of the most beautiful results in all of physics, and it is waiting in the next subtopic.

BERNOULLI'S PRINCIPLE: DERIVATION AND MEANING

This is the section the entire chapter has been building toward. Every concept we have developed so far: pressure, surface tension, and the equation of continuity has been preparing the ground for what comes next. Bernoulli's equation is to fluid mechanics what Newton's second law is to particle mechanics: it is the central organising principle, the equation that ties everything together, and one of the most beautiful results in all of physics.

The idea behind it is deceptively simple. When a fluid speeds up, its pressure drops. When it slows down, its pressure rises. Speed and pressure trade with each other, like two ends of a seesaw. This trade is not random; it follows an exact mathematical law, and that law is Bernoulli's equation.

Physical Intuition Before the Mathematics

Before we touch a single equation, let us build the intuition that will make the mathematics feel natural rather than mysterious.

Think about energy. You already know from mechanics that energy is conserved: it can change form (kinetic to potential, potential to kinetic), but the total cannot increase or decrease unless an external force does work. A ball rolling down a hill converts potential energy into kinetic energy. A ball rolling up a hill does the reverse. At every point, the total mechanical energy is the same.

Now apply this thinking to a fluid flowing through a pipe. A small parcel of fluid moving through the pipe carries three forms of energy:

- 1) **Kinetic energy**, because it is moving. The faster it moves, the more kinetic energy it has.
- 2) **Gravitational potential energy**, because it may be at some height above a reference level. The higher it is, the more potential energy it has.
- 3) **Pressure energy**, and this is the new idea. In a fluid, the surrounding fluid exerts pressure on the parcel from all sides. This pressure can do work on the parcel (pushing it forward) or the parcel can do work against the pressure (pushing surrounding fluid out of the way). The energy associated with this pressure is called pressure energy, and it equals the pressure multiplied by the volume of the parcel.

Here is the key insight. If the fluid is ideal (incompressible, non-viscous, and flowing steadily), no energy is lost to friction and no energy is added by a pump. Consequently, the total energy per unit volume which is the sum of pressure energy, kinetic energy, and gravitational potential energy must remain constant along any streamline. If the kinetic energy increases (because the fluid speeds up), something else must decrease to compensate. That something is the pressure energy. Hence, where the fluid moves faster, the pressure is lower.

Think of it as an energy budget. The fluid has a fixed total budget of energy per unit volume. If it spends more on speed (kinetic energy), it has less to spend on pressure. If it spends more on height (potential energy), it has less for both speed and pressure. The budget is fixed; only the allocation changes.

This is Bernoulli's principle:

For a steady, incompressible, non-viscous flow, the sum of the pressure, the kinetic energy per unit volume, and the gravitational potential energy per unit volume is constant along a streamline.

Now let us derive it rigorously.

Full Derivation of Bernoulli's Equation

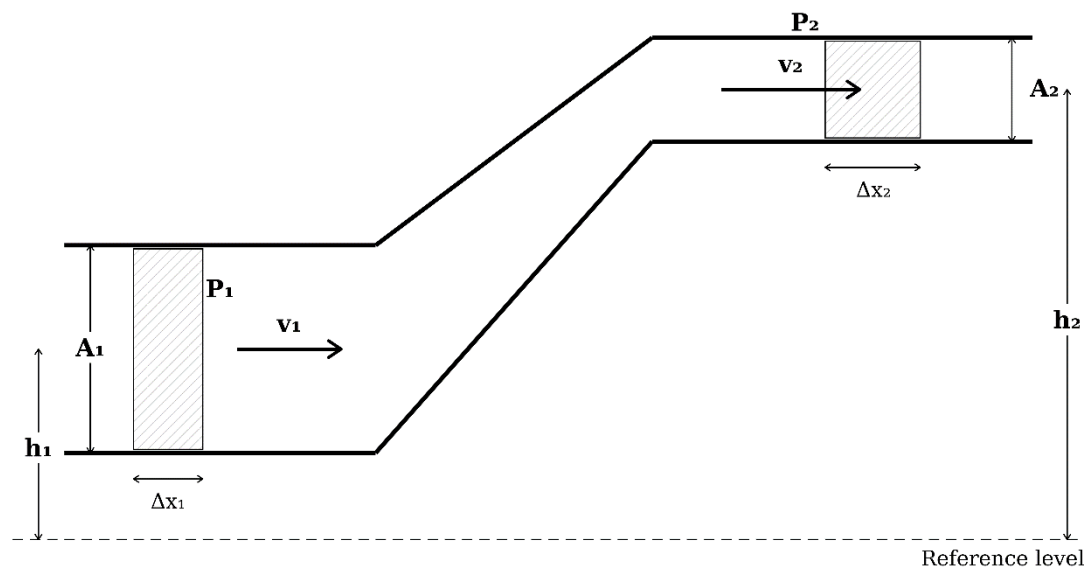


Figure: Fluid flowing through a non-uniform pipe that narrows and rises. At the lower section (area A_1 , velocity v_1 , pressure P_1 , height h_1), a fluid element of length Δx_1 is shown. At the upper section (area A_2 , velocity v_2 , pressure P_2 , height h_2), the corresponding element has length Δx_2 . Heights are measured from the horizontal reference level.

Consider an ideal fluid (incompressible, non-viscous, irrotational) flowing steadily through a pipe of varying cross-section that is not horizontal. At the lower end, the pipe has cross-sectional area A_1 , and the fluid flows with velocity v_1 at pressure P_1 , at a height h_1 above a chosen reference level. At the upper end, the corresponding quantities are A_2 , v_2 , P_2 , and h_2 .

In a small time interval Δt , a small volume of fluid ΔV enters the pipe at the lower end and an equal volume ΔV leaves at the upper end. By the equation of continuity, these volumes are equal because the fluid is incompressible:

$$\Delta V = A_1 v_1 \Delta t = A_2 v_2 \Delta t$$

The mass of this fluid element is:

$$\Delta m = \rho \Delta V$$

Work done on the fluid by the surrounding pressure:

At the lower end, the fluid behind the element pushes it forward with a force $F_1 = P_1 A_1$. This force moves the element through a distance $\Delta x_1 = v_1 \Delta t$. The work done by this force is:

$$W_1 = P_1 A_1 \times \Delta x_1 = P_1 A_1 \times v_1 \Delta t = P_1 \Delta V$$

At the upper end, the fluid ahead of the element resists the incoming fluid with a force $F_2 = P_2 A_2$, directed backward. The element moves forward through $\Delta x_2 = v_2 \Delta t$, doing work against this resistance:

$$W_2 = P_2 A_2 \times \Delta x_2 = P_2 A_2 \times v_2 \Delta t = P_2 \Delta V$$

The net work done on the fluid element by the pressure forces is:

$$W_{\text{net}} = W_1 - W_2 = P_1 \Delta V - P_2 \Delta V = (P_1 - P_2) \Delta V$$

Change in kinetic energy of the fluid element:

$$\Delta KE = \frac{1}{2} \Delta m v_2^2 - \frac{1}{2} \Delta m v_1^2 = \frac{1}{2} \rho \Delta V v_2^2 - \frac{1}{2} \rho \Delta V v_1^2 = \frac{1}{2} \rho \Delta V (v_2^2 - v_1^2)$$

Change in gravitational potential energy of the fluid element:

$$\Delta PE = \Delta m g h_2 - \Delta m g h_1 = \rho \Delta V g h_2 - \rho \Delta V g h_1 = \rho \Delta V g (h_2 - h_1)$$

Applying the work-energy theorem:

The net work done on the fluid element equals the total change in its mechanical energy (kinetic plus potential):

$$W_{\text{net}} = \Delta KE + \Delta PE$$

$$(P_1 - P_2) \Delta V = \frac{1}{2} \rho \Delta V (v_2^2 - v_1^2) + \rho \Delta V g (h_2 - h_1)$$

Dividing every term by ΔV (which is non-zero):

$$P_1 - P_2 = \frac{1}{2} \rho (v_2^2 - v_1^2) + \rho g (h_2 - h_1)$$

Rearranging so that all quantities at point 1 are on the left and all quantities at point 2 are on the right:

$$P_1 + \frac{1}{2} \rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g h_2$$

Since points 1 and 2 are any two points along the same streamline, this equation holds for all points. We can therefore write it in its general form:

$$P + \frac{1}{2} \rho v^2 + \rho g h = \text{constant (along a streamline)}$$

This is **Bernoulli's equation**. Each term has the dimensions of pressure (Nm^{-2} or Pa), and each has a clear physical meaning:

P is the **static pressure**, which is the actual fluid pressure at that point.

$\frac{1}{2} \rho v^2$ is the **dynamic pressure**, which is the kinetic energy per unit volume. It represents the pressure equivalent of the fluid's motion.

$\rho g h$ is the **hydrostatic pressure**, which is the gravitational potential energy per unit volume. It represents the pressure equivalent of the fluid's height.

Bernoulli's equation says that the sum of these three pressures is constant along a streamline. If one increases, at least one of the others must decrease to maintain the balance.

Special case: horizontal flow. If the pipe is horizontal, $h = 0$ ($h_1 = h_2$), the gravitational terms cancel, and Bernoulli's equation reduces to:

$$P + \frac{1}{2} \rho v^2 = \text{constant}$$

This is the form you will use most often. It says, directly and unambiguously: *in a horizontal flow, where the speed is high, the pressure is low; where the speed is low, the pressure is high*. This single statement explains an astonishing range of phenomena, from how aeroplanes fly to why shower curtains attack you.

Conditions and Limitations

Bernoulli's equation is powerful, but it is not universal. It was derived under specific assumptions, and it is valid only when those assumptions are reasonably satisfied. Understanding the limitations is just as important as knowing the equation itself, because applying Bernoulli's equation where it does not apply leads to wrong answers and wrong physics.

1. The fluid must be incompressible

The density ρ must remain constant throughout the flow. This is an excellent assumption for liquids and for gases at speeds well below the speed of sound (roughly below 100ms^{-1} for air). For high-speed gas flows (aircraft near or above the speed of sound), compressibility effects become important, and Bernoulli's equation in this simple form no longer applies.

2. The fluid must be non-viscous

The derivation assumed no energy loss due to internal friction. In reality, all fluids have some viscosity, and viscous losses cause the pressure to drop along the pipe even at constant speed. For fluids with low viscosity (water, air) flowing through short pipes, the viscous losses are small and Bernoulli's equation gives good results. For highly viscous fluids (honey, engine oil) or very long pipes, viscous losses dominate and Bernoulli's equation significantly overestimates the pressure.

3. The flow must be steady

The velocity at each point must not change with time. Bernoulli's equation does not apply to rapidly changing flows, such as the flow during the opening or closing of a valve, or the pulsating flow of blood from a beating heart (although it can be applied approximately to the average flow between heartbeats).

4. The flow must be irrotational

The fluid elements must not rotate about their own axes as they move along. This condition is usually satisfied in flows far from solid boundaries, but is violated near walls and in vortices.

6. The equation applies along a streamline

Bernoulli's equation relates quantities at two points on the same streamline. It does not, in general, relate quantities at points on different streamlines, unless additional conditions (such as irrotationality throughout the entire flow) are met.

A common misconception: Students sometimes state that “Bernoulli's equation proves that faster fluid always has lower pressure.” This is an overstatement. Bernoulli's equation says that along a streamline in a steady, incompressible, non-viscous flow, higher speed corresponds to lower pressure. It does not apply to compressible flows, unsteady flows, or comparisons between different streamlines. A supersonic aircraft does not have vacuum around its wings, despite the very high air speed, because at those speeds the air is compressible and the simple form of Bernoulli's equation breaks down.

With the equation derived and its boundaries understood, let us now see it in action.

BINDER Example 22

Water enters a house through a horizontal pipe of inner diameter 2.0cm at an absolute pressure of $4.0 \times 10^5\text{Pa}$ and a flow velocity of 4.0ms^{-1} . The pipe leads to a bathroom on the second floor, 5.0m higher, where the inner diameter narrows to 1.0cm. Find:

- The flow velocity in the bathroom pipe.
- The water pressure in the bathroom.

Take the density of water as 1000kgm^{-3} and $g = 9.8\text{ms}^{-2}$.

Solution

- By the equation of continuity:

$$A_1 v_1 = A_2 v_2$$

For circular pipes, $A = \pi \left(\frac{d}{2}\right)^2$, so the area ratio equals the square of the diameter ratio:

$$\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2} = \frac{(2.0\text{cm})^2}{(1.0\text{cm})^2} = 4$$

$$v_2 = v_1 \times \frac{A_1}{A_2} = 4.0\text{ms}^{-1} \times 4 = 16\text{ms}^{-1}$$

Therefore, the flow velocity in the bathroom pipe is 16ms^{-1} .

(b) Applying Bernoulli's equation between the entry point (point 1, ground level) and the bathroom (point 2, height 5.0m):

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2$$

Taking the ground level as the reference ($h_1 = 0$, $h_2 = 5.0\text{m}$) and rearranging for P_2 :

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) + \rho g(h_1 - h_2)$$

Substituting values:

$$P_2 = 4 \times 10^5\text{Pa} + \frac{1}{2} \times 1000\text{kgm}^{-3} \times ((4\text{ms}^{-1})^2 - (16\text{ms}^{-1})^2) + 1000\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times (0 - 5\text{m})$$

$$P_2 = 2.31 \times 10^5\text{Pa}$$

Therefore, the water pressure in the bathroom is $2.31 \times 10^5\text{Pa}$.

Making Sense of the Answer: The pressure dropped from $4.0 \times 10^5\text{Pa}$ to $2.31 \times 10^5\text{Pa}$, a reduction of about 42%. Two effects caused this drop: the increase in speed (from 4.0ms^{-1} to 16ms^{-1} , which consumed $1.20 \times 10^5\text{Pa}$ of pressure energy) and the increase in height (5.0m, which consumed $0.49 \times 10^5\text{Pa}$). The speed increase was the dominant factor. This is why water pressure on upper floors of buildings is noticeably lower than on the ground floor, especially when multiple taps are open simultaneously.

Think Like a Physicist: This problem uses both the continuity equation (to find v_2) and Bernoulli's equation (to find P_2). Many Bernoulli problems require both equations working together: continuity provides the velocity, and Bernoulli provides the pressure. Always start with continuity to find the unknown velocity before applying Bernoulli.

BINDER Example 23

Water flows steadily through a horizontal pipe that narrows from a cross-sectional area of 25cm^2 to 10cm^2 . The pressure in the wider section is $6.0 \times 10^4\text{Pa}$, and the velocity there is 2.0ms^{-1} . Find the pressure at the constriction. Take $\rho = 1000\text{kgm}^{-3}$.

Solution

First, find the velocity at the constriction using the continuity equation:

$$v_2 = v_1 \times \frac{A_1}{A_2} = 2.0\text{ms}^{-1} \times \frac{25\text{cm}^2}{10\text{cm}^2} = 5.0\text{ms}^{-1}$$

Since the pipe is horizontal ($h_1 = h_2$), Bernoulli's equation reduces to:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Rearranging for P_2 :

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2)$$

$$P_2 = 6.0 \times 10^4\text{Pa} + \frac{1}{2} \times 1000\text{kgm}^{-3} \times ((2.0\text{ms}^{-1})^2 - (5.0\text{ms}^{-1})^2) = 4.95 \times 10^4\text{Pa}$$

Therefore, the pressure at the constriction is $4.95 \times 10^4\text{Pa}$.

Making Sense of the Answer: The speed increased from 2.0ms^{-1} to 5.0ms^{-1} , and the pressure dropped from $6.0 \times 10^4\text{Pa}$ to $4.95 \times 10^4\text{Pa}$. Speed up, pressure down; exactly as Bernoulli predicts. The pressure drop ($1.05 \times 10^4\text{Pa}$) is about 17.5% of the original pressure, which is significant but not dramatic. In a more extreme constriction, the pressure drop would be much larger.

Think Like a Physicist: For horizontal pipes, Bernoulli's equation simplifies to $P + \frac{1}{2}\rho v^2 = \text{constant}$. This is the form to reach for whenever the pipe is horizontal or the height difference is negligible. The gravitational term only matters when there is a significant change in elevation.

REAL Example 24

Kipute notices something strange every time she takes a shower. When she turns on the water, the shower curtain moves inward toward her, as if being sucked in. It does not blow outward, as she would expect from the spray hitting it. The harder the water flows, the more the curtain moves inward.

Explain why a shower curtain moves inward when the shower is running.

Solution

When the shower is running, a stream of water and entrained air moves rapidly downward between Kipute and the curtain. By Bernoulli's equation for horizontal flow ($P + \frac{1}{2}\rho v^2 = \text{constant}$), the rapidly moving air on the inner side of the curtain has a lower pressure than the still air on the outer side.

This creates a pressure difference across the curtain: higher pressure on the outside (where the air is stationary), lower pressure on the inside (where the air is moving with the shower stream). Consequently, the net force on the curtain is directed inward, pushing it toward the shower. The faster the water flows, the greater the air speed on the inner side, the greater the pressure difference, and the more strongly the curtain is pushed inward.

Therefore, the curtain is not being “sucked” inward. It is being **pushed** inward by the higher atmospheric pressure on the outer side, because the moving air on the inner side has reduced pressure.

Making Sense of the Answer: This is Bernoulli's equation at work in your bathroom. The moving air has lower pressure; the still air has higher pressure. The pressure difference pushes the curtain toward the region of lower pressure. Many people describe this as “suction,” but there is no suction force in physics. There is only a pressure difference, and the net force always pushes from high pressure toward low pressure.

Think Like a Physicist: Whenever you see an object being “pulled toward” a fast-moving fluid, ask: **is the pressure on the fast side lower than on the slow side?** If yes, the object is being pushed by the pressure difference, not pulled by some mysterious suction. This reasoning explains why two ships sailing close together in parallel tend to be drawn toward each other, why a flag flutters, and why roofs blow off during storms (the fast wind above creates lower pressure than the still air inside the house, so the roof is pushed upward from below).

HOT Example 25

Water flows through a pipe system consisting of three sections. In the first section, the cross-sectional area is 30cm^2 , the velocity is 3.0ms^{-1} , and the pressure is $2.0 \times 10^5\text{Pa}$. The pipe then narrows to 15cm^2 in the second section, which is at the same height. Finally, the pipe rises 4.0m and widens to 20cm^2 in the third section.

Find the velocity and pressure in the second and third sections. Take $\rho = 1000\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Solution

Finding the velocity in the second section:

$$v_2 = v_1 \times \frac{A_1}{A_2} = 3.0\text{ms}^{-1} \times \frac{30\text{cm}^2}{15\text{cm}^2} = 6.0\text{ms}^{-1}$$

Finding the pressure in the second section:

The first and second sections are at the same height, so Bernoulli's equation reduces to:

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2)$$

$$P_2 = 2.0 \times 10^5\text{Pa} + \frac{1}{2} \times 1000\text{kgm}^{-3} \times ((3.0\text{ms}^{-1})^2 - (6.0\text{ms}^{-1})^2) = 1.865 \times 10^5\text{Pa}$$

Therefore, in the second section, the velocity is 6.0ms^{-1} and the pressure is $1.865 \times 10^5\text{Pa}$.

Finding the velocity in the third section:

$$v_3 = v_1 \times \frac{A_1}{A_3} = 3.0\text{ms}^{-1} \times \frac{30\text{cm}^2}{20\text{cm}^2} = 4.5\text{ms}^{-1}$$

Finding the pressure in the third section:

Applying Bernoulli's equation between section 1 (height $h_1 = 0$) and section 3 (height $h_3 = 4.0\text{m}$):

$$P_3 = P_1 + \frac{1}{2}\rho(v_1^2 - v_3^2) + \rho g(h_1 - h_3)$$

$$P_3 = 2 \times 10^5 \text{ Pa} + \frac{1}{2} \times 1000 \text{ kg m}^{-3} \times ((3 \text{ ms}^{-1})^2 - (4.5 \text{ ms}^{-1})^2) + 1000 \text{ kg m}^{-3} \times 9.8 \text{ ms}^{-2} \times (0 - 4 \text{ m})$$

$$P_3 = 1.552 \times 10^5 \text{ Pa}$$

Therefore, in the third section, the velocity is 4.5 ms^{-1} and the pressure is $1.552 \times 10^5 \text{ Pa}$.

Making Sense of the Answer: Follow the pressure as it drops through the system: $2.0 \times 10^5 \text{ Pa} \rightarrow 1.865 \times 10^5 \text{ Pa} \rightarrow 1.552 \times 10^5 \text{ Pa}$. Each drop has a clear physical cause. The first drop (from section 1 to 2) is caused entirely by the speed increase (the sections are at the same height). The second drop (from section 1 to 3) is caused by both the speed increase and the height increase. The height contribution (39200 Pa) is actually larger than the speed contribution (5625 Pa) in this case, which shows that elevation changes can dominate over speed changes when the pipe rises significantly.

Think Like a Physicist: In multi-section problems, always apply Bernoulli's equation between section 1 and each subsequent section directly, rather than going step by step from 1 to 2 and then from 2 to 3. Going directly from 1 to 3 avoids accumulating rounding errors from intermediate calculations. The constant in Bernoulli's equation is the same at all points on the streamline, so you can compare any two points directly.

That completes the derivation and first applications of Bernoulli's equation. But the true power of Bernoulli's equation lies in its applications to real devices and real phenomena. In the next section, we apply it to five celebrated devices. Each application takes the abstract equation and turns it into something you can see, build, and use.

APPLICATIONS OF BERNOULLI'S PRINCIPLE

Bernoulli's equation would be beautiful even if it had no practical applications. But it does. It has so many that an entire engineering discipline (fluid dynamics) is built on its foundations. In this section, we take the abstract equation from the previous section and put it to work in five celebrated devices and phenomena. Each one demonstrates a different face of the same principle: where speed goes up, pressure goes down.

Torricelli's Theorem (Fluid Flowing from a Tank)

Imagine a large open tank filled with water to a height h above a small hole (orifice) in the side wall near the bottom. When the plug is removed, water rushes out of the hole. How fast does it emerge?

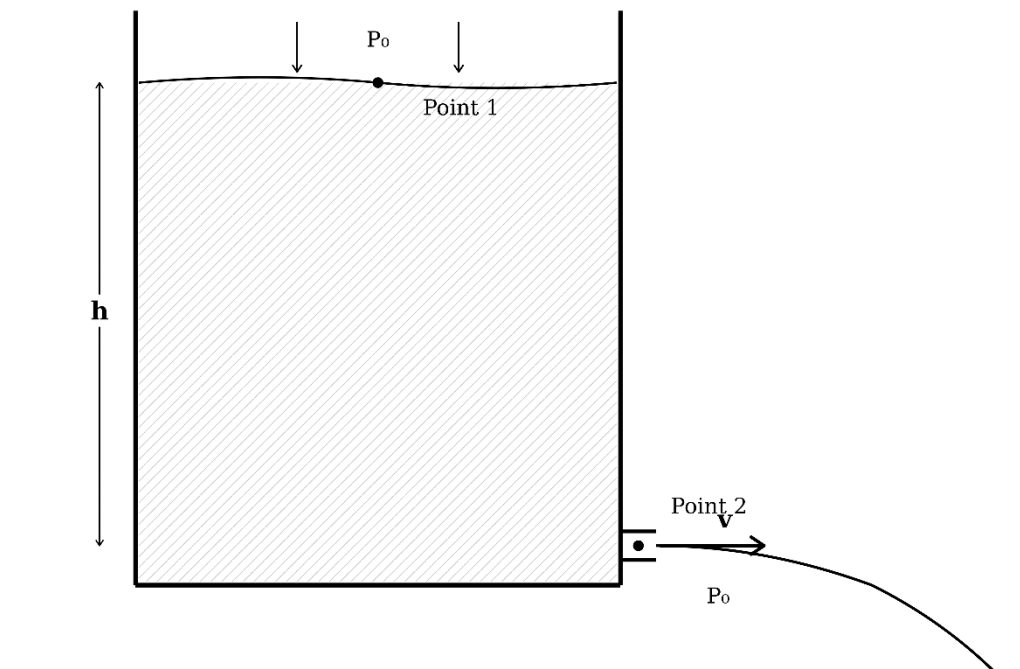


Figure: A tank of water with an orifice near the base. Point 1 is at the water surface (velocity ≈ 0 , pressure P_0). Point 2 is at the orifice (velocity v , pressure P_0). Both points are open to the atmosphere. The depth of water above the orifice is h .

Apply Bernoulli's equation between the water surface (point 1) and the orifice (point 2). Both the surface and the orifice are open to the atmosphere, so $P_1 = P_2 = P_0$ (atmospheric pressure). Take the orifice as the reference level, so $h_2 = 0$ and $h_1 = h$.

$$P_0 + \frac{1}{2} \rho v_1^2 + \rho gh = P_0 + \frac{1}{2} \rho v_2^2 + 0$$

Now here is a crucial simplification. The tank is large and the orifice is small. Consequently, the water level drops very slowly, so $v_1 \approx 0$. The atmospheric pressure terms cancel on both sides. What remains is:

$$\rho gh = \frac{1}{2} \rho v_2^2$$

Cancelling ρ and solving for v_2 :

$$v_2 = \sqrt{2gh}$$

Hence, the speed of efflux is:

$$v = \sqrt{2gh}$$

This is **Torricelli's theorem**. It states that *the speed of efflux of a liquid from an orifice at depth h below the surface is the same as the speed that a freely falling body would acquire after falling through the same height h* . This is a remarkable result: the water shoots out of the hole as fast as if it had fallen from the surface in free fall.

Time to empty the tank

If the tank has a cross-sectional area A and the orifice has area a , we can find how long it takes to empty the tank completely. By the equation of continuity, the rate at which water leaves through the orifice equals the rate at which the water level drops: $Av_1 = av_2$ with $A_1 = A, A_2 = a$.

$$A \frac{dh}{dt} = -a\sqrt{2gh}$$

The negative sign indicates that h decreases with time. Separating variables and integrating from initial height h to zero:

$$-\frac{A}{a\sqrt{2g}} \int_h^0 h^{-1/2} dh = \int_0^t dt$$

$$t = \frac{2A}{a} \sqrt{\frac{h}{2g}} = \frac{A}{a} \sqrt{\frac{2h}{g}}$$

Notice that the water emerging from the orifice is a horizontal projectile. It leaves the orifice with horizontal velocity $v = \sqrt{2gh}$ and then follows a parabolic trajectory under gravity, exactly like the projectiles we studied in Chapter 6. The horizontal range of the water jet can be calculated using projectile equations once the height of the orifice above the ground is known.

The Venturi Meter

The venturi meter is a device used to measure the speed of a fluid flowing through a pipe. It works by creating a constriction in the pipe, measuring the resulting pressure difference, and using Bernoulli's equation to calculate the flow speed.

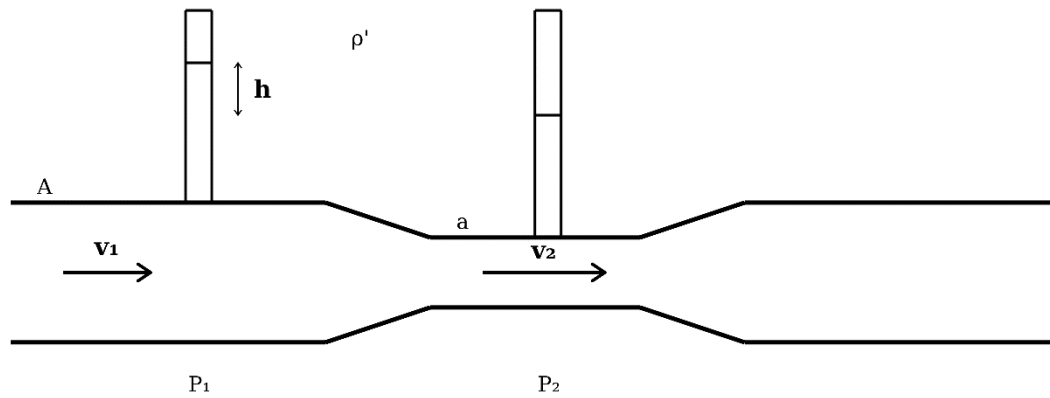


Figure: A venturi meter. Fluid flows from the wide section (area A , velocity v_1 , pressure P_1) through the narrow throat (area a , velocity v_2 , pressure P_2). The manometer liquid of density ρ' shows a level difference h , from which the flow speed is calculated.

A liquid of density ρ flows through a horizontal pipe from a wide section of area A (where the velocity is v_1 and the pressure is P_1) to a narrow throat of area a (where the velocity is v_2 and the pressure is P_2). A U-shaped manometer containing a liquid of density ρ' is connected between the two sections.

By the equation of continuity:

$$v_2 = v_1 \times \frac{A}{a}$$

By Bernoulli's equation for horizontal flow:

$$P_1 - P_2 = \frac{1}{2} \rho (v_2^2 - v_1^2)$$

The pressure difference $P_1 - P_2$ is measured by the manometer. If the manometer liquid levels differ by a height h :

$$P_1 - P_2 = \rho' gh$$

Substituting $v_2 = v_1 A/a$ into the Bernoulli result and equating with the manometer reading:

$$\rho' gh = \frac{1}{2} \rho \left(v_1^2 \times \frac{A^2}{a^2} - v_1^2 \right) = \frac{1}{2} \rho v_1^2 \left(\frac{A^2}{a^2} - 1 \right) = \frac{1}{2} \rho v_1^2 \left(\frac{A^2 - a^2}{a^2} \right)$$

Solving for v_1 :

$$v_1 = a \sqrt{\frac{2\rho' gh}{\rho(A^2 - a^2)}}$$

The volume flow rate is then:

$$Q = Av_1 = Aa \sqrt{\frac{2\rho' gh}{\rho(A^2 - a^2)}}$$

The venturi meter is widely used in practice. Water suppliers, such as DAWASCO uses venturi meters to measure water flow through Dar es Salaam's distribution pipes. Fuel injection systems in engines use the venturi principle to mix air and fuel in the correct proportions. Medical devices use miniature venturi meters to measure airflow in respiratory equipment.

Aerofoil Lift (Dynamic Lift)

This is perhaps the most celebrated application of Bernoulli's principle: the explanation of how an aeroplane generates the lift force that holds it in the air.

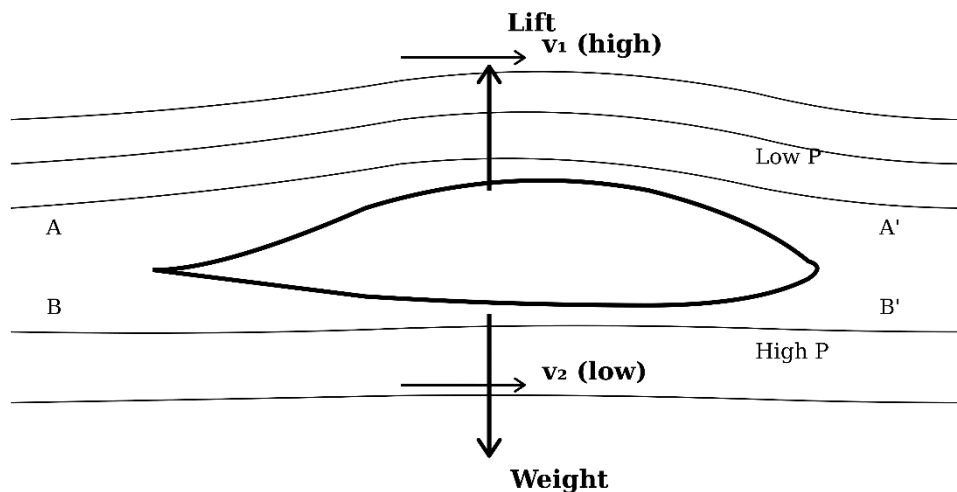


Figure: Cross-section of an aerofoil. Streamlines above the wing are closer together (faster air, lower pressure). Streamlines below are more widely spaced (slower air, higher pressure). The pressure difference creates an upward lift force.

An aircraft wing is shaped so that its upper surface is longer and more curved than its lower surface. When the wing moves through the air (or equivalently, when air flows past the wing), the air stream splits at the front edge. The air travelling over the longer upper surface must cover a greater distance in the same time as the air travelling under the shorter lower surface.

By the equation of continuity, if the air above must move faster, the streamlines above the wing are more closely spaced (equivalent to a narrower “pipe”). Consequently, $v_1 > v_2$, where v_1 is the speed above the wing and v_2 is the speed below.

By Bernoulli’s equation ($P + \frac{1}{2}\rho v^2 = \text{constant}$), the faster air above the wing has lower pressure (P_1) than the slower air below (P_2). This pressure difference creates a net upward force on the wing. This force is called **dynamic lift**.

If the wing has area A , the lift force is:

$$F = (P_2 - P_1) \times A = \frac{1}{2} \rho A (v_1^2 - v_2^2)$$

For the aircraft to fly level, the lift force must equal the weight of the aircraft:

$$F = Mg$$

$$\frac{1}{2} \rho A (v_1^2 - v_2^2) = Mg$$

This is why heavily loaded aircraft need longer runways: greater weight requires greater lift, which requires higher speed, which requires more distance to achieve during takeoff. It is also *why aircraft fly faster at higher altitudes:* the air density ρ is lower at high altitude, so the speed must be greater to generate the same lift.

The Magnus effect (spinning ball)

When a ball spins as it moves through the air, it drags the air around it in the direction of spin. On one side, the air dragged by the spin adds to the oncoming airflow, increasing the air speed. On the opposite side, the spin opposes the airflow, decreasing the air speed. By Bernoulli’s principle, the side with faster air has lower pressure than the side with slower air. Consequently, a net force pushes the ball sideways, toward the low-pressure side.

This is the **Magnus effect**, and it explains *why a spinning football curves in flight*, *why a cricket bowler can make the ball swing*, and *why a table tennis player can make the ball dip or swerve with topspin or sidespin*.

The Atomiser (Sprayer)

An atomiser converts a liquid into a fine spray using Bernoulli’s principle. It is used in perfume bottles, insecticide sprayers, paint spray guns, and carburettors.

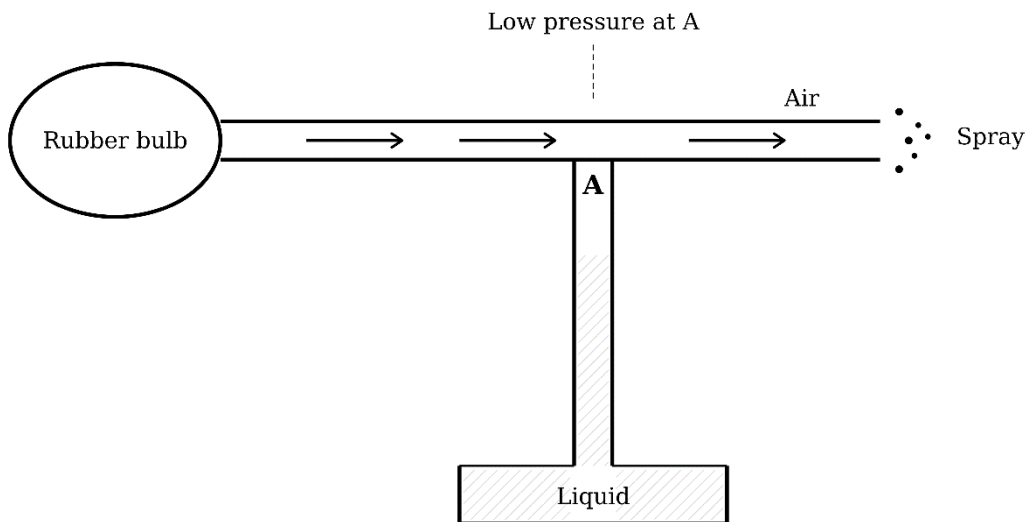


Figure: An atomiser. Fast-moving air from the rubber bulb passes over point A, creating low pressure. Atmospheric pressure on the liquid surface pushes the liquid up the vertical tube, where it emerges as a fine spray.

When the rubber bulb is squeezed, air rushes through the narrow horizontal tube at high speed. At point A, where the vertical tube meets the horizontal tube, the high-speed air creates a region of low pressure (by Bernoulli's principle). The atmospheric pressure acting on the liquid surface in the container is now greater than the pressure at point A. As a result, this pressure difference pushes the liquid up the vertical tube. When it reaches point A, the fast-moving air stream breaks it into fine droplets, producing a spray.

The Pitot Tube

The pitot tube measures the velocity of a fluid (or the speed of an object moving through a fluid) by comparing the stagnation pressure with the static pressure.

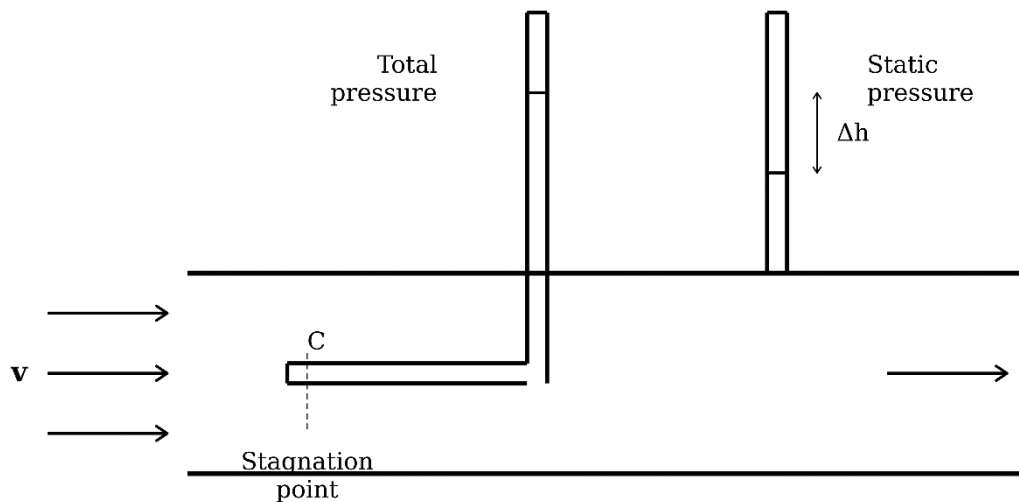


Figure: A pitot tube in a fluid stream. The stagnation tube faces the flow and measures total pressure. The static tube is perpendicular to the flow and measures static pressure. The height difference Δh gives the fluid velocity.

At the stagnation point (where the fluid is brought to rest), the velocity is zero. Applying Bernoulli's equation between a point in the undisturbed flow (velocity v , static pressure P) and the stagnation point (velocity 0, total pressure P_T):

$$P + \frac{1}{2}\rho v^2 = P_T$$

$$P_T - P = \frac{1}{2}\rho v^2$$

If the pressure difference is measured by a manometer showing a height difference Δh :

$$\frac{1}{2}\rho v^2 = \rho' g \Delta h$$

$$v = \sqrt{\frac{2\rho' g \Delta h}{\rho}}$$

where ρ' is the density of the manometer liquid and ρ is the density of the flowing fluid. If the flowing fluid itself is used as the manometer liquid ($\rho' = \rho$), this simplifies to:

$$v = \sqrt{2g\Delta h}$$

Pitot tubes are used in aircraft to measure airspeed, in weather stations to measure wind speed, and in industrial pipelines to monitor flow rates.

The theory has been served; now it is time for the worked examples to earn their keep. The following problems bring Bernoulli's applications from the textbook into the real world, and they will test whether you have truly understood the principle or merely memorised the formula. Get your pen ready!

BINDER Example 26

A cylindrical water tank of radius 1.0m rests on a platform 5.0m above the ground. The tank is filled with water to a height of 5.0m. A small circular plug of area $1.0 \times 10^{-4} \text{m}^2$ is removed from an orifice at the bottom of the tank.

- Find the speed at which water flows from the orifice.
- Find the horizontal distance from the base of the platform to the point where the water jet strikes the ground. Take $g = 9.8 \text{ms}^{-2}$.

Solution

(a) By Torricelli's theorem, the speed of efflux from an orifice at depth h below the water surface is:

$$v = \sqrt{2gh}$$

$$v = \sqrt{2 \times 9.8 \text{ms}^{-2} \times 5.0 \text{m}} = 9.9 \text{ms}^{-1}$$

Therefore, the speed of efflux is 9.9ms^{-1} .

(b) Horizontal range of the jet:

The water leaves the orifice horizontally at $v = 9.9 \text{ms}^{-1}$, at a height of 5.0m above the ground (the height of the platform). This is a projectile problem. The time to fall to the ground is:

$$H = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 5.0 \text{m}}{9.8 \text{ms}^{-2}}} = 1.01 \text{s}$$

The horizontal distance is:

$$x = v \times t = 9.9 \text{ms}^{-1} \times 1.01 \text{s} = 10.0 \text{m}$$

Therefore, the water jet strikes the ground at a horizontal distance of approximately 10.0m from the base of the platform.

Making Sense of the Answer: The water shoots out at nearly 10ms^{-1} and lands 10m away. This is a substantial distance, roughly the length of a classroom. The problem beautifully connects Bernoulli's equation (Torricelli gives the exit speed) with Chapter 6 (projectile motion gives the range). Physics builds on itself.

Think Like a Physicist: Torricelli's theorem is a projectile problem in disguise. The exit speed $v = \sqrt{2gh}$ is identical to the speed an object would reach after falling freely through height h . Once the water leaves the orifice, it is a projectile. Always check whether a Torricelli problem has a second part involving projectile motion.

BINDER Example 27

A venturi meter is fitted in a horizontal pipe carrying water ($\rho = 1000\text{kgm}^{-3}$). The cross-sectional areas of the wide section and the throat are 25cm^2 and 10cm^2 respectively. A mercury manometer ($\rho = 13600\text{kgm}^{-3}$) connected between the two sections shows a level difference of 3.0cm . Find the flow speed in the wide section and the volume flow rate. Take $g = 9.8\text{ms}^{-2}$.

Solution

Given: $A = 25\text{cm}^2 = 25 \times 10^{-4}\text{m}^2$, $a = 10\text{cm}^2 = 10 \times 10^{-4}\text{m}^2$, $\rho = 1000\text{kgm}^{-3}$, $\rho' = 13,600\text{kgm}^{-3}$, $h = 3.0\text{cm} = 3.0 \times 10^{-2}\text{m}$

Using the venturi meter formula:

$$v_1 = a \sqrt{\frac{2\rho'gh}{\rho(A^2 - a^2)}} = \sqrt{\frac{2 \times 13,600\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 3.0 \times 10^{-2}\text{m}}{1000\text{kgm}^{-3}((25 \times 10^{-4}\text{m}^2)^2 - (10 \times 10^{-4}\text{m}^2)^2)}} = 1.23\text{ms}^{-1}$$

The volume flow rate is:

$$Q = Av_1 = 25 \times 10^{-4}\text{m}^2 \times 1.23\text{ms}^{-1} = 3.08 \times 10^{-3}\text{m}^3\text{s}^{-1}$$

Therefore, the flow speed in the wide section is 1.23ms^{-1} and the volume flow rate is $3.08 \times 10^{-3}\text{m}^3\text{s}^{-1}$ (or approximately 3.1 litres per second).

Making Sense of the Answer: A tiny mercury level difference of just 3.0cm reveals a flow speed of 1.23ms^{-1} and a flow rate of about 3 litres per second. The venturi meter works because mercury is 13.6 times denser than water, so a small height difference corresponds to a large pressure difference. This sensitivity is **why mercury manometers are preferred for measuring small pressure differences in water flow**.

Think Like a Physicist: The venturi meter formula looks intimidating, but it is built from just two equations you already know: the continuity equation ($Av_1 = av_2$) and Bernoulli's equation ($P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$). In an examination, you can derive the result from scratch rather than memorising the final formula. Derivation is always safer than memorisation.

BINDER Example 28

A pitot tube is placed in a stream of water ($\rho = 1000\text{kgm}^{-3}$) flowing through a horizontal pipe. The pitot tube is connected to a mercury manometer ($\rho = 13600\text{kgm}^{-3}$), which shows a height difference of 8mm. Calculate the speed of the water flow. Take $g = 9.8\text{ms}^{-2}$.

Solution

Given: $\rho = 1000\text{kgm}^{-3}$, $\rho' = 13,600\text{kgm}^{-3}$, $\Delta h = 8\text{mm} = 8 \times 10^{-3}\text{m}$, $g = 9.8\text{ms}^{-2}$

At the stagnation point, the fluid is brought to rest. Applying Bernoulli's equation between the undisturbed flow (velocity v , static pressure P) and the stagnation point (velocity 0, total pressure P_T):

$$P + \frac{1}{2}\rho v^2 = P_T$$

The pressure difference $P_T - P = \frac{1}{2}\rho v^2$ is measured by the mercury manometer:

$$\frac{1}{2}\rho v^2 = \rho'g\Delta h$$

Solving for v :

$$v = \sqrt{\frac{2\rho'g\Delta h}{\rho}} = \sqrt{\frac{2 \times 13,600\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 8 \times 10^{-3}\text{m}}{1000\text{kgm}^{-3}}} = 1.46\text{ms}^{-1}$$

Therefore, the speed of the water flow is approximately 1.5ms^{-1} .

Making Sense of the Answer: A tiny mercury difference of just 8mm reveals a flow speed of 1.5ms^{-1} . This high sensitivity is why pitot tubes use a denser manometer liquid: mercury amplifies small pressure

differences into readable height differences. If water were used as the manometer liquid instead, the same flow speed would produce a height difference of about 11cm which is readable, but less precise.

Think Like a Physicist: The pitot tube formula $v = \sqrt{2\rho'g\Delta h/\rho}$ looks similar to Torricelli's $v = \sqrt{2gh}$, and for good reason: both come from Bernoulli's equation with one point at rest. The difference is that Torricelli has the fluid itself providing the height, while the pitot tube uses a separate manometer liquid.

REAL Example 29

During a physics demonstration, Mr. Akilikubwa holds two sheets of paper parallel to each other, about 3cm apart, and blows a stream of air between them. Kipute expects the papers to fly apart, but instead they move **toward** each other. The whole class gasps.

Explain why the two paper sheets move toward each other when air is blown between them.

Solution

When Mr. Akilikubwa blows air between the sheets, the air between them moves at high speed. By Bernoulli's equation for horizontal flow ($P + \frac{1}{2}\rho v^2 = \text{constant}$), the rapidly moving air between the sheets has a lower pressure than the still air on the outer sides of the sheets.

This creates a pressure difference across each sheet: higher atmospheric pressure on the outer side (where the air is stationary) and lower pressure on the inner side (where the air is moving). Consequently, each sheet experiences a net inward force, pushing it toward the other sheet.

Therefore, the papers are not being “pulled” together. They are being **pushed** together by the higher atmospheric pressure on their outer surfaces, because the fast-moving air between them has reduced the pressure on their inner surfaces.

Making Sense of the Answer: This is one of the simplest and most convincing demonstrations of Bernoulli's principle. It requires only two sheets of paper and a pair of lungs. The same physics explains **why a shower curtain moves inward when the shower is running, why roofs blow off during storms** (fast wind above the roof creates lower pressure than the still air inside the house), and why **two ships sailing close together in parallel tend to drift toward each other**.

Think Like a Physicist: Whenever an object appears to be “attracted” toward a region of fast-moving fluid, the real mechanism is a pressure difference. The fast fluid has low pressure; the still fluid has high pressure. The net force pushes the object from the high-pressure side toward the low-pressure side. There is no “suction” force in physics, only pressure differences.

HOT Example 30

A cylindrical tank of radius 0.50m is filled with water to a height of 4.0m. A plug is removed from a small hole of area $2.0 \times 10^{-4}\text{m}^2$ at the base of the tank.

- Find the initial speed of efflux.
- Find the initial volume flow rate.
- Show that the time to empty the tank completely is given by $t = \frac{A}{a} \sqrt{\frac{2h}{g}}$ and calculate this time. Take $g = 9.8\text{ms}^{-2}$.

Solution

- (a) By Torricelli's theorem:

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8\text{ms}^{-2} \times 4.0\text{m}} = 8.85\text{ms}^{-1}$$

Therefore, the initial speed of efflux is 8.85ms^{-1} .

- (b) Initial volume flow rate:

$$Q = av = 2.0 \times 10^{-4}\text{m}^2 \times 8.85\text{ms}^{-1} = 1.77 \times 10^{-3}\text{m}^3\text{s}^{-1}$$

Therefore, the initial volume flow rate is $1.77 \times 10^{-3}\text{m}^3\text{s}^{-1}$ (about 1.8 litres per second).

(c) Let h be the height of water at any time t . The speed of efflux at this instant is $v = \sqrt{2gh}$. By the equation of continuity, the rate at which the volume decreases in the tank equals the rate at which water exits through the orifice:

$$-A \frac{dh}{dt} = a\sqrt{2gh}$$

The negative sign arises because h is decreasing with time. Separating variables:

$$\frac{dh}{\sqrt{h}} = -\frac{a}{A}\sqrt{2g} dt$$

Integrating from initial height $h_0 = h$ (at $t = 0$) to $h = 0$ (at t):

$$\int_h^0 h^{-1/2} dh = -\frac{a}{A}\sqrt{2g} \int_0^t dt$$

$$[2h^{1/2}]_h^0 = -\frac{a}{A}\sqrt{2g} \times t$$

$$0 - 2\sqrt{h} = -\frac{a}{A}\sqrt{2g} \times t$$

$$t = \frac{2A\sqrt{h}}{a\sqrt{2g}} = \frac{A}{a} \sqrt{\frac{2h}{g}}$$

Hence:

$$t = \frac{A}{a} \sqrt{\frac{2h}{g}}$$

Substituting values:

$$t = \frac{\pi \times (0.50\text{m})^2}{2.0 \times 10^{-4}\text{m}^2} \times \sqrt{\frac{2 \times 4.0\text{m}}{9.8\text{ms}^{-2}}} = 3548\text{s}$$

Therefore, the time to empty the tank completely is 3548s (or about 59 minutes).

Making Sense of the Answer: Nearly an hour to empty a tank that is only 4m tall through a hole the size of a small coin. This makes physical sense: as the tank empties, the water level drops, the pressure at the orifice decreases, the exit speed decreases (by Torricelli), and the flow rate slows down. The tank empties quickly at first and progressively more slowly toward the end. This is why the time is proportional to $\sqrt{h}$, not to h : the square root reflects the decelerating nature of the emptying process.

Think Like a Physicist: This problem combines Torricelli's theorem (from Bernoulli), the equation of continuity, and calculus (separation of variables and integration). It is exactly the kind of multi-step problem that examinations favour. The strategy is to write the differential equation first, then separate variables, then integrate, then substitute. Each step is straightforward; the challenge is knowing the sequence.

HOT Example 31

An aircraft has a total wing area of 120m^2 and a total mass of 20000kg . During level flight, the speed of air over the upper surface of the wings is 70ms^{-1} and under the lower surface is 60ms^{-1} . Take the density of air as 1.2kgm^{-3} and $g = 9.8\text{ms}^{-2}$.

- Find the dynamic lift on the wings.
- Determine whether the aircraft can maintain level flight.

Solution

(a) Dynamic lift is given by:

$$F = \frac{1}{2} \rho A (v_1^2 - v_2^2)$$

where $v_1 = 70\text{ms}^{-1}$ (upper surface, faster) and $v_2 = 60\text{ms}^{-1}$ (lower surface, slower).

$$F = \frac{1}{2} \times 1.2 \text{kgm}^{-3} \times 120 \text{m}^2 \times ((70 \text{ms}^{-1})^2 - (60 \text{ms}^{-1})^2) = 93600 \text{N}$$

Therefore, the dynamic lift on the wings is 93600N (or 93.6kN).

(b) The weight of the aircraft is:

$$W = Mg = 20,000 \text{kg} \times 9.8 \text{ms}^{-2} = 196000 \text{N}$$

Comparing lift and weight:

$$F = 93600 \text{N} < W = 196000 \text{N}$$

The dynamic lift (93.6kN) is less than the weight (196kN). Therefore, the aircraft **cannot** maintain level flight at these air speeds.

(The pilot would need to increase the speed, increase the angle of attack, or extend the wing flaps to generate additional lift.)

Making Sense of the Answer: The lift is only about 48% of the weight needed. A speed difference of 10ms^{-1} between the upper and lower surfaces is not enough for this heavy aircraft. If the upper surface speed were increased to about 95ms^{-1} (with the lower surface at 60ms^{-1}), the lift would match the weight. This shows why commercial aircraft need high takeoff speeds and long runways.

Think Like a Physicist: The lift force depends on the difference of the squares of the speeds, $(v_1^2 - v_2^2) = (v_1 + v_2)(v_1 - v_2)$, not simply on the difference of the speeds, $(v_1 - v_2)$. This means that if both speeds increase (the aircraft moves faster overall), the lift also increases because of the extra factor $(v_1 + v_2)$, even if the speed difference $(v_1 - v_2)$ stays the same. This is the physics behind the rule that **faster aircraft generate more lift**.

The worked examples have had their say, and Bernoulli's principle has proven itself across tanks, pipes, wings, and sprayers. In the next subtopic, we leave the ideal fluid behind and confront reality: fluids that resist flowing, fluids with internal friction, fluids with **viscosity**. The third hidden personality of fluids is about to step forward.

VISCOSITY AND NEWTON'S LAW OF VISCOSITY

In the previous subtopics, we treated fluids as though they had no internal friction. We assumed they flowed without resistance, without energy loss, without any layer dragging on any other layer. That was the ideal fluid, and it gave us Bernoulli's beautiful equation. But now it is time to face reality. Real fluids resist flowing. Honey pours slowly. Engine oil clings to your fingers. Even water, which seems to flow so freely, slows down and stops if you do not keep pushing it.

This resistance to flow is called **viscosity**, and it is the third hidden personality of fluids. We introduced the idea briefly when we discussed types of fluid flow. Now we give it a precise definition, a physical law, and a mathematical derivation.

The Concept of Viscosity

Pour honey from a spoon. It moves slowly, reluctantly, as though something inside it is fighting every millimetre of motion. Now pour water from the same spoon. It falls instantly, almost eagerly.

Both liquids are being pulled by the same gravitational force. The difference is that honey has much stronger **internal friction** between its layers. When honey flows, the layer touching the spoon does not move at all (it sticks to the surface). The layer just above it moves a tiny bit. The next layer moves a little faster, and so on. Each layer drags backward on the layer above it, resisting its motion. The result is that the whole body of honey moves slowly, as though every layer is fighting a battle with its neighbours.

This internal friction between adjacent layers of a moving fluid is called **viscous drag**, and the property of the fluid that measures how strongly its layers resist sliding over one another is called **viscosity**.

Water has viscosity too, but it is about 10,000 times weaker than that of honey. As a result, the layers of water slide over each other almost effortlessly, and the water flows quickly.

The Velocity Profile

When a fluid flows through a pipe, it does not move as a single block. Instead, it moves in layers, each at a different speed. Understanding this layered structure is essential before we can write any equation.

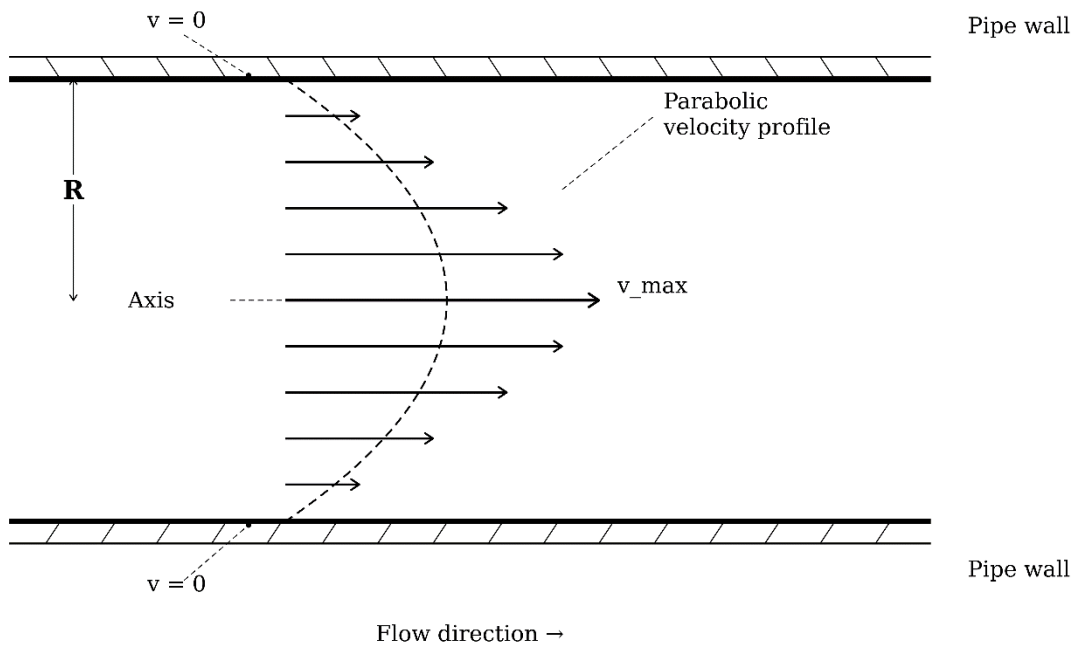


Figure: Velocity profile of a viscous fluid flowing through a pipe of radius R . The velocity is zero at the pipe wall (no-slip condition) and maximum at the centre. The arrows represent the velocity at each layer. The dashed curve connecting the arrow tips traces a parabola.

Here is what happens, layer by layer:

At the pipe wall: The fluid layer in direct contact with the wall has **zero velocity**. It is stuck to the wall and does not move at all. This is called the **no-slip condition**, and it is one of the most important facts in fluid mechanics. It holds for all real fluids on all solid surfaces.

At the centre: The fluid layer at the very centre of the pipe has the **maximum velocity**. No wall is nearby to slow it down.

Between the wall and the centre: Every layer has an intermediate velocity, increasing smoothly from zero at the wall to maximum at the centre.

This smooth variation of velocity from zero at the wall to maximum at the centre is called the **velocity profile**. For laminar flow in a circular pipe, the velocity profile has the shape of a **parabola**. The velocity at a distance r from the centre of a pipe of radius R is:

$$v(r) = v_{\max} \left(1 - \frac{r^2}{R^2} \right)$$

At the centre ($r = 0$), $v = v_{\max}$. At the wall ($r = R$), $v = 0$. The parabolic shape is a direct consequence of the viscous forces between the layers.

Newton's Law of Viscosity

Now we make the concept quantitative. We want to answer the question: *how large is the viscous force between two adjacent layers of fluid, and what does it depend on?*

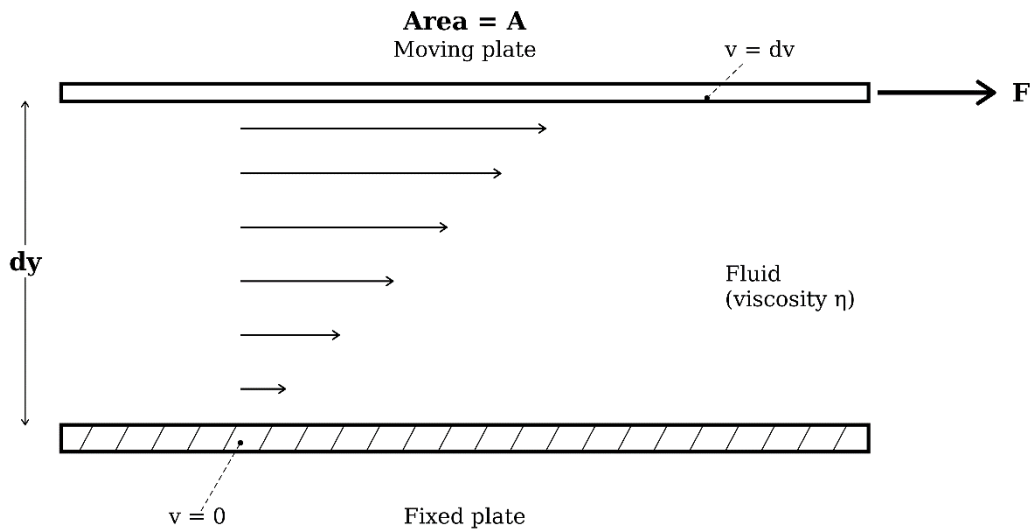


Figure: The setup for Newton's law of viscosity. A fluid of viscosity η fills the gap of thickness dy between a fixed plate (bottom, $v = 0$) and a moving plate (top, $v = dv$, area A). A force F is required to maintain the motion. The fluid layers between the plates have velocities increasing linearly from zero at the fixed plate to dv at the moving plate.

Consider two adjacent layers of fluid, each of area A , separated by a small perpendicular distance dy . The upper layer moves at velocity $v + dv$ and the lower layer moves at velocity v . The difference in their velocities is dv .

Defining the velocity gradient

The quantity that describes how rapidly the velocity changes as you move perpendicular to the flow is:

$$\frac{dv}{dy}$$

This is called the **velocity gradient** (or **rate of shear strain**). It tells you how fast the layers are sliding over one another. A large velocity gradient means the layers are sliding quickly; a small velocity gradient means they are sliding slowly. Its unit is $(\text{ms}^{-1})/\text{m} = \text{s}^{-1}$.

Identifying what the viscous force depends on

Newton observed experimentally that the viscous force F between two adjacent layers depends on two factors:

- 1) The **area of contact** A between the layers. If you double the area, you double the friction, just as a wider brake pad produces more friction than a narrow one. Therefore:

$$F \propto A$$

- 2) The **velocity gradient** $\frac{dv}{dy}$. If the layers are sliding faster over each other (larger $\frac{dv}{dy}$), the friction is greater. If they are sliding slowly (smaller $\frac{dv}{dy}$), the friction is weaker. Therefore:

$$F \propto \frac{dv}{dy}$$

Combining the proportionalities

Since F is proportional to both A and $\frac{dv}{dy}$:

$$F \propto A \times \frac{dv}{dy}$$

Introducing the constant of proportionality

To convert the proportionality into an equation, we introduce a constant η (the Greek letter eta) that depends on the nature of the fluid:

$$F = \eta A \frac{dv}{dy}$$

This equation is **Newton's law of viscosity**. It can be stated in words as follows:

The viscous force F between two adjacent layers of a fluid is directly proportional to the area of contact A between the layers and to the velocity gradient dv/dy perpendicular to the direction of flow.

The constant of proportionality η is called the **coefficient of viscosity** of the fluid.

Let us now examine each quantity in the equation carefully.

- F is the viscous force (in N) between two adjacent layers. It acts tangentially, parallel to the layers, opposing the relative motion between them.
- A is the area of contact between the layers (in m^2).
- dv/dy is the velocity gradient (in s^{-1}).
- η is the coefficient of viscosity of the fluid. It is a measure of how strongly the fluid resists the sliding of its layers.

Deriving the Units of the Coefficient of Viscosity

The unit of η can be derived from Newton's law by making η the subject of the formula.

Starting from:

$$F = \eta A \frac{dv}{dy}$$

Rearranging to isolate η :

$$\eta = \frac{F}{A \times \frac{dv}{dy}}$$

Now substituting the SI units of each quantity:

$$[\eta] = \frac{[F]}{[A] \times [dv/dy]}$$

$$[\eta] = \frac{N}{m^2 \times s^{-1}}$$

$$[\eta] = \frac{N}{m^2} \times s$$

$$[\eta] = N \cdot s \cdot m^{-2} = Nsm^{-2}$$

This can also be written as $Pa \cdot s$ (pascal-second), since $1Pa = 1Nm^{-2}$, or as $kgm^{-1}s^{-1}$, since $1N = 1kgms^{-2}$:

$$[\eta] = kgms^{-2} \times s \times m^{-2} = kgm^{-1}s^{-1}$$

In the older CGS system, the unit of viscosity is the **poise** (P), named after Jean-Léonard-Marie Poiseuille:

$$1 \text{ poise} = 1 \text{ dyne} \cdot s \cdot cm^{-2} = 0.1 Nsm^{-2}$$

The commonly used subunit is the **centipoise** (cP): $1 \text{ cP} = 10^{-3} Nsm^{-2}$.

Representative values at room temperature: Water has $\eta \approx 1.0 \times 10^{-3} Nsm^{-2}$ (about 1 centipoise), blood has $\eta \approx 3 \times 10^{-3} Nsm^{-2}$, glycerine has $\eta \approx 1.5 Nsm^{-2}$, and honey has $\eta \approx 2$ to $10 Nsm^{-2}$ depending on temperature. Air has $\eta \approx 1.8 \times 10^{-5} Nsm^{-2}$, which is roughly 50 times less viscous than water.

Newtonian and Non-Newtonian Fluids

Fluids that obey Newton's law of viscosity (where η is constant at a given temperature and does not depend on the velocity gradient) are called **Newtonian fluids**. Water, air, glycerine, and most simple liquids are Newtonian.

Fluids whose viscosity changes with the velocity gradient are called **Non-Newtonian fluids**. Blood, paint, toothpaste, and ketchup are non-Newtonian. Ketchup, for example, becomes less viscous when you shake

the bottle (this is called **shear thinning**), which is why tapping the bottle makes it pour more easily. Cornstarch mixed with water becomes more viscous when you stir it hard (this is called **shear thickening**), which is why you can run across a pool of cornstarch mixture but sink if you stand still.

For all problems in this chapter, we deal only with Newtonian fluids.

Factors Affecting Viscosity

The viscosity of a fluid is not a fixed universal constant. It depends on the conditions, and understanding these factors is essential for solving problems and interpreting experiments.

1. Temperature (the dominant factor)

For liquids, viscosity **decreases** with increasing temperature. As the temperature rises, molecules move faster and the intermolecular forces that cause the internal friction are more easily broken. Consequently, the layers slide over each other more easily, and the fluid flows more freely. This is why honey flows much faster when warmed, and why engine oil is thinner on a hot day than on a cold morning.

For gases, the opposite is true: viscosity **increases** with increasing temperature. In a gas, viscosity arises not from intermolecular forces (which are negligible) but from the transfer of momentum between layers by molecules bouncing back and forth. At higher temperatures, molecules move faster and transfer more momentum between layers, increasing the friction. Hence, hot air is slightly more viscous than cold air.

2. Nature of the fluid

Different fluids have vastly different viscosities at the same temperature. Liquids with strong hydrogen bonds (like glycerine) or long-chain molecules (like engine oil) have high viscosity because their molecules cling to each other and resist sliding. Liquids with weak intermolecular forces (like petrol or acetone) have low viscosity.

3. Pressure (usually negligible for liquids)

For most liquids, viscosity is nearly independent of pressure under normal conditions. For gases, viscosity is essentially independent of pressure at moderate pressures but may change at very high pressures.

The theory has been laid out with care. Now it is time to see whether it holds up under the pressure of a well-crafted problem. The following examples will test your ability to apply Newton's law of viscosity, and the last one will demand that you derive, integrate, and calculate all in one go.

BINDER Example 32

A flat glass plate of area 0.05m^2 is pulled with a constant velocity of 0.5ms^{-1} over a layer of glycerine 2mm thick on a flat surface. The coefficient of viscosity of glycerine is 1.5Nsm^{-2} . Calculate the force required to maintain the motion.

Solution

Given: $A = 0.05\text{m}^2$, $v = 0.5\text{ms}^{-1}$, $dy = 2\text{mm} = 2 \times 10^{-3}\text{m}$, $\eta = 1.5\text{Nsm}^{-2}$

The bottom layer of glycerine is in contact with the fixed flat surface, so its velocity is zero (no-slip condition). The top layer is in contact with the moving glass plate, so its velocity equals the plate velocity: 0.5ms^{-1} . The velocity gradient is therefore:

$$\frac{dv}{dy} = \frac{v_{\text{top}} - v_{\text{bottom}}}{dy} = \frac{0.5\text{ms}^{-1} - 0}{2 \times 10^{-3}\text{m}} = 250\text{s}^{-1}$$

Applying Newton's law of viscosity:

$$F = \eta A \frac{dv}{dy}$$

$$F = 1.5\text{Nsm}^{-2} \times 0.05\text{m}^2 \times 250\text{s}^{-1} = 18.75\text{N}$$

Therefore, the force required to pull the glass plate at constant velocity is 18.75N.

Making Sense of the Answer: Nearly 19 newtons to slide a plate over a 2mm layer of glycerine. That is roughly the weight of a 2kg bag of sugar. Glycerine is a highly viscous liquid ($\eta = 1.5\text{Nsm}^{-2}$), which is why the force is so substantial. If the glycerine were replaced by water ($\eta = 10^{-3}\text{Nsm}^{-2}$), the required force would be 1500 times smaller: about 0.013N, the weight of a single coin.

Think Like a Physicist: The velocity gradient dv/dy is the engine of this calculation. If the glycerine layer were thicker (say 4mm instead of 2mm), the velocity gradient would halve, and so would the force. The thicker the fluid layer, the more gently the velocity changes from zero to v , and the weaker the viscous drag. This is **why thick layers of lubricant reduce friction more effectively than thin layers**.

REAL Example 33

On a cold morning in Moshi, Kipanga tries to pour cooking oil from a bottle. The oil barely moves. His mother heats the bottle gently in warm water for a minute, and the oil pours freely.

Explain why warming the oil makes it flow more easily.

Solution

The viscosity of a liquid decreases with increasing temperature. At low temperature, the molecules of the cooking oil have relatively low kinetic energy. The intermolecular attractive forces between the oil molecules are strong enough to resist the sliding of adjacent layers over one another. Consequently, the viscous drag between layers is large, the velocity gradient is small for a given applied force, and the oil flows slowly.

When the bottle is warmed, the temperature of the oil rises. The oil molecules gain kinetic energy and move more vigorously. As a result, the intermolecular forces are more easily broken, the resistance between adjacent layers decreases, and the coefficient of viscosity η drops. The same gravitational force now produces a larger velocity gradient, and the oil flows faster.

Therefore, warming the oil reduces its viscosity, which reduces the internal friction between layers, which allows the oil to flow more freely under the same gravitational force.

Making Sense of the Answer: This is **why mechanics warm engine oil before draining it from a car**: warm oil flows out quickly; cold oil clings stubbornly to the engine walls. It is also **why the chocolate industry controls temperature precisely during manufacturing**: at the wrong temperature, chocolate flows too slowly to fill moulds, or too quickly to hold its shape.

Think Like a Physicist: For liquids, higher temperature means lower viscosity. For gases, higher temperature means higher viscosity. The mechanisms are completely different. In liquids, viscosity comes from intermolecular forces (which weaken at higher temperature). In gases, viscosity comes from molecular momentum transfer (which increases at higher temperature). Always state which mechanism applies when explaining a temperature effect on viscosity.

HOT Example 34

A flat circular disc of radius 15cm rotates on a thin film of oil of thickness 1.5mm placed between the disc and a flat surface. The coefficient of viscosity of the oil is 0.80 Nsm^{-2} . If the torque required to maintain a steady rotation of 60 revolutions per minute is τ , show that:

$$\tau = \frac{\pi\eta\omega R^4}{2d}$$

and calculate τ .

Solution

The disc rotates on the oil film. The oil layer touching the fixed surface has zero velocity (no-slip condition). The oil layer touching the rotating disc moves with the disc. Between these two layers, the velocity varies linearly from zero at the bottom to $v = \omega r$ at the top (at radial distance r from the centre). We derive the torque by considering an infinitesimal annular ring and integrating.

Defining the annular ring

Consider a thin annular ring on the disc at radius r from the centre, with width dr . The area of this ring is:

$$dA = 2\pi r dr$$

Finding the velocity at radius r

At radius r , the linear velocity of the disc surface is:

$$v = \omega r$$

where ω is the angular velocity of the disc.

Finding the velocity gradient across the oil film

The oil beneath this ring has the disc moving at $v = \omega r$ on top and the fixed surface (velocity = 0) on the bottom, separated by oil film thickness d . The velocity gradient is:

$$\frac{dv}{dy} = \frac{\omega r - 0}{d} = \frac{\omega r}{d}$$

Applying Newton's law of viscosity to find the force on the ring

The viscous force on this annular ring is:

$$dF = \eta \times dA \times \frac{dv}{dy}$$

Substituting the expressions for dA and $\frac{dv}{dy}$ from above:

$$dF = \eta \times 2\pi r dr \times \frac{\omega r}{d}$$

Rearranging:

$$dF = \frac{2\pi\eta\omega}{d} \times r^2 dr$$

Finding the torque from this ring

This viscous force acts tangentially at radius r . The torque (moment of force) contributed by this ring about the axis of rotation is:

$$\begin{aligned} d\tau &= r \times dF \\ d\tau &= r \times \frac{2\pi\eta\omega}{d} \times r^2 dr \\ d\tau &= \frac{2\pi\eta\omega}{d} \times r^3 dr \end{aligned}$$

Integrating over the entire disc

The total torque is obtained by integrating from $r = 0$ (centre) to $r = R$ (edge):

$$\tau = \int_0^R \frac{2\pi\eta\omega}{d} r^3 dr$$

The constants $\frac{2\pi\eta\omega}{d}$ come outside the integral:

$$\tau = \frac{2\pi\eta\omega}{d} \int_0^R r^3 dr = \frac{2\pi\eta\omega}{d} \left[\frac{r^4}{4} \right]_0^R = \frac{2\pi\eta\omega}{d} \left(\frac{R^4}{4} - \frac{0^4}{4} \right) = \frac{\pi\eta\omega R^4}{2d}$$

Hence:

$$\tau = \frac{\pi\eta\omega R^4}{2d}$$

Numerical calculation:

$$\omega = 2\pi \times \frac{60}{60} = 2\pi \text{ rad s}^{-1} = 6.283 \text{ rad s}^{-1}$$

Substituting values:

$$\tau = \frac{\pi \times 0.80 \text{ Nsm}^{-2} \times 6.283 \text{ rad s}^{-1} \times (0.15 \text{ m})^4}{2 \times 1.5 \times 10^{-3} \text{ m}} = 2.665 \text{ Nm}$$

Therefore, the torque required to maintain the steady rotation is approximately 2.7 Nm.

Making Sense of the Answer: A torque of 2.7 Nm to spin a disc on a thin oil film. For comparison, the torque needed to open a typical door handle is about 1 to 2 Nm. The oil film is only 1.5 mm thick, which

creates a steep velocity gradient and a substantial viscous resistance. A thicker oil film would reduce the torque; a more viscous oil would increase it.

Think Like a Physicist: This derivation follows a beautiful pattern that appears throughout physics: divide the object into infinitesimal elements (here, annular rings), write the expression for each element, and integrate over the whole object. You met this exact technique with moment of inertia in rotational mechanics. The integration step is where calculus earns its keep in physics.

That completes the formal treatment of viscosity. But viscosity does more than drag on plates and discs. When it acts inside a cylindrical pipe, it controls the rate at which a viscous fluid flows through the pipe. The equation that describes this is Poiseuille's formula, and it is waiting in the next subtopic. After that, we meet Stokes' law, which describes the viscous drag on a sphere falling through a fluid, and which leads to one of the most elegant ideas in fluid mechanics: **terminal velocity**.

POISEUILLE'S FORMULA

In the previous subtopic, we learned that viscosity is the internal friction that resists the flow of a fluid. We derived Newton's law of viscosity, which tells us the force between two adjacent layers. But a natural question follows: *if you know the viscosity of a fluid and the dimensions of the pipe, can you predict how much fluid will flow through the pipe per second?*

The answer is yes, and the equation that does this is called **Poiseuille's formula** (sometimes called the Hagen-Poiseuille equation). It is one of the most practically useful results in fluid mechanics, and it connects the flow rate to the pipe radius, the pressure difference driving the flow, the viscosity of the fluid, and the length of the pipe. Doctors use it to understand blood flow. Engineers use it to design pipelines. Chemists use it to measure viscosity. By the end of this section, you will be able to do all three.

Setting up the problem

Consider a viscous, incompressible fluid flowing steadily through a horizontal cylindrical pipe of internal radius R and length L . The pressure at the entry end is P_1 and at the exit end is P_2 , with $P_1 > P_2$. The pressure difference $\Delta P = P_1 - P_2$ is what drives the flow.

We want to find the total volume of fluid that passes through the pipe per unit time (the volume flow rate Q). To do this, we will consider a thin cylindrical shell of fluid inside the pipe, find the velocity of that shell, and then integrate over all shells from the centre to the wall.

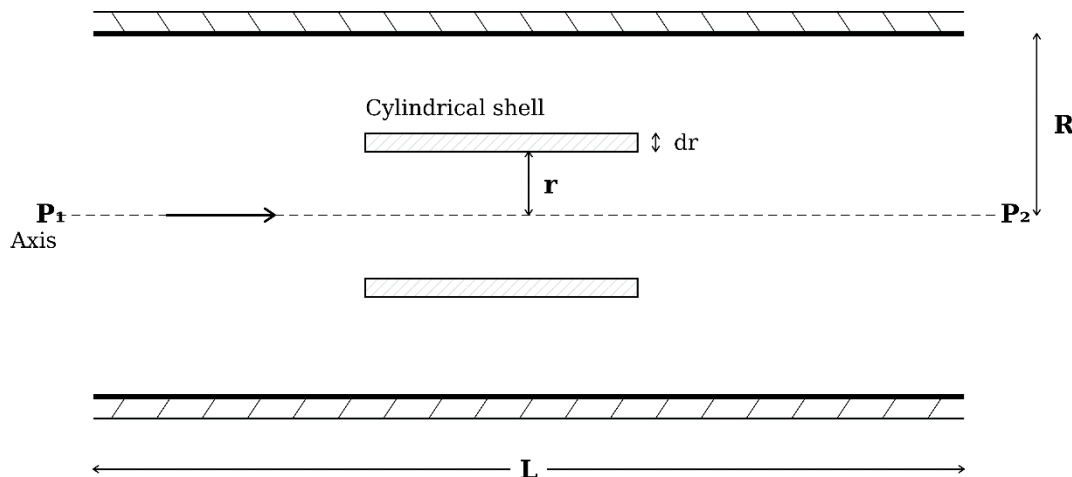


Figure: Cross-section (side view) of a horizontal cylindrical pipe of radius R and length L . The fluid flows from left (pressure P_1) to right (pressure P_2). A thin cylindrical shell of fluid at distance r from the axis, with thickness dr , is highlighted. The velocity profile is parabolic: maximum at the axis, zero at the wall.

Deriving the velocity profile

Identifying the cylindrical shell

Inside the pipe, consider a thin cylindrical shell of fluid at distance r from the central axis, with thickness dr and length L . This shell is a hollow cylinder of inner radius r and outer radius $r + dr$.

Writing the driving force on the shell

The pressure difference $\Delta P = P_1 - P_2$ acts on the cross-sectional area πr^2 of the fluid cylinder of radius r . The net forward force due to the pressure difference on this inner cylinder is:

$$F_{\text{pressure}} = (P_1 - P_2) \times \pi r^2 = \Delta P \times \pi r^2$$

Writing the viscous (retarding) force on the shell

The viscous force acts on the curved surface of this cylinder. By Newton's law of viscosity, the viscous force on a surface of area A is $F = \eta A \left(\frac{dv}{dy} \right)$. For a cylindrical surface at radius r , the area is $A = 2\pi rL$, and the velocity gradient in the radial direction is $\frac{dv}{dr}$ (note: $\frac{dv}{dr}$ is negative because v decreases as r increases from the centre to the wall). The viscous retarding force is:

$$F_{\text{viscous}} = -\eta \times 2\pi rL \times \frac{dv}{dr}$$

The negative sign ensures that F_{viscous} is positive (opposing the flow), since $\frac{dv}{dr}$ is negative.

Applying the condition for steady flow

For steady (non-accelerating) flow, the driving force equals the retarding force:

$$\begin{aligned} F_{\text{pressure}} &= F_{\text{viscous}} \\ \Delta P \times \pi r^2 &= -\eta \times 2\pi rL \times \frac{dv}{dr} \end{aligned}$$

Simplifying and separating variables

Dividing both sides by πr :

$$\Delta P \times r = -\eta \times 2L \times \frac{dv}{dr}$$

Rearranging to isolate dv :

$$dv = -\frac{\Delta P}{2\eta L} \times r \, dr$$

Integrating to find the velocity at radius r

Integrating both sides. The left side integrates from some velocity v (at radius r) to 0 (at the wall, $r = R$, where the no-slip condition gives $v = 0$). The right side integrates from r to R :

$$\begin{aligned} \int_v^0 dv &= -\frac{\Delta P}{2\eta L} \int_r^R r \, dr \\ 0 - v &= -\frac{\Delta P}{2\eta L} \left[\frac{r^2}{2} \right]_r^R = -\frac{\Delta P}{2\eta L} \times \frac{R^2 - r^2}{2} = -\frac{\Delta P(R^2 - r^2)}{4\eta L} \\ -v &= -\frac{\Delta P(R^2 - r^2)}{4\eta L} \\ v(r) &= \frac{\Delta P}{4\eta L} (R^2 - r^2) \end{aligned}$$

This is the **velocity profile** for laminar flow in a cylindrical pipe. It confirms that the profile is parabolic as v depends on r^2 . At the centre ($r = 0$), the velocity is maximum:

$$v_{\text{max}} = \frac{\Delta P R^2}{4\eta L}$$

At the wall ($r = R$), the velocity is zero, as required by the no-slip condition.

Deriving the volume flow rate (Poiseuille's formula)

Now we calculate the total volume of fluid flowing through the pipe per unit time by integrating the velocity over the entire cross-section.

Finding the volume flow rate through the thin shell

The thin cylindrical shell at radius r with thickness dr has cross-sectional area:

$$dA = 2\pi r dr$$

The volume of fluid passing through this shell per unit time is:

$$dQ = v(r) \times dA = \frac{\Delta P}{4\eta L} (R^2 - r^2) \times 2\pi r dr$$

$$dQ = \frac{\pi \Delta P}{2\eta L} (R^2 r - r^3) dr$$

Integrating over the entire cross-section

The total volume flow rate is obtained by integrating from $r = 0$ to $r = R$:

$$Q = \int_0^R \frac{\pi \Delta P}{2\eta L} (R^2 r - r^3) dr$$

Taking the constants outside the integral and integrating the rest:

$$Q = \frac{\pi \Delta P}{2\eta L} \int_0^R (R^2 r - r^3) dr = \frac{\pi \Delta P}{2\eta L} \left(R^2 \left[\frac{r^2}{2} \right]_0^R - \left[\frac{r^4}{4} \right]_0^R \right) = \frac{\pi \Delta P}{2\eta L} \left(\frac{R^4}{2} - \frac{R^4}{4} \right) = \frac{\pi \Delta P}{2\eta L} \times \frac{R^4}{4}$$

Hence:

$$Q = \frac{\pi \Delta P R^4}{8\eta L}$$

This is **Poiseuille's formula**. It can be stated in words as follows:

The volume flow rate Q of a viscous, incompressible fluid in steady laminar flow through a horizontal cylindrical pipe of radius R and length L is directly proportional to the pressure difference ΔP and to the fourth power of the radius, and inversely proportional to the viscosity η and the length of the pipe.

Physical meaning and consequences

Poiseuille's formula deserves careful study. Every factor in the equation tells a physical story.

1. **The R^4 dependence is extraordinary.** The flow rate depends on the **fourth power** of the radius. This means that a small change in radius produces a dramatic change in flow rate. If you double the radius of the pipe, the flow rate increases by a factor of $2^4 = 16$. If you halve the radius, the flow rate drops to $1/16$ of its original value. This is why even a slight narrowing of a blood vessel (as in atherosclerosis) causes a devastating reduction in blood flow.
2. **The dependence on ΔP is linear.** Double the pressure difference, and the flow rate doubles. This is the simplest factor and the most intuitive: push harder, get more flow.
3. **The dependence on η is inverse.** A more viscous fluid flows more slowly through the same pipe under the same pressure. This is why honey drips slowly from a tap while water gushes.
4. **The dependence on L is inverse.** A longer pipe means more viscous resistance, so less flow. Double the length, halve the flow rate.

Average velocity: The average velocity of the fluid across the entire cross-section can be found by dividing the volume flow rate by the cross-sectional area:

$$v_{\text{avg}} = \frac{Q}{\pi R^2} = \frac{\Delta P R^2}{8\eta L}$$

Comparing this with the maximum velocity at the centre:

$$v_{\text{max}} = \frac{\Delta P R^2}{4\eta L} = 2 \times v_{\text{avg}}$$

Therefore, in Poiseuille flow, the maximum velocity is exactly **twice** the average velocity.

Exponential Decay of Fluid Level

Here is a problem that combines Poiseuille's formula with calculus in a way that produces one of the most elegant results in the entire chapter. It is worth savouring.

A cylindrical tank of cross-sectional area A_T is filled with liquid to an initial height h_0 . The liquid drains out slowly through a long, narrow horizontal tube (a capillary) of radius R and length L attached at the base of the tank. The liquid has viscosity η . We want to find how the liquid level h changes with time.

This is different from the Torricelli emptying problem we solved earlier. In that problem, the orifice was short and wide, so viscosity was negligible and the exit speed depended only on height (Torricelli's theorem). Here, the tube is long and narrow, so viscous resistance dominates and the flow rate is governed by Poiseuille's formula.

Derivation

Writing the flow rate through the capillary

At any instant, the liquid in the tank stands at height h above the tube. The pressure difference driving the flow through the capillary is the hydrostatic pressure at the base of the tank:

$$\Delta P = \rho gh$$

By Poiseuille's formula, the volume flow rate through the capillary is:

$$Q = \frac{\pi \Delta P R^4}{8 \eta L} = \frac{\pi \rho gh R^4}{8 \eta L}$$

Relating the flow rate to the rate of change of height

The volume of liquid leaving the tank per unit time equals the rate at which the liquid level drops multiplied by the tank area:

$$Q = -A_T \frac{dh}{dt}$$

The negative sign appears because h is decreasing as liquid drains out.

Equating the two expressions for Q

$$-A_T \frac{dh}{dt} = \frac{\pi \rho g R^4}{8 \eta L} \times h$$

Separating variables

$$\frac{dh}{h} = -\frac{\pi \rho g R^4}{8 \eta L A_T} dt$$

The group of constants on the right is a single positive number. Let us call it c :

$$c = \frac{\pi \rho g R^4}{8 \eta L A_T}$$

So the equation becomes:

$$\frac{dh}{h} = -c dt$$

Integrating both sides

The left side is integrated from h_0 (at $t = 0$) to h (at time t). The right side is integrated from 0 to t :

$$\int_{h_0}^h \frac{dh'}{h'} = -c \int_0^t dt'$$

$$[\ln h']_{h_0}^h = -c[t']_0^t$$

$$\ln h - \ln h_0 = -ct$$

$$\ln\left(\frac{h}{h_0}\right) = -ct$$

$$\log_e\left(\frac{h}{h_0}\right) = -ct$$

$$h = h_0 e^{-ct}$$

where $c = \frac{\pi \rho g R^4}{8 \eta L A_T}$.

This is an **exponential decay**. The liquid level does not drop linearly, nor does it follow the square-root law we found for Torricelli emptying. Instead, it decays exponentially: the level drops quickly at first (when h is large and the driving pressure is high) and progressively more slowly as h approaches zero (when the driving pressure weakens).

In principle, the tank never fully empties: the exponential function approaches zero but never reaches it. In practice, the level becomes negligibly small after a few time constants ($t \approx \frac{3}{c}$ to $\frac{5}{c}$).

The quantity $\frac{1}{c}$ is the **time constant** of the system:

$$\tau = \frac{1}{c} = \frac{8 \eta L A_T}{\pi \rho g R^4}$$

After one time constant ($t = \tau$), the level has dropped to $\frac{h_0}{e} \approx 0.37h_0$, or about 37% of its initial value.

Why this matters: This result connects fluid mechanics to the same exponential decay that appears in radioactive decay, RC circuit discharge, and Newton's law of cooling. The mathematics is identical because the underlying structure is the same: the rate of decrease of a quantity is proportional to its current value. Whenever you see $\frac{dh}{dt} \propto -h$, the solution is always $h = h_0 e^{-ct}$.

Conditions for Poiseuille's Formula to Apply

Poiseuille's formula was derived under specific assumptions, and it is valid only when these are satisfied:

1. The flow must be laminar

Poiseuille's formula does not apply to turbulent flow. The Reynolds number must be below 2000 for Poiseuille's formula to be applicable. This means for any speed below critical velocity v_c , the Poiseuille's formula applies. For speeds above v_c , the flow is turbulent and Poiseuille's formula is no longer valid.

2. The fluid must be incompressible

The density must be constant throughout the pipe.

3. The fluid must be Newtonian

The viscosity η must be constant and independent of the velocity gradient.

4. The flow must be steady

The velocity at each point must not change with time.

5. The pipe must be rigid and cylindrical

Flexible or irregularly shaped pipes are not described by this formula.

6. The pipe must be long enough

Near the entrance of a pipe, the velocity profile is still developing and is not yet parabolic. Poiseuille's formula applies only in the fully developed region, far enough from the entrance. This entry length is typically about 50 to 100 pipe diameters.

It is time to put this powerful formula to work. The following examples range from straightforward substitution to clinical medicine, and each one reveals a different face of Poiseuille's equation.

BINDER Example 35

Water ($\eta = 1.0 \times 10^{-3} \text{ Nsm}^{-2}$) flows through a horizontal pipe of radius 5mm and length 2m under a pressure difference of 400Pa. Calculate:

- (a) The volume flow rate.
- (b) The velocity of the water at the centre of the pipe.
- (c) The average velocity of the water.

Solution

Given: $\eta = 1.0 \times 10^{-3} \text{ Nsm}^{-2}$, $R = 5\text{mm} = 5 \times 10^{-3}\text{m}$, $L = 2\text{m}$, $\Delta P = 400\text{Pa}$

(a) Applying Poiseuille's formula:

$$Q = \frac{\pi \Delta P R^4}{8 \eta L}$$

$$Q = \frac{\pi \times 400\text{Pa} \times (5 \times 10^{-3}\text{m})^4}{8 \times 1.0 \times 10^{-3} \text{ Nsm}^{-2} \times 2\text{m}} = 4.909 \times 10^{-5} \text{ m}^3 \text{ s}^{-1}$$

Therefore, the volume flow rate is $4.909 \times 10^{-5} \text{ m}^3 \text{ s}^{-1}$ (about 49 millilitres per second).

(b) The velocity at the centre of the pipe is the maximum velocity which is given by:

$$v_{\text{max}} = \frac{\Delta P R^2}{4 \eta L} = \frac{400\text{Pa} \times (5 \times 10^{-3}\text{m})^2}{4 \times 1.0 \times 10^{-3} \text{ Nsm}^{-2} \times 2\text{m}} = 1.25 \text{ ms}^{-1}$$

Therefore, the velocity at the centre of the pipe is 1.25 ms^{-1} .

(c) The average velocity is given by:

$$v_{\text{avg}} = \frac{v_{\text{max}}}{2} = \frac{1.25 \text{ ms}^{-1}}{2} = 0.625 \text{ ms}^{-1}$$

Alternatively, from the flow rate:

$$v_{\text{avg}} = \frac{Q}{\pi R^2} = \frac{4.91 \times 10^{-5} \text{ m}^3 \text{ s}^{-1}}{\pi \times (5 \times 10^{-3}\text{m})^2} = 0.625 \text{ ms}^{-1}$$

Both methods give the same answer, confirming the result.

Therefore, the average velocity is 0.625 ms^{-1} .

Making Sense of the Answer: The maximum velocity (1.25 ms^{-1}) is exactly twice the average (0.625 ms^{-1}), as Poiseuille's theory predicts. The flow rate of about 49 millilitres per second through a pipe only 1cm in diameter is substantial. If the radius were halved to 2.5mm (keeping everything else the same), the flow rate would drop by a factor of $2^4 = 16$, from 49 to about 3 millilitres per second. The fourth-power dependence is ruthless.

Think Like a Physicist: It is always helpful to verify your answer by confirming that $v_{\text{max}} = 2v_{\text{avg}}$. If this relationship does not hold, you have made an error somewhere.

BINDER Example 36

Two pipes of equal length are connected in series. Pipe A has radius R and pipe B has radius $2R$. If the same fluid flows through both pipes under steady conditions, find the ratio of the pressure drop across pipe A to the pressure drop across pipe B.

Solution

In series connection, the volume flow rate Q is the same through both pipes (by conservation of mass: what flows out of A flows into B).

From Poiseuille's formula, rearranged to give the pressure drop:

$$\Delta P = \frac{8 \eta L Q}{\pi R^4}$$

For pipe A (radius R):

$$\Delta P_A = \frac{8 \eta L Q}{\pi R^4}$$

For pipe B (radius $2R$):

$$\Delta P_B = \frac{8\eta LQ}{\pi(2R)^4} = \frac{8\eta LQ}{\pi \times 16R^4} = \frac{8\eta LQ}{16\pi R^4}$$

Taking the ratio:

$$\frac{\Delta P_A}{\Delta P_B} = \frac{\frac{8\eta LQ}{\pi R^4}}{\frac{8\eta LQ}{16\pi R^4}} = \frac{16\pi R^4}{\pi R^4} = 16$$

Therefore, the pressure drop across pipe A is **16 times** the pressure drop across pipe B.

Making Sense of the Answer: Pipe A is narrower, so it offers far more resistance to flow. The factor of 16 comes directly from the fourth-power dependence: $(2R/R)^4 = 16$. In a series system, the narrow pipe is the bottleneck. Most of the total pressure drop occurs across the narrowest section, just as most of the voltage drop in a series electrical circuit occurs across the largest resistor.

Think Like a Physicist: Poiseuille's formula has a beautiful analogy with Ohm's law in electricity. The pressure difference ΔP is analogous to voltage, the flow rate Q is analogous to current, and the quantity $8\eta L/(\pi R^4)$ plays the role of resistance. Pipes in series add their resistances; pipes in parallel share the flow. This analogy is widely used in medical physics to model the circulatory system.

REAL Example 37

Kipute reads in a biology textbook that atherosclerosis (the buildup of fatty plaque inside arteries) can reduce the internal radius of an artery by 25%. She calculates the effect on blood flow using Poiseuille's formula and is shocked by the result.

Assuming all other factors remain constant, what percentage decrease in blood flow did Kipute find, and why was it so surprising?

Solution

Let the original radius be R and the reduced radius be $R' = 0.75R$ (a 25% reduction).

By Poiseuille's formula, $Q \propto R^4$ when all other factors are constant. The ratio of the new flow rate to the original is:

$$\frac{Q'}{Q} = \frac{R'^4}{R^4} = \frac{(0.75R)^4}{R^4} = (0.75)^4 = 0.316$$

Therefore:

$$\text{Percentage decrease} = \frac{Q - Q'}{Q} \times 100\% = \left(\frac{Q - 0.316Q}{Q} \right) \times 100\% = 68.4\%$$

The percentage decrease in blood flow is 68.4%.

Reason: Kipute expected a moderate decrease since the radius only reduced by 25%. However, the reduction in blood flow much higher than 25% suggesting that the blood flow depends on the **fourth power of the radius**, not just the radius itself.

Making Sense of the Answer: A 25% narrowing of the artery causes a 68% drop in blood flow. This is the devastating power of the fourth-power law. A modest narrowing produces a catastrophic reduction in flow. This is precisely why atherosclerosis is so dangerous: by the time the patient notices symptoms (reduced blood supply to the heart or brain), the artery may have already lost more than half its flow capacity. Poiseuille's formula explains why cardiologists take even small blockages very seriously.

Think Like a Physicist: The R^4 dependence means that percentage changes in radius are amplified dramatically in the flow rate. A 10% reduction in radius gives $(0.90)^4 = 0.656$, a 34% drop in flow. A 50% reduction gives $(0.50)^4 = 0.0625$, a 94% drop. This is why the body has powerful mechanisms to regulate artery diameter: even tiny changes have enormous consequences.

HOT Example 38

A viscous liquid flows through a horizontal tube of length 50cm and internal radius 1mm under a pressure difference of $3 \times 10^4 \text{ Pa}$. The volume of liquid collected in 10 minutes is 500 cm^3 .

- Calculate the coefficient of viscosity of the liquid.
- Find the velocity of the liquid at a distance of 0.5mm from the axis.

(c) Find the velocity at the wall.

Solution

(a) Given: $L = 50\text{cm} = 0.50\text{m}$, $R = 1\text{mm} = 1.0 \times 10^{-3}\text{m}$, $\Delta P = 3 \times 10^4\text{Pa}$

Volume collected: $V = 500\text{cm}^3 = 500 \times 10^{-6}\text{m}^3 = 5.0 \times 10^{-4}\text{m}^3$

Time: $t = 10\text{min} = 600\text{s}$

Volume flow rate:

$$Q = \frac{V}{t} = \frac{5.0 \times 10^{-4}\text{m}^3}{600\text{s}} = 8.33 \times 10^{-7}\text{m}^3\text{s}^{-1}$$

Rearranging Poiseuille's formula to make η the subject:

$$Q = \frac{\pi \Delta P R^4}{8 \eta L}$$

$$\eta = \frac{\pi \Delta P R^4}{8 Q L}$$

Substituting:

$$\eta = \frac{\pi \times 3 \times 10^4\text{Pa} \times (1.0 \times 10^{-3}\text{m})^4}{8 \times 8.33 \times 10^{-7}\text{m}^3\text{s}^{-1} \times 0.50\text{m}} = 2.83 \times 10^{-2}\text{Nsm}^{-2}$$

Therefore, the coefficient of viscosity of the liquid is $2.83 \times 10^{-2}\text{Nsm}^{-2}$.

(b) Using the velocity profile formula:

$$v(r) = \frac{\Delta P}{4 \eta L} (R^2 - r^2)$$

With $r = 0.5\text{mm} = 5.0 \times 10^{-4}\text{m}$:

$$v = \frac{3 \times 10^4\text{Pa}}{4 \times 2.83 \times 10^{-2}\text{Nsm}^{-2} \times 0.50\text{m}} \times ((1.0 \times 10^{-3}\text{m})^2 - (5.0 \times 10^{-4}\text{m})^2) = 0.398\text{ms}^{-1}$$

Therefore, the velocity at 0.5mm from the axis is approximately 0.40ms^{-1} .

(c) At the wall, $r = R$:

$$v(R) = \frac{\Delta P}{4 \eta L} (R^2 - R^2) = \frac{\Delta P}{4 \eta L} \times 0 = 0$$

Therefore, the velocity at the wall is **zero**.

Making Sense of the Answer: The viscosity of about 0.0283Nsm^{-2} places this liquid between water (0.001Nsm^{-2}) and glycerine (1.5Nsm^{-2}); perhaps a light oil or a dilute sugar solution. The velocity at mid-radius (0.40ms^{-1}) is 75% of the maximum velocity. This makes sense from the parabolic profile: at half the radius, $r^2/R^2 = 0.25$, so $v = v_{\text{max}}(1 - 0.25) = 0.75 \times v_{\text{max}}$. And the velocity at the wall is exactly zero, as the no-slip condition demands.

Think Like a Physicist: Poiseuille's formula is frequently used in reverse: measure the flow rate, and calculate the viscosity. This is the principle behind the capillary viscometer, one of the standard laboratory instruments for measuring viscosity. The key experimental quantities are the volume collected, the time, the pressure difference, and the tube dimensions. The tube radius must be measured very carefully, because it enters the formula as R^4 : a 5% error in measuring R produces a 20% error in η .

That completes Poiseuille's formula. In the next subtopic, we turn to a different problem: *what happens when a solid object moves through a viscous fluid?* The answer is **Stokes' law**, and it leads to the concept of **terminal velocity**, where gravity and viscous drag reach a perfect, elegant balance.

STOKES' LAW AND TERMINAL VELOCITY

Drop a steel ball into a jar of honey. It sinks, but slowly, far more slowly than it would fall through air. Drop the same ball into water. It sinks faster, but still not as fast as it would in free fall. Now drop a tiny grain of sand into the same honey. It sinks so slowly that you can barely see it move.

In every case, the fluid pushes back against the falling object, resisting its motion. This resistance is viscous drag, and the equation that describes it for a sphere moving through a viscous fluid is called **Stokes' law**. It leads to one of the most elegant ideas in fluid mechanics: **terminal velocity**, the constant speed at which an object falls when the drag force exactly balances the net downward force.

Stokes' Law

Derivation

When a sphere moves through a viscous fluid, the fluid must part to let it through and then close up again behind it. This disturbs many layers of fluid near the sphere, and all of these layers exert viscous drag on the sphere. The total drag force F depends on three quantities:

- 1) The **coefficient of viscosity** η of the fluid; because a more viscous fluid exerts more drag.
- 2) The **radius** r of the sphere; because a larger sphere disturbs more fluid.
- 3) The **velocity** v of the sphere; because a faster sphere shears the fluid layers more rapidly.

We can write:

$$F = k \eta^a r^b v^c$$

where k is a dimensionless constant, and a , b , c are powers to be determined. We find these powers using **dimensional analysis**.

Writing the dimensions of each quantity:

$$[F] = \text{MLT}^{-2}, [\eta] = \text{ML}^{-1}\text{T}^{-1}, [r] = \text{L}, [v] = \text{LT}^{-1}$$

Substituting into the dimensional equation:

$$\begin{aligned} [F] &= [\eta]^a [r]^b [v]^c \\ \text{MLT}^{-2} &= (\text{ML}^{-1}\text{T}^{-1})^a \times \text{L}^b \times (\text{LT}^{-1})^c \\ \text{MLT}^{-2} &= \text{M}^a \text{L}^{-a} \text{T}^{-a} \times \text{L}^b \times \text{L}^c \text{T}^{-c} \\ \text{MLT}^{-2} &= \text{M}^a \times \text{L}^{-a+b+c} \times \text{T}^{-a-c} \end{aligned}$$

Equating the powers of M , L , and T on both sides:

For M : $1 = a$

For T : $-2 = -a - c$; $-2 = -1 - c$; $c = 1$

For L : $1 = -a + b + c$; $1 = -1 + b + 1$; $b = 1$

Substituting the powers back into the formula:

$$F = k \eta^1 r^1 v^1 = k \eta r v$$

Dimensional analysis tells us that the drag force is proportional to $\eta r v$, but it cannot determine the dimensionless constant k . The full mathematical solution (due to Sir George Gabriel Stokes, 1851, obtained by solving the fluid flow equations around a sphere) gives:

$$k = 6\pi$$

Therefore:

$$\mathbf{F} = 6\pi\eta r v$$

This is **Stokes' law**. It can be stated in words as follows:

The viscous drag force on a sphere moving slowly through a viscous fluid is directly proportional to the coefficient of viscosity of the fluid, the radius of the sphere, and the velocity of the sphere.

The force F acts in the direction opposite to the motion of the sphere: if the sphere is falling downward, the drag force acts upward.

Conditions for Stokes' law to apply

1. The sphere must be smooth and rigid

Rough or deformable objects experience different drag.

2. The flow must be laminar

Stokes' law applies only when the Reynolds number for the sphere is small (typically $Re < 1$). At higher speeds, the flow around the sphere becomes turbulent and the drag increases much faster than Stokes' law predicts.

3. The fluid must extend far in all directions

The walls of the container must be far enough from the sphere that they do not affect the flow pattern. If the sphere is falling through a narrow tube, wall effects modify the drag.

4. The sphere must not be accelerating rapidly

Stokes' law describes the drag in steady or quasi-steady motion.

Terminal Velocity

To understand terminal velocity, study the following figure:

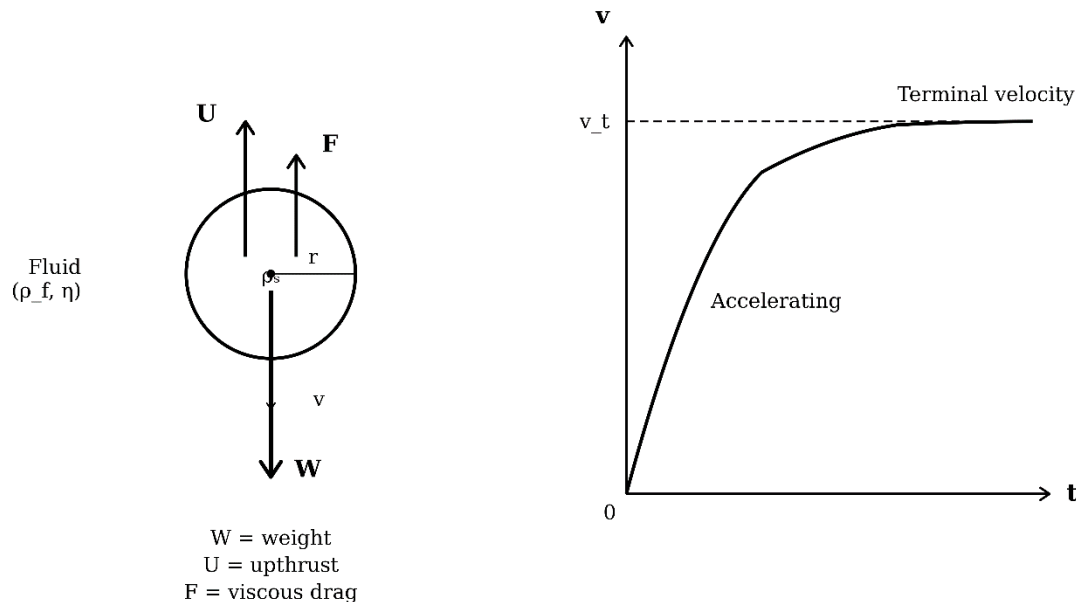


Figure: (Left) A sphere of radius r and density ρ_s falling through a viscous fluid of density ρ_f and viscosity η . Three forces act on the sphere: weight W (downward), upthrust U (upward), and viscous drag F (upward). At terminal velocity, $W = U + F$. **(Right)** The velocity-time graph for the falling sphere. The velocity increases from zero, with decreasing acceleration, and approaches the terminal velocity v_t asymptotically.

When a sphere is released from rest in a viscous fluid, three forces act on it as it falls:

1) Weight W , acting downward:

$$W = \frac{4}{3} \pi r^3 \rho_s g$$

where ρ_s is the density of the sphere.

2) Upthrust (buoyancy) U , acting upward. By Archimedes' principle, the upthrust equals the weight of fluid displaced:

$$U = \frac{4}{3} \pi r^3 \rho_f g$$

where ρ_f is the density of the fluid.

3) Viscous drag F , acting upward (opposing the downward motion). By Stokes' law:

$$F = 6\pi\eta r v$$

The net downward force at any instant is:

$$F_{\text{net}} = W - U - F = \frac{4}{3} \pi r^3 \rho_s g - \frac{4}{3} \pi r^3 \rho_f g - 6\pi\eta r v$$

$$F_{\text{net}} = \frac{4}{3} \pi r^3 (\rho_s - \rho_f) g - 6\pi\eta r v$$

Now watch what happens as the sphere falls:

At the moment of release ($v = 0$): The drag force is zero (because $v = 0$). The net downward force is $W - U = \frac{4}{3}\pi r^3(\rho_s - \rho_f)g$, which is the effective weight. The sphere accelerates downward.

As the sphere speeds up: The velocity v increases, so the drag force $F = 6\pi\eta rv$ increases. The net downward force decreases. The acceleration decreases.

At terminal velocity: The sphere reaches a speed v_t at which the drag force has grown large enough to exactly balance the effective weight. The net force is zero. The acceleration is zero. The sphere continues to fall at a constant velocity. This constant velocity is the **terminal velocity**.

Derivation of the terminal velocity formula:

At terminal velocity, $F_{\text{net}} = 0$:

$$W - U - F = 0$$

$$\frac{4}{3}\pi r^3 \rho_s g - \frac{4}{3}\pi r^3 \rho_f g - 6\pi\eta r v_t = 0$$

Rearranging to isolate the drag term:

$$6\pi\eta r v_t = \frac{4}{3}\pi r^3 (\rho_s - \rho_f)g$$

Dividing both sides by $6\pi\eta r$:

$$v_t = \frac{4\pi r^3 (\rho_s - \rho_f)g}{3 \times 6\pi\eta r} = \frac{4r^3 (\rho_s - \rho_f)g}{18\eta r} = \frac{4r^2 (\rho_s - \rho_f)g}{18\eta} = \frac{2r^2 (\rho_s - \rho_f)g}{9\eta}$$

Hence:

$$v_t = \frac{2r^2 (\rho_s - \rho_f)g}{9\eta}$$

This is the **terminal velocity formula** for a sphere falling through a viscous fluid.

Let us examine what it tells us.

1. **The terminal velocity is proportional to r^2 .** Larger spheres fall faster. A sphere of twice the radius has four times the terminal velocity. This is why large raindrops fall faster than small ones, and why fine sand settles much more slowly than coarse gravel in water.
2. **The terminal velocity is proportional to $(\rho_s - \rho_f)$.** The greater the density difference between the sphere and the fluid, the faster the sphere falls. A steel ball sinks quickly through water because the density difference is large. A plastic ball barely sinks because the density difference is small. If $\rho_s = \rho_f$, the terminal velocity is zero: the sphere is neutrally buoyant and does not sink at all.
3. **The terminal velocity is inversely proportional to η .** The more viscous the fluid, the slower the sphere falls. A steel ball sinks slowly through honey but quickly through water.

Measuring Viscosity Using Stokes' Law

The terminal velocity formula can be rearranged to measure the viscosity of a liquid:

$$\eta = \frac{2r^2 (\rho_s - \rho_f)g}{9v_t}$$

This is the basis of the **falling sphere viscometer**. A sphere of known radius r and density ρ_s is dropped into a tall column of the liquid whose viscosity is to be measured. The sphere is allowed to accelerate until it reaches terminal velocity. The terminal velocity v_t is then measured by timing the sphere as it falls through a known distance between two marks on the column (placed far enough from the top and bottom to ensure the sphere is at terminal velocity and not affected by the ends). The density ρ_f of the liquid is measured separately. The viscosity η is then calculated from the formula.

Sources of error include: the sphere may not be perfectly spherical (gives incorrect r), the sphere may not have reached terminal velocity when timing begins (gives v_t too high), wall effects from the container (use a container at least 10 times wider than the sphere), temperature variations during the experiment (viscosity changes with temperature), and air bubbles clinging to the sphere (changes effective density and radius).

Now the equations have been derived, the physics has been explained, and all three forces have been identified. What remains is to see whether these ideas can survive contact with real numbers. The following examples will test your understanding of Stokes' law and terminal velocity, and the last one carries a twist that separates those who have memorised the formula from those who have understood the physics.

BINDER Example 39

A steel ball of radius 2mm and density 7800kgm^{-3} is dropped into glycerine of density 1260kgm^{-3} and viscosity 1.5Nsm^{-2} . Calculate the terminal velocity of the ball. Take $g = 9.8\text{ms}^{-2}$.

Solution

Given: $r = 2\text{mm} = 2 \times 10^{-3}\text{m}$, $\rho_s = 7800\text{kgm}^{-3}$, $\rho_f = 1260\text{kgm}^{-3}$, $\eta = 1.5\text{Nsm}^{-2}$, $g = 9.8\text{ms}^{-2}$

Applying the terminal velocity formula:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

$$v_t = \frac{2 \times (2 \times 10^{-3}\text{m})^2 \times (7800 - 1260)\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 1.5\text{Nsm}^{-2}} = 0.038\text{ms}^{-1}$$

Therefore, the terminal velocity of the steel ball in glycerine is approximately 0.038ms^{-1} (or 3.8cms^{-1}).

Making Sense of the Answer: About 3.8cm per second, walking speed for an ant, not for a steel ball. Glycerine is extremely viscous ($\eta = 1.5\text{Nsm}^{-2}$, about 1500 times more viscous than water), so even a dense steel ball sinks very slowly through it. In water ($\eta = 10^{-3}\text{Nsm}^{-2}$), the same ball would have a terminal velocity about 1500 times larger (roughly 57ms^{-1}), which is far too fast for Stokes' law to apply (the flow would be turbulent). This illustrates an important point: Stokes' law works best for small spheres in highly viscous fluids.

Think Like a Physicist: Always check whether the Stokes' law conditions are satisfied. The Reynolds number for this ball is $R_e = \rho_f v_t (2r) / \eta = 1260 \times 0.038 \times 0.004 / 1.5 \approx 0.13$, which is well below 1. The flow is laminar, and Stokes' law is valid. If R_e came out much larger than 1, the answer would be unreliable.

BINDER Example 40

In a falling sphere viscometer experiment, a glass sphere of radius 1.5mm and density 2500kgm^{-3} is dropped into a tall column of oil of density 900kgm^{-3} . The sphere reaches terminal velocity and falls through 20cm in 4.2s. Calculate the coefficient of viscosity of the oil. Take $g = 9.8\text{ms}^{-2}$.

Solution

Given: $r = 1.5\text{mm} = 1.5 \times 10^{-3}\text{m}$, $\rho_s = 2500\text{kgm}^{-3}$, $\rho_f = 900\text{kgm}^{-3}$, $d = 20\text{cm} = 0.20\text{m}$, $t = 4.2\text{s}$

First, find the terminal velocity:

$$v_t = \frac{d}{t} = \frac{0.20\text{m}}{4.2\text{s}} = 0.0476\text{ms}^{-1}$$

Rearranging the terminal velocity formula to find η :

$$\eta = \frac{2r^2(\rho_s - \rho_f)g}{9v_t}$$

$$\eta = \frac{2 \times (1.5 \times 10^{-3}\text{m})^2 \times (2500 - 900)\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 0.0476\text{ms}^{-1}} = 0.165\text{Nsm}^{-2}$$

Therefore, the coefficient of viscosity of the oil is approximately 0.17Nsm^{-2} .

Making Sense of the Answer: A viscosity of 0.17Nsm^{-2} is typical of a moderately viscous oil; thicker than water (10^{-3}Nsm^{-2}) but much thinner than glycerine (1.5Nsm^{-2}). The terminal velocity of about 4.8cms^{-1} through 20cm in 4.2s is slow enough for the flow to be laminar, confirming that Stokes' law is applicable.

Think Like a Physicist: In the viscometer experiment, the timing must begin only after the sphere has reached terminal velocity. If you start timing too early (while the sphere is still accelerating), the measured velocity will be lower than the true terminal velocity, and the calculated viscosity will be too high. This is

why the two timing marks are placed well below the surface of the liquid, leaving a run-up distance for the sphere to reach terminal velocity.

REAL Example 41

During a rainy evening in Dar es Salaam, Kipanga and Kipute stand on the balcony watching the rain. Kipanga notices that some drops are tiny and seem to float down gently, while others are large and hit the ground hard. He asks Kipute: “If gravity is the same for all drops, why do the big ones fall so much faster?”

Kipute smiles. She has just finished studying Stokes’ law. What was her explanation?

Solution

Each raindrop reaches a terminal velocity at which the viscous drag from the air exactly balances its effective weight. By the terminal velocity formula:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

The density difference ($\rho_s - \rho_f$), the gravitational acceleration g , and the viscosity η of air are the same for all raindrops. The only factor that varies from drop to drop is the radius r .

Since the terminal velocity is proportional to r^2 , a drop with twice the radius has four times the terminal velocity.

The r^2 dependence of terminal velocity is explained by the fact that the weight of a sphere grows as r^3 , while the Stokes drag grows only as r ($\frac{r^3}{r} = r^2$), so larger spheres need a higher velocity before the drag can balance the weight.

Making Sense of the Answer: A fine mist drop ($r = 0.1\text{mm}$) has a terminal velocity of about 0.3ms^{-1} in air, gentle enough to feel like it is floating. A heavy raindrop ($r = 2\text{mm}$) has a terminal velocity of about 6 to 9ms^{-1} which is fast enough to sting your skin. The r^2 law explains the entire spectrum of raindrop behaviour, from hovering mist to pelting monsoon rain.

Think Like a Physicist: Strictly speaking, Stokes’ law applies only for very small Reynolds numbers ($Re < 1$). Large raindrops fall fast enough that the flow around them becomes turbulent, and the actual drag is higher than Stokes’ law predicts. In practice, this means large raindrops have slightly lower terminal velocities than the Stokes formula gives. Nevertheless, the qualitative conclusion that, bigger drops fall faster because v_t increases with r remains correct.

HOT Example 42

Two spheres, P and Q, are made of the same material and are dropped simultaneously into a large tank of viscous liquid. Sphere P has radius r and sphere Q has radius $3r$. Both spheres eventually reach terminal velocity.

- Find the ratio of the terminal velocity of Q to that of P.
- If the viscous drag on sphere P at terminal velocity is F_p , find the viscous drag on sphere Q at terminal velocity in terms of F_p .
- Sphere P reaches terminal velocity after falling 5cm. Estimate whether sphere Q reaches terminal velocity in a shorter or longer distance, and explain why.

Solution

(a) Both spheres are made of the same material, so ρ_s is the same. Both fall through the same fluid, so ρ_f and η are the same. By the terminal velocity formula:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

Since all factors except r are common:

$$v_t \propto r^2$$

$$\frac{v_Q}{v_P} = \frac{(3r)^2}{r^2} = \frac{9r^2}{r^2} = 9$$

Therefore, the terminal velocity of Q is **9 times** that of P.

(b) At terminal velocity, the drag force equals the effective weight (weight minus upthrust):

$$F_{\text{drag}} = \frac{4}{3}\pi r^3(\rho_s - \rho_f)g$$

This is proportional to r^3 :

$$\frac{F_Q}{F_P} = \frac{(3r)^3}{r^3} = \frac{27r^3}{r^3} = 27$$

Therefore, the viscous drag on Q at terminal velocity is $27F_P$.

(c) Sphere Q is heavier (27 times the effective weight of P) and has a higher terminal velocity (9 times that of P). It accelerates under a net force that starts at 27 times the effective weight of P and decreases as the drag builds up. Although the drag force grows as the sphere speeds up, Q must accelerate to a much higher terminal velocity ($9v_P$) before equilibrium is reached.

The distance needed to reach terminal velocity depends on the acceleration phase. A heavier, faster sphere needs more distance to reach its higher terminal velocity, (for the same reason that a heavy truck needs a longer distance to reach cruising speed than a bicycle).

Therefore, sphere Q reaches terminal velocity after falling a **longer distance** than sphere P.

Making Sense of the Answer: Part (a) gives the r^2 law for velocity. Part (b) gives the r^3 law for drag at terminal velocity. Part (c) requires physical reasoning beyond the formula: the question is not about the magnitude of the terminal velocity but about how long it takes to get there. This distinction separates formula-users from physics-thinkers.

Think Like a Physicist: Part (b) has a subtle trap. Students sometimes use Stokes' law directly ($F = 6\pi\eta r v$) and substitute the terminal velocities, getting $F_Q/F_P = (3r \times 9v_P)/(r \times v_P) = 27$. This gives the correct answer, but the cleaner approach is to recognise that at terminal velocity, drag equals effective weight, and effective weight is proportional to r^3 . Both methods must give the same answer, if they do not, you have made an error.

HOT Example 43

A thin square metal plate of side 10cm and mass 50g is released from rest between two large, fixed, vertical surfaces. The gap between each surface and the plate is 0.5mm, and each gap is filled with oil of viscosity 0.20Ns m^{-2} . The plate slides vertically downward under gravity.

Kipanga watches it descend and expects it to accelerate like any falling object. But after a very short time, the plate stops accelerating and falls at a constant speed. **What is this speed, and why does the plate reach a constant velocity?**

Solution

The plate falls under gravity and is resisted by viscous drag from the oil on both sides. After a brief initial acceleration, the plate reaches a constant velocity (terminal velocity) when the net force on it is zero.

Three forces act on the plate:

Weight (downward): $W = mg = 0.050\text{kg} \times 9.8\text{ms}^{-2} = 0.49\text{N}$

Viscous drag from the left oil layer (upward): By Newton's law of viscosity, the drag force from one oil layer is $F_1 = \eta A \frac{v}{d}$, where A is the area of the plate, v is the plate velocity, and d is the gap thickness. The velocity gradient is $\frac{v}{d}$ because the plate moves at v and the fixed wall is at rest.

Viscous drag from the right oil layer (upward): $F_2 = \eta A \frac{v}{d}$

Since there are two oil layers (one on each side of the plate), the total viscous drag is:

$$F_{\text{total}} = 2\eta A \frac{v}{d}$$

At terminal velocity, the total drag equals the weight:

$$W = 2\eta A \frac{v_t}{d}$$

Rearranging for v_t :

$$v_t = \frac{Wd}{2\eta A}$$

Substituting values:

$$A = (0.10\text{m})^2 = 0.01\text{m}^2, d = 0.5\text{mm} = 5 \times 10^{-4}\text{m}, \eta = 0.20\text{Nsm}^{-2}$$

$$v_t = \frac{0.49\text{N} \times 5 \times 10^{-4}\text{m}}{2 \times 0.20\text{Nsm}^{-2} \times 0.01\text{m}^2}$$

$$v_t = \frac{2.45 \times 10^{-4}}{4.0 \times 10^{-3}} = 0.061\text{ms}^{-1}$$

Therefore, the terminal velocity of the plate is approximately 0.061ms^{-1} (about 6.1cms^{-1}).

The plate reaches a constant velocity because the viscous drag force is proportional to velocity ($F \propto v$). As the plate accelerates from rest, v increases, so the drag increases. The acceleration decreases. Eventually, the drag grows large enough to exactly balance the weight. At that point, the net force is zero, the acceleration is zero, and the plate falls at constant terminal velocity.

Making Sense of the Answer: About 6cm per second through a 0.5mm oil gap. This is extremely slow for a falling object, which shows how effectively a thin layer of viscous oil resists motion. If the gaps were doubled to 1mm, the velocity gradient would halve, the drag would halve, and the terminal velocity would double to about 12cms^{-1} . This is the same physics behind viscous dampers used in door closers and shock absorbers.

Think Like a Physicist: This problem is the flat-plate version of Stokes' terminal velocity for a sphere. The logic is identical: identify all forces, set the net force to zero, solve for velocity. The factor of 2 in the drag comes from having oil on both sides of the plate. Missing this factor of 2 is the most common error in this type of problem.

That completes the treatment of Stokes' law and terminal velocity. With this subtopic, the three hidden personalities of fluids: surface tension, the pressure-speed trade-off of Bernoulli, and viscosity have all been revealed. Let us now assemble the key applications.

APPLICATIONS OF FLUID MECHANICS

If you have made it this far, you have earned something remarkable: you now understand the three hidden personalities of fluids. You know that surface tension makes drops spherical, needles float, and liquids climb up narrow tubes. You know that Bernoulli's equation trades speed for pressure, lifting aircraft, emptying tanks, and attacking shower curtains. You know that viscosity makes honey crawl, blood flow obey Poiseuille, and falling spheres reach terminal velocity.

This section gathers all the major applications in one place. It is not a new theory section. It is a concise, organised reference that pulls together everything you have already learned, application by application. These applications appear frequently in examinations, and having them assembled in one place will save you time during revision.

Applications of Surface Tension

1. Insects walking on water (pond skaters)

A pond skater's legs are coated with a waxy, hydrophobic substance. When the insect stands on water, its legs press down on the surface but do not break through. The surface tension of water acts along the contact line around each leg, providing an upward force that supports the insect's weight. The insect is light enough that this upward force is sufficient. A human, with far greater weight relative to foot area, would break the surface instantly.

2. Floating needle

A steel needle placed gently on the surface of still water floats, even though steel is about eight times denser than water. The needle is supported not by buoyancy (which would be far too small) but by surface tension

acting along the contact line where the water surface meets the needle. The water surface bends downward slightly under the needle, and the vertical components of the surface tension forces along this contact line balance the needle's weight. Adding a drop of detergent reduces the surface tension, and the needle sinks immediately.

3. Spherical liquid drops

In the absence of external forces (or when forces are balanced, as in free fall or in space), liquid drops are perfectly spherical. This is because surface tension pulls every part of the surface inward, minimising the surface area. Among all shapes with a given volume, a sphere has the smallest surface area. Consequently, a sphere is the shape of lowest surface energy, and surface tension drives every drop toward it.

4. Detergents and soap

Detergent molecules have a hydrophilic (water-attracting) head and a hydrophobic (water-repelling) tail. When dissolved in water, they wedge themselves into the surface layer with their tails pointing outward. This disrupts the cohesive forces between water molecules and dramatically reduces the surface tension (from about 0.073Nm^{-1} to about 0.025Nm^{-1}). As a result, the soapy water spreads more easily, wets surfaces more completely, and penetrates into grease and dirt, loosening them for removal.

5. Waterproofing of fabrics

Waterproof fabrics are treated with hydrophobic coatings that increase the angle of contact between water and the fabric fibres to well above 90° . Consequently, instead of being absorbed by capillary action (which requires $\theta < 90^\circ$), water beads up on the surface and rolls off. The same principle is at work on waxed car surfaces and on the leaves of lotus plants.

6. Capillary action in plants

Water rises from the roots to the leaves of plants partly through capillary rise in the narrow xylem vessels. The adhesive forces between water and the hydrophilic walls of the xylem pull the water upward. Capillary rise alone, however, can account for only about 1 metre of rise. In tall trees, the dominant mechanism is transpiration pull: evaporation of water from the leaf surfaces creates a negative pressure at the top of the water column, pulling it upward like a continuous chain.

7. Water absorption by soil

Water from rainfall or irrigation is held in the tiny spaces between soil particles by capillary forces. Without this, water would simply drain downward under gravity, and plant roots would have no moisture to absorb between rainfalls. Farmers sometimes break up the top layer of soil (tilling) to increase the size of the air gaps, which reduces capillary rise from the wet subsoil and prevents evaporation from the surface.

8. Oil spreading on water

When a drop of oil falls on a water surface, it spreads into a thin film. Water has high surface tension (0.073Nm^{-1}) and oil has low surface tension ($\approx 0.03\text{Nm}^{-1}$). Replacing the high-energy water-air interface with a lower-energy oil-air interface reduces the total surface energy. Consequently, spreading is energetically favourable, and it continues until the oil forms a very thin film.

9. Lung surfactants

The alveoli (tiny air sacs) in the lungs are lined with a thin film of liquid. During exhalation, the alveoli shrink, and without intervention, the surface tension of this film would collapse them completely. The body produces a substance called **pulmonary surfactant** that reduces the surface tension of the alveolar fluid, preventing collapse and allowing the alveoli to re-inflate easily on the next breath. Premature babies sometimes lack sufficient surfactant, leading to respiratory distress syndrome.

10. Blotting paper and paper towels

The narrow spaces between the plant fibres in paper act as capillary tubes. When the paper touches a liquid, capillary rise draws the liquid into these spaces rapidly. The fibres are hydrophilic, so the angle of contact is small and the capillary rise is strong. Waxed paper, by contrast, has hydrophobic fibres, so the contact angle exceeds 90° and the liquid is not absorbed.

Applications of Fluid Flow

1. Torricelli's theorem and tank drainage

The speed of efflux from an orifice in a tank is $v = \sqrt{2gh}$, where h is the depth of liquid above the orifice. This result is used in the design of water tanks, dams, and irrigation systems to predict flow rates and drainage times. Combined with projectile motion, it predicts the horizontal range of the water jet emerging from the orifice.

2. Venturi meter

A constriction in a pipe creates a speed increase (by the equation of continuity) and a corresponding pressure decrease (by Bernoulli's equation). By measuring the pressure difference between the wide section and the throat using a manometer, the flow speed and volume flow rate can be calculated. Venturi meters are used by water utilities (such as DAWASCO in Dar es Salaam) to monitor water distribution, in fuel injection systems to mix air and fuel, and in respiratory equipment to measure airflow.

3. Aerofoil lift (dynamic lift)

An aircraft wing is shaped so that air flows faster over the curved upper surface than over the flatter lower surface. By Bernoulli's equation, the pressure above the wing is lower than below. This pressure difference creates an upward lift force $F = \frac{1}{2}\rho A(v_1^2 - v_2^2)$. The same principle applies to helicopter rotor blades, wind turbine blades, and racing car spoilers (which are inverted aerofoils designed to push the car downward for better grip).

4. Magnus effect (spinning ball)

A spinning ball moving through air drags the air around it in the direction of spin. On one side, the spin adds to the oncoming airflow (higher speed, lower pressure). On the opposite side, the spin opposes the airflow (lower speed, higher pressure). The pressure difference deflects the ball sideways. This is how footballers curve free kicks, cricket bowlers swing the ball, and table tennis players create topspin and sidespin.

5. Atomiser and sprayer

Fast-moving air blown through a horizontal tube creates a low-pressure region at the junction with a vertical tube dipping into a liquid reservoir. The atmospheric pressure on the liquid surface pushes the liquid up the vertical tube, where it meets the fast airstream and is broken into a fine spray. This principle is used in perfume bottles, insecticide sprayers, paint spray guns, and carburettors in older engines.

6. Pitot tube

A tube facing directly into a fluid stream brings the fluid to rest at its tip (the stagnation point). By Bernoulli's equation, the pressure at this point equals the static pressure plus $\frac{1}{2}\rho v^2$. Comparing the stagnation pressure with the static pressure (measured by a side opening) gives the fluid velocity. Pitot tubes are mounted on aircraft to measure airspeed, in weather stations to measure wind speed, and in industrial pipelines to monitor flow rates.

7. Chimney effect (stack effect)

Hot gases inside a chimney are less dense than the cooler air outside. The pressure difference drives the hot gases upward through the chimney. By continuity, fresh air is drawn in at the base to replace them. A taller chimney creates a larger pressure difference and a stronger draft. This same effect ventilates buildings with tall atriums and drives the airflow in cooling towers.

8. Blood flow in the circulatory system

Blood flow through arteries and veins obeys the equation of continuity and Poiseuille's formula. The total cross-sectional area of capillaries is much larger than that of the aorta, so blood flows slowly in capillaries (allowing time for gas exchange) even though it flows quickly in the aorta. Arterial narrowing (atherosclerosis) reduces flow dramatically because of the R^4 dependence in Poiseuille's formula.

9. Hydraulic systems (Pascal's principle)

Pressure applied to a confined fluid is transmitted equally in all directions. This is the basis of hydraulic brakes, hydraulic lifts, and hydraulic presses. A small force applied to a small piston creates the same

pressure as a large force on a large piston. Consequently, the force is amplified by the ratio of the piston areas. Every daladala bus in Tanzania uses hydraulic brakes that rely on this principle.

10. Sedimentation and particle separation

In a mixture of particles of different sizes suspended in a fluid, larger particles reach higher terminal velocities (because $v_t \propto r^2$ by Stokes' law) and settle to the bottom faster. This is used in mining to separate mineral particles by size, in blood tests (erythrocyte sedimentation rate, ESR, measures how fast red blood cells settle in a tube), and in water treatment plants to remove suspended solids.

That completes the applications assembly. This section serves as your quick-reference map: when an examination question asks you about a specific application, you know exactly where to find the theory and the worked example that goes with it.

Two sections remain. The miscellaneous worked examples will test your ability to combine concepts from different parts of the chapter. The Digging Deeper exercise will push you further. The finish line is in sight.

MISCELLANEOUS WORKED EXAMPLES ON FLUID MECHANICS

Example 44

- Distinguish between cohesive forces and adhesive forces. Using these concepts, explain why water forms a concave meniscus in a glass tube while mercury forms a convex meniscus.
- A soap bubble of radius 5cm and surface tension 0.03Nm^{-1} is blown at the end of a tube at sea level where atmospheric pressure is $1.01 \times 10^5\text{Pa}$. Calculate the total pressure inside the bubble.

Solution

- Cohesive forces are attractive forces between molecules of the same substance while adhesive forces are attractive forces between molecules of different substances.

In a glass tube containing water, the adhesive forces between water molecules and glass molecules are stronger than the cohesive forces between water molecules. Consequently, water molecules are pulled toward the glass wall, causing the liquid to climb upward along the wall. This produces a concave meniscus with an angle of contact less than 90° .

In a glass tube containing mercury, the cohesive forces between mercury atoms are stronger than the adhesive forces between mercury and glass. Hence, mercury molecules are pulled away from the glass wall, and the liquid curves downward near the wall. This produces a convex meniscus with an angle of contact greater than 90° .

- A soap bubble has two surfaces. The excess pressure is:

$$\Delta P = \frac{4\gamma}{r} = \frac{4 \times 0.03\text{Nm}^{-1}}{0.05\text{m}} = 2.4\text{Pa}$$

The total pressure inside the bubble is:

$$P = P_{\text{atm}} + \Delta P = 1.01 \times 10^5\text{Pa} + 2.4\text{Pa} = 1.01024 \times 10^5\text{Pa}$$

Example 45

- A student applies Bernoulli's equation to calculate the pressure in a pipe carrying hot, thick engine oil at high speed through a very long industrial pipeline. Explain three reasons why Bernoulli's equation may give an inaccurate result in this situation.
- Water flows through a horizontal pipe that narrows from an internal diameter of 8cm to 4cm. The pressure in the wider section is $3.0 \times 10^5\text{Pa}$ and the flow speed there is 1.5ms^{-1} . Find the pressure in the narrower section. Take $\rho = 1000\text{kgm}^{-3}$.

Solution

- Bernoulli's equation assumes three conditions that are violated in this situation:

First, the fluid must be non-viscous. Engine oil is highly viscous, so significant energy is lost to internal friction between layers as it flows. Consequently, the pressure drops along the pipe even without any change in speed or height, which Bernoulli's equation does not account for.

Second, the fluid must be incompressible. Hot engine oil may experience density changes due to temperature variations along the long pipeline, violating the constant-density assumption.

Third, the flow must be steady. In a long industrial pipeline, fluctuations in pump output or valve positions may cause unsteady flow, where velocities at a given point change with time.

Therefore, Bernoulli's equation is unreliable for this situation because the fluid is viscous, possibly compressible at varying temperatures, and the flow may not be steady.

(b) By the continuity equation:

$$v_2 = v_1 \times \frac{d_1^2}{d_2^2} = 1.5 \text{ms}^{-1} \times \frac{(8\text{cm})^2}{(4\text{cm})^2} = 6.0 \text{ms}^{-1}$$

For horizontal flow, Bernoulli's equation gives:

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) = 3.0 \times 10^5 \text{Pa} + \frac{1}{2} \times 1000 \text{kgm}^{-3} \times ((1.5 \text{ms}^{-1})^2 - (6.0 \text{ms}^{-1})^2)$$

$$P_2 = 2.83 \times 10^5 \text{Pa}$$

Example 46

- (a) Explain why a large spherical drop of liquid has less surface energy per unit volume than a small drop of the same liquid.
- (b) A liquid drop of radius R and surface tension γ is sprayed into n identical smaller drops. Derive an expression for the work done in the spraying process. If a drop of mercury of radius 1mm ($\gamma = 0.5\text{Nm}^{-1}$) is broken into 8 equal drops, calculate the work done.

Solution

- (a) The surface area of a sphere is $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$. The surface-to-volume ratio is:

$$\frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{r}$$

This ratio is inversely proportional to r . A larger drop has a smaller surface-to-volume ratio, and therefore less surface energy per unit volume. A smaller drop has a larger ratio and more surface energy per unit volume. This is why energy must be supplied to break a large drop into smaller ones.

- (b) By conservation of volume: $\frac{4}{3}\pi R^3 = n \times \frac{4}{3}\pi r^3$, giving $r = R/n^{1/3}$.

The work done equals the increase in surface energy:

$$W = \gamma(n \times 4\pi r^2 - 4\pi R^2) = 4\pi\gamma \left(\frac{nR^2}{n^{2/3}} - R^2 \right) = 4\pi\gamma R^2(n^{1/3} - 1)$$

Hence:

$$W = 4\pi\gamma R^2(n^{1/3} - 1)$$

For the mercury drop: $R = 1 \times 10^{-3}\text{m}$, $\gamma = 0.5\text{Nm}^{-1}$, $n = 8$

$$W = 4\pi \times 0.5\text{Nm}^{-1} \times (10^{-3}\text{m})^2 \times (8^{1/3} - 1) = 4\pi \times 0.5\text{Nm}^{-1} \times 10^{-6}\text{m}^2 \times (2 - 1)$$

$$W = 6.28 \times 10^{-6}\text{J}$$

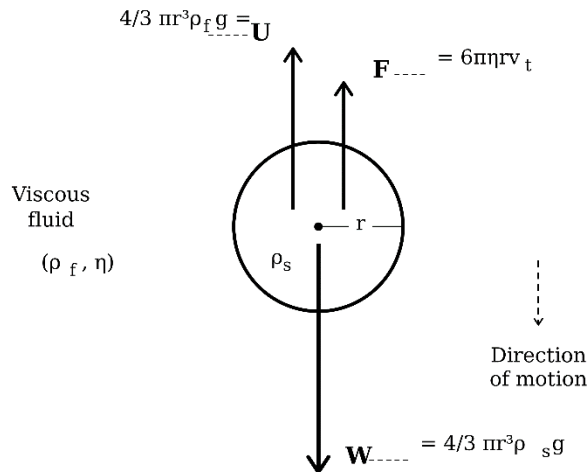
Example 47

- (a) A steel ball is dropped into a tall jar of glycerine. Initially it accelerates, but after a short distance it moves at constant velocity. Explain why the ball stops accelerating, and draw a well labelled diagram showing all the forces acting on the ball when it has reached constant velocity.
- (b) A tank of water rests on a platform 3m above the ground. The water surface is 4m above the base of the tank. A small hole is opened at the base. Find the speed of the emerging water and the horizontal distance from the platform where the jet hits the ground. Take $g = 9.8\text{ms}^{-2}$.

Solution

(a) When the ball is first released, its velocity is zero and the viscous drag force ($F = 6\pi\eta rv$) is also zero. The net downward force is the weight minus the upthrust, so the ball accelerates. As the ball speeds up, the viscous drag increases (because drag is proportional to velocity). Consequently, the net downward force decreases and the acceleration decreases. Eventually, the drag becomes large enough to exactly balance the effective weight. At that point, the net force is zero, the acceleration is zero, and the ball falls at a constant velocity called the **terminal velocity**.

At terminal velocity, three forces act: weight $W = \frac{4}{3}\pi r^3 \rho_s g$ downward, upthrust $U = \frac{4}{3}\pi r^3 \rho_f g$ upward, and viscous drag $F = 6\pi\eta r v_t$ upward. The condition is $W = U + F$. So the diagram is:



At terminal velocity:

$$W = U + F$$

(b) By Torricelli's theorem:

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 4} = \sqrt{78.4} = 8.85 \text{ ms}^{-1}$$

The jet emerges horizontally from a height $H = 3\text{m}$ above the ground. The time to fall:

$$t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 3\text{m}}{9.8 \text{ ms}^{-2}}} = \sqrt{0.612 \text{ s}^2} = 0.782 \text{ s}$$

Horizontal range:

$$x = vt = 8.85 \text{ ms}^{-1} \times 0.782 \text{ s} = 6.92 \text{ m}$$

Therefore, the jet hits the ground approximately 6.9m from the base of the platform.

Example 48

- (a) When a flat plate is dragged across a thin layer of oil on a surface, the force required increases if the plate area increases, the plate speed increases, or the oil layer becomes thinner. Using Newton's law of viscosity, explain why each of these three changes increases the required force.
- (b) Water ($\eta = 1.0 \times 10^{-3} \text{ Nsm}^{-2}$, $\rho = 1000 \text{ kgm}^{-3}$) flows through a horizontal pipe of length 2m and internal radius 3mm at a volume flow rate of $2.0 \times 10^{-5} \text{ m}^3 \text{ s}^{-1}$. Find the pressure difference between the ends of the pipe and the maximum velocity of the water.

Solution

(a) By Newton's law of viscosity, the viscous force is $F = \eta A \left(\frac{dv}{dy} \right)$, where A is the contact area, $\frac{dv}{dy}$ is the velocity gradient, and η is the viscosity.

Increasing plate area: The force is directly proportional to A. A larger plate has more surface in contact with the oil, so more fluid layers are sheared simultaneously. Consequently, the total viscous resistance increases.

Increasing plate speed: The velocity gradient $\frac{dv}{dy} = \frac{v}{d}$ increases when the plate speed v increases (with oil thickness d unchanged). A steeper velocity gradient means the layers are sliding over each other faster, producing greater friction.

Thinner oil layer: Reducing the thickness d increases the velocity gradient $\frac{dv}{dy} = \frac{v}{d}$ because the same velocity change occurs over a shorter distance. A steeper gradient means stronger friction between layers.

(b) From Poiseuille's formula, $Q = \frac{\pi \Delta P R^4}{8\eta L}$, rearranging:

$$\Delta P = \frac{8\eta L Q}{\pi R^4} = \frac{8 \times 1.0 \times 10^{-3} \text{ Nsm}^{-2} \times 2\text{m} \times 2.0 \times 10^{-5} \text{ m}^3 \text{ s}^{-1}}{\pi \times (3 \times 10^{-3} \text{ m})^4}$$

$$\Delta P = 1257 \text{ Pa}$$

The maximum velocity (at the centre):

$$v_{\text{max}} = \frac{\Delta P R^2}{4\eta L} = \frac{1257 \text{ Pa} \times (3 \times 10^{-3} \text{ m})^2}{4 \times 10^{-3} \text{ Nsm}^{-2} \times 2\text{m}} = 1.41 \text{ ms}^{-1}$$

Example 49

- (a) A student claims that a soap bubble with twice the radius has twice the excess pressure. Identify the error in this claim and provide the correct relationship.
- (b) Two soap bubbles of radii 2cm and 4cm coalesce under isothermal conditions to form a new bubble. Find the radius of the new bubble and the change in surface energy. Take $\gamma = 0.025 \text{ Nm}^{-1}$.

Solution

(a) The student's claim is incorrect. The excess pressure inside a soap bubble is $\Delta P = 4\gamma/r$, which is **inversely** proportional to the radius. A bubble with twice the radius has **half** the excess pressure, not twice. The student has confused direct and inverse proportionality.

(b) Under isothermal conditions, surface energy is conserved:

$$2 \times 4\pi r_1^2 \gamma + 2 \times 4\pi r_2^2 \gamma = 2 \times 4\pi R^2 \gamma$$

$$R^2 = r_1^2 + r_2^2 = (0.02\text{m})^2 + (0.04\text{m})^2 = 4 \times 10^{-4} \text{ m}^2 + 16 \times 10^{-4} \text{ m}^2 = 20 \times 10^{-4} \text{ m}^2$$

$$R = \sqrt{20 \times 10^{-4} \text{ m}^2} = 4.47 \times 10^{-2} \text{ m} = 4.47 \text{ cm}$$

Change in surface area (two surfaces):

$$\Delta A = 8\pi(R^2 - r_1^2 - r_2^2) = 8\pi(20 - 4 - 16) \times 10^{-4} \text{ m}^2 = 0$$

Since $\Delta A = 0$, the change in surface energy is **zero** under isothermal conditions, (which is consistent with the energy conservation assumption used in the derivation).

Example 50

- (a) Explain what is meant by "streamline flow" and why streamlines can never cross each other.
- (b) A venturi meter is designed to measure the flow rate of water in a pipe of internal diameter 10cm. The available manometer uses mercury ($\rho = 13600 \text{ kgm}^{-3}$). If the throat diameter is chosen as 4cm and the mercury level difference reads 5cm, calculate the volume flow rate. Take $\rho = 1000 \text{ kgm}^{-3}$ and $g = 9.8 \text{ ms}^{-2}$.

Solution

(a) Streamline flow (laminar flow) is a type of fluid flow in which every particle passing through a given point follows exactly the same path as every previous particle that passed through that point. The velocity at any fixed point remains constant with time.

Streamlines cannot cross because at a crossing point, a fluid particle would have two velocities simultaneously (one along each streamline). This is physically impossible. Therefore, the non-crossing of streamlines is a requirement of steady flow.

(b) Given: $A = \pi(0.05)^2 = 7.854 \times 10^{-3} \text{m}^2$, $a = \pi(0.02)^2 = 1.257 \times 10^{-3} \text{m}^2$, $h = 0.05 \text{m}$

$$A^2 - a^2 = (7.854 \times 10^{-3} \text{m}^2)^2 - (1.257 \times 10^{-3} \text{m}^2)^2 = 6.011 \times 10^{-5} \text{m}^4$$

$$v_1 = a \sqrt{\frac{2\rho'gh}{\rho(A^2 - a^2)}} = 1.257 \times 10^{-3} \text{m}^2 \times \sqrt{\frac{2 \times 13600 \text{kgm}^{-3} \times 9.8 \text{ms}^{-2} \times 0.05 \text{m}}{1000 \text{kgm}^{-3} \times 6.011 \times 10^{-5} \text{m}^4}}$$

$$v_1 = 0.592 \text{ms}^{-1}$$

$$Q = Av_1 = 7.854 \times 10^{-3} \text{m}^2 \times 0.592 \text{ms}^{-1} = 4.65 \times 10^{-3} \text{m}^3 \text{s}^{-1}$$

Therefore, the volume flow rate is $4.65 \times 10^{-3} \text{m}^3 \text{s}^{-1}$.

Example 51

- (a) A student uses the falling sphere method to measure the viscosity of glycerine. She uses a large steel ball (radius 5mm) and gets a very different answer from her classmate who uses a small ball (radius 0.5mm). Explain which student is more likely to get an accurate result, and why the other student's result is unreliable.
- (b) Two spheres, X and Y, are made of materials with densities 8000kgm^{-3} and 3000kgm^{-3} respectively. Sphere X has radius r and sphere Y has radius $2r$. Both are dropped into the same viscous liquid of density 1000kgm^{-3} . Find the ratio of their terminal velocities.

Solution

(a) The student using the small ball (0.5mm) is more likely to get an accurate result. Stokes' law ($F = 6\pi\eta rv$), on which the falling sphere method is based, is valid only when the flow around the sphere is laminar, which requires the Reynolds number to be less than 1.

A large ball falls faster (terminal velocity is proportional to r^2) and has a larger characteristic length. Both factors increase the Reynolds number. Consequently, the flow around the large ball may become turbulent, in which case Stokes' law underestimates the actual drag, the measured terminal velocity is lower than Stokes' law predicts, and the calculated viscosity is incorrect.

The small ball, with its lower speed and smaller radius, maintains a low Reynolds number and keeps the flow laminar. Therefore, its result is more reliable.

(b) Terminal velocity: $v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$

For sphere X: $v_X = \frac{2r^2(8000 - 1000)g}{9\eta} = \frac{2r^2 \times 7000g}{9\eta}$

For sphere Y: $v_Y = \frac{2(2r)^2(3000 - 1000)g}{9\eta} = \frac{2 \times 4r^2 \times 2000g}{9\eta} = \frac{8r^2 \times 2000g}{9\eta}$

$$\frac{v_X}{v_Y} = \frac{2 \times 7000}{8 \times 2000} = \frac{7}{8}$$

Therefore, $v_X : v_Y = 7 : 8$.

Example 52

- (a) Explain, using Bernoulli's principle, why it is dangerous to stand close to the edge of a railway platform when a high-speed train passes through.
- (b) Water is supplied to a building through a horizontal pipe of radius R and length L at a certain flow rate Q . The water company needs to double the flow rate but can only increase the pressure by 50%. By what factor must the pipe radius be increased? State the engineering implication of your answer.

Solution

(a) When a high-speed train passes, the air between the person and the train is dragged along and moves at high speed. By Bernoulli's equation, fast-moving air has lower pressure than the still air behind the person.

This pressure difference creates a net force pushing the person toward the train. The faster the train, the greater the speed difference, the larger the pressure drop, and the stronger the force.

(b) From Poiseuille's formula: $Q = \frac{\pi \Delta P R^4}{8 \eta L}$

Since η and L are unchanged, $Q \propto \Delta P \times R^4$.

Let the new radius be R' and the new pressure be $1.5\Delta P$:

$$\frac{Q'}{Q} = \frac{1.5\Delta P \times R'^4}{\Delta P \times R^4}$$

$$2 = 1.5 \times \left(\frac{R'}{R}\right)^4$$

$$\left(\frac{R'}{R}\right)^4 = \frac{2}{1.5} = \frac{4}{3}$$

$$\frac{R'}{R} = \left(\frac{4}{3}\right)^{1/4} = 1.075$$

The radius must increase by only about **7.5%**. This illustrates the extraordinary sensitivity of the R^4 law: a tiny increase in pipe radius produces a huge increase in flow capacity. So the engineering implication is that replacing old pipes with slightly wider ones is far more cost-effective than installing more powerful pumps to increase pressure.

Example 53

- (a) A student fills a capillary tube with water and holds it horizontally. The water does not flow out. Explain why, using the concept of surface tension.
- (b) A capillary tube of internal radius 0.3mm is dipped vertically into a liquid of surface tension 0.05Nm^{-1} , density 800kgm^{-3} , and angle of contact 30° . Calculate the height of the liquid in the tube. If the tube is only 15cm long and open at both ends, describe what happens. Take $g = 9.8\text{ms}^{-2}$.

Solution

(a) At each open end of the horizontal capillary tube, the water surface forms a meniscus. Surface tension acts along the circumference of this meniscus, creating a pressure difference (excess pressure) across the curved surface. This excess pressure is sufficient to support the weight of the small column of water inside the narrow tube against gravity. The narrower the tube, the stronger this effect. Consequently, the water remains held in place.

(b) Using the capillary rise formula:

$$h = \frac{2\gamma \cos\theta}{\rho g r} = \frac{2 \times 0.05\text{Nm}^{-1} \times \cos 30^\circ}{800\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 3 \times 10^{-4}\text{m}} = 0.0368\text{m}$$

$$h = 3.68\text{cm}$$

Since the tube is 15cm long (much longer than 3.68cm), the liquid rises to 3.68cm and stops. The meniscus forms normally at this height.

Understand! If the tube were shorter than 3.68cm, the liquid would rise to the top of the tube but would **not overflow**. Instead, the meniscus at the top would adjust its radius of curvature (becoming flatter) until the excess pressure across the meniscus is just sufficient to support the liquid column equal to the tube length.

The angle of contact would increase from 30° to a value that satisfies $\cos\theta' = \frac{\rho g r L_{\text{tube}}}{2\gamma}$.

Example 54

- (a) Explain, using Bernoulli's principle, how an atomiser works.
- (b) An aeroplane of mass 5000kg has wings of total area 40m^2 . During takeoff, the speed of air over the upper surface is 80ms^{-1} . Find the minimum speed of air under the lower surface for the plane to become airborne. Take $\rho_{\text{air}} = 1.2\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Solution

(a) When the rubber bulb is squeezed, air rushes through the horizontal tube at high speed. At the junction with the vertical tube, the fast-moving air creates a region of low pressure (by Bernoulli's principle: higher speed means lower pressure). The atmospheric pressure on the liquid surface in the container is now greater than the pressure at the junction. This pressure difference pushes the liquid up the vertical tube. When it reaches the junction, the fast airstream breaks it into fine droplets, producing a spray.

(b) For the plane to become airborne, the lift force must equal the weight:

$$F = Mg$$

$$\frac{1}{2} \rho A (v_1^2 - v_2^2) = Mg$$

$$v_1^2 - v_2^2 = \frac{2Mg}{\rho A} = \frac{2 \times 5000 \text{ kg} \times 9.8 \text{ ms}^{-2}}{1.2 \text{ kg m}^{-3} \times 40 \text{ m}^2} = 2042 \text{ m}^2 \text{ s}^{-2}$$

$$v_2 = \sqrt{v_1^2 - 2042 \text{ m}^2 \text{ s}^{-2}} = \sqrt{(80 \text{ ms}^{-1})^2 - 2042 \text{ m}^2 \text{ s}^{-2}} = 66 \text{ ms}^{-1}$$

Therefore, the minimum speed of air under the wing is approximately 66 ms^{-1} .

Example 55

- (a) A laboratory technician measures the viscosity of an oil using the falling sphere method and obtains $\eta = 0.8 \text{ N s m}^{-2}$. However, she notices that the glass sphere she used was not perfectly spherical and that the oil temperature rose by 3°C during the experiment. Explain how each of these factors would affect her result, and whether the true viscosity is likely to be higher or lower than her measured value.
- (b) Glycerine ($\eta = 1.5 \text{ N s m}^{-2}$, $\rho = 1260 \text{ kg m}^{-3}$) flows through a horizontal pipe of radius 5 mm and length 0.5 m. If the pressure difference between the ends is $2 \times 10^4 \text{ Pa}$. Calculate the volume flow rate, the average velocity, and the Reynolds number. Comment on the type of flow.

Solution

(a) **Non-spherical shape:** Stokes' law assumes a perfect sphere. A non-spherical object experiences greater drag than a sphere of the same volume because its irregular shape disrupts the flow more. Greater drag means a lower terminal velocity. Since $\eta = \frac{2r^2(\rho_s - \rho_f)g}{9v_t}$, a lower v_t gives a higher calculated viscosity. Therefore, the non-spherical shape causes the measured viscosity to be **too high**.

Temperature rise: The viscosity of a liquid decreases with increasing temperature. If the oil warmed by 3°C during the experiment, the viscosity at the end was lower than at the start. The sphere fell through oil of varying viscosity, and the measured terminal velocity reflects an average viscosity that is lower than the viscosity at the initial temperature. Therefore, the measured value of 0.8 N s m^{-2} is likely **lower** than the true viscosity at the starting temperature.

The two errors act in opposite directions: the shape error makes the result too high, while the temperature error makes it too low. The net effect depends on which error dominates.

(b) By Poiseuille's formula:

$$Q = \frac{\pi \Delta P R^4}{8 \eta L} = \frac{\pi \times 2 \times 10^4 \text{ Pa} \times (5 \times 10^{-3} \text{ m})^4}{8 \times 1.5 \text{ N s m}^{-2} \times 0.5 \text{ m}}$$

$$Q = 6.54 \times 10^{-6} \text{ m}^3 \text{ s}^{-1}$$

The average velocity:

$$v_{\text{avg}} = \frac{Q}{\pi R^2} = \frac{6.54 \times 10^{-6} \text{ m}^3 \text{ s}^{-1}}{\pi \times (5 \times 10^{-3} \text{ m})^2} = 0.0833 \text{ ms}^{-1}$$

Reynolds number:

$$R_e = \frac{\rho_f v_{\text{avg}} \times 2R}{\eta} = \frac{1260 \text{ kg m}^{-3} \times 0.0833 \text{ ms}^{-1} \times 0.01 \text{ m}}{1.5 \text{ N s m}^{-2}} = 0.70$$

Since $R_e = 0.70$, which is far below 2000, the flow is **laminar**.

Example 56

- (a) Explain why the viscosity of liquids decreases with increasing temperature while the viscosity of gases increases with increasing temperature.
- (b) A water tank has two identical small holes, one at depth h below the surface and another at depth $4h$. Find the ratio of: (i) the speeds of efflux, and (ii) the volume flow rates from the two holes.

Solution

(a) In liquids, viscosity arises from intermolecular attractive forces that resist the sliding of adjacent layers. As temperature increases, molecules gain kinetic energy and overcome these forces more easily. Consequently, the resistance between layers decreases and viscosity drops.

In gases, viscosity arises from the transfer of momentum between layers by molecules moving randomly back and forth. As temperature increases, molecules move faster and transfer more momentum between layers. As a result, the friction between layers increases and viscosity rises.

The two mechanisms are fundamentally different: intermolecular forces in liquids versus momentum transfer in gases.

(b) By Torricelli's theorem, $v = \sqrt{2gh'}$ where h' is the depth below the surface.

$$(i) \frac{v_1}{v_2} = \frac{\sqrt{2gh}}{\sqrt{2g(4h)}} = \frac{\sqrt{h}}{\sqrt{4h}} = \frac{1}{2}$$

(ii) Both holes are identical (same area a), so $Q = av$:

$$\frac{Q_1}{Q_2} = \frac{v_1}{v_2} = \frac{1}{2}$$

The deeper hole has twice the efflux speed and twice the volume flow rate.

Example 57

- (a) A rectangular wire frame with a sliding wire is dipped in a soap solution, forming a thin film. When the sliding wire is released, it moves inward (toward the frame) on its own. Explain why the wire moves, and show mathematically that the force per unit length pulling the wire equals the work done per unit area in stretching the film.
- (b) A soap film is formed on a rectangular wire frame with a sliding wire of length 5cm. The surface tension of the soap solution is 0.03Nm^{-1} . The sliding wire is pulled outward through 2cm. Calculate the work done and the force required to hold the wire in the new position.

Solution

(a) The soap film has two surfaces, and surface tension acts along each surface, pulling every part of the film toward minimum area. The sliding wire is free to move, so the surface tension forces on it are unopposed. These forces pull the wire inward, reducing the film area and hence the surface energy. The wire moves because the system spontaneously reduces its surface energy.

To show the equivalence: the surface tension force on the wire of length l (two surfaces) is $F = 2\gamma l$. If the wire moves a distance dx , the work done is $dW = F dx = 2\gamma l dx$. The increase in surface area (two surfaces) is $dA = 2l dx$. Therefore, the work done per unit area is:

$$\frac{dW}{dA} = \frac{2\gamma l dx}{2l dx} = \gamma$$

This equals the surface tension (force per unit length), confirming that surface tension and surface energy are numerically equal.

(b) The soap film has two surfaces. The increase in total area:

$$\Delta A = 2 \times l \times x = 2 \times 0.05\text{m} \times 0.02\text{m} = 2.0 \times 10^{-3}\text{m}^2$$

Work done:

$$W = \gamma \times \Delta A = 0.03\text{Nm}^{-1} \times 2.0 \times 10^{-3}\text{m}^2 = 6.0 \times 10^{-5}\text{J}$$

Force required (two surfaces):

$$F = 2\gamma l = 2 \times 0.03 \text{ Nm}^{-1} \times 0.05 \text{ m} = 3.0 \times 10^{-3} \text{ N}$$

Example 58

Kipanga has been thinking. He announces to **Mr. Akilikubwa** that he has calculated the perfect speed at which a person could swim through honey without getting tired, because the viscous drag at low speed is small.

Mr. Akilikubwa raises an eyebrow. “Tell me your calculation.”

Kipanga says: “Honey has viscosity $\eta \approx 5 \text{ Nsm}^{-2}$ and density $\rho_f = 1400 \text{ kgm}^{-3}$. A person of mass 70 kg , volume 0.07 m^3 , and effective radius 0.15 m can be modelled as a sphere. By Stokes’ law, the drag at $v = 0.1 \text{ ms}^{-1}$ would be about $F = 6\pi\eta rv = 6\pi \times 5 \times 0.15 \times 0.1 = 1.41 \text{ N}$. This is a tiny force. So swimming slowly through honey should be easy.”

Mr. Akilikubwa smiles. “You have made at least three errors. Find them.”

Identify Kipanga’s three errors and, for one of them, calculate the correct value to show how it changes the conclusion.

Solution

Error 1: A human body is not a sphere. Stokes’ law applies to smooth spheres. A human body has arms, legs, and an irregular shape that creates far more drag than a sphere of the same volume. The drag on a real human body is many times larger than Stokes’ formula predicts.

Error 2: The buoyancy changes everything. The density of the human body ($\rho_s = m/V = 70/0.07 = 1000 \text{ kgm}^{-3}$) is less than the density of honey (1400 kgm^{-3}). Consequently, the person would float in honey, not sink. There is no need to swim; the upthrust exceeds the weight. The net upward force is:

$$F_{\text{net}} = V(\rho_f - \rho_s)g = 0.07 \text{ m}^3 \times (1400 - 1000) \text{ kgm}^{-3} \times 9.8 \text{ ms}^{-2} = 274 \text{ N}$$

This is a substantial upward force, much larger than the person’s weight. So the person would bob on the surface of the honey.

Hence, the buoyancy error changes the entire conclusion.

Error 3: Stokes’ law requires laminar flow with $R_e < 1$. The Reynolds number for a person moving at 0.1 ms^{-1} through honey is:

$$R_e = \frac{\rho_f v (2r)}{\eta} = \frac{1400 \text{ kgm}^{-3} \times 0.1 \text{ ms}^{-1} \times 0.30 \text{ m}}{5 \text{ Nsm}^{-2}} = 8.4$$

This is well above 1, so Stokes’ law does not apply. The actual drag would be much larger than Kipanga calculated.

Kipanga’s plan fails on three counts: the wrong shape, the wrong density comparison, and the wrong flow regime. Physics does not reward wishful thinking.

If you have survived all 15 miscellaneous examples with your confidence intact, congratulations! You have been squeezed through constrictions like water in a venturi meter, broken into smaller drops like mercury under pressure, and dragged through viscous resistance like a steel ball in glycerine. But you emerged faster, stronger, and with lower pressure on your ego.

Now comes the Digging Deeper exercise. Fifty-five questions. No worked solutions to peek at. No Mr. Akilikubwa to rescue you. Just you, the formulas, and whatever understanding you have built over the course of this chapter. Kipanga would panic. Kipute would calmly prepare her pen and calculator, with a confident smile. Be like Kipute.

DIGGING DEEPER EXERCISE 10

EXERCISE 10A: BINDER QUESTIONS

Question 1

Using the molecular theory of surface tension, explain why a molecule at the surface of a liquid has higher potential energy than a molecule deep inside the liquid. Hence explain why liquid surfaces tend to contract to the smallest possible area.

Question 2

A student places a steel needle flat on water and it floats. She then adds a drop of liquid soap to the water and the needle sinks immediately. Explain both observations using the concept of surface tension.

Question 3

Two glass tubes of different radii are dipped vertically into the same beaker of water. The water rises to different heights in the two tubes. Explain why the water rises higher in the narrower tube, and state the relationship between the height of rise and the tube radius.

Question 4

Explain the difference between a concave meniscus and a convex meniscus. In each case, state whether adhesive or cohesive forces dominate, and give one example of a liquid-solid combination that produces each type.

Question 5

A fluid flows steadily through a pipe that narrows at one section. Explain, using the equation of continuity, why the fluid speeds up at the narrow section even though no additional force is applied.

Question 6

Using Bernoulli's equation, explain why the pressure of a fluid decreases when its speed increases in a horizontal pipe. Your explanation should refer to the energy budget of the fluid.

Question 7

Explain why a ball spinning about a vertical axis curves sideways as it moves through the air. Name this effect and identify which side of the ball has lower pressure.

Question 8

Using the concept of viscosity, explain why the velocity of a fluid in a pipe is zero at the wall and maximum at the centre. Include a sketch of the velocity profile for laminar flow in a cylindrical pipe.

Question 9

Explain why the viscosity of a liquid decreases with increasing temperature while the viscosity of a gas increases with increasing temperature. In each case, identify the molecular mechanism responsible.

Question 10

A steel ball is dropped into a tall jar of glycerine. Describe, with reference to the forces acting on the ball, how its velocity changes from the moment of release until it reaches the bottom of the jar. Sketch the velocity-time graph.

EXERCISE 10B: REAL QUESTIONS

Question 11

A pond skater (an insect) can walk on the surface of still water, but a coin placed gently on the same surface sinks. Explain why the insect is supported but the coin is not, even though both are denser than water.

Question 12

During the rainy season in Dar es Salaam, water droplets on a freshly waxed car form small, nearly spherical beads, but on an unwaxed car they spread into flat patches. Explain the difference using the concepts of angle of contact and surface tension.

Question 13

A farmer in Dodoma notices that after tilling the top layer of soil, the soil beneath stays moist for longer than untilled soil. Using the concept of capillarity, explain why breaking up the top layer helps retain moisture.

Question 14

During a storm in Mtwara, the roof of a house is blown upward even though the wind blows horizontally across the top of the roof. Explain how a horizontal wind can produce a vertical (upward) force on the roof.

Question 15

A mechanic in Dar es Salaam pours engine oil through a narrow funnel on a hot afternoon. His colleague in Moshi, working in cold morning air, pours the same brand of oil through an identical funnel but finds it flows much more slowly. Using the concepts of viscosity and Poiseuille's formula, explain why the same oil flows at different rates in the two locations.

Question 16

Explain why commercial aircraft fly faster at high altitude (where air is thinner) than at low altitude, even though the engines produce roughly the same thrust.

Question 17

A doctor listens to a patient's chest with a stethoscope and hears a faint "whooshing" sound (murmur) in one of the arteries. Using the concepts of Reynolds number and fluid flow, explain what this sound indicates about the condition of that artery.

Question 18

Kipanga notices that during a heavy downpour, large raindrops hit the ground hard while fine drizzle seems to float down gently. He claims that gravity must be pulling the large drops harder. Kipute disagrees and says the difference is about terminal velocity, not gravitational force. Evaluate both claims and explain who is correct.

Question 19

A person standing on a railway platform feels a strong pull toward a high-speed train as it passes. Explain this effect using Bernoulli's principle and suggest why safety lines are painted well back from the platform edge.

Question 20

In a hospital, an intravenous (IV) drip delivers saline solution to a patient through a thin tube. Explain why raising the IV bag higher above the patient increases the flow rate, and identify which formula(s) governs the flow through the tube.

Question 21

Explain why a soap bubble always takes a spherical shape, regardless of how it was blown. Your answer should refer to surface energy.

Question 22

A student sips a thick milkshake through a narrow straw and finds it very difficult. She switches to a wider straw and the milkshake flows much more easily. Using Poiseuille's formula, explain why the wider straw makes such a large difference.

Question 23

Oil tanker ships sometimes spill oil onto the ocean surface. Explain, using the concept of surface energy, why the oil spreads to form a thin film rather than staying as a thick blob.

Question 24

Fine dust particles from a construction site remain suspended in the air for hours, while coarse sand grains settle to the ground within seconds. Using Stokes' law and terminal velocity, explain this difference.

EXERCISE 10C: HOT QUESTIONS**Question 25**

An air bubble of radius 1mm is formed at the bottom of a lake at a depth of 10m. The surface tension of water is 0.073Nm^{-1} and atmospheric pressure is $1.01 \times 10^5\text{Pa}$. Calculate:

- (a) the excess pressure inside the bubble, and
- (b) the total pressure inside the bubble.

Question 26

Two soap bubbles of radii 4cm and 6cm ($\gamma = 0.04\text{Nm}^{-1}$) come into contact and share a common surface. Calculate:

- (a) the radius of curvature of the common surface, and
- (b) the pressure difference across the common surface.

Question 27

A spherical drop of mercury of radius 3mm ($\gamma = 0.5\text{Nm}^{-1}$, $\rho = 13,600\text{kgm}^{-3}$, $c = 140\text{Jkg}^{-1}\text{K}^{-1}$) breaks into 27 identical smaller drops. Calculate:

- (a) the radius of each small drop,
- (b) the work done in the process, and
- (c) the temperature rise of the mercury if all the work appears as heat.

Question 28

A capillary tube of internal radius 0.4mm is dipped vertically into water ($\gamma = 0.072\text{Nm}^{-1}$, $\rho = 1000\text{kgm}^{-3}$).

- (a) Calculate the height of rise.
- (b) If the capillary tube is only 3cm long, explain what happens to the water level and the meniscus shape. Take $g = 9.8\text{ms}^{-2}$.

Question 29

Mercury ($\gamma = 0.5\text{Nm}^{-1}$, $\rho = 13,600\text{kgm}^{-3}$, $\theta = 138^\circ$) is poured into a glass U-tube with one arm of internal radius 2mm and the other of internal radius 6mm. Calculate the difference in mercury levels in the two arms. Take $g = 9.8\text{ms}^{-2}$.

Question 30

Water flows through a horizontal pipe that narrows from a diameter of 12cm to 6cm. The pressure in the wider section is $2.5 \times 10^5\text{Pa}$ and the flow speed there is 2ms^{-1} . Calculate:

- (a) the flow speed in the narrow section,
- (b) the pressure in the narrow section, and
- (c) the volume flow rate. Take $\rho = 1000\text{kgm}^{-3}$.

Question 31

A large open water tank has its base 8m above the ground. The water level is 5m above the base. A small hole is opened at the base of the tank. Calculate:

- (a) the speed of efflux,
- (b) the time for the water jet to reach the ground, and
- (c) the horizontal distance from the base of the tank where the jet strikes the ground. Take $g = 9.8\text{ms}^{-2}$.

Question 32

A venturi meter has a pipe diameter of 20cm and a throat diameter of 8cm. Water ($\rho = 1000\text{kgm}^{-3}$) flows through the pipe. A mercury manometer ($\rho = 13600\text{kgm}^{-3}$) connected between the pipe and the throat shows a level difference of 4cm. Calculate the volume flow rate in litres per second. Take $g = 9.8\text{ms}^{-2}$.

Question 33

A student claims: “By Bernoulli’s equation, a supersonic aircraft flying at 600ms^{-1} must have near-zero pressure around its wings, since the dynamic pressure $\frac{1}{2}\rho v^2$ at that speed is enormous.” Identify two errors in this reasoning and explain why the student’s conclusion is wrong.

Question 34

Water ($\eta = 1.0 \times 10^{-3}\text{Nsm}^{-2}$) flows through a horizontal pipe of internal radius 2cm and length 5m under a pressure difference of 500Pa. Calculate:

- (a) the volume flow rate, and
- (b) the new flow rate if the radius is doubled while all other factors remain constant.

Question 35

Two horizontal pipes of equal length L and radii R and $2R$ are connected in parallel. Find the ratio of the volume flow rate through the wider pipe to that through the narrower pipe.

Question 36

A steel ball of radius 1.5mm ($\rho = 7800\text{kgm}^{-3}$) falls through oil ($\rho = 900\text{kgm}^{-3}$) and reaches terminal velocity. The ball is timed falling through 25cm in 5.2s. Calculate:

- (a) the terminal velocity,
- (b) the coefficient of viscosity of the oil, and
- (c) the Reynolds number.

Comment on whether Stokes’ law is valid. Take $g = 9.8\text{ms}^{-2}$.

Question 37

Two students each measure the excess pressure inside a bubble. Student A measures an air bubble of radius 2mm inside water ($\gamma = 0.073\text{Nm}^{-1}$). Student B measures a soap bubble of the same radius in air ($\gamma = 0.025\text{Nm}^{-1}$). Student A gets a higher value and claims her bubble has more surface tension. Evaluate this claim by calculating both excess pressures and identifying the real reason for the difference.

Question 38

A pitot tube is mounted on an aircraft flying through air of density 0.9kgm^{-3} . The pitot tube is connected to a water manometer ($\rho = 1000\text{kgm}^{-3}$) which shows a level difference of 18cm. Calculate the airspeed of the aircraft.

Question 39

A water company supplies water through a pipe of radius R at a flow rate Q . Due to population growth, the company needs to triple the flow rate. If the pressure and pipe length cannot be changed:

- (a) By what factor must the radius increase?
- (b) If replacing the pipe costs Tsh 50,000 per centimetre of diameter increase, and the original diameter is 10cm, estimate the cost of the upgrade.

Question 40

A soap film is formed on a rectangular frame with a sliding wire of length 8cm. The wire is in equilibrium under the action of the surface tension force and a weight hanging from it. If $\gamma = 0.03\text{Nm}^{-1}$, calculate:

- (a) the force exerted by the film on the wire, and
- (b) the mass of the weight required to keep the wire in equilibrium. Take $g = 9.8\text{ms}^{-2}$.

Question 41

Water flows upward through a vertical pipe that widens from radius 2cm at the bottom to 4cm at the top. The bottom and top sections are 3m apart. The flow speed at the bottom is 4ms^{-1} and the pressure there is $3 \times 10^5\text{Pa}$. Find the pressure at the top. Take $\rho = 1000\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 42

A cylindrical tank of radius 0.6m is filled with water to a height of 2m. Water drains through a small hole of area 1cm^2 at the bottom. Derive an expression for the time to empty the tank and calculate this time. Take $g = 9.8\text{ms}^{-2}$.

Question 43

An aircraft of mass 15,000kg has a total wing area of 50m^2 . The air speed over the upper surface of the wings is 100ms^{-1} . Calculate the minimum speed of air under the lower surface required for the aircraft to maintain level flight. Take $\rho_{\text{air}} = 1.2\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 44

Glycerine ($\eta = 1.5\text{Nsm}^{-2}$) is poured onto a flat horizontal surface. A circular disc of radius 20cm and mass 2kg is placed on the glycerine layer, which has a thickness of 1mm. The disc is pushed horizontally at a constant velocity of 0.3ms^{-1} . Calculate the force required to maintain this velocity.

Question 45

Three identical soap bubbles, each of radius r , coalesce under isothermal conditions to form a single bubble. Show that the radius of the new bubble is $R = r\sqrt{3}$, and find the ratio of the excess pressure in the original bubbles to that in the new bubble.

Question 46

Water rises to a height of 4.5cm in a capillary tube of radius r_1 . When the same tube is dipped into a different liquid of density 800kgm^{-3} , surface tension 0.052Nm^{-1} , and angle of contact 30° , the liquid rises to 6.2cm. Find:

- the radius r_1 of the tube, and
- the surface tension of water if its angle of contact with glass is 0° . Take $\rho_{\text{water}} = 1000\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 47

A sphere of radius r and density 1.5ρ is released from rest in a liquid of density ρ and viscosity η . (a) Show that the terminal velocity is $v_t = \frac{r^2\rho g}{9\eta}$.

- Find the ratio of the terminal velocity of this sphere to that of a sphere of radius $2r$ and density 3ρ falling through the same liquid.

Question 48

A horizontal pipe carries water at 3ms^{-1} . At one point, the pipe develops a small leak (a hole of area 0.5cm^2 in the top surface). Calculate the speed at which water emerges from the leak and the volume of water lost per minute. Take the pressure inside the pipe as $2 \times 10^5\text{Pa}$ and atmospheric pressure as $1.01 \times 10^5\text{Pa}$. Take $\rho = 1000\text{kgm}^{-3}$.

Question 49

A viscous liquid flows through a horizontal pipe of radius R and length L . The flow rate is Q . The same liquid is then passed through a system of four identical pipes, each of radius $R/2$ and length L , connected in parallel. Compare the total flow rate through the parallel system with Q , assuming the same pressure difference drives the flow in both cases.

Question 50

A flat circular disc of radius 10cm is placed on a thin film of oil ($\eta = 0.5\text{Nsm}^{-2}$, film thickness 0.8mm) on a flat surface. The disc is rotated at 90 revolutions per minute. Calculate the torque required to maintain the rotation.

Question 51

A ball of mass 0.2kg is projected horizontally at 30ms^{-1} from a height of 25m . Ignoring air resistance, calculate where it lands. Now explain qualitatively how viscous drag from the air would change:

- (i) the horizontal distance, and
- (ii) the time of flight, compared to the no-drag case.

Question 52

A patient's artery has an internal radius of 2.5mm . Due to plaque buildup, the radius at one section is reduced to 1.5mm . Blood ($\rho = 1060\text{kgm}^{-3}$) flows at 0.3ms^{-1} in the healthy section where the pressure is $1.2 \times 10^4\text{Pa}$. Calculate:

- (a) the blood speed in the narrowed section,
- (b) the pressure in the narrowed section, and
- (c) the percentage reduction in volume flow rate through a long section of the narrowed artery compared to the healthy artery. Comment on the medical significance of your answers.

EXAMINATION QUESTIONS FOR PART THREE**Question 1**

- (a) A solid cylinder and a hollow cylinder of the same mass and radius are released simultaneously from rest at the top of an inclined plane. Both roll without slipping. The solid cylinder reaches the bottom first. Account for this observation.
- (b) A wet umbrella is spun rapidly and water droplets fly off the edges. Explain why the droplets leave the umbrella and why they travel in straight lines after leaving.
- (c) A uniform solid disc of mass 4.0kg and radius 0.30m is mounted on a frictionless axle. A light cord is wrapped around the rim and pulled with a constant force of 12N. Calculate the angular acceleration of the disc and the angular velocity after 5.0s from rest.

Question 2

- (a) An ice skater spins slowly with arms outstretched, then pulls her arms close to her body and spins much faster. No external torque acts on her. Account for the increase in angular velocity.
- (b) Water rises higher in a narrow glass tube than in a wide one when both are dipped into the same beaker. Explain why.
- (c) A soap bubble of radius 3.0cm ($\gamma = 0.025\text{Nm}^{-1}$) is blown at the end of a tube in a room where atmospheric pressure is $1.01 \times 10^5\text{Pa}$. Calculate the total pressure inside the bubble.

Question 3

- (a) A flywheel in a machine continues to rotate for a long time after the motor is switched off. A bicycle wheel with the same mass but smaller radius stops much sooner under similar frictional conditions. Account for this difference.
- (b) During a storm in Bagamoyo, a corrugated iron roof is lifted off a house even though the wind blows horizontally. Explain how a horizontal wind produces a vertical force on the roof.
- (c) A steel ball of radius 2.0mm ($\rho = 7800\text{kgm}^{-3}$) is dropped into glycerine ($\rho = 1260\text{kgm}^{-3}$, $\eta = 1.5\text{Nsm}^{-2}$). Calculate the terminal velocity of the ball. Take $g = 9.8\text{ms}^{-2}$.

Question 4

- (a) A rotating platform slows down when a person walks from the centre to the edge, but speeds up when the person walks back to the centre. Explain why.
- (b) A pond skater walks on water, but a small coin placed gently on the same surface sinks. Both are denser than water. Explain why the insect is supported but the coin is not.
- (c) A uniform rod of mass 3.0kg and length 1.2m is pivoted at one end and released from a horizontal position. Calculate the angular velocity of the rod as it passes through the vertical position. Take $g = 9.8\text{ms}^{-2}$.

Question 5

- (a) A ball rolling on a flat surface reaches the edge of a table and falls off. Even while falling, the ball continues to spin. Explain why the spin persists during the fall.
- (b) A doctor listens with a stethoscope and hears a faint murmur in a patient's artery. Explain what physical change in the artery causes this sound.
- (c) Water ($\eta = 1.0 \times 10^{-3}\text{Nsm}^{-2}$) flows through a horizontal pipe of internal radius 4mm and length 1.5m under a pressure difference of 600Pa. Calculate the volume flow rate and the maximum velocity at the centre of the pipe.

Question 6

- (a) Two gears of different radii are meshed together. When the smaller gear is turned, the larger gear rotates more slowly but with greater torque. Account for this observation.
- (b) Oil spilled from a tanker spreads rapidly on the ocean surface rather than remaining as a thick blob. Explain why, using the concept of surface energy.

- (c) A solid sphere of mass 2.0kg and radius 0.10m rolls without slipping down an incline of height 3.0m . Calculate the speed of the sphere at the bottom. Take $g = 9.8\text{ms}^{-2}$.

Question 7

- (a) A spinning top remains upright while spinning fast but falls over when it slows down. Account for this behaviour using the concept of angular momentum.
- (b) A capillary tube dipped in mercury shows a depression rather than a rise. Explain why mercury behaves differently from water in a glass tube.
- (c) A flat circular disc of radius 12cm rotates on a thin film of oil ($\eta = 0.80\text{Nsm}^{-2}$, thickness 1.0mm) at 120 revolutions per minute. Calculate the torque required to maintain the rotation.

Question 8

- (a) A solid sphere and a thin hollow spherical shell of equal mass and radius are given the same angular velocity. More work is needed to stop the shell than to stop the solid sphere. Account for this.
- (b) When Kipute blows air between two sheets of paper held parallel to each other, the sheets move toward each other instead of apart. Explain this observation.
- (c) A cylindrical tank of cross-sectional area 0.50m^2 is filled with water to a height of 4.0m . A small orifice of area 2.0cm^2 is opened at the base. Calculate the initial speed of efflux and the initial volume flow rate. Take $g = 9.8\text{ms}^{-2}$.

Question 9

- (a) A door is much easier to open by pushing near the handle than near the hinge, even though the same force is applied. Account for this observation.
- (b) A newly waxed car repels rainwater, which forms small beads on the surface. An unwaxed car allows water to spread into flat patches. Explain the difference.
- (c) Two spheres P and Q are made of the same material. Sphere P has radius r and sphere Q has radius $3r$. Both are dropped into the same viscous liquid. Find the ratio of their terminal velocities.

Question 10

- (a) A helicopter's tail rotor prevents the body of the helicopter from spinning. Explain why the body would spin without the tail rotor.
- (b) Fine clay particles stirred into a glass of water remain suspended for hours, while coarse sand grains settle within seconds. Explain this difference.
- (c) A uniform solid cylinder of mass 5.0kg and radius 0.20m rotates about its axis at 10rad s^{-1} . A braking torque of 0.50Nm is applied. Calculate the time taken for the cylinder to stop and the number of revolutions it makes before stopping.

Question 11

- (a) A uniform disc can be made to rotate about an axis through its centre or about an axis through its edge. The disc requires more torque to achieve the same angular acceleration when rotated about the edge. Account for this difference.
- (b) When a fast-flowing river passes under a bridge with narrow arches, the water level drops slightly as it passes through the arches. Explain why the water level decreases at the narrow section.
- (c) A venturi meter in a water pipeline has a pipe diameter of 15cm and a throat diameter of 5cm . A mercury manometer ($\rho = 13600\text{kgm}^{-3}$) shows a height difference of 3.0cm . Calculate the volume flow rate of water ($\rho = 1000\text{kgm}^{-3}$). Take $g = 9.8\text{ms}^{-2}$.

Question 12

- (a) A solid ball and a hollow ball of the same mass and radius roll along a flat surface at the same speed. The hollow ball is harder to stop. Account for this.
- (b) A student holds a funnel connected to a tap and lets water flow out of the wide end gently. She then flips the funnel so water exits the narrow end. The water shoots out much faster. Explain why.

- (c) A spherical drop of mercury ($\gamma = 0.5\text{Nm}^{-1}$, $\rho = 13,600\text{kgm}^{-3}$, $c = 140\text{Jkg}^{-1}\text{K}^{-1}$) of radius 2mm breaks into 8 identical smaller drops. Calculate the work done and the temperature rise if all the work is converted to heat.

Question 13

- (a) A ball rolling without slipping on a rough surface has both translational and rotational kinetic energy. If the same ball were sliding without rolling at the same speed on a frictionless surface, it would have less total kinetic energy. Account for this difference.
- (b) Honey pours slowly from a jar on a cold morning but flows freely when warmed. Explain why heating reduces the resistance to flow.
- (c) An aircraft of mass 8000kg has wings of total area 30m^2 . The air speed over the upper surface is 90ms^{-1} . Calculate the minimum speed of air under the lower surface for the aircraft to become airborne. Take $\rho_{\text{air}} = 1.2\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 14

- (a) When a car engine is started, the crankshaft must overcome the rotational inertia of the flywheel before the engine runs smoothly. Explain why a heavier flywheel requires more energy to start but provides smoother operation once running.
- (b) A person standing near the edge of a railway platform feels pulled toward a passing express train. Explain this effect.
- (c) A capillary tube of internal radius 0.5mm is dipped vertically into water ($\gamma = 0.073\text{Nm}^{-1}$, $\rho = 1000\text{kgm}^{-3}$, $\theta = 0^\circ$). Calculate the height of rise. If the tube is only 2cm long, explain what happens. Take $g = 9.8\text{ms}^{-2}$.

Question 15

- (a) A merry-go-round spins freely. A child standing at the edge jumps off tangentially. The merry-go-round speeds up slightly after the child leaves. Account for this.
- (b) A soap bubble is always spherical regardless of how it was blown. Explain why.
- (c) A uniform disc of mass 8.0kg and radius 0.40m rotates at 6.0rad s^{-1} about its axis. A 2.0kg lump of clay is dropped vertically onto the disc at the rim and sticks. Calculate the new angular velocity.

Question 16

- (a) A tightrope walker carries a long horizontal pole. Explain how the pole helps maintain balance.
- (b) Water flows through a horizontal pipe that widens at one section. At the wider section, the water moves more slowly. Yet the pressure at the wider section is higher than at the narrow section. Explain why slower-moving fluid has higher pressure.
- (c) A flat plate of area 0.04m^2 is pulled at constant velocity of 0.6ms^{-1} over a layer of oil ($\eta = 0.30\text{Nsm}^{-2}$) of thickness 2.0mm on a flat surface. Calculate the force required.

Question 17

- (a) A hammer thrown through the air tumbles, but one specific point inside the hammer follows a smooth parabolic path. Explain what this point is and why only this point follows the parabolic trajectory.
- (b) A water company needs to triple the flow rate through a pipe but can only increase the pressure by 50%. Explain why a small increase in pipe radius would be more effective than a large increase in pressure.
- (c) A solid sphere of mass 0.50kg and radius 0.05m rolls without slipping along a horizontal surface at 4.0ms^{-1} and then rolls up an incline. Calculate the maximum height reached on the incline. Take $g = 9.8\text{ms}^{-2}$.

Question 18

- (a) Two identical discs rotate about the same axis. One spins at ω and the other is stationary. They are pressed together and reach a common angular velocity. Explain why kinetic energy is lost in this process even though angular momentum is conserved.

- (b) An atomiser produces a fine spray when air is blown across the top of a vertical tube dipping into a liquid. Explain how the spray is produced.
- (c) A glass ball of radius 1.0mm and density 2500kgm^{-3} is timed falling through oil ($\rho_f = 800\text{kgm}^{-3}$). It takes 6.5s to fall 30cm at terminal velocity. Calculate the viscosity of the oil. Take $g = 9.8\text{ms}^{-2}$.

Question 19

- (a) A ball is thrown with backspin (spinning backward while moving forward). As it bounces, it reverses direction. Account for this observation.
- (b) A uniform rod pivoted at one end swings faster than the same rod pivoted at its centre when released from the same angular displacement. Account for this.
- (c) A turntable of moment of inertia 0.80kgm^2 rotates at 3.0rad s^{-1} . A ring of mass 2.0kg and radius 0.25m is gently placed on the turntable coaxially. Calculate the new angular velocity and the kinetic energy lost.

Question 20

- (a) A solid cylinder rolls down a rough incline without slipping, but the same cylinder slides down a smooth incline without rolling. In both cases, the cylinder starts from rest at the same height. The sliding cylinder reaches the bottom faster. Explain why.
- (b) Blood flows slowly in capillaries despite moving quickly in the aorta. Explain how the total cross-sectional area of capillaries accounts for this.
- (c) Two soap bubbles of radii 2.0cm and 5.0cm ($\gamma = 0.04\text{Nm}^{-1}$) are connected by a tube. Explain what happens and calculate the radius of curvature of the common interface.

Question 21

- (a) When a spinning gyroscope is placed on a pedestal with one end of its axle supported, it does not fall but instead moves sideways in a slow circle. Account for this behaviour.
- (b) A glass capillary tube is dipped into water and then slowly lifted out while the bottom end remains submerged. The water inside the tube remains higher than the water outside. Explain why the water does not drain out.
- (c) A string is wound around a solid cylinder of mass 4.0kg and radius 0.15m. The cylinder is free to rotate about a horizontal axis. When the string is released, a 2.0kg mass hangs from it and falls. Calculate the acceleration of the falling mass and the tension in the string. Take $g = 9.8\text{ms}^{-2}$.

Question 22

- (a) Two rods of the same mass and length are welded into a T-shape and an L-shape respectively. Both are rotated about an axis through one end. One has a larger moment of inertia than the other. Explain why the distribution of mass relative to the axis determines which shape is harder to rotate.
- (b) Water rises in a paper towel when one end is dipped in water, but does not rise in a waxed paper strip. Explain the difference.
- (c) Water flows upward through a vertical pipe that widens from radius 3cm at the bottom to 6cm at the top. The sections are 4m apart. The flow speed at the bottom is 6ms^{-1} and the pressure there is $2.5 \times 10^5\text{Pa}$. Find the pressure at the top. Take $\rho = 1000\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 23

- (a) A uniform metre ruler balanced at its centre with a heavy mass on one end tips over. The same ruler with the mass at the centre remains balanced. Explain why the position of the mass affects the stability of the ruler.
- (b) Explain why a spinning cricket ball curves sideways during flight.
- (c) A hollow sphere of mass M and radius R rolls without slipping along a horizontal surface with speed v . Show that the total kinetic energy is $\frac{5}{6}Mv^2$.

Question 24

- (a) An Atwood machine with a massive pulley takes longer to accelerate than one with a light pulley, even though the hanging masses are the same. Account for this.
- (b) A large raindrop falls faster than a small one during a downpour. Explain why, using the concept of terminal velocity.
- (c) A wire frame with a sliding wire of length 6cm has a soap film ($\gamma = 0.03\text{Nm}^{-1}$). The sliding wire is pulled outward by 4cm. Calculate the work done and the force on the sliding wire.

Question 25

- (a) A solid sphere rolling at speed v on a horizontal surface has more total kinetic energy than a block sliding at the same speed v on a frictionless surface. Both have the same mass. Account for the extra energy in the rolling sphere.
- (b) Explain why the excess pressure inside a small soap bubble is greater than inside a large soap bubble, even though both are made from the same solution.
- (c) A solid cylinder of mass 10kg and radius 0.25m is mounted horizontally. A rope is wound around it and a 5.0kg mass is attached to the free end. The system is released from rest. Calculate the speed of the hanging mass after it has fallen 2.0m. Take $g = 9.8\text{ms}^{-2}$.

Question 26

- (a) A car's engine is connected to the wheels through a gearbox. In low gear, the wheels turn slowly but with great torque. In high gear, the wheels turn fast but with less torque. Explain this trade-off.
- (b) A pitot tube mounted on an aircraft measures airspeed using a pressure difference. Explain how the speed of the aircraft is related to this pressure difference.
- (c) Glycerine ($\eta = 1.5\text{Nsm}^{-2}$) flows through a horizontal tube of radius 3mm and length 0.8m. The pressure difference is $5 \times 10^4\text{Pa}$. Calculate the volume flow rate and the Reynolds number. Is the flow laminar? Take $\rho = 1260\text{kgm}^{-3}$.

Question 27

- (a) A rigid body rotating about an axis has all its particles moving at different linear speeds but the same angular velocity. Account for this.
- (b) A water tank has two identical small holes at different depths. The water from the lower hole hits the ground further from the tank than water from the upper hole when both are open. Explain why.
- (c) A uniform solid disc of mass 6.0kg and radius 0.30m rotates at 8.0rad s^{-1} . Calculate the rotational kinetic energy and the torque required to bring it to rest in 4.0s.

Question 28

- (a) A mechanic uses two hands to turn a steering wheel. Each hand pushes in opposite directions on opposite sides of the wheel. Explain why this is more effective than pushing with one hand on one side.
- (b) A soap film formed on a wire frame with a sliding wire pulls the wire inward when released. Explain why the film exerts a force on the wire.
- (c) A sphere of density 2ρ and radius r is released in a liquid of density ρ and viscosity η . Show that the terminal velocity is $v_t = \frac{2r^2\rho g}{9\eta}$.

Question 29

- (a) A solid ball rolls up an incline and comes momentarily to rest before rolling back down. At the instant it is at rest, it has zero translational velocity but still has potential energy. Account for where the kinetic energy went.
- (b) Two flywheels of the same mass rotate at the same angular velocity. One is a solid disc and the other is a thin ring. The ring stores more kinetic energy. Account for this difference.

- (c) A solid cylinder and a solid sphere of the same mass and radius start from rest and roll without slipping down an incline of height h . Find the ratio of their speeds at the bottom.

Question 30

- (a) A rolling ball on a rough surface gradually slows down and eventually stops. Explain the role of friction in both maintaining the rolling motion and eventually stopping it.
- (b) A cook notices that fresh cooking oil flows easily from the bottle, but the same oil after repeated use in a deep fryer becomes thick and pours slowly. Explain what has changed in the oil.
- (c) A pulley of mass 3.0kg and radius 0.20m is treated as a uniform disc. Masses of 5.0kg and 3.0kg hang on opposite sides. Calculate the acceleration of the system and the tensions in the two strings. Take $g = 9.8\text{ms}^{-2}$.

Question 31

- (a) A constant torque applied to a wheel for a short time produces a small change in angular velocity. The same torque applied for twice as long produces twice the change. Account for this using the rotational analogue of Newton's second law.
- (b) When a garden hose with a nozzle is used, the water exits at high speed and the hose recoils backward. Explain how Bernoulli's principle and Newton's third law each contribute to this observation.
- (c) An air bubble of radius 0.5mm is formed at the bottom of a lake at a depth of 20m . Calculate the total pressure inside the bubble. Take $\gamma = 0.073\text{Nm}^{-1}$, $\rho = 1000\text{kgm}^{-3}$, $P_{\text{atm}} = 1.01 \times 10^5\text{Pa}$, and $g = 9.8\text{ms}^{-2}$.

Question 32

- (a) A uniform rod is easier to balance vertically on a finger if it is long than if it is short. Account for this.
- (b) A thick milkshake is much harder to sip through a narrow straw than through a wide one. Explain why.
- (c) A uniform solid sphere of mass 3.0kg and radius 0.10m is rotating at 20rad s^{-1} about an axis through its centre. Calculate its angular momentum and the torque needed to stop it in 5.0s .

Question 33

- (a) A ball placed on a smooth incline slides without rotating, but the same ball on a rough incline both rotates and translates simultaneously. Explain the role of friction in producing rotation during rolling without slipping.
- (b) Explain why a liquid drop in free fall (such as inside the International Space Station) is perfectly spherical, while a drop resting on a table is flattened.
- (c) A large open tank has water filled to 6m . A hole is opened at a depth of 2m below the surface. The jet emerges horizontally. Calculate the horizontal distance from the tank where the jet hits the ground. Take $g = 9.8\text{ms}^{-2}$.

Question 34

- (a) A flat circular disc has a moment of inertia about an axis perpendicular to its face that equals the sum of the moments about two perpendicular diameters. Explain physically why the resistance to rotation about the face axis is greater than about a diameter.
- (b) A farmer in Singida breaks up the top layer of dry soil to prevent water from evaporating. Explain how breaking the top layer reduces water loss.
- (c) A disc of radius 0.20m is placed on a thin oil film ($\eta = 0.40\text{Nsm}^{-2}$, thickness 0.5mm) and pushed at constant velocity 0.50ms^{-1} . The disc has area 0.04m^2 . Calculate the force needed.

Question 35

- (a) A cannon mounted on a wheeled cart fires a ball. The cart rolls backward. Explain why the angular momentum of the system is not conserved about a horizontal axis through the cannon's barrel, but the linear momentum is conserved.
- (b) A water tank with a tap at the bottom delivers water more slowly as the tank empties. Explain why the flow rate decreases as the water level drops.

- (c) A uniform solid cylinder of mass M and radius R rolls without slipping down an incline angled at θ to the horizontal. Show that the acceleration is $a = \frac{2g\sin\theta}{3}$.

Question 36

- (a) A long-handled spanner makes it easier to loosen a tight bolt than a short-handled one. Explain why, using the relationship between torque, force, and the distance from the pivot.
- (b) Raindrops are approximately spherical, not teardrop-shaped as commonly depicted. Explain why surface tension produces this shape.
- (c) A 2.0kg sphere ($\rho = 7800\text{kgm}^{-3}$) falls through oil ($\rho = 900\text{kgm}^{-3}$, $\eta = 0.50\text{Nsm}^{-2}$) and reaches terminal velocity. Calculate the terminal velocity. Take $g = 9.8\text{ms}^{-2}$.

Question 37

- (a) A playground roundabout is pushed by a child running alongside it. Even after the child stops pushing, the roundabout continues to accelerate briefly before beginning to decelerate. Explain why the deceleration does not begin immediately.
- (b) Two ships sailing parallel to each other in close proximity are drawn toward each other. Explain this effect.
- (c) A body of moment of inertia 0.60kgm^2 rotates at 10rad s^{-1} . A constant frictional torque reduces it to 4rad s^{-1} in 8s . Calculate the frictional torque, the number of revolutions in those 8s , and the energy dissipated.

Question 38

- (a) A coin spinning on a table eventually wobbles and falls flat. Explain why the coin cannot maintain its spinning motion indefinitely.
- (b) A child blows a soap bubble and notices that the bubble slowly shrinks when the blowing tube is left open. Explain why the bubble contracts rather than remaining the same size.
- (c) Two horizontal capillary tubes of equal length are connected in series. The first has radius R and the second has radius $2R$. Find the ratio of the pressure drop across the first tube to that across the second tube.

Question 39

- (a) A solid sphere and a solid cylinder of equal mass and radius roll at the same speed on a horizontal surface. Both experience the same rolling friction. The cylinder takes longer to stop. Account for this.
- (b) A factory uses a long, narrow pipe to transport oil. In winter, the flow rate drops dramatically even though the pump pressure has not changed. Explain why.
- (c) A motor drives a grinding wheel of moment of inertia 0.40kgm^2 from rest to 50rad s^{-1} in 10s at constant angular acceleration. Calculate the torque exerted by the motor and the power delivered at the end of the 10s .

Question 40

- (a) Two identical flywheels rotate at the same angular velocity. One is suddenly coupled to a stationary identical flywheel. Explain what happens to the angular velocity and the kinetic energy of the system.
- (b) During a flood, water rushes through a narrow gap in a wall much faster than in the wide river behind it. Explain why.
- (c) A pitot tube connected to a water manometer ($\rho = 1000\text{kgm}^{-3}$) measures the speed of air ($\rho = 1.2\text{kgm}^{-3}$) flowing through a duct. The manometer shows a height difference of 8.0cm . Calculate the air speed. Take $g = 9.8\text{ms}^{-2}$.

Question 41

- (a) A ball thrown with topspin bounces at a lower angle than a ball thrown without spin. Account for this.

- (b) A swimming pool overflow system uses a wide channel to carry water. Replacing it with a narrower channel reduces the flow significantly, even though the height difference driving the flow remains the same. Explain why.
- (c) A solid sphere of mass 2.5kg and radius 0.08m is suspended from a string and acts as a pendulum. Using the parallel axis theorem, calculate the moment of inertia about the pivot point (at the top of the string, 0.60m from the centre of the sphere).

Question 42

- (a) Two soap bubbles of different sizes are connected by a tube. The smaller bubble gets smaller and the larger bubble gets larger until only one remains. Explain why.
- (b) A diver at depth experiences pressure from all directions, not just from above. Explain why fluid pressure at a point is the same in all directions.
- (c) A solid sphere of mass 1.5kg and radius 0.06m rolls without slipping at 5.0ms^{-1} on a horizontal surface and then rolls up a 30° incline. Calculate the distance along the incline the sphere travels before stopping. Take $g = 9.8\text{ms}^{-2}$.

Question 43

- (a) A rotating platform is used in a physics experiment. When sand is gently sprinkled onto the rotating platform from above, the platform slows down. Explain why.
- (b) A pipe carrying water develops a small leak at the top. Water squirts upward from the leak. Explain why the water emerges from the leak instead of air being drawn in.
- (c) A uniform rod of mass 2.0kg and length 0.80m is pivoted at its centre and initially at rest. A 0.05kg bullet travelling at 200ms^{-1} horizontally strikes one end of the rod and embeds itself. Calculate the angular velocity of the rod immediately after impact.

Question 44

- (a) A gyroscope spinning inside an aircraft helps maintain the aircraft's orientation. Explain how the gyroscope resists changes in direction.
- (b) During a severe storm, windows sometimes shatter outward rather than inward, even though the wind is blowing against the outside of the building. Explain why.
- (c) Three identical spheres A, B, and C have the same radius but are made of different materials with densities 8000kgm^{-3} , 4000kgm^{-3} , and 2000kgm^{-3} respectively. All three are dropped simultaneously into the same viscous liquid of density 1000kgm^{-3} . Rank them in order of terminal velocity from highest to lowest, and find the ratio $v_A : v_B : v_C$.

Question 45

- (a) A rotating disc on a viscous oil film gradually slows down and eventually stops. Explain the role of viscosity in removing the disc's rotational kinetic energy.
- (b) A dam is built much thicker at the base than at the top. Explain the physics behind this design.
- (c) A solid ball of mass 0.30kg and radius 0.02m rolls without slipping along a table at 3.0ms^{-1} . It rolls off the edge and falls 1.5m to the ground. Calculate the speed of the ball just before it hits the ground. Take $g = 9.8\text{ms}^{-2}$.

Question 46

- (a) A spinning hard-boiled egg is briefly touched to stop it and then released. It stays still. A raw egg treated the same way starts spinning again after the finger is removed. Account for this difference.
- (b) A laboratory technician drops identical steel balls into two tall jars of different liquids. The ball in one jar takes much longer to reach the bottom. Explain how this observation reveals which liquid has higher viscosity.
- (c) In a blood transfusion, the bottle is positioned so that the blood level is 1.3m above a needle of internal diameter 0.36mm and length 30mm. If 4.5cm^3 of blood passes through the needle in one minute, calculate the viscosity of blood. Take $\rho_{\text{blood}} = 1060\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 47

- (a) A solid sphere and a solid cylinder of equal mass roll without slipping from rest down the same incline. Without calculation, explain which reaches the bottom with greater translational speed and why.
- (b) A Bunsen burner draws air into its barrel through a small side hole when gas flows through the narrow jet at the base. Explain how the flowing gas draws air into the burner.
- (c) A water tower maintains a constant water level at a height of 10m above a circular outlet at the base. Calculate the diameter of the outlet needed to deliver water at a rate of 26.4m^3 per minute. Take $g = 9.8\text{ms}^{-2}$.

Question 48

- (a) A ball rolling along a surface strikes a low step. If the ball is large enough that its centre is above the top of the step, it rolls over. If the ball's centre is below the top, it bounces back. Account for this difference using the concept of torque about the contact point.
- (b) When Kipute pours water gently from a bottle, it flows smoothly down the side. When she tilts the bottle steeply and pours fast, the flow becomes noisy and irregular. Explain what causes this transition.
- (c) An air bubble of radius 1.0cm rises at a steady speed of 0.20cms^{-1} through a liquid of density 1470kgm^{-3} . Calculate the viscosity of the liquid. Take $\rho_{\text{air}} = 1.2\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 49

- (a) A bicycle wheel with light spokes and a heavy rim has a larger moment of inertia than a wheel with heavy spokes and a light rim, even though both have the same total mass. Account for this.
- (b) A patient's blood pressure is always measured at the upper arm, level with the heart. Explain why measuring at the ankle would give a higher reading that does not reflect the heart's actual pumping pressure.
- (c) A pipe carrying liquid develops deposits on its inner walls, reducing the effective radius by 8%. Calculate the factor by which the pressure difference must increase to maintain the same flow rate.

Question 50

- (a) A disc rolling without slipping on a rough surface reaches the base of a smooth (frictionless) incline and begins to climb. Explain what happens to the translational and rotational motion of the disc as it moves up the smooth incline.
- (b) Water from a tap flows as a smooth, clear stream near the tap but breaks into separate droplets further down. Explain why the stream breaks up as it falls.
- (c) A spherical raindrop of radius 2.0mm and density 1000kgm^{-3} falls from a height of 500m. It accelerates with decreasing acceleration until reaching terminal velocity at a height of 250m, then falls at constant speed for the remaining distance. Calculate the work done by gravity on the drop in each half of the journey. Take $g = 9.8\text{ms}^{-2}$.

Question 51

- (a) A water tank has a small hole in its wall and is continuously refilled to maintain a constant water level. A plumber suggests enlarging the hole to get more water. Explain why the speed of the emerging water does not increase when the hole is enlarged, but the volume of water collected per second does increase.
- (b) Campers are advised never to touch the inside of a wet canvas tent during rain. Explain why touching the inside causes water to drip through.
- (c) Water flows steadily through a horizontal tube of cross-sectional area 25cm^2 . The static pressure is $1.3 \times 10^5\text{Pa}$ and the total pressure is $1.4 \times 10^5\text{Pa}$. Calculate the velocity of the water and the mass of water flowing past a point per second. Take $\rho = 1000\text{kgm}^{-3}$.

Question 52

- (a) A thin uniform rod is held horizontally at one end and released. The free end accelerates downward faster than a freely falling object. Explain how a point on the rod can accelerate faster than g even though the only forces acting are gravity and the pivot reaction.

- (b) A skydiver falls much faster before opening the parachute than after. Explain how the parachute reduces the terminal velocity.
- (c) In a Millikan-type experiment, an uncharged oil drop of radius $2.0 \times 10^{-5}\text{m}$ and density $1.2 \times 10^3\text{kgm}^{-3}$ falls through air of viscosity $1.8 \times 10^{-5}\text{Nsm}^{-2}$. Neglecting the buoyancy of air, calculate the terminal speed of the drop and the viscous force at that speed. Take $g = 9.8\text{ms}^{-2}$.

Question 53

- (a) A mechanic applies a large force to a wrench trying to loosen a seized bolt, but the bolt does not move. Despite the large torque, no work is done on the bolt. Explain why.
- (b) A large bottle is emptied through a capillary siphon. The same bottle filled with petrol empties 2.5 times faster than when filled with water through the same siphon. Explain what this tells us about the relative viscosities of water and petrol.
- (c) Water flows through a horizontal pipe that narrows from a diameter of 0.60m at section A to 0.20m at section B. The pressure difference between A and B is equivalent to 1.0m column of water. Calculate the volume of water flowing per second. Take $g = 9.8\text{ms}^{-2}$ and $\rho = 1000\text{kgm}^{-3}$.

Question 54

- (a) A ball rolling without slipping on a rough incline slows down and eventually comes to rest with both $v = 0$ and $\omega = 0$ at the same instant. Explain why the translational and rotational motions stop simultaneously rather than one stopping before the other.
- (b) A glass rod dipped in water and withdrawn carries a thin film of water clinging to it. The same rod dipped in mercury and withdrawn comes out perfectly clean and dry. Explain this difference.
- (c) Water ($\eta = 1.0 \times 10^{-3}\text{Nsm}^{-2}$, $\rho = 1000\text{kgm}^{-3}$) flows through a pipe of diameter 2.0cm. Calculate the maximum volume flow rate before the flow becomes turbulent. Take the critical Reynolds number as 2000.

Question 55

- (a) A child pushes a playground roundabout through two full revolutions. Another child pushes with the same force but through only half a revolution. Explain why the first child transfers more energy to the roundabout.
- (b) A glass capillary tube dipped in paraffin shows a smaller rise than the same tube dipped in water. Explain why different liquids rise to different heights in the same capillary tube.
- (c) A glass capillary tube is dipped vertically into paraffin ($\gamma = 0.026\text{Nm}^{-1}$, $\rho = 800\text{kgm}^{-3}$, angle of contact = 26°). The paraffin rises 4.5cm. Calculate the internal diameter of the tube. Take $g = 9.8\text{ms}^{-2}$.

Question 56

- (a) A horizontal pipe of uniform cross-section carries water from one end to the other. According to Bernoulli's theorem, the pressure should remain constant along the pipe since the speed and height do not change. However, pressure measurements reveal a steady decrease from inlet to outlet. Explain this observation.
- (b) A flag mounted on a pole flutters vigorously when strong winds blow but hangs limply on a calm day. Explain why moving air causes the flag to flutter rather than remain flat.
- (c) A pressure gauge attached to a closed water pipe reads $2.5 \times 10^5\text{Pa}$. When the valve is opened and water begins to flow, the reading drops to $2.0 \times 10^5\text{Pa}$. Calculate the speed of water flowing through the pipe. Take $\rho = 1000\text{kgm}^{-3}$.

Question 57

- (a) A parachutist jumps from a height of 2000m and descends gently, while a stone released from the same height crashes into the ground at high speed. Explain why the two objects behave so differently despite falling from the same height.

- (b) Water emerges horizontally from a small hole in the side of a tank, but the stream curves downward and hits the ground some distance from the base. Explain why the water follows a curved path rather than continuing in a straight horizontal line.
- (c) Two identical spherical drops of water fall through air, each at terminal velocity V . The drops collide and merge into a single spherical drop. Determine the terminal velocity of the combined drop in terms of V .

Question 58

- (a) A student claims that replacing a water tank with a wider one, while keeping the water level the same, will reduce the efflux velocity from a hole at the base. Explain why this claim is incorrect.
- (b) Water flows steadily through a horizontal pipe that narrows from a wide section to a narrow section. Measurements show that water moves faster in the narrow section. Explain why the speed increases even though nothing is pushing the water harder.
- (c) Water is maintained at a constant height of 10m in a large tank. A small hole in the side of the tank is 3.0m below the surface. The jet emerges horizontally. Calculate the speed of efflux and the horizontal distance from the tank at which the jet strikes the ground. Take $g = 9.8\text{ms}^{-2}$.

Question 59

- (a) Soap solution is a more effective cleansing agent than plain water. Explain why, using the concept of surface tension.
- (b) Explain why the excess pressure inside a soap bubble of a given radius is exactly twice that inside an air bubble of the same radius in water.
- (c) The excess pressure inside a soap bubble of radius 1.0cm is balanced by the pressure due to a column of oil of specific gravity 0.8 and height 2.0mm. Calculate the surface tension of the soap solution. Take $g = 9.8\text{ms}^{-2}$.

Question 60

- (a) A large open tank has a small circular hole at its base. When the tank is full, the water jet from the hole shoots out forcefully. As the tank drains and the water level falls, the jet becomes progressively weaker. Explain this observation.
- (b) A container resting on a smooth surface has a small hole in one of its walls. When the container is filled with liquid and the hole is unplugged, the liquid flows out and the container begins to move in the opposite direction. Explain why the container experiences this backward thrust.
- (c) A large cylindrical tank contains water to a depth of 0.30m. A hole of cross-sectional area 6.5cm^2 at the bottom allows water to drain. Calculate the initial drainage rate. At a point 30cm below the hole, determine the cross-sectional area of the falling water stream. Take $g = 9.8\text{ms}^{-2}$.

Question 61

- (a) A daladala mechanic in Dar es Salaam squirts oil horizontally from a can held at waist height. The oil lands further away than when the can is held near the ground. Explain why the height at which the can is held affects the horizontal distance the oil travels before hitting the ground.
- (b) A horizontal pipe carrying water has a narrow constriction in the middle. Small vertical tubes open to the atmosphere are attached at the wide section and at the constriction. The water level in the vertical tube at the constriction is lower than at the wide section. Explain this observation.
- (c) A syringe containing water is held horizontally with its nozzle at a height $h = 1.25\text{m}$ above the ground. The diameter of the piston is 5 times that of the nozzle. The piston is pushed at a constant speed of 0.20ms^{-1} . Calculate the speed at which the water hits the ground. Take $g = 9.8\text{ms}^{-2}$.

Question 62

- (a) A nurse at Muhimbili National Hospital notices that replacing an IV needle with one of slightly smaller radius causes the drip rate to fall dramatically. Explain why a small decrease in needle radius produces such a large drop in flow rate.

- (b) Two capillary tubes of different radii but the same length are connected in series, and liquid flows through them under streamline conditions. The pressure drop is not shared equally between the two tubes. Explain why the narrower tube carries the larger share of the total pressure drop.
- (c) Three capillaries of the same length but of internal radii $3r$, $4r$, and $5r$ are connected in series. Liquid flows through them under streamline conditions. If the pressure difference across the third capillary is 8.1mm of the liquid, find the pressure difference across the first capillary.

Question 63

- (a) On a still morning in Dar es Salaam, tiny fog droplets seem to hang almost motionless in the air, while larger raindrops during the afternoon downpour fall rapidly. Explain why very small drops fall so much more slowly than large ones.
- (b) A glass sphere falls through glycerine and reaches terminal velocity. When the experiment is repeated with the glycerine warmed to a higher temperature, the same sphere reaches a higher terminal velocity. Explain this observation.
- (c) An oil drop of density 900kgm^{-3} falls through air ($\eta = 1.8 \times 10^{-5}\text{Nsm}^{-2}$, $\rho_{\text{air}} = 1.2\text{kgm}^{-3}$) at a terminal velocity of $3.2 \times 10^{-2}\text{ms}^{-1}$. Calculate the radius of the drop. If the drop splits into a smaller drop of half the original radius, calculate the new terminal velocity. Take $g = 9.8\text{ms}^{-2}$.

Question 64

- (a) The accumulation of snow or ice on an aeroplane wing during flight may reduce the lift. Explain how this surface change affects the wing's ability to generate lift.
- (b) When a liquid flows through a narrow constriction in a pipe, its speed increases. Explain where the liquid gets the extra kinetic energy for this increased speed.
- (c) Water flows steadily through a horizontal pipe made of two sections joined end to end. The first section is 21cm long with a diameter of 0.225cm , and the second is 7cm long with a diameter of 0.075cm . If the pressure difference across the entire combination is 14cm of water, find the pressure difference across each section. Take $g = 9.8\text{ms}^{-2}$ and $\rho = 1000\text{kgm}^{-3}$.

Question 65

- (a) Automobile manufacturers recommend using oils of different viscosities in different seasons. Explain why the same oil is not suitable for both hot and cold weather.
- (b) A car engine runs more freely and smoothly as it heats up after a cold start. Explain why warming up reduces the resistance to motion inside the engine.
- (c) In an experiment to determine the coefficient of viscosity of motor oil, a glass sphere of mass $1.2 \times 10^{-4}\text{kg}$ and diameter $4.0 \times 10^{-3}\text{m}$ reaches a terminal velocity of $5.4 \times 10^{-2}\text{ms}^{-1}$. The density of the motor oil is 900kgm^{-3} . Calculate the coefficient of viscosity. Take $g = 9.8\text{ms}^{-2}$.

Question 66

- (a) A water tank at a school in Mwanza is continuously fed by a supply pipe at the top. Students open a tap at the base and notice that the water level drops at first but then remains steady, even though the tap is still running. Explain why the water level stabilises at a particular height.
- (b) Under a constant pressure head, a liquid flows through a pipe at rate Q . The pipe is replaced by one that is twice as long and half the diameter. Explain qualitatively why the new flow rate is dramatically less than Q .
- (c) Under a given pressure head, the rate of flow of liquid through a pipe is Q . If the length of the pipe is doubled and its diameter is halved, calculate the new rate of flow in terms of Q .

Question 67

- (a) Most people assume an aeroplane can only fly right-side up, yet stunt pilots routinely fly upside down. Explain how an aeroplane generates lift even when it is inverted.
- (b) During takeoff and landing, pilots extend wing flaps that increase both the wing area and the angle of attack. Explain how extending flaps helps the aircraft fly safely at lower speeds.

- (c) Water is supplied to a house at ground level through a pipe of inner diameter 1.5cm at an absolute pressure of $6.5 \times 10^5 \text{ Pa}$ and a velocity of 5 ms^{-1} . The pipe leading to the second floor bathroom 8m above has an inner diameter of 0.75cm. Calculate the flow velocity and the pressure at the second floor. Take $\rho = 1000 \text{ kg m}^{-3}$ and $g = 9.8 \text{ ms}^{-2}$.

Question 68

- (a) A horizontal pipeline increases uniformly from 0.080m diameter to 0.160m diameter in the direction of flow. A student predicts that the pressure gauge reading at the wider section should be lower because “more space means less pressure.” Explain why this prediction is incorrect.
- (b) A large cylindrical tank of cross-sectional area A contains water to a height H_1 above a small hole of area a at the base, where $a \ll A$. The hole is opened and the tank drains. Show that the time for the water level to fall from H_1 to H_2 is:

$$t = \frac{A}{a} \sqrt{\frac{2}{g}} (\sqrt{H_1} - \sqrt{H_2})$$

- (c) Water flows through the pipeline described in (a) at 96 litres per second. A pressure gauge at the 0.080m section reads $3.5 \times 10^5 \text{ Pa}$. Calculate the reading at the 0.160m section. Take $\rho = 1000 \text{ kg m}^{-3}$.

Question 69

- (a) Water flows into a tank of large cross-sectional area at a constant rate while draining out through a hole at the base. The water level initially rises but eventually stabilises at a fixed height. Explain why the level stops rising.
- (b) Two small spheres of the same material fall through a viscous liquid. The diameter of one sphere is three times that of the other. After both have reached terminal velocity, explain which sphere takes longer to fall the same vertical distance and why.
- (c) Water flows into a tank at a rate of $2 \times 10^{-4} \text{ m}^3 \text{ s}^{-1}$ and drains out through a hole of area 2 cm^2 at the base. Calculate the equilibrium height of water in the tank. Take $g = 9.8 \text{ ms}^{-2}$.

Question 70

- (a) It is much easier to throw a curve with a tennis ball than with a cricket ball. Explain why.
- (b) A horizontal cylinder filled with water of volume V is fitted with a piston at one end and a small orifice of cross-sectional area a at the other, where a is much less than the piston area. A constant force pushes the piston and squeezes all the water out through the orifice in time t . Neglecting friction and viscosity, show that the work done by the force is $W = \frac{\rho V^3}{2a^2 t^2}$.
- (c) A steel ball of radius 2.0mm and density 7800 kg m^{-3} is dropped into glycerine of density 1260 kg m^{-3} and viscosity 0.83 N s m^{-2} . Calculate the terminal velocity. Take $g = 9.8 \text{ ms}^{-2}$.

Question 71

- (a) Water leaks horizontally from a hole in the side of a large tank. A plumber plugs the hole and drills a new one at a lower position on the same wall. The jet from the lower hole emerges faster but hits the ground closer to the tank. Explain why a faster jet can land closer to the base.
- (b) An oil drop falls through air with a terminal velocity of $5 \times 10^{-4} \text{ ms}^{-1}$. Neglecting the density of air compared to that of oil, explain why a drop of half the radius would have a terminal velocity exactly one quarter of this value.
- (c) Water leaks horizontally from a small hole in the side of a large open tank. The hole is 80cm below the water surface and 120cm above the ground. Calculate the speed at which water emerges from the hole and the horizontal distance from the tank at which the stream hits the ground. Take $g = 9.8 \text{ ms}^{-2}$.

Question 72

- (a) A high diver can perform more somersaults before entering the water if she tucks her body tightly rather than keeping it outstretched. Explain why tucking allows more somersaults in the same fall time.

- (b) There is no single, unique value of the moment of inertia for a given rigid body. Explain why and identify what determines the value.
- (c) A disc of moment of inertia $2.5 \times 10^{-4} \text{kgm}^2$ rotates freely about an axis through its centre at 20 revolutions per minute. A piece of wax of mass 0.04kg is dropped gently onto the disc at a distance of 0.05m from the axis. Calculate the new angular velocity in revolutions per minute.

Question 73

- (a) A flywheel is designed so that most of its mass is concentrated at the rim rather than near the hub. Explain why this design is preferred for a flywheel whose purpose is to store rotational energy and smooth out speed fluctuations.
- (b) A uniform solid sphere has the same moment of inertia about every axis passing through its centre. Explain why a sphere has this property while a cylinder does not.
- (c) A grinding wheel of radius 0.40m and mass 3.0kg rotates at 3600 revolutions per minute. Calculate the torque required to bring it to rest in 10s. Treat the wheel as a uniform disc.

Question 74

- (a) A rotating object must always have a non-zero moment of inertia about its axis of rotation. Explain why it is physically impossible for a real object to rotate with zero moment of inertia.
- (b) A solid cylinder rolls without slipping down an inclined plane. Explain why the acceleration of the cylinder down the incline is less than $g \sin \theta$.
- (c) A solid cylinder of mass M and radius of gyration k rolls without slipping from rest down an inclined plane of angle α and length ℓ . Find the speed at the bottom in terms of α , ℓ , k , g , and the radius R .

Question 75

- (a) A hollow cylinder and a solid cylinder have the same mass and the same outer radius. Both are set spinning about their central axes to the same angular velocity. Explain which one requires more work to reach that angular velocity and why.
- (b) A uniform semicircular hoop of mass M and radius R lies in a plane. By integration, show that the centre of mass of the hoop is located at a distance $\frac{2R}{\pi}$ from the centre of the full circle along the axis of symmetry.
- (c) A thin circular ring of mass M and radius r rotates about its axis with angular velocity ω_1 . Two small objects, each of mass m , are gently attached to the ring at diametrically opposite points. Calculate the new angular velocity in terms of M , m , and ω_1 .

Question 76

- (a) A rotating object does not necessarily have a net torque acting on it. Explain why an object can rotate at constant angular velocity with zero net torque, and give a physical example.
- (b) Mechanics sometimes extend the handle of a wrench by slipping a section of pipe over it to loosen a seized bolt. Explain why this procedure makes the bolt easier to turn but risks damaging it.
- (c) Masses $m_1 = 0.20\text{kg}$ and $m_2 = 0.25\text{kg}$ are suspended from a light cord that passes over a wheel of radius 0.15m and moment of inertia 0.12kgm^2 . The masses start at the same height. Calculate the speed of each mass and the angular velocity of the wheel when the vertical distance between them is 0.30m. Take $g = 9.8\text{ms}^{-2}$.

Question 77

- (a) A ball sliding without rotating on a frictionless surface has translational kinetic energy but no rotational kinetic energy. If it then moves onto a rough surface and begins to roll without slipping, its translational speed decreases. Explain where the “lost” translational energy has gone.
- (b) The Earth moves faster when closer to the Sun and slower when farther away. Explain why, using conservation of angular momentum.
- (c) If the Earth were to suddenly contract to half its present radius without any external torque acting on it, calculate the new length of the day. Assume the Earth is a uniform solid sphere.

Question 78

- (a) Opening a door requires pushing at a single point far from the hinge, while twisting open a bottle cap requires gripping with thumb and fingers on opposite sides. Explain why different techniques are needed for these two tasks.
- (b) A stationary object can have a non-zero angular acceleration at the instant a torque is first applied to it. Explain why an object at rest can simultaneously have $\omega = 0$ and $\alpha \neq 0$.
- (c) The Earth has a radius of $6.37 \times 10^6 \text{ m}$ and a mean density of 5450 kg m^{-3} . Calculate the moment of inertia of the Earth about its axis of rotation. Treat the Earth as a uniform solid sphere.

Question 79

- (a) When building a house, masons plaster brick walls with cement. Explain how the plastering improves the strength and water resistance of the wall, referring to the role of surface tension and adhesion.
- (b) Molecules at the surface of a liquid have more potential energy than molecules deep within the liquid. Explain why a molecule at the surface is in a higher energy state than one surrounded by neighbours on all sides.
- (c) During a rainstorm, 64 identical small raindrops, each of radius r , combine to form a single large drop. Calculate the ratio of the total surface energy of the 64 small drops to the surface energy of the single large drop.

Question 80

- (a) A barometer uses a narrow tube, which introduces an error in the reading because capillary rise adds to the true liquid column height. Explain how using two tubes of different radii and comparing their readings allows the true pressure to be determined.
- (b) A thin glass plate resting on the surface of water requires a measurable downward force, beyond its own weight, to detach it from the surface. Explain where this extra resistance comes from and why it depends on the perimeter of the plate rather than its area.
- (c) A barometer contains two uniform capillary tubes of radii $6.5 \times 10^{-4} \text{ m}$ and $1.24 \times 10^{-3} \text{ m}$. The height of water in the narrow tube is 0.20 m more than in the wide tube. Calculate the true pressure difference. Take $\gamma = 0.073 \text{ N m}^{-1}$, $\theta = 0^\circ$, $\rho = 1000 \text{ kg m}^{-3}$, and $g = 9.8 \text{ m s}^{-2}$.

Question 81

- (a) Two soap bubbles of unequal radii are blown at the two ends of a tube and then connected by opening a valve between them. Explain what happens and why.
- (b) Surface tension may be defined as a force per unit length or as energy per unit area. By considering the work done in increasing the surface area of a liquid film, show that these two definitions are dimensionally and physically equivalent.
- (c) A spherical drop of mercury of radius 5 mm falls on the ground and breaks into 1000 identical droplets. Calculate the work done in breaking the drop. Take $\gamma = 0.50 \text{ N m}^{-1}$.

Question 82

- (a) Hot soup releases flavour molecules more readily than cold soup of the same recipe. Explain how the temperature of the soup affects the transport of flavour molecules to your sense of smell and taste, referring to the role of surface tension.
- (b) When writing with a fountain pen, ink flows from the reservoir down through the nib and onto the paper, even when the pen is held at an angle against gravity. Explain how capillary action enables the ink to reach the paper.
- (c) Calculate the work done against surface tension in blowing a soap bubble of diameter 1.0 cm . Take $\gamma = 0.030 \text{ N m}^{-1}$.

Question 83

- (a) Two pieces of paper joined by glue are virtually impossible to pull apart cleanly. Explain why the bond formed by the glue is so strong, referring to adhesive forces and surface contact.

- (b) The coefficient of surface tension of a liquid does not change when the surface area is doubled. Explain why surface tension is an intrinsic property of the liquid and not a function of surface area.
- (c) Two spherical mercury droplets of radii 0.1cm and 0.2cm coalesce into a single spherical drop. Calculate the energy released. Take $\gamma = 0.50\text{Nm}^{-1}$.

Question 84

- (a) The angle of contact of mercury with glass is obtuse, while that of water with glass is acute. Explain why the same glass surface produces opposite wetting behaviour in these two liquids.
- (b) When blowing soap bubbles, it is noticeably harder to start a very small bubble than to inflate it further once it has grown. Explain why more effort is needed to form a small bubble than to enlarge an existing one.
- (c) Water rises in a capillary tube to a height of 2.0cm. Calculate the height to which water rises in another capillary tube whose radius is one-third that of the first tube.

Question 85

- (a) During the rainy season in Moshi, rainwater soaks into a cotton towel left outside but rolls off a plastic tarpaulin in nearly spherical droplets. Explain this difference in behaviour.
- (b) The pressure inside a very small air bubble submerged in water is significantly greater than the pressure in the surrounding water at the same depth, while for a large bubble the difference is negligible. Explain why the excess pressure depends on the size of the bubble.
- (c) 1000 spherical droplets of water, each of diameter 10^{-8}m , coalesce to form a single large spherical drop. Calculate the energy liberated. Take $\gamma = 0.073\text{Nm}^{-1}$.

Question 86

- (a) A farmer breaks up the top layer of dry soil to prevent water from evaporating from the moist soil below. Explain how breaking the soil reduces water loss, referring to capillary action.
- (b) A number of small spherical water droplets, each of radius r , coalesce to form a single spherical drop of radius R . The surface energy released heats the combined drop. Show that the rise in temperature is $\Delta\theta = \frac{3\gamma}{\rho c} \left(\frac{1}{r} - \frac{1}{R} \right)$.
- (c) The xylem tubes in the actively growing outer layer of a tree are uniform cylinders with a contact angle of 45° and surface tension $5 \times 10^{-2}\text{Nm}^{-1}$. Calculate the maximum radius of the xylem tubes for sap to rise by capillarity alone to a height of 20m. Take $\rho = 1000\text{kgm}^{-3}$ and $g = 9.8\text{ms}^{-2}$.

Question 87

- (a) A soap bubble of radius r is enclosed in a sealed piston chamber at initial air pressure P_0 . The piston is slowly pulled out until the bubble doubles in radius to $2r$. Assuming isothermal conditions, show that the new air pressure in the chamber is $P = \frac{P_0}{8} - \frac{3\gamma}{2r}$.
- (b) A capillary tube is cut to a length shorter than the expected rise height and dipped vertically in water. The water rises to the top but does not overflow. Explain what happens to the meniscus and the angle of contact when the tube is too short for the full equilibrium rise.
- (c) Calculate the excess pressure inside a soap bubble of radius 1.0cm, given $\gamma = 2.5 \times 10^{-2}\text{Nm}^{-1}$. If an air bubble of the same radius is formed at a depth of 40cm inside a container of the same soap solution (relative density 1.2), calculate the total pressure inside the air bubble. Take $P_{\text{atm}} = 1.013 \times 10^5\text{Pa}$ and $g = 9.8\text{ms}^{-2}$.

Question 88

- (a) When a small loop of thread is placed on an intact soap film and the film inside the loop is punctured with a pin, the thread immediately springs into a perfect circle. Explain why the thread takes this particular shape.
- (b) Water rises in a glass capillary tube to a height of 9.0cm, while mercury in the same tube is depressed by 3.4cm. Explain why the same tube produces opposite behaviour for these two liquids.

- (c) Using the data in (b), with angles of contact 0° for water and 135° for mercury, determine the ratio of the surface tension of mercury to that of water. Take $\rho_w = 1000\text{kgm}^{-3}$ and $\rho_{\text{Hg}} = 13,600\text{kgm}^{-3}$.

Question 89

- (a) A stationary bicycle is much harder to balance on than one that is moving. Explain how the spinning wheels help the rider maintain balance.
- (b) A uniform metre rule is balanced on a pivot at its centre. Two equal masses hung at equal distances from the pivot keep the rule balanced. If both masses are moved further from the pivot by the same amount, the rule remains balanced but is noticeably harder to tilt. Explain why increasing the distance makes the system more resistant to angular acceleration.
- (c) A thin rectangular sheet of aluminium of mass 0.032kg has a length of 0.25m and a width of 0.10m . Calculate its moment of inertia about an axis in the plane of the sheet, parallel to the length and passing through its centre of mass. Then calculate the moment of inertia about an axis parallel to the width and passing through its centre of mass.

ANSWERS TO DIGGING DEEPER EXERCISES

EXERCISE 1A

1. (a) A physical quantity is a property that can be measured and expressed as a number with a unit. The number tells you how much; the unit tells you of what. Without a unit, the number carries no information. “I ran 5” could mean 5 metres, 5 kilometres, or 5 hours. The measurement is meaningless until the unit is specified.

(b) In 1999, the Mars Climate Orbiter spacecraft was destroyed because one engineering team reported thrust data in pound-force seconds while NASA’s navigation software expected newton seconds. The number was correct; the missing unit conversion cost 327 million US dollars.

2. A fundamental quantity is one that cannot be expressed in terms of simpler physical quantities. Length is fundamental because no combination of other quantities defines what length is. Velocity is derived because it can be expressed as length divided by time. The classification depends on whether the quantity can be decomposed into simpler quantities, not on whether it can be measured directly with an instrument.

3. Dimensional correctness is necessary but not sufficient for physical correctness. For example, $v^2 = u^2 + 3as$ is dimensionally correct (every term has dimensions L^2T^{-2}), but the coefficient 3 is wrong. The correct equation has coefficient 2. Dimensional analysis catches equations that are definitely wrong; it cannot confirm that an equation is definitely right.

4. (1) Checking formulas: verify that $s = ut + \frac{1}{2}at^2$ is dimensionally homogeneous by confirming that all terms have dimensions L.

(2) Deriving relationships: show that the period of a simple pendulum must have the form $T = k\sqrt{l/g}$ by equating powers of M, L, and T in the assumed relationship $T = kl^a g^b$.

(3) Converting units between systems: convert the gravitational constant G from CGS ($6.67 \times 10^{-8} \text{g}^{-1} \text{cm}^3 \text{s}^{-2}$) to SI ($6.67 \times 10^{-11} \text{kg}^{-1} \text{m}^3 \text{s}^{-2}$) by raising each base-unit conversion factor to the power given by the dimensional formula $M^{-1}L^3T^{-2}$.

5. (1) Cannot determine dimensionless constants: dimensional analysis gives $T = k\sqrt{l/g}$ but cannot find $k = 2\pi$. The constant must come from a full derivation or experiment.

(2) Cannot distinguish quantities with the same dimensions: work and torque both have dimensions ML^2T^{-2} , yet they are physically different quantities. Dimensional analysis treats them as identical.

(3) Cannot handle trigonometric, logarithmic, or exponential functions: the formula $x = A \sin(\omega t + \phi)$ cannot be derived by dimensional analysis because the sine function requires a dimensionless argument that is invisible to the method.

6. (a) A systematic error shifts all readings in the same direction by the same amount or proportion. It is consistent and repeatable. A random error varies unpredictably in both magnitude and direction from one reading to the next.

(b) Systematic errors are not reduced by repetition. Example: a balance with a zero error of +0.2 g adds 0.2 g to every reading. Averaging 50 readings still gives a mean that is 0.2 g too high. The cure is recalibration.

Random errors are reduced by repetition and averaging. Example: slight variations in pendulum timing scatter readings above and below the true value. The mean of 20 readings is closer to the true period than any single reading because the high and low deviations tend to cancel.

7. When two nearly equal quantities are subtracted, the absolute errors add, but the result is much smaller than either original measurement. The relative error of the difference is therefore much larger than the relative error of either individual measurement.

8. (a) Yes, the measurements are highly precise. They cluster tightly between 5.01 g and 5.02 g, a spread of only 0.01 g.

(b) No, they are not accurate. The mean is 5.014 g, which is displaced from the true value (5.50 g) by 0.486 g.

(c) A systematic error is present, most likely a calibration error or zero error in the balance. The instrument consistently gives readings that are approximately 0.5 g too low.

9. When timing a single oscillation of period roughly 2 s, the total timing uncertainty comes from reaction time at start and stop, roughly 0.2 s each in the worst case, giving about 0.2 s total uncertainty. The percentage error is approximately $0.2/2 = 10\%$.

When timing 20 oscillations, the same reaction time uncertainty (about 0.2 s) applies to the total time of about 40 s. The percentage error in the total time is $0.2/40 = 0.5\%$. Dividing by 20 gives the period with the same percentage error of 0.5%.

The percentage error drops from 10% to 0.5%, an improvement by a factor of 20, simply by timing more oscillations in a single run.

10. First thing: The uncertainty from a ruler is not necessarily exactly 1 mm. When measuring a length, there is an uncertainty at both ends (where the object starts and where it ends). If each end has an uncertainty of 0.5 mm (half the smallest division), the total uncertainty from the ruler alone is up to 1 mm. But if one end is carefully aligned at the zero mark, the uncertainty at that end may be smaller, making the total less than 1 mm.

Second thing: The total uncertainty includes more than just the instrument resolution. Parallax, the object not lying perfectly straight, temperature expansion, and human judgement all add additional uncertainty. The actual uncertainty in the measurement may be larger than 1 mm even though the ruler's resolution is 1 mm.

EXERCISE 1B

11. (a) A systematic error. Every measurement is consistently too short, because the stretched section makes the first 5 cm of the tape cover more than 5 cm of actual length. All readings are shifted in the same direction by the same proportion.

(b) No. Averaging cannot reduce a systematic error. Every repetition carries the same shift, so the mean of three wrong readings is still wrong by the same amount. The tailor needs a new tape measure or must calibrate the old one against a known standard length.

12. The readings differ because of **random errors**: slight variations in cuff pressure, the patient's blood flow between heartbeats, small differences in arm positioning, and the doctor's judgement in reading the gauge. These fluctuations are unpredictable and vary from one reading to the next. Averaging reduces their effect because the deviations above the true value tend to cancel those below it. The mean of three readings is closer to the true blood pressure than any single reading.

13. (a) Kipute is more precise. Her readings are perfectly consistent (zero spread), while Kipanga's scatter over a range of 0.07 m.

(b) Accuracy cannot be determined without knowing the true length of the table. Without the true value, we can assess precision but not accuracy.

(c) Kipanga is wrong because the quality of a set of measurements depends on both precision and accuracy, not just the mean value. Kipute's measurements are far more precise: any single reading from Kipute is reliable, while any single reading from Kipanga could be anywhere from 1.20 m to 1.27 m. In an experiment where individual readings matter (for example, plotting a graph), Kipute's data is far superior.

14. (a) No manufacturing process can produce a part with zero uncertainty. Every cutting tool, lathe, and mould introduces some variation. Demanding exactly 10.000 cm would require infinite precision, which is physically impossible.

(b) A piston does not need to be exactly the right size; it needs to be close enough to function correctly. A piston within the tolerance range 10.000 ± 0.005 cm will fit properly, seal against combustion gases, and move freely. Any dimension within this range gives acceptable performance, so exact perfection is unnecessary.

15. The danger depends on the consequences of the error, not just its percentage. A 5% error in a drug dose for a medication with a narrow therapeutic window can push the dose from effective to toxic, risking organ damage or death. A 5% error in a classroom length (say 0.5 m in a 10 m room) has no safety, medical, or structural consequence. The same percentage error carries completely different stakes depending on what is being measured and what depends on it.

16. (a) Systematic errors: (1) The wristwatch may run consistently fast or slow, adding the same bias to every timing. (2) The student's reaction time at the start is roughly the same on every trial, creating a consistent delay.

(b) Random errors: (1) The student's reaction time varies slightly from one trial to the next, sometimes pressing the button a fraction early, sometimes late. (2) Small air currents cause the pendulum's amplitude and period to fluctuate unpredictably between swings.

17. Absolute error has units and tells you the size of the uncertainty, but not its significance. Relative error is dimensionless and expresses the uncertainty as a fraction of the measured value, allowing fair comparison between different measurements.

Example: Measurement A has an absolute error of 0.1 mm on a 2 mm wire diameter (relative error 5%). Measurement B has an absolute error of 10 m on a 2 km bridge length (relative error 0.5%). Measurement B has a much larger absolute error but is far more precise in relative terms. Relative error reveals this difference; absolute error hides it.

18. (a) Kipanga can confirm precision. His five readings are identical (37.0°C every time), so the spread is zero and the measurements are highly precise.

(b) To check accuracy (the other property), Kipanga would need to compare his thermometer against a known standard. For example, he could measure the temperature of pure melting ice (should read 0°C) or steam above boiling water at standard atmospheric pressure (should read 100°C). If the thermometer reads correctly at these reference points, it is accurate. If not, it has a systematic error, and all his 37.0°C readings are displaced from the true temperature by the same amount.

19. (a) At a scale of 1:1,000,000, each millimetre on the map represents $1,000,000 \text{ mm} = 1000 \text{ m} = 1 \text{ km}$ in reality. So an error of 1 mm on the map corresponds to an error of 1 km on the ground.

(b) In cartography, even tiny measurement errors are amplified enormously by the scale factor. A slight wobble of a pen or a 1 mm misplacement of a point translates to a kilometre-scale error in the real world. This is why cartographers work with extremely high precision and why modern maps rely on GPS systems that measure to within centimetres.

EXERCISE 1C

20 (a) $\text{kgm}^{-1}\text{s}^{-2}$ (b) $\text{kgm}^2\text{s}^{-2}$ (c) $\text{kgm}^{-1}\text{s}^{-1}$ (d) kg s^{-2}

21.

Left side: $[v] = \text{LT}^{-1}$

Right side:

$$\frac{[r^2][\rho_s - \rho_f][g]}{[\eta]} = \frac{L^2 \times ML^{-3} \times LT^{-2}}{ML^{-1}T^{-1}} = \frac{ML^0T^{-2}}{ML^{-1}T^{-1}} = LT^{-1}$$

Both sides have dimensions LT^{-1} . The equation is dimensionally homogeneous.

22. Assume $h = k\gamma^a \rho^b r^c g^d$

Dimensions: $[h] = L$, $[\gamma] = MT^{-2}$, $[\rho] = ML^{-3}$, $[r] = L$, $[g] = LT^{-2}$

$$L = (MT^{-2})^a (ML^{-3})^b (L)^c (LT^{-2})^d = M^{a+b} L^{-3b+c+d} T^{-2a-2d}$$

For M: $0 = a + b \dots$ (i)

For T: $0 = -2a - 2d$, so $d = -a \dots$ (ii)

For L: $1 = -3b + c + d \dots$ (iii)

From (i): $b = -a$. Substituting into (iii): $1 = 3a + c - a = 2a + c$

Four unknowns but three equations: the system is underdetermined. Taking $a = 1$ (simplest case): $b = -1$, $d = -1$, $c = 1 - 2(1) = -1$.

$$h = k \frac{\gamma}{\rho r g}$$

23.(a) $[E] = ML^2T^{-2}$

$$[mgh^2] = M \times LT^{-2} \times L^2 = ML^3T^{-2}$$

Since $ML^2T^{-2} \neq ML^3T^{-2}$, the formula is dimensionally incorrect.

(b) Let $E = mgh^n$: $ML^2T^{-2} = M \times LT^{-2} \times L^n = ML^{1+n}T^{-2}$

Equating powers of L: $2 = 1 + n$, giving $n = 1$.

(d) Dimensional analysis gives $E = kmgh$ but cannot determine the dimensionless constant k .

24. (a) $t = k \sqrt{\frac{h}{g}}$ (b) $k = \sqrt{2}$; The full formula is $t = \sqrt{\frac{2h}{g}}$.

25. (a) 9950 kg m^{-3} (b) From m: 0.4%. From l: 0.2%. From d: 2.5%. (c) 3.1%

(d) The diameter is the smallest measured quantity (0.80 mm), so even a small absolute error (0.01 mm) gives a large relative error (1.25%). The power of 2 then doubles this to 2.5%. The smallest measurement, raised to the highest power, always dominates.

(e) The diameter contributes 2.5% out of 3.1%. To bring the total below 3%, its contribution must drop below about 2.4%. Since $2 \times \Delta d/d < 2.4\%$, we need $\Delta d < 0.0096$ mm. A micrometer with 0.005 mm resolution would achieve this, reducing the diameter contribution to $2 \times 0.005/0.80 = 1.25\%$ and the total to $0.4\% + 0.2\% + 1.25\% = 1.85\%$.

26. (a) $f = (60 \pm 9) \text{ cm}$

(b) Each measurement has only about 2% error. But the denominator $(u - v) = 10 \text{ cm}$ is much smaller than either u or v , while its absolute error (1.0 cm) is the sum of both individual errors. This gives 10% relative error in the denominator alone. Subtracting two similar quantities destroys the precision of the original measurements.

27. (a) 8.3%, $g = (10.0 \pm 0.8) \text{ ms}^{-2}$

(b) The simple pendulum formula $g = 4\pi^2 l/T^2$ contains no subtraction of nearly equal quantities. Its total error is typically below 1%. The Atwood machine formula requires $(m_1 - m_2)$, which contributes 5% error from the subtraction alone, even when each mass is measured to only 0.3%. This catastrophic cancellation is an inherent weakness of the Atwood method when the two masses are close in value.

28. (a) 9.70 ms^{-2} (b) $\Delta a = 0.65 \text{ ms}^{-2} \approx 0.7 \text{ ms}^{-2}$, $a = 9.7 \pm 0.7 \text{ ms}^{-2}$ (c) 6.7%

29.

(a) Measurements: mass of stone m (using balance), volume of stone V (by displacement, measure water level in measuring cylinder before and after submerging the stone).

(b) Balance (least count 0.1 g or 0.01 g), measuring cylinder (least count 1 ml or 0.5 ml).

(c) Sources of error:

(1) Systematic error: trapped air bubbles on the stone's surface increase the apparent volume. Minimise by tilting the stone as it enters the water and tapping the cylinder.

(2) Systematic error: the stone may absorb water, increasing the apparent mass if massed after submersion. Minimise by measuring mass of the dry stone before submersion.

(3) Random error: reading the meniscus level varies slightly between attempts. Minimise by reading at eye level, perpendicular to the scale, and repeating three times.

(d) $\rho = \frac{m}{V}$. The percentage uncertainty is $\frac{\Delta\rho}{\rho} = \frac{\Delta m}{m} + \frac{\Delta V}{V}$, where $\Delta V = \Delta V_1 + \Delta V_2$ (the uncertainties in the two water level readings add because volume is found by subtraction).

30. (a) From $g = 4\pi^2 l T^{-2}$:

$$\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2 \frac{\Delta T}{T}$$

(b) $\frac{\Delta l}{l} = \frac{1\text{mm}}{1000\text{mm}} = 0.1\%$

(c) The total must be below 1%:

$$0.1\% + \frac{2\Delta T}{T} \leq 1\%$$

$$\frac{2\Delta T}{T} \leq 0.9\%$$

$$\frac{\Delta T}{T} \leq 0.45\%$$

The maximum allowable percentage error in T is 0.45%.

(d) For $l = 1.000\text{ m}$: $T \approx 2\pi \sqrt{\frac{1.000}{9.8}} = 2.007\text{ s}$

If the student times n oscillations, the total time is nT and the percentage error in T is:

$$\frac{\Delta T}{T} = \frac{0.2\text{s}}{n \times 2.007\text{s}} \leq 0.45\%$$

$$n \geq \frac{0.2}{0.0045 \times 2.007} = 22.1$$

The student should time at least 23 oscillations.

EXERCISE 2A

1. (a) $v = 0\text{m/s}$ (b) $a = 0\text{m/s}^2$

Explanation

At rest means the velocity is zero. If the velocity does not change with time, then acceleration (rate of change of velocity) is zero too.

2. Not correct.

Explanation

$a = 0\text{m/s}^2$ means velocity is constant; it can be 0m/s or non-zero.

3. Distance is the total path length while displacement is the change in position with direction.

Example: moving 6m east then 6m west means distance = 12m, while displacement = 0m.

4. The final position is opposite to the chosen positive direction relative to the origin.

5. (a) $a = 0\text{m/s}^2$ (b) Net force = 0N

6. The acceleration is in the negative direction (opposite to the chosen positive direction).

Examples:

First situation: moving forward and slowing down (velocity positive but decreasing in magnitude).

Second situation: moving backward and speeding up (velocity negative and increasing in magnitude).

7.

- Equal when motion is in one direction only.
- Different when motion involves a change of direction over the interval.

8. (a) positive (b) 2m/s^2

9. (a) $v = 0\text{m/s}$ (b) No, the acceleration is not necessarily zero.

Explanation:

Acceleration depends on the **rate of change of velocity**, not on the value of velocity itself. At the instant the body is momentarily at rest, the velocity is 0m/s , but it may still be **changing direction or magnitude**. Therefore, the acceleration can be non-zero even when velocity is zero.

10. (a) 0m/s (b) $a = 0\text{m/s}^2$ during that interval.

11. (a) 55m (b) 25m east

12. (a) 2m/s^2 (b) The velocity increases by 2m/s every 1s .

EXERCISE 2B

13. Motion is relative. Relative to the moving bus, stationary trees have velocity in the opposite direction, so they appear to move backward.

14. Distance measures total path length (400m), while displacement depends only on initial and final positions. Returning home makes displacement = 0m .

15. Average speed depends on total distance and time, so it is non-zero. Average velocity depends on displacement and time; for an out-and-back trip, displacement = 0m , so average velocity = 0m/s .

16. The steady driver has nearly uniform motion, while the other has non-uniform motion with changing velocity. Repeated speeding up and slowing down causes the “jerky” feeling.

17. Both stones experience the same downward acceleration due to gravity, but the thrown stone starts with a non-zero initial downward velocity, so it reaches the ground first.

18. Ignoring air resistance, all objects fall with the same acceleration due to gravity, g , regardless of mass. In air, the effect of air resistance depends on how large the resistive force is **compared to the object’s weight**. For a leaf, air resistance is large compared with its small weight, so it greatly reduces the acceleration and velocity. For a coin, the weight is much larger compared with air resistance, so the motion is only slightly affected.

19. Acceleration is the rate of change of velocity. With constant acceleration due to gravity, velocity increases by equal amounts in equal times, so velocity increases even though acceleration remains constant.

20. Velocity depends on the chosen reference direction. If the chosen positive direction is opposite to the car’s motion, the velocity is negative even though the car is moving forward.

21. On the way up, velocity is upward while acceleration is downward. On the way down, both velocity and acceleration are downward.

22. No, the student was not correct.

Explanation

At the highest point, the velocity of the ball is momentarily zero, but acceleration depends on the rate of change of velocity, not on the value of velocity itself. Gravity continues to act downward, so the ball still has a downward acceleration even when its velocity is zero.

EXERCISE 2C

23.

- (a) 16m/s
- (b) 256m
- (c) 22s

24.

- (a) 16s
- (b) 192m

25.

- (a) 2.45s
- (b) 29.4m
- (c) 4.90s
- (d) Motion is symmetric under constant gravitational acceleration.

26. 3.58s

27.

- (a) -6.0m/s^2
- (b) 15m
- (c) 39m
- (d) Direction reversal increases distance without increasing displacement.

28.

- (a) $+12.1\text{m/s}$
- (b) 27.1m/s

29.

- (a) 0.993s and 4.11s
- (b) 0.993s
- (c) Velocity is positive at the earlier (smaller) time and negative at the later (larger) time.

30.

- (a) 2.67s
- (b) 45.2m
- (c) Thrown-up stone: $+3.87\text{m/s}$; dropped stone: -26.1m/s

31.

- (a) 4.22s
- (b) 31.3m/s

32.

- (a) 1.50s
- (b) 19.0m

33.

- (a) 1.83s
- (b) 16.5m
- (c) $+0.08\text{m/s}$ and $+15.9\text{m/s}$

34.

- (a) 36.5m
- (b) 4.26s
- (c) 26.7m/s
- (d) Negative time would correspond to an event before

the chosen start instant and is therefore rejected as unphysical.

35.

- (a) 98.0m
- (b) 19.6m/s

(c) After Stone B is released, both stones have the same acceleration, so their relative acceleration is zero. Since Stone A has been accelerating for longer, it already has a larger downward velocity; therefore, the separation increases at a constant rate.

EXERCISE 3A

1. Since the velocity is constant, acceleration is zero. By Newton's first law, a body with zero acceleration must have zero resultant force. This conclusion depends only on the absence of velocity change because acceleration measures change of velocity, not its magnitude.

2. In both cases the acceleration is zero. By Newton's first law, zero acceleration requires zero resultant force whether the velocity is zero (rest) or a non-zero constant value.

3. A resultant force is required to change velocity, not to maintain it. By Newton's second law, the net force is proportional to acceleration, not velocity.

The everyday idea seems true because presence of resistive forces on Earth oppose motion, so an applied force is needed to balance them.

4. "No force acts" means the body is completely isolated. "Resultant force is zero" means forces act but cancel.

Real motion on Earth usually involves balanced forces, not absence of forces because there is always interaction between bodies.

5. The error is assuming action–reaction forces act on the same body. By Newton's third law, action–reaction forces act on different bodies and therefore cannot cancel to prevent acceleration of a single body.

6. Each student experiences a force due to interaction with the other. **Because their masses may differ**, the same force produces different accelerations according to $a = F/m$.

7. Acceleration depends on the direction of the resultant force, not on the direction of motion (velocity direction).

Slowing down while moving upward means acceleration is downward.

8. Although heavier objects have larger weight (W), they also have proportionally larger mass. By Newton's Second Law, $a = W/m$, so gravitational acceleration is independent of mass when resistive forces are negligible.

9. Constant velocity implies zero acceleration which in turn means zero resultant force in accordance with Newton's first law.

This implies that the applied force balances friction exactly; so friction is present, not absent.

10. The normal reaction can differ from weight if there are vertical components of other forces or vertical acceleration. $N = mg$ only when acceleration perpendicular to the surface is zero and no other perpendicular forces act.

11. By Newton's third law, forces during collision are equal and opposite. By Newton's second law, different masses experience different accelerations, leading to different velocity changes over the same time interval.

12. Increasing stopping time reduces acceleration. By Newton's second law, smaller acceleration produces smaller force, reducing injury risk during braking.

EXERCISE 3B

13. Kipute's body continues with its forward velocity because of inertia. By Newton's first law, her motion will not change unless a resultant force acts on her. When the daladala brakes, her feet are slowed by contact with the floor, but her upper body tends to keep moving forward briefly, so she lurches forward relative to the vehicle.

14. When Kipanga stops pedalling, the bicycle is no longer receiving a driving force, but it keeps moving due to inertia. It slows down because there is a backward resultant force due to resistive forces such as air resistance and friction. By Newton's second law, a backward resultant force produces a backward acceleration, so the velocity decreases gradually.

15. For the phone to move with the car, a forward frictional force must act on the phone to give it the same acceleration as the dashboard. If the friction available is not large enough, the phone cannot gain the car's acceleration, so it slips and appears to move backward relative to the accelerating car.

16. Before the desk moves, static friction adjusts to match the applied push, keeping the resultant force zero so the desk stays at rest, consistent with Newton's first law. When the push becomes large enough to exceed the maximum possible static friction, the desk starts moving. Once it is moving, friction becomes kinetic friction, which is usually smaller than the maximum static friction, so motion becomes easier to maintain.

17. When the pickup accelerates forward, the bag tends to maintain its original state of motion due to inertia. If friction between the bag and the pickup bed is not sufficient to provide the needed forward acceleration to the bag, the bag accelerates less than the pickup and therefore slides backward relative to the vehicle.

18. Feeling heavier means that the normal reaction on Kipute is greater than her weight, while feeling lighter means that the normal reaction is smaller than her weight. When the lift accelerates upward, Kipute must have an upward resultant force, so

by Newton's second law the normal reaction must be greater than her weight. When the lift accelerates downward, the resultant force is downward, so the normal reaction is less than her weight.

19. Carrying a passenger increases the total mass of the system being accelerated. If the driving force is roughly unchanged, Newton's second law shows that a larger mass experiences a smaller acceleration, so the motorcycle speeds up more slowly.

20. Bending the knees increases the time over which momentum is reduced to zero. For the same change in momentum, Newton's second law shows that increasing stopping time reduces the average force, lowering injury risk.

21. The book is at rest, so its acceleration is zero and the resultant force is zero. The forces acting are its weight downward and the normal reaction upward, which balance. Therefore, the book is not force-free; it is in equilibrium because forces act and cancel, not because no forces act.

22. On muddy ground, resistive forces are larger, reducing the resultant forward force. On smooth tarmac, resistive forces are smaller, so the same driving effort produces a larger acceleration according to Newton's second law.

23. When the bodaboda stops suddenly, the passenger tends to continue moving forward due to inertia. A backward resultant force from contact with the seat or rider is required to stop the passenger safely.

24. Although the filled bottle has greater inertia, its larger weight increases the normal reaction and hence friction. The larger resistive force produces a larger deceleration, allowing it to stop sooner in real conditions.

25. When the car hits a pothole, the vertical motion of the car changes very rapidly. This produces a sudden change in the passengers' acceleration. By Newton's second law, a sudden change in acceleration requires a sudden change in the resultant force, so the normal reaction on the passengers increases sharply for a short time, producing the strong upward jolt.

EXERCISE 3C

26. (a) 546N (b) The passenger feels lighter. **Justification:** This is because although the lift is moving upward, it is slowing down, so the acceleration (and hence the resultant force) is downward. This makes the normal reaction smaller than the passenger's weight.

27. (a) 19.6N (b) 0.98m/s^2

28. (a) 3125N (b) 8.0s (c) Stopping distance increases. **Explanation:** With the same braking force, increasing the mass reduces the deceleration of the car according to Newton's second law. Since the car still has to reduce its velocity from the same initial value to zero, a smaller deceleration means the car takes longer and travels a greater distance before stopping.

29. (a) 2.67m/s (b) 3:2

30. (a) 100N (b) Opposite to initial motion

EXERCISE 4A

1. For equilibrium, the vector sum of all forces must be zero. The first two ropes create a resultant force (found by vector addition). The third rope must provide a force equal in magnitude but opposite in direction to this resultant. Therefore, the third rope's direction is uniquely determined as opposite to the resultant of the first two forces.

2. The tension is not equal to either weight because both masses are accelerating. For the heavier mass (accelerating downward), tension must be less than its weight to allow net downward force. For the lighter mass (accelerating upward), tension must be greater than its weight to allow net upward force. The tension has a single value throughout the string that is intermediate between the two weights, allowing both masses to accelerate at the same rate in opposite directions.

3. The normal reaction equals the component of weight perpendicular to the incline surface, which is $mg\cos\theta$. This is less than the full weight mg because the weight vector is resolved into two components: one perpendicular to the surface ($mg\cos\theta$) and one parallel to the surface ($mg\sin\theta$). The perpendicular component determines the normal reaction. As the angle increases, $\cos\theta$ decreases, so the normal reaction becomes even smaller.

4. Your actual weight (gravitational force = mg) remains constant because your mass and Earth's gravity do not change. What changes is your apparent weight (the normal reaction from the lift floor). When the lift accelerates upward, the floor must push you upward with force greater than your weight to accelerate you. This increased normal reaction is what you feel as "feeling heavier." The sensation of weight comes from normal reaction, not from gravitational force.

5. The applied force F must accelerate both blocks (total mass $m_1 + m_2$). However, the string tension only needs to accelerate one block (the one not directly receiving force F). Since tension accelerates less mass than F does, the tension is smaller. Specifically, if F pulls the first block, tension = $(m_2/(m_1 + m_2)) \times F$, which is always less than F .

6. Friction opposes the tendency of motion, not actual motion. The weight component $mg\sin\theta$ acts down the slope, creating a tendency for the block to slide downward. Static friction opposes this tendency by acting up the slope. Friction adjusts its magnitude (up to maximum $\mu_s mg\cos\theta$) to exactly balance the downward component, preventing motion. No external upward force is needed because friction provides the balancing force.

7. "Light" means the string has negligible mass compared to the objects connected. This ensures the string does not require force to accelerate itself, so tension is the same throughout its length. "Inextensible" means the string does not stretch, so when one end moves a certain distance, the other end moves the same distance. This ensures connected objects have the same magnitude of acceleration (though possibly in different directions). Both assumptions simplify analysis by ensuring uniform tension and equal acceleration magnitudes.

8. At constant velocity (zero acceleration), the tension only needs to overcome the resistive forces on the trailer (friction, air resistance). When accelerating, the tension must provide both the force to overcome resistance **and** the force to accelerate the trailer's mass ($T = ma + \text{resistive forces}$). Since the acceleration term adds to the required tension, it is greater during acceleration than at constant velocity.

9. Counter-example: a load supported by two strings at angles. Each tension is not equal to weight because only the vertical components add to balance the weight.

10. The string closer to horizontal must have the larger tension because it has a smaller vertical component per unit tension. To supply enough vertical support, it must be pulled harder.

EXERCISE 4B

11. Each side of the rope has both a vertical and horizontal component. Only the vertical components support the bag's weight. Since each rope is at an angle (not vertical), its vertical component is less than the full tension (vertical component = $T\cos\theta$). To provide enough vertical support to balance the weight, the tension must be greater than simply half the weight. The more the rope deviates from vertical, the greater the tension required.

12. Vibrations from the road cause the rope fibers to settle and compress slightly, and the knots to tighten, effectively making the rope shorter between fixed points. Additionally, temperature changes (the rope heats during day, cools at night) cause expansion and contraction. Even small stretching under initial load redistributes tension. These effects accumulate to create slack in the rope, requiring periodic re-tightening to maintain secure load.

13. As the ramp angle increases, the component of the sack's weight parallel to the ramp ($mgsin\theta$) increases. This component acts down the ramp and must be overcome to push the sack up. At steeper angles, $sin\theta$ is larger, so more force is needed to balance this component and any friction. While the distance is shorter, the required force is greater, making the work (force $\times$ distance) roughly the same but the task more difficult due to the larger force requirement.

14. When the rope is jerked suddenly, the bucket must accelerate rapidly from rest or from a lower velocity to a higher velocity. By Newton's second law ($F = ma$), rapid acceleration requires a large force, which means very high tension in the rope. This tension can exceed the rope's breaking strength. Pulling smoothly means smaller acceleration, which requires less force (tension), keeping the rope safely below its breaking point while still raising the bucket efficiently.

15. Static friction must act up the slope to balance the component of weight down the slope. Even at rest, that component tends to cause sliding, so friction is required to maintain equilibrium.

16. Pressing the bottom harder increases the normal reactions at the contacts, which increases the maximum possible static friction. With larger available friction, the plank can stay in equilibrium without slipping.

17. The rope tension has components in different directions. Horizontal component opposes sideways slipping, while the downward vertical component increases the normal reaction, which increases friction and thus helps prevent sliding.

18. The hinge provides a reaction but cannot supply the necessary balancing effect on its own. The cable provides an additional force with suitable components that help balance the weight and keep the signboard in equilibrium; without it, the forces cannot balance and the signboard rotates/falls.

EXERCISE 4C

19. (a) $T = 15\text{N}$ (b) $F = 17.2\text{N}$

20. (a) $a = 2.67 \text{ m/s}^2$ (b) $T = 49.9\text{N}$ (c) $s = 5.34\text{m}$

21. (a) $a = 3.5 \text{ m/s}^2$ (b) $T = 14\text{N}$ (c) $a = 0.56\text{m/s}^2$

22. (a) Block slides down ($mgsin\theta = 33.1\text{N} > f_{\text{max}} = 31.9\text{N}$) (b) $a = 0\text{m/s}^2$

23. (a) $m_2 = 3.37\text{kg}$ (b) $T = 24.6\text{N}$ (c) $t = 1.1\text{s}$

24. (a) 12204N (b) 10584N (c) 8424N

25. (a) $a = 3.68\text{m/s}^2$ (b) $v = 3.83\text{m/s}$ (c) $s = 2\text{m}$

26. (a) $a = 5\text{m/s}^2$ (b) $T = 35\text{N}$ (c) $T = 20\text{N}$

27. (a) $T = 33.1\text{N}$ (b) $P = 23\text{N}$ acting up the plane on A.

28. (a) $T = 58.8\text{N}$ (b) $f = 19.6\text{N}$ down the plane (c) $\mu = 0.289$

29. (a) The string making 45° with the horizontal has the greater tension, because it is more steeply inclined and must provide a larger vertical component to support the load. (b) Tension in 30° string = 87.8N , Tension in 45° string = 107.6N .

30. (a) $a = 1.96\text{m/s}^2$ upward (b) $R = 882\text{N}$

EXERCISE 5A

1. The truck takes longer to stop because it has much greater momentum than the car. Momentum $p = mv$, so with equal velocities but much larger mass, the truck has much larger momentum. Since impulse (Force $\times$ time) equals change in

momentum, and both vehicles apply the same braking force, the truck requires much longer time to achieve the same magnitude momentum change. From $\text{Impulse} = F\Delta t = \Delta p$, if F is the same but Δp (truck) $\gg \Delta p$ (car), then Δt (truck) $\gg \Delta t$ (car).

2. (a) The coefficient of restitution ($e = 0.87$ for this case) tells us that the collision is inelastic but not perfectly inelastic. The value $e = 0.87$ means the separation speed is 87% of the approach speed, indicating most but not all kinetic energy is conserved. Values of e between 0 and 1 indicate varying degrees of inelasticity.

(b) The "missing" 25% of energy was converted into other forms during the collision: heat (warming the ball and floor slightly due to internal friction in the deforming materials), sound (the bounce noise), and permanent microscopic deformation of the ball's material structure. Energy was not destroyed—it transformed from organized kinetic energy into disorganized thermal energy and other non-recoverable forms.

3. (a) The rocket accelerates forward due to conservation of momentum, not by "pushing" on anything. Initially, the rocket-fuel system is at rest (total momentum = 0). When exhaust gases are ejected backward at high velocity, they carry backward momentum. Since no external forces act in space, total momentum must remain zero. Therefore, the rocket must gain equal forward momentum to balance the exhaust's backward momentum. The rocket and exhaust push on each other (Newton's third law), creating the forward acceleration.

(b) Thrust force $F = u_{\text{rel}}(dm/dt)$ remains constant if exhaust velocity and fuel burn rate are constant. However, acceleration $a = F/m$, and mass m continuously decreases as fuel burns. As m decreases while F stays constant, acceleration a increases. Additionally, in space (no gravity), all thrust goes toward acceleration rather than fighting weight, further emphasizing the increasing acceleration effect.

4. (a) The boat moves backward due to conservation of momentum. Initially, both person and boat are at rest (total momentum = 0). When the person throws the object forward, the object gains forward momentum. Since no external horizontal forces act on the person-boat-object system, total momentum must remain zero. To balance the object's forward momentum, the person and boat must acquire equal backward momentum, causing the boat to move backward.

(b) **Yes**, the person could make the boat move forward by throwing the object backward. If the object is thrown backward, it gains backward momentum. To keep total momentum at zero, the person and boat must gain equal forward momentum, causing the boat to move forward. The direction the object is thrown determines the boat's direction of motion (opposite to the thrown object).

5. The bouncing collision would cause more severe injuries. In the bouncing collision, each car reverses direction (velocity changes from $+v$ to $-v$), giving momentum change of $2mv$. In the sticking collision, each car just stops (velocity changes from $+v$ to 0), giving momentum change of only mv . Since the bouncing collision has **twice the momentum change** and occurs in **shorter time** (elastic rebounds are quick), so from $F = \Delta p/\Delta t$, it produces much larger forces.

6. The motor must provide force because the belt's mass continuously increases as sand lands on it. The falling sand initially has zero horizontal velocity. To maintain the belt's constant velocity, each kilogram of sand must be accelerated horizontally from 0 to the belt's velocity. Every second, new sand lands and must be accelerated. This continuous acceleration of incoming mass requires continuous force from the motor: $F = v(dm/dt)$, even though the belt itself isn't accelerating (v is constant).

7. We cannot use energy conservation for the entire process because the bullet-block collision is perfectly inelastic; the bullet embeds in the block, converting most of its kinetic energy to heat, sound, and deformation. Energy is **not** conserved during inelastic collisions. However, we **can** use energy conservation for the swing portion only because after the collision, the block (with embedded bullet) swings like a pendulum with only conservative forces acting (gravity and tension). During the swing, kinetic energy converts to gravitational potential energy and back, with total mechanical energy remaining constant. Hence, we must use momentum conservation for the collision stage and energy conservation separately for the swing stage.

8. (a) The bullet and gun experience equal forces during firing due to Newton's third law. When the gun's firing mechanism pushes the bullet forward, the bullet simultaneously pushes the gun backward with equal magnitude force. These are action-reaction pairs. The forces are equal in magnitude, opposite in direction, and act for the same duration (while bullet is in the barrel).

(b) The bullet moves much faster than the gun because it has much smaller mass. Both experience the same force for the same time, so both gain the same magnitude of momentum ($\text{impulse} = F\Delta t = \Delta p$). From $p = mv$, if momentum magnitudes are equal but masses are vastly different, the smaller mass must have much greater velocity.

(c) The **bullet** has more kinetic energy. Although they have equal magnitude momenta (p), kinetic energy $KE = p^2/(2m)$. For equal momentum, kinetic energy is inversely proportional to mass. Since the bullet has much smaller mass, it has much more kinetic energy.

9. (a) **Upward journey:** Momentum **decreases** in magnitude and **remains upward in direction**. Initially, the ball has large upward momentum. Gravity exerts constant downward force, therefore causing constant downward change in momentum ($dp/dt = -mg$). The upward momentum magnitude steadily decreases until it reaches zero at the peak.

(b) **Downward journey:** Momentum increases in magnitude but is downward in direction. At the peak, momentum is zero. Gravity continues exerting downward force, now increasing downward momentum. The momentum magnitude (in downward direction) increases steadily as the ball falls.

(c) **Total change:** The total change in momentum from throw to return equals twice the initial momentum, downward. If thrown upward with momentum $+p$, it returns with momentum $-p$ (same magnitude, opposite direction). The total change is $\Delta p = (-p) - (+p) = -2p$, or $2p$ downward. This equals the impulse from gravity: $\text{Impulse} = (mg)(\text{total time}) = \Delta p$.

EXERCISE 5B

10. (a) The daladala's velocity decreases each time a passenger boards (each boarding event reduces the vehicle's velocity slightly).

(b) This happens due to conservation of momentum. Initially, the daladala has momentum $p = mv$ (mass $\times$ velocity). Each passenger initially has zero horizontal momentum (standing still). When a passenger boards, the total momentum remains constant (no external horizontal forces during the brief boarding). The combined mass increases (daladala + passenger), so to keep momentum constant, velocity must decrease: if $p = (m + \Delta m)v'$, then $v' < v$.

(c) It is easier to maintain velocity if passengers run alongside first because they already have horizontal momentum matching (or close to) the daladala's velocity. When they board with matching velocity, the momentum exchange is minimal because the combined mass increases but velocity changes insignificantly. If passengers jump from rest, the daladala must "share" its momentum with stationary mass, causing significant slowdown that the driver must compensate for with extra power.

11. (a) The boat moves backward (opposite to the fisherman's walking direction). As the fisherman walks forward, the boat slides backward through the water.

(b) This happens because the conservation of momentum of the fisherman-boat system (no external horizontal forces). Initially, both are at rest with zero momentum. When the fisherman walks forward, he gains forward momentum. To keep total momentum at zero, the boat must gain equal backward momentum. The fisherman's forward momentum is balanced by the boat's backward momentum, making the boat to move backward.

(c) When the fisherman walks back to his original position (relative to the boat), the boat returns to its starting position in the water. Any momentum the system gained in one direction is exactly reversed when the fisherman walks back. Since total momentum must remain zero throughout (no external forces), the boat must end up exactly where it started.

12. (a) Trolley A would stop completely when the trolleys have equal masses and collision is elastic. For elastic collisions between equal masses (one initially at rest), the moving object transfers all its velocity to the stationary object and stops. This is a special case where $v_1 = 0$ and $v_2 = u_1$ exactly. The moving trolley's momentum and kinetic energy are completely transferred to the stationary trolley.

(b) The collisions sound different because of energy conservation differences. The sharp 'click' indicates almost an elastic collision where kinetic energy is conserved, in which the trolleys bounce apart quickly with minimal energy converted to sound and heat. The dull 'thud' indicates an inelastic collision where significant kinetic energy converts to sound (the thud itself), heat, and deformation of the bumpers. More energy dissipated as sound creates a louder, duller sound.

(c) The 'click' collision is more likely to be closer to elastic. The sharp, brief sound indicates the trolleys bounced apart rapidly with minimal energy loss. Elastic collisions are characterized by quick rebounds with little deformation. The 'thud' indicates prolonged contact with significant deformation and energy dissipation, characteristic of inelastic collisions. So the sound itself is evidence of the energy partition; less sound energy means more kinetic energy retained (elastic), more sound means more kinetic energy lost (inelastic).

13. The farmer's reasoning is **incorrect**. While good grains are heavier (more mass), what matters for bounce height is the coefficient of restitution (e), not the mass. Both heavy and light grains fall from the same height, so both have the same velocity just before impact ($v = \sqrt{2gh}$). The bounce height depends on e , which is determined by material properties (elasticity), not mass. Good grains have lower e (less elastic), so they bounce less regardless of being heavier. Mass does not determine bounce height when drop height is the same.

14. (a) The truck's **momentum decreases** as water leaks out. Momentum $p = mv$, and mass m continuously decreases as water drips away. So if velocity v stays constant (driver maintains constant velocity), the decreasing mass means decreasing momentum.

(b) The driver needs to apply **less force** to maintain constant velocity compared to when full. With constant velocity on a level road, the driving force equals the total resistive forces (air resistance + friction). As mass decreases, friction (which depends on normal reaction $= mg$) decreases proportionally. Therefore, less driving force is needed to balance the reduced friction.

(c) If water sprays backward horizontally (like a jet), the driver needs **less force** than the leaking case. If the water sprays backward horizontally, it carries backward momentum. By conservation of momentum, the truck receives a forward reaction force (forward thrust like a rocket thrust). Therefore, the driver needs to apply less force to maintain the same constant velocity because the water jet provides forward thrust that helps maintain velocity and hence the jet effect assists the engine.

15. In a collision, the passenger's momentum must change from a moving state to rest (the same Δp). Crumple zones are designed to compress and deform during impact, thereby increasing the collision time. This larger Δt reduces the peak force experienced by the passengers ($F = \Delta p / \Delta t$ with larger Δt). Thus, crumple zones protect passengers by increasing the collision time and reducing the force during impact.

16. (a) The effect is small because the mass of the zebra is extremely large compared with the mass of the dart. When the dart embeds in the zebra, the collision is inelastic and momentum is conserved:

$$m_{\text{zebra}}v_{\text{zebra}} + m_{\text{dart}}v_{\text{dart}} = (m_{\text{dart}} + m_{\text{zebra}}) \times v_{\text{final}}$$

Since $m_{\text{zebra}} \gg m_{\text{dart}}$, the total mass after the collision is almost the same as the zebra's mass. Thus, the dart's momentum (small mass $\times$ high velocity) is very small compared to the zebra's momentum (huge mass $\times$ moderate velocity). When shared among the combined mass, the zebra's velocity decreases by less than 1%.

(b) The dart penetrates the zebra's tissue, doing work against resistance forces (tearing muscle, skin, etc.). This work converts the dart's kinetic energy into: (1) heat in both dart and surrounding tissue (tissue warms slightly), (2) sound of impact, (3) permanent deformation and tearing of tissue.

(c) The effect on a small antelope is more noticeable because the antelope has much smaller mass than the zebra. So in this case, the dart's momentum represents a larger fraction of the antelope's total momentum, so the velocity change is more significant.

EXERCISE 5C

17. (a) The boat moves due to conservation of momentum. Initially, both fisherman and boat are at rest (total momentum = 0). When the fisherman walks backward (relative to boat), he gains backward momentum. To keep total momentum at zero, the boat must gain equal forward momentum, so it moves forward.

(b) (i) 1.6m (forward) (ii) 2.4m (backward)

18. (a) (i) $e = 0.63$ for good grain (ii) $e = 0.84$ for chaff

(b) The coefficient of restitution depends on the materials of **both** colliding objects (grain/chaff and wooden board). Good grain is denser and harder, with different internal structure than chaff. When they impact the wood, the energy losses due to deformation, heat, and sound are different. The harder, denser grain deforms less elastically, losing more energy. The lighter, more flexible chaff compresses more elastically, retaining more energy. Therefore, different materials produce different e values even on the same surface.

19. (a) (i) -630kgm/s (or 630kgm/s opposite to initial direction) (ii) 4200N (wall pushes mkokoteni backward)

(b) The wall exerts a large force because the mkokoteni undergoes a large momentum change (from $+450\text{ kg}\cdot\text{m/s}$ to -180 kgm/s , total change 630kgm/s) in a very short time (0.15s). From $F=\Delta p/\Delta t$, the short collision time results in large force. This is because, the mkokoteni not only stops but rebounds, which always increases much the momentum change compared to just stopping.

20. (a) 8.64m/s (b) 25.8m

(c) When a passenger jumps onto a moving boda-boda from rest, the collision is perfectly inelastic with large velocity difference. This creates large forces during the brief impact, risking injury and loss of balance. If the passenger runs alongside first to match the boda-boda's velocity, the relative velocity at boarding becomes zero or very small, reducing impact forces significantly.

21. (a) The impulse (change in momentum) is the same in both case. However, impulse = Force $\times$ time. On concrete, the short stopping time (0.05s) requires very large force. On sand, the longer stopping time (0.25s) means smaller force. The large force on concrete exceeds the coconut's structural strength, causing it to crack. Sand extends the collision time, reducing force below the breaking point.

(b) (i) 240N on concrete (ii) 48N on sand (iii) Ratio = 5:1 (concrete force is 5 times larger than sand force)

22. (a) No, the two balls do **not** have the same momentum at any instant (except $t = 0$).

Explanation:

The projected ball has both horizontal momentum ($p_x = m \times 15$, constant) and vertical momentum ($p_y = mgt$, increasing as t increases). The dropped ball has only vertical momentum ($p_y = mgt$). At $t = 0$, both have zero momentum. After $t = 0$, the projected ball has total momentum magnitude $\sqrt{p_x^2 + p_y^2}$ while the dropped ball has only p_y . These are never equal except at $t = 0$.

(b) (i) $15m\text{ kgm/s}$ (where m is ball mass in kg) (ii) $19.6m\text{ kgm/s}$ for both balls (where m is ball mass in kg) (iii) $24.7m\text{ kgm/s}$ (where m is ball mass in kg)

(c) The horizontal momentum remains constant because no horizontal forces act on the ball (air resistance neglected), so horizontal momentum is conserved. The vertical momentum changes because gravity exerts a constant downward force, causing continuous increase in vertical momentum.

23. (a) 3.05m/s at 71° below its original direction.

(b) Inelastic collision since initial KE is not equal to final KE.

(c) Practical situations:

1. Billiard/pool balls colliding at angles.
2. Car accidents at intersections (vehicles approaching from different directions).
3. Football/soccer players colliding while running in different directions.
4. Collision between two vehicles at a T-junction.

24. (a) (i) 160000N (ii) 10.2m/s^2 (iii) 22.2m/s^2

(b) The rocket's acceleration increases because thrust remains constant (depends on dm/dt and u_{rel} , both constant) while mass continuously decreases as fuel burns. From $a = F_{net}/m$, as m decreases and F_{thrust} stays constant, acceleration a increases. Additionally, weight (mg) also decreases, further increasing net upward force.

(c) (i) **Rocket's momentum:** After fuel exhaustion, only gravity acts (external force). **Momentum is not conserved;** it decreases as gravity does negative work, slowing the rocket until it reaches maximum height, at which point momentum becomes zero.

(ii) **Kinetic energy:** Also **not conserved.** Gravity does negative work on the upward-moving rocket, continuously removing kinetic energy and converting it to gravitational potential energy until the rocket stops at maximum height ($KE = 0$).

(d) **Advantage:** Higher burn rate means shorter burn time ($6000/100 = 60s$ vs $6000/50 = 120s$). This reduces gravity losses during powered ascent, as less time is spent in overcoming gravity.

Disadvantage: Higher burn rate means higher thrust force (320000 vs 160000), requiring stronger structural components to handle increased force. This adds mass to the rocket, reducing the effective mass ratio and potentially decreasing final velocity.

25. (a) (i) 0.333kg/s (ii) 2.67N downward (iii) 2.31N

(b) (i) **Force effect:** If rainfall rate doubles, dm/dt doubles. Since $F = (dm/dt) \times v$, **force also doubles.**

(ii) **Coefficient of restitution:** e remains unchanged. The coefficient depends on materials (water and roof surface), not on how much rain falls or how hard it rains. Doubling the rainfall rate increases force but does not change the elasticity of each individual raindrop's collision.

(c) Perpendicular force would be **greater** because all the vertical force acts perpendicular to a horizontal surface, whereas on an inclined surface only the component perpendicular to the surface acts normal to it.

26. (a) Since the ice is frictionless, there is no external horizontal force on the two skaters, so the total momentum of the system remains zero. As they pull each other, they move toward each other with equal and opposite momenta: $m_A v_A = m_B v_B$. Since skater B has the larger mass (80kg), skater A (60kg) must move with a greater velocity to keep momenta equal. Therefore, A travels a greater distance than B, and they meet at a point closer to skater B.

(b) (i) A moves 6.86m , B moves 5.14m (ii) $v_A = 1.72\text{m/s}$, $v_B = 1.29\text{m/s}$ (toward each other)

(c) (i) $v_B = 2.25\text{m/s}$ (opposite direction to A) (ii) $KE_{total} = 472.5\text{J}$

(d) The kinetic energy came from **chemical energy in the skaters' muscles**. When they pushed away from each other, their muscles did work (converted chemical energy from food into mechanical work). This work increased the kinetic energy of both skaters from zero to 472.5J total. (Similar to an explosion where chemical energy converts to kinetic energy), the skaters' muscles acted as the energy source for this perfectly inelastic "reverse collision."

EXERCISE 6A

1. In the horizontal direction, there is no force acting on the stone (ignoring air resistance), so by Newton's first law, the horizontal velocity remains constant. In the vertical direction, gravity acts continuously downward with constant acceleration $g = 9.8\text{m/s}^2$, causing the vertical velocity to increase steadily.

2. The trajectory is parabolic because horizontal displacement increases linearly with time ($x = ut$) while vertical displacement varies with the square of time ($y = ut - \frac{1}{2}gt^2$). When we eliminate time t from these equations, we get $y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$, which is a quadratic equation in x . Any quadratic relationship between y and x represents a parabola. Thus, the parabolic shape results from the combination of uniform horizontal motion and uniformly accelerated vertical motion.

3. Launching from height h increases the total flight time. The projectile first completes its normal parabolic arc (as if on level ground), then continues falling through the additional height h before landing. This extra descent time allows more horizontal distance to accumulate since horizontal velocity remains constant. This is supported by formula $t = \frac{u \sin \theta + \sqrt{(u \sin \theta)^2 + 2gh}}{g}$, which shows that flight time increases with h , and since $R = (u \cos \theta)t$, greater time means greater range (horizontal distance).

4. At maximum height, only the vertical velocity component becomes zero ($v_y = 0$), but the horizontal velocity component remains unchanged throughout flight ($v_x = u \cos \theta$). So the total velocity vector at maximum height is purely horizontal with magnitude $v = u \cos \theta$. Consequently, the projectile continues moving forward even at its peak because no horizontal force acts to stop it; it simply has no upward or downward motion at that instant.

5. Throwing downward adds initial downward velocity, causing faster descent and thus reduced time of flight. Additionally, the horizontal velocity component is reduced to $u \cos \theta < u$ for any downward angle. Both factors work against range: shorter time of flight and slower horizontal velocity combine to produce less horizontal distance. Horizontal throw maximizes horizontal velocity (all velocity directed forward) and maximizes flight time among, giving maximum horizontal range.

6. The horizontal throw hits first. Both start at the same height, but the angled throw has an upward vertical component that makes it rise first before falling, while the horizontal throw has zero initial vertical velocity and begins falling immediately. Since both experience the same downward acceleration g , the one that starts descending immediately (horizontal throw) reaches ground sooner. The angled throw must first decelerate upward, stop at maximum height, then accelerate downward, taking longer overall.

7. Both maximum height $H = \frac{u^2 \sin^2 \theta}{2g}$ and range $R = \frac{u^2 \sin 2\theta}{g}$ are proportional to u^2 , so doubling velocity quadruples their amounts. Physically, greater initial velocity provides more kinetic energy. In the vertical direction, this converts to greater potential energy (peak of greater height). In the horizontal direction, increased initial velocity means greater velocity maintained throughout flight ($v_x = u \cos \theta$ unchanged) combined with longer flight time (from greater height) produces much greater range.

8. Range formula $R = \frac{u^2 \sin 2\theta}{g}$ shows R is proportional to $\sin 2\theta$. The maximum value of sine function is 1, which is achieved when $2\theta = 90^\circ$, giving $\theta = 45^\circ$. At this angle, $R = \frac{u^2}{g}$, which is the maximum possible range for given velocity u . Any other angle gives $\sin 2\theta < 1$, resulting in $R < \frac{u^2}{g}$. This represents a fundamental limit imposed by the combination of initial kinetic energy and gravitational acceleration; no choice of angle can overcome this limit.

EXERCISE 6B

9. When the water leaves the nozzle, it has an initial horizontal velocity. After leaving the nozzle, gravity acts downward while the horizontal motion continues unchanged (neglecting air resistance). The combination of constant horizontal motion and downward acceleration produces a parabolic path.

10. For any target distance R less than maximum range ($R < \frac{u^2}{g}$), the range equation $R = \frac{u^2 \sin 2\theta}{g}$ has two solutions for 2θ : one acute angle, and its supplement. These correspond to two launch angles that are complementary (summing to 90°). One angle is shallow (below 45°), giving a low, fast arc; the other is steep (above 45°), giving a high, slow arc. Both reach the same distance but follow different paths. Fountain designers typically **choose** the steeper angle for aesthetic appeal as the higher arc is more visually dramatic and impressive.

11. Although both angles reach the same horizontal distance (complementary angles), the 30° angle has several practical advantages.

1. The water arrives faster because it travels a lower, shorter arc, allowing quicker response to the fire.
2. The lower trajectory means water arrives with greater horizontal velocity component, providing more impact force to penetrate smoke and reach the base of flames.
3. The flatter arc is easier to aim and control as the firefighter can see the water path more clearly and make quick adjustments.
4. The 30° jet wastes less water on excessive height (water spends less time in the air and thus experiences less loss due to air resistance and dispersion), directing more energy toward forward reach rather than vertical climb.

12. When the ball approaches the rim while rising or moving nearly horizontally, it has little or upward vertical velocity. If it strikes the rim, this vertical motion tends to deflect the ball upward and away from the basket. When the ball approaches while descending, it has a downward vertical velocity. If it hits the rim, this downward momentum helps guide the ball into the basket rather than causing it to bounce away. Additionally, a rising ball that misses the hoop continues upward and away, whereas a descending ball that is close to the rim may still drop through the basket.

13. Aiming downward reduces range because of two compounding effects. First, the downward angle reduces the horizontal component of velocity. Second, adding downward initial velocity makes the water hit the ground faster, reducing flight time. Slower horizontal velocity and shorter time of flight combine to produce much less horizontal distance.

14. When a package is released from a helicopter, it retains the horizontal velocity it had at the moment of release. Even if the helicopter appears to hover, it usually has small horizontal motions due to wind or position adjustments. The package therefore continues with this horizontal motion while gravity pulls it downward, producing a parabolic path that may land away from the release point.

15. Both observers are correct because they describe motion from different reference frames. The bridge observer is stationary relative to the stone's release point. Since the stone is simply dropped (zero horizontal velocity relative to the bridge), it falls straight down under gravity; pure vertical motion.

The boat observer moves horizontally relative to the bridge. From the boat's reference frame, the stone has horizontal velocity equal and opposite to the boat's motion at the moment of release. The boat observer sees the stone combine this horizontal component with vertical acceleration due to gravity, creating a parabolic trajectory.

16. Shot putters release below 45° because they do not launch from ground level; they release from shoulder height. When launching from an elevated position, the optimal angle for maximum range is less than 45° . The additional height provides extra flight time even without much upward velocity, so directing more velocity horizontally (flatter angle) produces greater range than 45° would. Additionally, in real conditions air resistance cannot be neglected and it (air resistance) reduces the horizontal range more strongly for higher angles due to longer time of flight at larger angle. As a result, the optimal angle becomes smaller than 45° .

EXERCISE 6C

17. (a) $y = x \tan \theta - \frac{g x^2}{2u^2 \cos^2 \theta}$ (b) 45.2°

18. $\theta_1 = 41^\circ$, $\theta_2 = 80^\circ$

19. (a) 28.3m (b) 32.7m

20. (a) 2s (b) $y_1 = 20.4\text{m}$, $y_2 = 20.4\text{m}$ (c) Since both balls have the same horizontal and vertical position, the **balls collide**.

21. Since heights differ, **collision does not occur** (At $t = 1.89\text{s}$; $y_A (29.8\text{m}) \neq y_B (22.6\text{m})$)

22. $x = 45.75\text{m}$

23. (a) From $R = \frac{u^2 \sin 2\theta}{g}$, equal ranges require $\sin 2\theta_1 = \sin 2\theta_2$. Since $\theta_1 \neq \theta_2$:

$$2\theta_1 = 180^\circ - 2\theta_2 \Rightarrow 2\theta_1 + 2\theta_2 = 180^\circ \Rightarrow \theta_1 + \theta_2 = 90^\circ$$

$$(b) T_1 = \frac{2u \sin \theta_1}{g} \text{ and } T_2 = \frac{2u \cos \theta_1}{g};$$

$$\frac{T_1}{T_2} = \frac{\sin \theta_1}{\cos \theta_1} = \tan \theta_1$$

$$(c) H_1 = \frac{u^2 \sin^2 \theta_1}{2g} \text{ and } H_2 = \frac{u^2 \cos^2 \theta_1}{2g};$$

$$\frac{H_1}{H_2} = \frac{\sin^2 \theta_1}{\cos^2 \theta_1} = \tan^2 \theta_1$$

(d) The product:

$$H_1 \times H_2 = \frac{u^2 \sin^2 \theta_1}{2g} \times \frac{u^2 \cos^2 \theta_1}{2g} = \frac{u^4 \sin^2 \theta_1 \cos^2 \theta_1}{4g^2} = \frac{u^4 \sin^2 2\theta_1}{16g^2}$$

$$\text{Since } R = \frac{u^2 \sin 2\theta_1}{g};$$

$$R^2 = \frac{u^4 \sin^2 2\theta_1}{g^2}$$

Therefore:

$$H_1 H_2 = \frac{R^2}{16}$$

$$24. (a) \text{ Maximum height: } H = \frac{u^2 \sin^2 \theta}{2g}$$

$$\text{Range: } R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin \theta \cos \theta}{g}$$

Setting $H = R$:

$$\frac{u^2 \sin^2 \theta}{2g} = \frac{2u^2 \sin \theta \cos \theta}{g}$$

Dividing both sides by $u^2 \sin \theta / g$:

$$\frac{\sin \theta}{2} = 2 \cos \theta$$

$$\tan \theta = 4$$

(b) 76° (c) 43.2m

25. (a) 14.43ms^{-1} (b) 2.08s , 4.8m (c) 45m from Player A.

EXERCISE 7A

1. (a) Constant. (b) Changing (direction changes continuously). (c) Changing (magnitude is constant but direction changes continuously, always pointing toward the centre). (d) Changing (same reason as centripetal acceleration — constant magnitude but changing direction).

2. (a) Acceleration is the rate of change of velocity, which is a vector. Although the speed (magnitude) is constant, the direction of the velocity changes continuously as the body moves around the circle. A change in direction of velocity constitutes an acceleration. (b) The acceleration is directed toward the centre of the circular path (centripetal).

3. (a) The stone moves in a straight line along the tangent to the circle at the point where the string broke. (b) When the string breaks, the centripetal force vanishes. With no net force acting on it, the stone continues moving at constant velocity in the direction it was travelling at the instant of the break, in accordance with Newton's first law.

4. Angular velocity (ω) is the rate of change of angular displacement, measured in rad/s. Linear velocity (v) is the rate of change of linear displacement along the circular path, measured in m/s. They are related by $v = \omega r$, where r is the radius of the circular path.

5. (a) $a = \frac{v^2}{r}$ (b) $a = \omega^2 r$ (c) Since $v = \omega r$, substituting into $\frac{v^2}{r}$ gives $\frac{(\omega r)^2}{r} = \omega^2 r$. Hence, both are equivalent forms of the same physical quantity expressed using different variables.

6. (a) Static friction between the tyres and the road. (b) Gravitational force of the Earth on the satellite. (c) The component of tension directed toward the centre (assisted or opposed by gravity depending on position). (d) The horizontal component of the normal reaction from the banked road surface. (e) The magnetic force on the moving charged particle.

7. The required centripetal force is $\frac{mv^2}{r}$ and the maximum friction available is μmg . Both are proportional to mass m . When equated, the mass cancels, giving $v_{\max} = \sqrt{\mu gr}$, which is independent of mass. A heavier car needs more centripetal force but also has proportionally more friction available.

8. (a) At the bottom. The tension must support the weight (acting away from centre) and provide the centripetal force (toward centre) simultaneously, so $T_{\text{bottom}} = mg + \frac{mv^2}{r}$, which is the largest value.

(b) At the top. Both tension and weight act toward the centre, so gravity assists the centripetal force and the string needs to supply less: $T_{\text{top}} = \frac{mv^2}{r} - mg$, which is the smallest value.

9. The design speed is the speed at which a vehicle can negotiate the banked curve without any friction. At this speed, the horizontal component of the normal reaction alone provides the required centripetal force.

(a) Faster than design speed: the banking alone cannot provide enough centripetal force. Friction must act down the slope (inward) to supplement the centripetal force. Without sufficient friction, the car slides outward (up the slope).

(b) Slower than design speed: the banking provides more centripetal force than needed. Friction must act up the slope (outward) to prevent the car from sliding inward (down the slope). Without friction, the car slides inward.

10. At the top of a hill, the centripetal acceleration is directed downward (toward the centre of curvature below the road). Part of the driver's weight is "used" to provide this centripetal acceleration, so the normal reaction (which the driver feels as weight) is reduced: $R = mg - \frac{mv^2}{r}$. The driver feels lighter because the support force is less than the actual weight.

The car loses contact with the road when $R = 0$, which occurs at speed $v = \sqrt{gr}$.

11. (a) Inside a loop: the critical speed at the top is a **minimum** speed. Below $v = \sqrt{gr}$, the object loses contact with the track and falls away from the circular path. The normal reaction is zero at the critical speed.

(b) Outside a curve (hill): the critical speed at the top is a **maximum** speed. Above $v = \sqrt{gr}$, the object loses contact with the road and becomes airborne. The normal reaction is zero at the critical speed.

In both cases, the critical speed has the same magnitude $\sqrt{gr}$, but for the inside loop it is a minimum to maintain contact, while for the outside curve it is a maximum before contact is lost.

EXERCISE 7B

12. When the bus turns right, friction between the seats and the passengers provides the centripetal force needed to follow the curved path. However, the passengers' bodies, due to inertia, tend to continue in a straight line (Newton's first law). This makes them feel as though they are being pushed to the left. No real force pushes them to the left; they simply resist the change in direction while the bus turns beneath them.

13. The recommended speed is based on the maximum speed at which a vehicle can safely negotiate the curve without sliding. This depends on the radius of the curve and the coefficient of friction between the tyres and the road: $v_{\max} = \sqrt{\mu gr}$. A sharp curve (small radius) has a lower safe speed than a gentle curve (large radius). Different curves have different radii, so their recommended speeds differ.

14. When a motorcycle turns, friction at the contact point provides the centripetal force horizontally, while the weight acts vertically through the centre of mass high above the ground. If the rider does not lean, these two forces create a net torque (turning effect of a force) that would topple the motorcycle outward. Leaning inward shifts the centre of mass so that the resultant of weight and friction passes through the contact point, eliminating the toppling torque. Without leaning, the motorcycle would tip outward and crash.

15. At the top of the hump, the bus follows a curved path whose centre of curvature is below the road. Part of the gravitational force is used to provide the centripetal acceleration downward, so the normal reaction from the seat on Kipanga is reduced: $R = mg - \frac{mv^2}{r}$. Kipanga feels lighter because the support force is less than his actual weight. The "stomach dropping" sensation is caused by this reduced contact force.

16. On a straight road, friction is mainly needed for braking and accelerating. On a curve, friction must provide the centripetal force to keep the vehicle on the circular path. When the road is wet, the coefficient of friction decreases. On a straight road, this mainly affects stopping distance. On a curve, it directly reduces the maximum safe speed ($v_{\max} = \sqrt{\mu gr}$), and a vehicle travelling at a speed that was safe on a dry road may now exceed the reduced safe speed and slide off the curve.

17. During the spin cycle, the drum wall provides the centripetal force that keeps the clothes moving in a circle. The water trapped in the fabric requires the same centripetal force to follow the circular path. However, the adhesive forces holding the water to the fabric are too weak to supply the required centripetal force at high speed. The water, unable to maintain circular motion, moves tangentially outward through the holes in the drum, following Newton's first law. The clothes, pressed firmly against the drum wall, remain inside.

18. When a vehicle rounds a curve, friction at ground level provides the centripetal force, while the centre of gravity is above the ground. This creates a torque that tends to tip the vehicle outward about the outer wheels. A tall vehicle has a high centre of gravity, which increases the arm of the overturning torque. A low, wide vehicle has a low centre of gravity and large wheel

separation, which increases the restoring torque resisting overturning. The maximum speed before overturning is $v = \sqrt{\frac{arg}{2h}}$, which is smaller for tall vehicles (large h , small a) and larger for low, wide vehicles (small h , large a).

19. The satellite is indeed falling toward the Earth due to gravity. However, it also has a large horizontal velocity. As it falls toward the Earth, it moves forward fast enough that the Earth's surface curves away beneath it at the same rate. The satellite continuously falls toward the Earth but never reaches it because the curvature of its path matches the curvature of the Earth. Gravity provides the centripetal force that bends the satellite's path into a circle. Mr. Akilikubwa's description is accurate: orbiting is continuous free fall where the falling object keeps "missing" the ground.

20. On a banked curve, the road surface is tilted so that the normal reaction has a horizontal component directed toward the centre of the curve. This component provides centripetal force without relying on friction. The advantage is that the vehicle can negotiate the curve safely even on a wet or icy surface, as long as it travels near the design speed.

However, banking alone is sufficient only at the design speed ($v = \sqrt{rg \tan \theta}$). At higher speeds, the banking cannot supply enough centripetal force, and friction is needed to make up the difference. At lower speeds, the banking provides too much inward force, and friction is needed to prevent the vehicle from sliding inward. Therefore, banking reduces dependence on friction but cannot eliminate it at all speeds.

21. Gravity does act on the astronauts and the space station. The astronauts appear weightless not because gravity is absent, but because both the astronauts and the station are in free fall together. Gravity provides the centripetal force for the orbital motion, and since everything inside the station accelerates toward the Earth at the same rate, there is no relative force between the astronauts and the station walls. With no contact force (normal reaction), the astronauts feel weightless. Weightlessness is the absence of a contact force, not the absence of gravity.

22. (a) 9375N (b) 7056N (c) Since the required force (9375N) exceeds the maximum friction (7056N), the car **cannot** safely negotiate the curve. It will skid outward.

23. (a) 34.1° (b) 4.73N (c) $r = 0.673\text{m}$, $v = 2.11\text{m/s}$

24. (a) The speed is different at different positions in a vertical circle. The student used v_{bottom} in the formula for the top, but the correct speed at the top (v_{top}) must first be found using energy conservation.

(b) Since v_{top}^2 is negative, the ball **never reaches the top**. (b) 114.9

25. (a) 18.9m/s (b) $7.52\text{m/s} \leq v \leq 27\text{m/s}$

26. (a) 12.25N (b) $v_{\text{top}} = 2.93\text{m/s}$, $T_{\text{top}} = 0.49\text{N}$ (c) 73.9°

27. (a) Since $h(1.5\text{m}) > h_{\text{min}}(1.25\text{m})$, the ball completes the loop. (b) $v_{\text{top}} = 3.13\text{m/s}$, $R_{\text{top}} = 0.98\text{N}$ (c) $R_{\text{bottom}} = 6.86\text{N}$

28. (a) 19.4m/s

(b) Since expected speed (16.7m/s) < maximum speed (19.4m/s), the curve is safe for the expected traffic speed on a dry road.

(c) 22.1° (d) 29.1m/s

29. (a) 26700N (b) 1.82 (c) 22.1m/s

30. (a) 2.39N (b) 20.03N (c) 7.16m/s

31. (a) For 0.2kg ball: 6.26m/s, For 0.5kg ball: 6.26m/s

(b) The student's claim is **wrong**. The minimum speed does not depend on mass. The formula $v_{\text{min}} = \sqrt{5gr}$ contains no mass term. This is because both the centripetal force requirement and the gravitational force increase in equal proportion with mass, so mass cancels.

32. (a) 10.6m/s (b) 1.13N (c) 66.4°

33. (a) 18.1m/s (b) 16.6m/s (c) the car **skids first** (Hint: compare speeds or coefficient of friction and $\frac{a}{2h}$)

(d) Inner wheel: 1.525m, outer wheel: 8.275m; where m is the mass of the car in kg.

34. (a) Since $v_{\text{bottom}}(5.5\text{m/s}) < v_{\text{min, bottom}}(7\text{m/s})$, the ball does **not** complete the circle. (b) 111.3° (c) 1.88m/s

35. (a) 9.39m/s (b) 16.5m (c) $\frac{R_{\text{bottom}}}{mg} = 6.5$. The structure must withstand 6.5mg at the bottom, which is greater than 6mg. Hence, the engineer must design for forces exceeding 6mg.

36. (a) $\theta = 48.2^\circ$, $v = 4.43\text{m/s}$ (b) 2m (c) Horizontal component: 2.95m/s, Vertical component: 3.3m/s

EXERCISE 8A

1. The gravitational force on each ball is $F = \frac{GMm}{r^2}$, where m is the mass of the ball. By Newton's second law, the acceleration is $a = \frac{F}{m} = \frac{GM}{r^2}$. Because the mass m appears in both the gravitational force (which pulls the ball down) and the inertia (which

resists acceleration), these two effects cancel exactly. As a result, the acceleration depends only on M (the Earth's mass) and r (the distance from the centre), not on m . Therefore, both balls fall with the same acceleration g and, since they start from the same height with zero initial velocity, they reach the ground at the same time.

2. The gravitational force between two people is $F = \frac{Gm_1m_2}{r^2}$. For two people the product of mass is so small that the gravitational pull is negligible. This force is because the gravitational constant G is extraordinarily tiny (6.67×10^{-11}). Gravity only becomes noticeable when at least one of the masses is extremely large (like a planet), because a large mass compensates for the smallness of G . Since neither person has a mass anywhere close to a planet, the force between them is far too weak to feel.

3. By Newton's third law, the gravitational forces are equal in magnitude. However, by Newton's second law, the acceleration produced is $a = \frac{F}{m}$, which depends on the mass of the body being accelerated. Because the Earth is about 81 times more massive than the Moon, the Earth's acceleration is 81 times smaller than the Moon's for the same force. As a result, the Moon moves in a large, visible orbit, while the Earth merely wobbles slightly.

4. Gravitational potential is defined relative to infinity, where $V = 0$. As a mass moves from infinity toward a planet, gravity does positive work on it (pulling it inward), so the mass loses potential energy. Because the potential energy at infinity is zero and the mass has less energy at any point near the planet, the potential must be negative. The closer to the planet, the more work gravity has done, and therefore the more potential energy has been lost. Thus the potential becomes more negative with decreasing distance. Physically, a more negative potential means the object is more tightly bound (attracted) to the planet, and more energy would be needed to free it and return it to infinity.

5. Above the surface, g decreases because the distance from the centre of the Earth increases. Since $g = \frac{GM_E}{r^2}$, a larger r gives a smaller g . The entire mass of the Earth still contributes to the gravitational pull, but the inverse square law weakens it with distance.

Below the surface, g decreases for a completely different reason. As you descend, the shell of mass above you exerts zero net gravitational force on you (by the Shell Theorem). Therefore, only the mass below you contribute to g . Because this effective mass decreases as you go deeper, g decreases as well. At the centre, all the mass is distributed symmetrically around you, giving zero net force and hence $g = 0$.

Thus above the surface, g decreases because distance increases. Below the surface, g decreases because the effective mass decreases.

6. For a satellite in circular orbit, gravity provides the centripetal force: $\frac{mv^2}{R} = \frac{GM_E m}{R^2}$. The satellite's mass m appears on both sides and cancels, giving $v = \sqrt{\frac{GM_E}{R}}$. Because v depends only on M_E and R , all satellites at the same orbital radius R have the same speed. Since the period is $T = \frac{2\pi R}{v}$, and both R and v are independent of m , the period is also the same for all satellites at the same altitude.

8. A negative total energy means the satellite is gravitationally bound to the Earth. The satellite has kinetic energy (positive) and gravitational potential energy (negative), but the magnitude of the potential energy exceeds the kinetic energy, so the sum is negative. As a result, the satellite does not have enough energy to escape to infinity; it remains trapped in orbit.

If the total energy became exactly zero, the satellite would have just enough energy to reach infinity with zero residual speed. This is the threshold of escape. The satellite would follow a parabolic trajectory, never returning but barely escaping. If the total energy became positive, the satellite would escape with speed to spare, following a hyperbolic trajectory. Therefore, negative total energy means a bound orbit, zero means the boundary of escape, and positive means the object is free.

8. The escape velocity is derived from conservation of energy: $\frac{1}{2}mv_E^2 - \frac{GMm}{r} = 0$. This is a scalar equation involving only the magnitude of the speed, not its direction. Because energy has no direction (it is a scalar quantity), the total kinetic energy is the same whether the object moves vertically, horizontally, or at any angle. Therefore, as long as the speed equals v_E and the object does not collide with the planet's surface on its way out, it will escape regardless of the direction of launch.

9. The escape velocity from the Moon (2.4km/s) is much lower than the escape velocity from the Earth (11.2km/s), because the Moon has much less mass. Gas molecules in any atmosphere are in constant random motion, and their speeds depend on the temperature. At the temperatures found on the Moon's surface (which can exceed 100°C in direct sunlight), the fastest gas molecules reach speeds above 2.4km/s and escape into space. Over billions of years, this gradual loss has stripped the Moon of all its atmospheric gases.

On the Earth, the average molecular speed at typical surface temperatures is about 0.3 to 0.6km/s, which is far below the escape velocity of 11.2km/s. As a result, virtually no atmospheric molecules have enough speed to escape, and the Earth retains its atmosphere comfortably.

10. When a satellite loses energy, its total energy $E = -\frac{GM_E m}{2R}$ becomes more negative. Because E is inversely proportional to R , a more negative E corresponds to a smaller orbital radius. The satellite therefore drops to a lower orbit.

At this lower orbit, the orbital speed is $v = \sqrt{\frac{GM_E}{R}}$. Since R has decreased, v increases. Thus the satellite moves faster despite having lost total energy. This happens because the fall to a lower orbit converts gravitational potential energy into kinetic energy. The gain in kinetic energy exceeds the energy lost to drag, so the net effect is an increase in speed.

EXERCISE 8B

11. The reading in Iringa will be slightly **lower**. A digital scale measures the gravitational force (weight) on the object and converts it to a mass reading using a calibrated value of g . Because Iringa is at a higher altitude than Dar es Salaam, the gravitational field strength g is slightly weaker there (since g decreases with altitude). As a result, the gravitational force on the gold is slightly less, and the scale displays a slightly lower reading.

However, the actual mass of the gold has not changed at all. Mass is an intrinsic property of the object and does not depend on location. Only the weight (the gravitational force) changes with altitude.

12. This belief is incorrect because the gravitational field at 400km altitude is still about 94% as strong as on the ground. Gravity is very much present at the altitude of the ISS.

The astronauts float because both they and the spacecraft are in **free fall** toward the Earth at exactly the same rate. Since gravity accelerates everything equally, the astronaut and the floor of the spacecraft fall together. There is no relative acceleration between the astronaut and the spacecraft, and therefore no contact force (no normal reaction from the floor, no tension from any support). With zero contact force, the astronaut feels weightless.

13. The Moon's gravity pulls on the Earth, but the pull is not uniform. It is strongest on the side of the Earth closest to the Moon (because that side is nearer) and weakest on the side farthest from the Moon (because that side is more distant). This difference in gravitational pull, called the tidal force, stretches the Earth's oceans along the Earth-Moon line.

On the near side, the Moon's pull is stronger than the average, so water is pulled toward the Moon, creating a bulge (high tide). On the far side, the Moon's pull is weaker than the average, so the water is "left behind" relative to the solid Earth, creating a second bulge on the opposite side. As the Earth rotates on its axis once per day, any coastal location passes through both bulges, experiencing two high tides and two low tides approximately every 24 hours.

14. First error: Kipanga says gravity is enormous at the centre. This is wrong. As you descend into the Earth, the shell of mass above you exerts zero net force on you (Shell Theorem, Part 2). Only the mass below you pulls you downward. Because this effective mass decreases as you approach the centre, g decreases as well. At the centre, all the mass is distributed symmetrically around you, so the net gravitational force is zero. Therefore, far from being crushed, Kipanga would experience zero gravity at the centre.

Second error: Kipanga says he would fall faster and faster all the way to the centre. This is also wrong. He would accelerate as he falls (because g still points toward the centre), but g is decreasing as he descends. He would reach maximum speed at the centre (where $g = 0$ and the acceleration changes direction). After passing the centre, gravity would pull him back, decelerating him. He would oscillate back and forth through the centre, like a mass on a spring, never stopping at the centre unless resistive forces like air resistance slowed him down.

15. The Earth rotates from west to east (in the eastward direction). A point on the equator moves (at about 465m/s) due to the Earth's rotation. Because the rocket is already moving eastward (with Earth) at this speed before launch, launching in the eastward direction adds this speed to the rocket's velocity for free, reducing the amount of fuel needed to reach orbital speed.

The equator provides the maximum benefit because the rotational speed is greatest there (the equator has the largest distance from the rotation axis). At higher latitudes, the rotational speed is smaller (since $v = \omega r_E \cos\theta$), so the "free" speed boost is reduced. Therefore, launching from an equatorial site in the eastward direction gives the most efficient use of fuel.

16. The total energy of a satellite in circular orbit is $E = -\frac{GM_E m}{2R}$. The geostationary satellite has a much larger orbital radius R , so its total energy is less negative (closer to zero). This means it is less tightly bound to the Earth.

Although the geostationary satellite has less kinetic energy (because it moves more slowly), it has much more gravitational potential energy (because it is much farther from the Earth). The increase in potential energy far exceeds the decrease in kinetic energy. As a result, the total energy of the geostationary satellite is higher (less negative) than that of the low-orbit satellite. This is consistent with the fact that more energy must be supplied to place a satellite in a higher orbit.

17. Reason 1 (Shape): The Earth is not a perfect sphere; it is slightly flattened at the poles and bulges at the equator (an oblate spheroid). The polar radius is shorter (about 21km shorter) than the equatorial radius. Since $g \propto \frac{1}{r^2}$, the smaller radius at the poles means the surface is closer to the centre, and therefore g is slightly larger there.

Reason 2 (Rotation): The Earth rotates, and a body on the surface requires centripetal acceleration to move in a circle. At the equator, the centripetal acceleration is $\omega^2 r_E$ (0.034m/s^2), which is directed inward (toward the rotation axis). Because part of the gravitational pull goes toward providing this centripetal acceleration, the effective g measured at the equator is reduced. At the poles, the body is on the rotation axis, so no centripetal acceleration is needed and the full gravitational pull is felt.

Both effects work together: they make g smaller at the equator and larger at the poles.

18. If the Sun's gravitational field suddenly vanished, the Earth would immediately move in a straight line tangent to its orbit at the instant of disappearance, at its current orbital speed (about 30km/s). It would not continue to curve, and it would not "orbit for a while."

The reason is Newton's first law. An object continues in a straight line at constant speed unless acted upon by a net force. The only force keeping the Earth in a curved orbit is the Sun's gravity. The moment that force disappears, there is nothing to deflect the Earth from a straight path. Therefore, the Earth would fly off along the tangent to its orbit at that instant.

This also shows that circular (or elliptical) motion requires a continuous centripetal force. The orbit is not a natural state; it is a constant forced deflection from straight-line motion. Remove the force, and the natural straight-line motion resumes immediately.

19. In a gravity assist, the spacecraft approaches a planet, is deflected by the planet's gravity, and leaves with a different speed and direction. From the planet's reference frame, the spacecraft's speed relative to the planet is the same before and after the encounter (it is like an elastic collision). However, from the Sun's reference frame, the spacecraft gains speed because the planet itself is moving.

The key is that the planet is orbiting the Sun at high speed. As the spacecraft swings around the planet, it is briefly "dragged along" by the planet's gravity in the direction of the planet's motion. This transfers a tiny amount of the planet's orbital kinetic energy to the spacecraft. Because the planet is enormously more massive than the spacecraft, the planet's speed decreases by an unmeasurably small amount, while the spacecraft gains a significant speed boost. Hence the spacecraft accelerates without burning fuel, at the expense of a negligible slowing of the planet.

20. Kepler's Third Law, derived from Newton's gravitation, gives $T^2 = \frac{4\pi^2}{GM} r^3$, where M is the mass of the central body. The constant of proportionality $\frac{4\pi^2}{GM}$ depends on M . For all planets orbiting the Sun, M is the same (the Sun's mass), so they all share the same constant, and $T^2 \propto r^3$ holds with a single constant.

However, if we compare an object orbiting the Sun with an object orbiting the Earth, the two central masses are different. Because $M_{\text{Sun}} \gg M_{\text{Earth}}$, the constant $\frac{4\pi^2}{GM}$ is much smaller for the Sun, meaning that for the same orbital radius, the period around the Sun would be much shorter. Therefore the T^2 - r^3 relationship holds within each system (planets around the Sun, moons around Jupiter, satellites around the Earth), but with a different constant for each central body. A more massive central body gives a smaller constant, hence shorter periods for the same radius.

EXERCISE 8C

21. (a) $6.02 \times 10^{24} \text{ kg}$ (b) 1023 m/s (or 1.02 km/s) (c) 9.80 m/s^2 . This agrees with the accepted value of 9.8 m/s^2 , confirming the consistency of the data.

22. (a) $-6.25 \times 10^7 \text{ J/kg}$ (b) $4.5 \times 10^7 \text{ J}$ (or 45 MJ) (c) 4.0 N/kg

23. (a) 2651 km (b) 39.6% (c) The height, h was significantly large compared to the radius of the earth.

24. (a) 4218 m/s (b) $v_{\text{orb}} = 2982 \text{ m/s}$, Ratio: $\frac{v_E}{v_{\text{orb}}} = \frac{4218}{2982} = 1.414 = \sqrt{2}$. Confirmed.

(c) The probe's speed (5000 m/s) exceeds the escape velocity (4218 m/s). Because the probe has more than enough kinetic energy to overcome the gravitational potential well, it will escape the planet entirely and fly off into space. It will not orbit, because orbiting requires a specific speed less than escape velocity.

25. (a) $3.68 \times 10^8 \text{ m}$ (b) $-1.481 \times 10^6 \text{ J/kg}$

26. (a) 14.7 GJ (b) 16.6 GJ

(c) The student's answer (14.7 GJ) **underestimates** the true value (16.6 GJ). The student accounted for the kinetic energy needed in orbit but forgot that the satellite must also be lifted from the surface to the orbital height. Lifting requires increasing the gravitational potential energy, which costs additional energy ($\Delta U = 1.9 \text{ GJ}$ in this case). The correct calculation includes both the kinetic energy and the change in potential energy automatically, because it uses the total energy difference between the surface and the orbit.

27. (a) 3.69 m/s^2 (b) 5.0 km/s (c) 30.5 m , 8.13 s

28. (a) $R = 16780 \text{ km}$, $h = 10380 \text{ km}$ (b) 4.88 km/s (c) 20240 MJ or $(2.024 \times 10^{10} \text{ J})$

(f) Fuel cost: $20240 \text{ MJ} \times \text{Tsh } 200,000/\text{MJ} = \text{Tsh } 4.048 \times 10^9 \approx \text{Tsh } 4 \text{ billion}$.

As a fraction of total cost: $\frac{4}{50} = 8\%$. The fuel cost is a relatively small fraction of the total satellite cost. The satellite itself (with its instruments, solar panels, and communication equipment) is far more expensive than the fuel needed to launch it. This is why making satellites lighter is less about fuel savings and more about reducing the size (and cost) of the rocket.

29. (a) $2.618 \times 10^{24} \text{ kg}$ (b) 6.98 m/s^2 (c) 4.19 m/s^2 (d) Surface: $-3.49 \times 10^7 \text{ J/kg}$, centre: $-5.24 \times 10^7 \text{ J/kg}$

30. (a) $v_{\text{orb}} = 7616 \text{ m/s}$, $v_E = 10770 \text{ m/s}$ (b) Speed factor $= \sqrt{2} = 1.414$ (c) Energy factor $= 2$ (d) $2.320 \times 10^{10} \text{ J}$ or 23.2 GJ

31. (a) $R = 42300 \text{ km}$, $v = 3.08 \text{ km/s}$, $T = 24 \text{ hours}$ (by design) (b) -5.68 GJ (c) 69.3 GJ

(d) Fuel energy needed (at 5% efficiency): $\frac{69.3 \text{ GJ}}{0.05} = 1386 \text{ GJ} = 1.386 \times 10^{12} \text{ J}$

Fuel mass: $\frac{1.386 \times 10^{12} \text{ J}}{3.0 \times 10^6 \text{ J/kg}} = 4.62 \times 10^5 \text{ kg} = 462 \text{ tonnes}$

The fuel mass (462 tonnes) is about 385 times the satellite mass (1200kg). This enormous fuel-to-payload ratio illustrates why reaching geostationary orbit is one of the most expensive feats in engineering.

32. (a) 11.8 years

(b) Semi-major axis of transfer orbit:

$$a = \frac{r_E + r_J}{2} = \frac{1.5 \times 10^{11} + 7.78 \times 10^{11}}{2} = \frac{9.28 \times 10^{11}}{2} = 4.64 \times 10^{11} \text{ m}$$

Period of the transfer orbit:

$$\frac{T_{\text{transfer}}^2}{T_E^2} = \frac{a^3}{r_E^3} = \frac{(4.64 \times 10^{11})^3}{(1.5 \times 10^{11})^3} = \frac{9.987 \times 10^{34}}{3.375 \times 10^{33}} = 29.59$$

$$T_{\text{transfer}} = \sqrt{29.59} \times 1 \text{ year} = 5.44 \text{ years}$$

Transfer time (half the orbit): $\frac{5.44}{2} = 2.72 \text{ years}$

A Hohmann transfer to Jupiter takes approximately 2.7 years.

33. (a) $6.19 \times 10^{-4} \text{ rad/s}$ (b) 8.52 (c) 2.82h (2 hours, 49 minutes).

34(a) The launch speed is $v = \frac{1}{2} v_E = \frac{1}{2} \sqrt{\frac{2GM_E}{r_E}}$

Using conservation of energy between the surface and the maximum height h (where $v = 0$):

$$\frac{1}{2}mv^2 - \frac{GM_E m}{r_E} = 0 - \frac{GM_E m}{r_E + h}$$

Substituting $v^2 = \frac{1}{4} \times \frac{2GM_E}{r_E} = \frac{GM_E}{2r_E}$:

$$\begin{aligned} \frac{1}{2}m \times \frac{GM_E}{2r_E} - \frac{GM_E m}{r_E} &= -\frac{GM_E m}{r_E + h} \\ \frac{GM_E m}{4r_E} - \frac{GM_E m}{r_E} &= -\frac{GM_E m}{r_E + h} \end{aligned}$$

Dividing through by $GM_E m$:

$$\begin{aligned} \frac{1}{4r_E} - \frac{1}{r_E} &= -\frac{1}{r_E + h} \\ \frac{1 - 4}{4r_E} &= -\frac{1}{r_E + h} \\ -\frac{3}{4r_E} &= -\frac{1}{r_E + h} \\ \frac{3}{4r_E} &= \frac{1}{r_E + h} \\ r_E + h &= \frac{4r_E}{3} \\ h &= \frac{4r_E}{3} - r_E = \frac{4r_E - 3r_E}{3} = \frac{r_E}{3} \end{aligned}$$

(b) $h = \frac{6400 \text{ km}}{3} = 2133 \text{ km}$

EXERCISE 9A

1. The claim is correct. The equation $\tau = I\alpha$ has exactly the same mathematical form as $F = ma$, with torque τ replacing force F , moment of inertia I replacing mass m , and angular acceleration α replacing linear acceleration a . Every algebraic manipulation that works for $F = ma$ works identically for $\tau = I\alpha$. The physics is different (forces push, torques twist), but the mathematical structure is the same.

2. (a) **The hollow sphere.** In a hollow sphere, all the mass is at the outer radius (far from the axis), while in a solid sphere, some mass is close to the axis. Since moment of inertia depends on mr^2 , mass farther from the axis contributes more. The hollow sphere has $I = \frac{2}{3}MR^2$ compared with $I = \frac{2}{5}MR^2$ for the solid sphere.

(b) **The solid sphere.** From $\tau = I\alpha$, a smaller I with the same torque gives a larger α . The solid sphere has the smaller moment of inertia and therefore accelerates angularly more easily.

3. (a) When the axis is at the centre, each half of the rod extends a maximum distance of $L/2$ from the axis. When the axis is at one end, the far half of the rod extends a maximum distance of L from the axis. Since moment of inertia depends on r^2 , the far half contributes much more when the axis is at the end. More mass is farther from the axis, so the moment of inertia is larger.

(b) $I_{\text{end}} = I_{\text{centre}} + Md^2$, where $d = L/2$ (distance from centre to end).

$I_{\text{end}} = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2$. Since $I_{\text{centre}} = \frac{1}{12}ML^2$, the ratio is $\frac{1/3}{1/12} = 4$. So $I_{\text{end}} = 4I_{\text{centre}}$.

4. From parallel axis theorem, $I = I_{\text{cm}} + Md^2$. Since Md^2 is always positive (M is positive, d^2 is positive), any axis at distance d from the centre of mass gives $I = I_{\text{cm}} + Md^2 > I_{\text{cm}}$. Only when $d = 0$ (the axis passes through the centre of mass) does $I = I_{\text{cm}}$. Therefore, I_{cm} is the minimum.

5. (a) The ring. The radius of gyration k represents the “effective distance” at which the entire mass would need to be concentrated to give the same moment of inertia. For the ring, all mass is at distance R from the axis, so $k_{\text{ring}} = R$. For the solid disc, mass is spread from the centre to R , so the effective distance is less: $k_{\text{disc}} = R/\sqrt{2} \approx 0.707R$. Thus the ring has the larger k because its mass is, on average, farther from the axis.

(b) The student is **incorrect**. The radius of gyration depends on shape and size, not on mass. For a solid disc of any mass, $k = R/\sqrt{2}$. Doubling the mass doubles I (from $\frac{1}{2}MR^2$ to $\frac{1}{2}(2M)R^2 = MR^2$), but $k = \sqrt{I/M} = \sqrt{MR^2/(2M)} = R/\sqrt{2}$, which is unchanged. The mass cancels.

6. (a) Torque equals force times perpendicular distance from the axis: $\tau = Fr$. The angular acceleration is determined by torque: $\alpha = \tau/I$. Pushing at the centre (small r) produces less torque for the same force than pushing at the handle (large r). To achieve the same torque (and therefore the same α), a larger force is needed at the centre.

(b) A single force produces both torque (causing rotation) and a net force (causing translation of the centre of mass). A couple consists of two equal and opposite forces, so the net force is zero (no translation), but the torques add because both forces twist in the same rotational direction. All the effort goes into rotation, with no energy wasted on translation.

7. The ratio of consecutive angular displacements is 2: 6: 10: 14 = 1: 3: 5: 7, which is the odd-number sequence. This sequence is the signature of uniform acceleration from rest (Galileo’s law). If the acceleration were not uniform, the ratios would deviate from 1: 3: 5: 7.

8. (a) Translational KE ($\frac{1}{2}Mv_{\text{cm}}^2$) is the energy due to the motion of the centre of mass while rotational KE ($\frac{1}{2}I\omega^2$) is the energy due to the spinning of the body about its centre of mass.

(b) A sliding body has only translational KE: $\frac{1}{2}Mv^2$. A rolling body has both translational KE and rotational KE: $\frac{1}{2}Mv^2(1 + k^2/R^2)$. Since $k^2/R^2 > 0$, the factor $(1 + k^2/R^2) > 1$, so the rolling body always has more KE.

9. (a) Both start with the same PE. At the bottom, this PE is shared between translational KE and rotational KE. The hollow cylinder has a larger k^2/R^2 ($= 1$) than the solid cylinder ($= 1/2$), so it diverts a larger fraction of its energy into rotation. With less energy going into translation, the hollow cylinder moves slower. The solid cylinder, with more translational KE, reaches the bottom first.

(b) On a frictionless incline, neither cylinder would rotate. Both would slide without rolling (like blocks), and both would reach the bottom at the same time with the same speed $v = \sqrt{2gh}$. Without friction, there is no torque to produce rotation, and the shape of the body is irrelevant.

10. (a) The contact point has two simultaneous velocities: the translational velocity v_{cm} (forward, shared by all points on the body) and the rotational velocity ωR (backward at the contact point, because the bottom of the wheel moves opposite to the direction of rolling). For pure rolling, $v_{\text{cm}} = \omega R$, so these two velocities cancel exactly at the contact point, giving zero net velocity.

(b) Near the top of the wheel. At the top, the translational and rotational velocities add to give $2v_{\text{cm}}$, which is the maximum speed on the entire wheel. The stone is flung off tangentially when the adhesion can no longer provide the centripetal force ($m\omega^2 R$) needed to keep it on the curved path. This force requirement is greatest where the speed is greatest, which is at the top.

11. (a) Work requires both force and displacement at the point where the force acts. During pure rolling, the contact point (where friction acts) is instantaneously at rest. Since the displacement of the contact point is zero at every instant, the work done by friction is zero, even though the friction force itself is not zero.

(b) The rotational KE comes from the gravitational PE. As the body descends, PE converts into both translational and rotational KE. Friction does not supply the energy; it redirects it. Friction provides the torque that spins the body, but because it does no work, it does not add or remove energy from the system. It merely determines how the available PE is partitioned between translational and rotational forms.

12. (a) No external torque acts on the system, so angular momentum is conserved: $I_1\omega_1 = I_2\omega_2$. The clay adds mass at some distance from the axis, increasing the moment of inertia ($I_2 > I_1$). To keep the product $I\omega$ (angular momentum) constant, the angular velocity ω must decrease.

(b) $KE = \frac{1}{2}I\omega^2 = \frac{1}{2}I\omega^2 \times \frac{1}{I} = \frac{(L\omega)^2}{2I} = \frac{L^2}{2I}$. So, with constant momentum L , KE varies inversely proportional to the moment of inertia I . Since L is conserved (momentum is constant) and I increases, $KE = \frac{L^2}{2I}$ decreases.

13. (a) A wheel that has been brought to rest by a braking torque at the instant it stops. At that instant, $\omega = 0$, but the braking torque is still acting, producing $\alpha \neq 0$. Another example: a pendulum at its maximum displacement has $\omega = 0$ momentarily but is angularly accelerating back toward equilibrium.

(b) Yes. A wheel rotating at constant angular velocity has $\omega \neq 0$ but $\alpha = 0$. No net torque acts on it, so the angular velocity does not change. Another example is the ceiling fan at steady speed.

14. (a) The acceleration is $a = \frac{g\sin\theta}{1+k^2/R^2}$. Both the driving force ($Mg\sin\theta$, proportional to M) and the effective inertia ($M(1+k^2/R^2)$, also proportional to M) increase with mass, so M cancels. The radius R cancels because k^2/R^2 is a pure number determined only by the shape. Only the shape factor k^2/R^2 remains.

(b) Fastest first: solid sphere ($k^2/R^2 = 2/5$), solid cylinder ($1/2$), hollow sphere ($2/3$), hollow cylinder (1).

Justification: Smaller k^2/R^2 means less energy diverted into rotation and more into translation, giving larger acceleration and faster arrival.

15. **Student B is correct.** When the shell is briefly stopped, the liquid inside continues to spin because no external torque has acted on it (the brief touch stops only the shell, not the liquid). The angular momentum of the liquid is not removed. When the shell is released, the still-spinning liquid exerts a torque on the shell through viscous friction, dragging it back into rotation. So the egg restarts because the liquid retained its angular momentum.

Student A's reasoning is vague. Newton's third law says the shell pushes back on the finger while being stopped, but this does not explain why the egg restarts after the finger is removed. The key physics is that the internal angular momentum of the liquid was never removed.

16. (a) Moment of inertia depends on how far each piece of mass is from the axis. For a uniform rod, rotating about the centre ($I = \frac{1}{12}ML^2$) places the mass closer to the axis on average than rotating about one end ($I = \frac{1}{3}ML^2$). The same rod has different values of I about different axes because the mass distribution relative to the axis is different.

(b) The perpendicular axis theorem: $I_z = I_x + I_y$, where x, y, z are three mutually perpendicular axes meeting at a point, with z perpendicular to the plane of the body. This theorem applies only to flat (planar) bodies such as discs, rings, and rectangular plates.

17. (a) The perpendicular axis theorem ($I_z = I_x + I_y$) is derived by considering the distance of each mass element from the z -axis in terms of its x and y coordinates: $r_z^2 = x^2 + y^2$. This decomposition works only when all mass elements lie in the x - y plane (i.e., have $z = 0$). For a three-dimensional body, mass elements are distributed in all three directions, and the simple decomposition fails.

(b) It is unnecessary because a sphere has perfect symmetry. The moment of inertia about any diameter is the same by symmetry: $I = \frac{2}{5}MR^2$. No theorem is needed to relate one diameter to another; they are all identical.

18. (a) The flywheel (ring shape). $KE = \frac{1}{2}I\omega^2$. The ring has larger moment of inertia $I = MR^2$, while the solid ball has smaller $I = \frac{2}{5}MR^2$. At the same ω , the ring stores $\frac{1}{2}MR^2\omega^2$ while the ball stores $\frac{1}{5}MR^2\omega^2$, which is 2.5 times less. So the ring having larger moment of inertia stores more energy because its mass is farther from the axis.

(b) The ball shape. From $\alpha = \tau/I$, a smaller I gives a larger angular acceleration for the same torque. The ball ($I = \frac{2}{5}MR^2$) reaches any given ω faster than the flywheel ($I = MR^2$) which has larger value of moment of inertia.

19. Rotational KE is $\frac{1}{2}I\omega^2$. For a given mass and radius, placing the mass at the rim (like a ring) maximises I to MR^2 , compared with $\frac{1}{2}MR^2$ for a solid disc. A heavy rim concentrates the mass far from the axis of rotation, increasing I and therefore increasing the energy stored at any given angular velocity.

20. (a) The friction force required for pure rolling is $f = \frac{Mg\sin\theta}{1+R^2/k^2}$, which increases with θ . The maximum available static friction is $\mu_s Mg\cos\theta$, which decreases with θ (because $\cos\theta$ decreases). As θ increases, the required friction grows while the available friction shrinks. At some critical angle, the required friction exceeds the maximum available, and the body begins to slip.

(b) The hollow cylinder. It requires more friction ($\frac{1}{2}Mg\sin\theta$) than the solid sphere ($\frac{2}{7}Mg\sin\theta$) for the same mass and angle. With the same available friction, the hollow cylinder reaches the slipping threshold at a lower angle. The solid sphere can roll on steeper surfaces without slipping.

EXERCISE 9B

21. Torque equals force times perpendicular distance from the axis: $\tau = Fr$. The hinges are the axis of rotation. Placing the handle at the far edge maximizes r , so a small force produces a large torque. If the handle were near the hinges (small r), a

much larger force would be needed to produce the same torque and the same angular acceleration. Door designers place the handle at maximum r to minimize the force a person needs to apply.

22. (a) A heavier wheel has a larger moment of inertia, which means it stores more rotational KE ($\frac{1}{2}I\omega^2$) for the same angular velocity. This stored energy keeps the wheel spinning against the friction of the clay being shaped. A light wheel would lose its speed quickly because it stores less energy and the clay's resistance would slow it down rapidly.

(b) The wheel has a large moment of inertia (large rotational inertia). So a heavy wheel takes a long time to stop because of its large rotational inertia (Just as a heavy truck takes a long time to stop because of its large linear inertia). The only torque slowing it is friction at the axle and the resistance of the clay, which are small compared to the wheel's angular momentum. By $\tau = dL/dt$, a small torque takes a long time to reduce a large angular momentum to zero.

23. (a) The torque produced is $\tau = Fr$, where r is the distance from the axis. Pushing near the centre (small r) produces a small torque for the same force. Pushing at the edge (large r) produces a much larger torque for the same effort. The younger brother applies the same force but at a smaller distance, contributing less torque and therefore less angular acceleration to the stone.

(b) The grinding stone is heavy and has a large moment of inertia. Each push stores rotational KE in the stone ($\frac{1}{2}I\omega^2$). Between pushes, this stored energy maintains the rotation against the friction of grinding. The stone acts as a flywheel: it absorbs energy during the push and releases it smoothly between pushes.

24. (a) When the skater pulls her arms in, her moment of inertia decreases (mass is brought closer to the rotation axis). No external torque acts about her vertical axis (friction at the ice is negligible). By conservation of angular momentum, $I\omega = \text{constant}$. If moment of inertia I decreases, angular momentum ω must increase to keep the product constant, and hence she spins faster.

(b) KE increases. The rotational KE is $\frac{1}{2}I\omega^2$. When I decreases and ω increases, the KE increases (because ω increases faster than I decreases, as $KE = \frac{L^2}{2I}$ and L is constant while I decreases). The extra KE comes from the work done by the skater's muscles as she pulls her arms inward against the centripetal acceleration. She does real physical work to bring her arms in, and this work becomes rotational KE.

25. (a) For a wheel to propel a car, it must roll without slipping. This requires static friction at the contact point to push the car forward. On mud, the coefficient of static friction is very low. The engine torque spins the wheel faster than $\frac{v_{cm}}{R}$ (violating the no-slip condition), so the wheel spins in place. Consequently, kinetic friction replaces static friction, which is weaker and cannot accelerate the car effectively.

(b) The wooden planks provide a rough surface with a much higher coefficient of static friction than mud. The increased μ_s restores the no-slip condition: the wheel can now grip the plank, establish pure rolling ($v_{cm} = \omega R$), and static friction pushes the car forward. The planks supply the friction that the mud cannot.

26. **Kipute is correct.** Spinning does not create an upward force or reduce the weight of the plate. The plate's weight mg remains the same throughout. What the spin provides is angular momentum along the vertical axis. A large angular momentum resists changes in the direction of the spin axis ($\tau = dL/dt$: changing the direction of L requires a torque). Small disturbances (wobbles) try to tilt the axis, but the large L resists these changes. When the plate spins fast (large L), the resistance to tilting is strong and the plate stays balanced. When the spin slows (small L), even a small disturbance can change the axis direction significantly, causing the plate to wobble and fall.

Relate! This is gyroscopic stability, the same physics that keeps a bicycle upright and a bullet flying nose-first.

27. (a) At any angle θ from the vertical during the fall, the gravitational torque about the pivot is $\tau = Mg\frac{L}{2}\sin\theta$, and the angular acceleration is:

$$\alpha = \frac{\tau}{I} = \frac{Mg\frac{L}{2}\sin\theta}{\frac{1}{3}ML^2} = \frac{3g\sin\theta}{2L}$$

At every corresponding angle θ , the longer rod (larger L) has a smaller angular acceleration. So the longer rod is always behind throughout the entire fall. Both rods sweep through the same angle (90°), but the longer rod accelerates more slowly at every stage, so it takes more time.

Alternative Solution for (a):

Both rods rotate about their lower ends under the torque of gravity. Using conservation of energy, when the rod falls from vertical to horizontal, the centre of mass drops by $L/2$:

$$Mg\frac{L}{2} = \frac{1}{2}I\omega^2 = \frac{1}{2} \times \frac{1}{3}ML^2 \times \omega^2$$

Solving: $\omega = \sqrt{\frac{3g}{L}}$. So the longer rod (larger L) reaches a smaller final angular velocity. Since both rods fall through the same angle (90°) but the longer rod rotates more slowly at the end, the longer rod takes more time to reach the ground.

(b) In free fall, all objects experience the same acceleration g regardless of mass or size, because the force (weight) and the inertia (mass) are both proportional to m . For a rotating rod, the torque is proportional to ML but the rotational inertia is proportional to ML^2 . The extra factor of L in the inertia means longer rods are harder to rotate, unlike free fall where length does not matter. Rotation introduces a size dependence that free fall does not have.

28. (a) The piece of cloth increases the coefficient of friction between the hand and the cap. With higher friction, the hand can apply a larger tangential force at the rim of the cap without slipping. Since torque = force $\times$ radius, a larger tangential force at the same radius produces a larger torque. Without the piece of cloth, the hand slips before enough torque is applied to overcome the cap's resistance.

(b) Pliers extend the distance r from the axis (the centre of the cap) at which the force is applied. Since $\tau = Fr$, doubling the handle length doubles the torque for the same hand force.

Relate! Long handles provide a mechanical advantage for rotation, just as a long lever provides mechanical advantage for translation.

29. (a) After leaving the board, no external torque acts on the diver about the centre of mass (gravity acts through the CM, producing zero torque about it). Angular momentum is therefore conserved throughout the flight. The diver controls the rotation rate by changing body shape: tucking (pulling knees to chest) reduces the moment of inertia, which increases ω to conserve L .

(b) Straightening out increases the moment of inertia dramatically (arms and legs are now far from the rotation axis). Since $I\omega = \text{constant}$, a large I requires a small ω . The diver enters the water with almost no rotation, allowing a clean, splash-free entry.

30. (a) The bucket's weight creates a torque about the drum's axis: $\tau_{\text{bucket}} = mg \times R_{\text{drum}}$. A larger drum radius R increases this opposing torque. The operator must apply a larger force at the handle to overcome this increased torque. The bucket weighs the same, but the moment arm is larger, so the torque is larger.

(b) The heavy wooden drum has a large moment of inertia, which means it stores angular momentum ($L = I\omega$) and rotational KE ($\frac{1}{2}I\omega^2$). When the operator stops, the stored angular momentum does not disappear instantly. The bucket's weight provides a torque that gradually decelerates the drum, but with large I , the deceleration is slow ($\alpha = \tau/I$ is small). The drum coasts forward briefly until friction and the bucket's torque bring it to rest.

31. (a) The concentrated mass at the far end gives the rungu a large moment of inertia about the grip point, and when swung, the knob has a large distance from the rotation axis ($r \approx L$). The KE at impact is $\frac{1}{2}I\omega^2$, and since I is large (most mass at maximum r), the KE delivered is much greater than a uniform stick where mass is spread evenly along the length. Additionally, the linear velocity of the knob is $v = \omega L$, which is the maximum possible for the given arm swing.

(b) The centre of mass of the rungu is closer to the knob (the heavy end), not at the geometric centre. When thrown, the rungu rotates about its centre of mass. Because the centre of mass is near the knob, the handle sweeps out a larger arc than the knob, giving the characteristic lopsided end-over-end tumble. If the mass were uniform, the rotation would be symmetric about the midpoint.

32. (a) When the child moves to the centre, the moment of inertia of the system (merry-go-round + child) decreases because the child's mass is now closer to the axis. No external torque acts on the system (the axle is frictionless for this argument). By conservation of angular momentum, $I\omega = \text{constant}$, so ω increases. The merry-go-round spins faster.

(b) When the child moves back to the edge, the moment of inertia increases again. By conservation of angular momentum, ω decreases. The merry-go-round slows back down.

33. In the ice-filled can, the ice is frozen solid and rotates together with the can as a single **rigid body**. The entire mass contributes to both the translational and rotational motion.

In the water-filled can, the liquid water does not rotate with the can. As the can rolls, the shell rotates but the water inside largely slides rather than spins. The can's shell must supply all the rotational inertia, but the water still contributes to the total mass that must be translated.

The ice-filled can behaves like a solid cylinder ($k^2/R^2 = 1/2$). The water-filled can behaves somewhere between a solid cylinder and a hollow cylinder, because the non-rotating water adds translational inertia without contributing useful rotational motion. Effectively, the water-filled can has a larger "effective k^2/R^2 " and therefore a smaller acceleration. Hence, the ice-filled can, with all its mass participating properly in the rolling, reaches the bottom first.

EXERCISE 9C

34. (a) 10.05 rad/s^2 (b) 50.3 rad/s (c) 45.2 rad (7.2 revolutions) (d) 758.1 m/s^2

35. (a) (1.0m, 2.0m) (b) 2.0m (c) (1.0m, 0), 2.0m downward.

36. (b) 0.104 kgm^2 (c) 0.416 kgm^2

37. (a) 0.10 kgm^2 (b) 3.0 Nm (c) 30 rad/s^2 (d) 90 rad/s (e) 21.5 revolutions.

38. (a) 493.4 J (b) $f = 20 \text{ N}$, $\tau = 5.0 \text{ Nm}$ (c) 20 rad/s^2 (d) 3.14s (e) 15.7 revolutions.

39. (a) Solid sphere: 5.29m/s, Hollow cylinder: 4.43m/s (b) $t_{\text{sphere}} = 1.512\text{s}$, $t_{\text{cylinder}} = 1.807\text{s}$ (c) 0.295s

40. (a) 4.20m/s²(b)12.60N (c) 0.28

(d) Since $f_{\text{max}}(4.50\text{N}) < f(12.60\text{N})$, the available friction is insufficient. The cylinder cannot roll without slipping. Instead, it slides and rotates simultaneously: translating faster than ωR (skidding). Kinetic friction acts up the slope, providing some torque that partially spins the cylinder, but the no-slip condition $v_{\text{cm}} = \omega R$ is never established. The cylinder accelerates faster than a pure rolling cylinder but slower than a frictionless sliding block.

41. (a) 324.0kgm² (b) 208.8kgm² (c) 3.10rad/s (d) 1.55

The final KE is 55% larger than the initial KE. The extra energy comes from the work done by the boy's muscles as he walks inward against the centripetal acceleration. Walking toward the centre on a spinning platform requires effort because the boy must overcome the tendency to be pushed outward. This muscular work is converted into rotational KE.

42. (a) 1.96m/s² (b) $T_1 = 62.72\text{N}$, $T_2 = 58.80\text{N}$ (c) 2.17m/s

43. (a) 0.32kgm² (b) 6.06rad/s (c) 4.85m/s (d) 18.4rad/s²(e) 0rad/s²

44. (a) 4.0Nm (b) 8.89rad/s² (c) 13.5s (d) 540W (e) 60W

45. (a) 30rad/s (c) 0.0239kgm² (d) 0.057Nm

46. (a) 12.6J (b) 0.643m (c) -2.96m/s² (d) 2.37N

(e) Friction acts **down the incline** when the sphere rolls up. As the sphere decelerates while rolling up, its translational speed decreases. To maintain the no-slip condition ($v_{\text{cm}} = \omega R$), the angular velocity must also decrease. Friction must provide a torque that decelerates the spin. The only way to produce a torque that slows the clockwise spin (for a sphere rolling up to the right) is for friction to act down the slope at the contact point. This is the opposite direction to rolling down, where friction acts up the slope to spin the body up.

47. (a) From $KE = \frac{1}{2}I\omega^2$:

$$I_{\text{min}} = \frac{2 \times KE}{\omega^2} = \frac{2 \times 5000\text{J}}{(300\text{rad/s})^2} = \frac{10000}{90000}\text{kgm}^2 = 0.111\text{kgm}^2$$

(b) For a solid disc: $I = \frac{1}{2}MR^2$, and $M = \rho\pi R^2t$. Substituting:

$$I = \frac{1}{2}\rho\pi R^2t \times R^2 = \frac{1}{2}\rho\pi tR^4$$

Making R the subject:

$$R^4 = \frac{2I}{\rho\pi t} = \frac{2 \times 0.111\text{kgm}^2}{7800\text{kg/m}^3 \times \pi \times 0.05\text{m}} = \frac{0.222}{1225.2}\text{m}^4 = 1.812 \times 10^{-4}\text{m}^4$$

$$R = (1.812 \times 10^{-4})^{1/4}\text{m} = 0.116\text{m}$$

The minimum radius is approximately 0.12m (12cm).

$$\begin{aligned} \text{(c) } M = \rho\pi R^2t &= 7800\text{kg/m}^3 \times \pi \times (0.116\text{m})^2 \times 0.05\text{m} = 7800 \times 3.1416 \times 0.01346 \times 0.05\text{kg} \\ &= 7800 \times 0.002114\text{kg} = 16.5\text{kg} \end{aligned}$$

(d) For a ring with inner radius $r = 0.8R$ and outer radius R :

$$\begin{aligned} I_{\text{ring}} &= \frac{1}{2}M(R^2 + r^2) = \frac{1}{2} \times 16.5\text{kg} \times [(0.116\text{m})^2 + (0.8 \times 0.116\text{m})^2] \\ &= \frac{1}{2} \times 16.5 \times [0.01346 + 0.008614]\text{kgm}^2 = \frac{1}{2} \times 16.5 \times 0.02207\text{kgm}^2 = 0.182\text{kgm}^2 \end{aligned}$$

$$\text{Energy stored: } KE = \frac{1}{2} \times 0.182\text{kgm}^2 \times (300\text{rad/s})^2 = 0.091 \times 90000 = 8190\text{J}$$

The ring stores 8190J compared to 5000J for the disc, using the same mass and outer radius. So the ring design is better for energy storage because it places mass farther from the axis, increasing I and hence storing more energy.

48. (a) 10rad/s (b) 2.5rad/s² (c) 2.5Nm (d) 5.5Nm, 55W

49. (a) 7.6kgm²s⁻¹ (b) 9.5rad/s (c) $\omega_f = +9.5\text{rad/s}$ (positive), so the system rotates in the direction of the first disc. (d) 73.5J, 67%

50. (a) The bullet travels horizontally and hits the free end at distance $L = 1.0\text{m}$ from the pivot:

$$L_{\text{bullet}} = mvL = 0.010\text{kg} \times 400\text{m/s} \times 1.0\text{m} = 4.0\text{kgm}^2\text{s}^{-1}$$

$$\text{(b) } I_{\text{rod}} = \frac{1}{3}ML^2 = \frac{1}{3} \times 2\text{kg} \times (1.0\text{m})^2 = 0.667\text{kgm}^2$$

$$I_{\text{bullet}} = mL^2 = 0.010\text{kg} \times (1.0\text{m})^2 = 0.010\text{kgm}^2$$

$$I_{\text{total}} = 0.667 + 0.010 = 0.677\text{kgm}^2$$

(c) Conservation of angular momentum (the impact is very brief, so no external torque has time to act):

$$\omega = \frac{L_{\text{bullet}}}{I_{\text{total}}} = \frac{4.0\text{kgm}^2\text{s}^{-1}}{0.677\text{kgm}^2} = 5.91\text{rad/s}$$

(d) After impact, the rod swings upward. Using conservation of energy from the moment after impact to the maximum angle θ :

$$\frac{1}{2}I_{\text{total}}\omega^2 = Mg\frac{L}{2}(1 - \cos\theta) + mgL(1 - \cos\theta)$$

The left side is the initial rotational KE. The right side is the PE gained: the rod's CM rises by $\frac{L}{2}(1 - \cos\theta)$ and the bullet rises by $L(1 - \cos\theta)$.

$$\frac{1}{2} \times 0.677\text{kgm}^2 \times (5.91\text{rad/s})^2 = (2\text{kg} \times 9.8\text{m/s}^2 \times 0.5\text{m} + 0.010\text{kg} \times 9.8\text{m/s}^2 \times 1.0\text{m})(1 - \cos\theta)$$

$$\frac{1}{2} \times 0.677 \times 34.93 = (9.8 + 0.098)(1 - \cos\theta)$$

$$11.82\text{J} = 9.898\text{N} \cdot \text{m} \times (1 - \cos\theta)$$

$$1 - \cos\theta = \frac{11.82}{9.898} = 1.194$$

$$\cos\theta = 1 - 1.194 = -0.194$$

$$\theta = \cos^{-1}(-0.194) = 101.2^\circ$$

The rod swings 101° past the vertical (slightly beyond horizontal).

51. (a) The student confuses two different types of circular motion. The sphere rolling down an incline involves rotation about its own centre, not motion in a circular path. The “centripetal acceleration” the student refers to does not apply here. Centripetal acceleration (v^2/r) applies to an object moving along a circular path (like a car rounding a bend). A sphere rolling down a straight incline follows a straight-line path; there is no centripetal acceleration of the centre of mass. The points on the rim do move in circles about the centre and have centripetal acceleration, but this is internal to the body and is provided by internal forces within the rigid body, not by friction.

(b) Friction at the contact point provides the torque about the centre of mass that causes the sphere to spin. Without friction, the sphere would slide without rotating. Friction converts part of the gravitational pull (which would otherwise go entirely into translation) into rotation. Friction does not provide centripetal acceleration for the centre of mass.

(c) No. On a flat horizontal surface at constant velocity, both v_{cm} and ω are constant. No angular acceleration is needed (no torque required), and no linear acceleration is needed (no net force required). In the ideal case, pure rolling on a flat surface at constant velocity requires zero friction. Friction is only needed when the body must accelerate or decelerate (change v_{cm} and ω).

52. (a) 75J (b) 3.34m (c) 7.49N (down the slope) (d) 5m/s

53. (a) Just before the collision, the sphere has:

Translational angular momentum about the edge: The centre of mass moves at v_0 and is at perpendicular distance $(R - h)$ from the edge (since the edge is at height h and the centre is at height R). So $L_{\text{trans}} = Mv_0(R - h)$.

Rotational angular momentum about its own centre: $L_{\text{rot}} = I_{\text{cm}}\omega = \frac{2}{5}MR^2 \times \frac{v_0}{R} = \frac{2}{5}MRv_0$.

Both angular momenta are in the same direction (the sphere rolls forward and spins in the same sense about the edge).

$$L = Mv_0(R - h) + \frac{2}{5}MRv_0 \quad \text{?}$$

(b) After collision, the sphere rotates about the edge with angular velocity ω' . The moment of inertia about the edge (using parallel axis theorem):

$$I_{\text{edge}} = I_{\text{cm}} + MR^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

Conservation of angular momentum about the edge:

$$Mv_0(R - h) + \frac{2}{5}MRv_0 = \frac{7}{5}MR^2\omega'$$

$$\omega' = \frac{Mv_0(R-h) + \frac{2}{5}MRv_0}{\frac{7}{5}MR^2} = \frac{5v_0(R-h) + 2Rv_0}{7R^2} = \frac{v_0(5R-5h+2R)}{7R^2} = \frac{v_0(7R-5h)}{7R^2}$$

(c) The sphere rotates about the edge. The CM starts at distance R from the edge (at ground level beside the step) and must swing up to be directly above the edge. The CM traces a circular arc of radius R about the edge.

The CM must rise by $R - (R - h) = h$.

Using energy conservation:

$$\begin{aligned}\frac{1}{2}I_{\text{edge}}\omega'^2 &\geq Mgh \\ \frac{1}{2} \times \frac{7}{5}MR^2 \times \left(\frac{v_0(7R-5h)}{7R^2}\right)^2 &\geq Mgh \\ \frac{7}{10}R^2 \times \frac{v_0^2(7R-5h)^2}{49R^4} &\geq gh \\ \frac{v_0^2(7R-5h)^2}{70R^2} &\geq gh \\ v_0 &\geq \frac{\sqrt{70ghR^2}}{7R-5h}\end{aligned}$$

The required minimum value of v_0 :

$$v_0 = \frac{R\sqrt{70gh}}{7R-5h}$$

54. (a) 200rad/s (b) 1200J (c) 15s (d) 318 revolutions.

55. (a) $KE = \frac{1}{2}Mv^2(1 + 2/5) = \frac{1}{2} \times 0.5\text{kg} \times (6\text{m/s})^2 \times \frac{7}{5} = 12.6\text{J}$

(b) At the top of the loop, the centre of the ball is at height $2R_{\text{loop}} - r \approx 2R_{\text{loop}} = 1.0\text{m}$ (since $r \ll R_{\text{loop}}$, we approximate).

Energy conservation:

$$\begin{aligned}\frac{1}{2}Mv_0^2\left(\frac{7}{5}\right) &= \frac{1}{2}Mv_{\text{top}}^2\left(\frac{7}{5}\right) + Mg(2R_{\text{loop}}) \\ 12.6\text{J} &= \frac{7}{10} \times 0.5\text{kg} \times v_{\text{top}}^2 + 0.5\text{kg} \times 9.8\text{m/s}^2 \times 1.0\text{m} \\ v_{\text{top}} &= 4.69\text{m/s}\end{aligned}$$

(c) At the top of the loop, the centripetal acceleration must be at least g for the ball to maintain contact:

$$\begin{aligned}\frac{v_{\text{top}}^2}{R_{\text{loop}}} &\geq g \\ \frac{22.0\text{m}^2/\text{s}^2}{0.50\text{m}} &= 44.0\text{m/s}^2\end{aligned}$$

Since $44.0\text{m/s}^2 \gg 9.8\text{m/s}^2$, the ball has more than enough speed. It completes the loop comfortably.

56. (a) $\alpha = \frac{\tau}{I} = \frac{6t-2}{0.4}\text{rad/s}^2 = (15t-5)\text{rad/s}^2$

Setting $\alpha = 0$: $15t - 5 = 0$, so $t = \frac{1}{3}\text{s} = 0.333\text{s}$.

(b) $\omega = \int_0^t \alpha \, dt = \int_0^3 (15t - 5) \, dt = \left[\frac{15t^2}{2} - 5t\right]_0^3 = \frac{15 \times 9}{2} - 5 \times 3 = 52.5\text{rad/s}$

(c) $L = I\omega = 0.4\text{kgm}^2 \times 52.5\text{rad/s} = 21.0\text{kgm}^2\text{s}^{-1}$

57. (a) $a = \frac{g\sin\theta}{1+k^2/R^2}$. For a hollow sphere, $k^2/R^2 = 2/3$:

$$a = \frac{g\sin\theta}{1+2/3} = \frac{g\sin\theta}{5/3} = \frac{3g\sin\theta}{5}$$

(b) $f = \frac{Mg\sin\theta}{1+R^2/k^2}$. For a hollow sphere, $R^2/k^2 = 3/2$:

$$f = \frac{Mg\sin\theta}{1+3/2} = \frac{Mg\sin\theta}{5/2} = \frac{2Mg\sin\theta}{5}$$

(c) For pure rolling: $f \leq \mu_s Mg \cos \theta$

$$\frac{2Mg \sin \theta}{5} \leq \mu_s Mg \cos \theta$$

$$\mu_s \geq \frac{2 \sin \theta}{5 \cos \theta} = \frac{2}{5} \tan \theta$$

Hence:

$$\mu_{s, \min} = \frac{2}{5} \tan \theta$$

(d) Required: $\mu_s \geq \frac{2}{5} \tan 45^\circ = \frac{2}{5} \times 1 = 0.40$

Available: $\mu_s = 0.35$

Since $0.35 < 0.40$, the sphere cannot roll without slipping. It slips while rolling up the ramp: the translational speed decreases faster than ωR (kinetic friction acts, providing less torque than needed for pure rolling). The sphere slides and rotates simultaneously, losing more energy to friction than in pure rolling. It reaches a lower maximum height than a purely rolling sphere would.

EXERCISE 10A

1. A molecule deep inside a liquid is surrounded by neighbouring molecules on all sides. The attractive forces from these neighbours act equally in all directions, so the net force on the interior molecule is zero and its potential energy is at a minimum.

A molecule at the surface has neighbours only below and to the sides. There are very few molecules above (only the sparse molecules of the vapour or air). Consequently, the surface molecule experiences a net inward pull, directed into the bulk of the liquid. To bring a molecule from the interior to the surface, work must be done against this inward pull. Therefore, surface molecules have higher potential energy than interior molecules.

Since the surface has higher energy, the liquid surface tends to contract to the smallest possible area, because this minimises the total surface energy.

2. **Needle floats:** When the needle is placed flat on the water surface, its weight is distributed along the length of the needle. The water surface behaves like a stretched elastic membrane due to surface tension. The vertical component of the surface tension force acting along the contact line around the needle supports the needle's weight. The surface is depressed slightly but not broken.

Needle sinks after soap is added: Liquid soap is a surfactant that dramatically reduces the surface tension of water (from about 0.073 Nm^{-1} to about 0.025 Nm^{-1}). With reduced surface tension, the upward surface tension force along the contact line is no longer sufficient to support the weight of the steel needle. Consequently, the surface breaks and the needle sinks.

3. By the capillary rise formula:

$$h = \frac{2\gamma \cos \theta}{\rho g r}$$

For the same liquid in the same type of tube, γ , θ , ρ , and g are all constant. Therefore:

$$h \propto \frac{1}{r}$$

The height of rise is inversely proportional to the radius of the tube. A narrower tube (smaller r) gives a larger height h , and a wider tube gives a smaller height. This is because in a narrower tube, the same surface tension force acts along a smaller circumference but supports a taller, thinner column of liquid.

4. **Concave meniscus:** The liquid surface curves upward near the wall of the container. This occurs when the adhesive forces (between the liquid and the solid) are stronger than the cohesive forces (between the liquid molecules). The liquid is attracted to the wall and climbs upward along it. The angle of contact is less than 90° . Example: water in a glass tube.

Convex meniscus: The liquid surface curves downward near the wall. This occurs when the cohesive forces are stronger than the adhesive forces. The liquid molecules are pulled away from the wall and toward each other. The angle of contact is greater than 90° . Example: mercury in a glass tube.

5. By the equation of continuity for an incompressible fluid:

$$A_1 v_1 = A_2 v_2$$

where A is the cross-sectional area and v is the fluid velocity. This equation is a statement of conservation of mass: the volume of fluid entering any section per unit time must equal the volume leaving per unit time.

At the narrow section, $A_2 < A_1$. For the product Av to remain constant, v_2 must be greater than v_1 . The fluid speeds up because the same volume of fluid must pass through a smaller area in the same time. No additional force is needed; the speed increase is an automatic consequence of mass conservation.

6. Bernoulli's equation for horizontal flow is:

$$P + \frac{1}{2}\rho v^2 = \text{constant}$$

The term P represents the pressure energy per unit volume, and $\frac{1}{2}\rho v^2$ represents the kinetic energy per unit volume. The sum of these two is constant along a streamline.

This can be understood as an energy budget: the fluid has a fixed total energy per unit volume. If the fluid speeds up (kinetic energy increases), the pressure energy must decrease to keep the total constant. Conversely, if the fluid slows down, the pressure increases.

Therefore, in a horizontal pipe, where the fluid moves faster (such as at a constriction), the pressure is lower, and where it moves slower (such as in a wider section), the pressure is higher.

7. When a ball spins about a vertical axis while moving forward through the air, it drags air around it in the direction of spin. On one side of the ball, the dragged air adds to the oncoming airflow, increasing the air speed on that side. On the opposite side, the dragged air opposes the oncoming flow, decreasing the air speed.

By Bernoulli's principle, the side with faster air has lower pressure, and the side with slower air has higher pressure. This pressure difference creates a net sideways force that deflects the ball toward the low-pressure side.

This is called the **Magnus effect**.

8. When a viscous fluid flows through a pipe, the layer in direct contact with the pipe wall adheres to the wall and has zero velocity. This is called the **no-slip condition**, and it applies to all real fluids on all solid surfaces.

The layer immediately above the wall is slowed by viscous drag from the stationary layer. The next layer is slowed less, and so on. Each layer is retarded by the one below it and pushed forward by the one above it. Consequently, the velocity increases smoothly from zero at the wall to a maximum at the centre, where no wall is nearby to exert drag.

Sketch: side view of pipe with horizontal arrows increasing in length from the wall (zero) to the centre (maximum), with a dashed parabolic envelope connecting the arrow tips.

9. **Liquids:** Viscosity in liquids arises from intermolecular attractive forces that resist the sliding of adjacent layers. As temperature increases, the molecules gain kinetic energy. The increased molecular motion weakens the intermolecular forces, making it easier for layers to slide over one another. Consequently, viscosity decreases.

Gases: Viscosity in gases does not arise from intermolecular bonds (which are negligible in gases). Instead, it arises from the transfer of momentum between layers by molecules moving randomly back and forth between them. At higher temperatures, gas molecules move faster and more vigorously, transferring more momentum between layers. As a result, the friction between layers increases and viscosity rises.

The two mechanisms are fundamentally different: intermolecular forces (liquids) versus molecular momentum transfer (gases).

10. **At the moment of release ($t = 0$):** The ball is at rest ($v = 0$). The only forces are the weight W acting downward and the upthrust U acting upward. Since the ball is denser than glycerine, $W > U$, so there is a net downward force. The ball accelerates downward.

As the ball speeds up: The viscous drag $F = 6\pi\eta r v$ increases with velocity. The net downward force becomes $W - U - F$, which decreases as v increases. Therefore, the acceleration decreases.

At terminal velocity: The drag has grown until $F = W - U$. The net force is zero and the acceleration is zero. The ball falls at a constant terminal velocity v_t .

For the remainder of the fall: The ball continues at constant velocity v_t until it reaches the bottom.

[Sketch: velocity-time graph. Curve starts at origin ($v = 0$ at $t = 0$), rises steeply at first, then the gradient decreases, and the curve levels off asymptotically at a horizontal dashed line labelled v_t]

EXERCISE 10B

11. The pond skater's weight is very small (typically less than 0.01N), and it is distributed across six legs, each of which presses on the water surface along a contact line. The surface tension force acts upward along the contact perimeter around each leg. Because the insect is light and the total contact perimeter is large relative to its weight, the upward surface tension force is sufficient to support it. The water surface bends slightly but does not break.

A coin has a much greater weight concentrated over a small area. The contact perimeter is short relative to the coin's weight. Consequently, the surface tension force along this perimeter is far too small to support the weight. The surface breaks immediately and the coin sinks.

The key factor is not density alone but the ratio of weight to contact perimeter. The pond skater has a favourable ratio; the coin does not.

12. On a freshly waxed car, the waxy coating is hydrophobic. The cohesive forces between water molecules are much stronger than the adhesive forces between water and wax. Consequently, the angle of contact is greater than 90° , and the water molecules are pulled together into compact, nearly spherical beads that sit on the surface.

On an unwaxed car, the paint surface is more hydrophilic. The adhesive forces between water and the paint are comparable to or stronger than the cohesive forces within the water. The angle of contact is small (well below 90°), so the water spreads out and wets the surface, forming flat patches.

The difference is entirely determined by the angle of contact, which depends on the relative strengths of adhesive and cohesive forces at the liquid-solid interface.

13. In untilled soil, the tiny spaces between soil particles act as capillary tubes. Water from the moist subsoil rises upward through these narrow channels by capillary action and reaches the surface, where it evaporates into the air. This continuous capillary rise followed by evaporation steadily dries the soil.

When the farmer tills the top layer, the soil structure is broken up into larger, irregular clumps with much wider air gaps. Capillary rise is inversely proportional to the gap width ($h \propto 1/r$). The wider gaps in the tilled layer have very weak capillary rise, so water from below cannot reach the surface easily. Consequently, the capillary connection between the moist subsoil and the dry air is broken, and the moisture is retained in the soil below the tilled layer.

14. When a strong wind blows horizontally across the top of a roof, the air above the roof moves at high speed. The air inside the house (below the roof) is stationary or nearly so.

By Bernoulli's equation, the fast-moving air above the roof has lower pressure than the still air below the roof. This pressure difference creates a net upward force on the roof:

$$F = (P_{\text{below}} - P_{\text{above}}) \times A_{\text{roof}} = \frac{1}{2} \rho v^2 \times A_{\text{roof}}$$

If this upward force exceeds the weight of the roof and the strength of its fastenings, the roof is lifted off.

15. Dar es Salaam is significantly hotter than Moshi in the early morning. The viscosity of a liquid decreases with increasing temperature because the higher kinetic energy of molecules weakens the intermolecular forces that resist the sliding of adjacent layers. Consequently, the oil in Dar es Salaam has a lower viscosity than the same oil in cold Moshi.

By Poiseuille's formula, $Q = \frac{\pi \Delta P R^4}{8 \eta L}$, the volume flow rate is inversely proportional to the viscosity η . Since the funnel dimensions, the oil level (which determines ΔP), and the geometry are identical in both cases, the lower viscosity in hot Dar es Salaam produces a higher flow rate. Hence, the oil flows faster in Dar es Salaam and slower in Moshi.

16. For an aircraft to maintain level flight, the lift force must equal the weight:

$$\frac{1}{2} \rho A (v_1^2 - v_2^2) = Mg$$

At high altitude, the air density ρ is lower than at sea level. For the lift force to remain equal to the weight (which has not changed), the speed difference ($v_1^2 - v_2^2$) must increase to compensate for the lower density. This requires higher airspeed.

Therefore, aircraft fly faster at high altitude because the thinner air generates less lift per unit speed. The aircraft must fly faster to produce the same lift force needed to support its weight.

17. In a healthy artery, blood flows in a laminar (smooth, orderly) pattern. So the Reynolds number is typically below the turbulent threshold of 2000-3000.

When an artery is partially blocked by plaque (atherosclerosis), the internal radius at the blockage is reduced. By the equation of continuity, the blood speed increases at the narrowed section. This increased speed raises the Reynolds number. If the Reynolds number exceeds approximately 2000-3000, the flow transitions from laminar to turbulent.

Turbulent flow produces chaotic, fluctuating pressure variations that create audible vibrations in the artery wall. These vibrations are the "murmur" the doctor hears through the stethoscope. Therefore, the murmur indicates that the artery is narrowed enough to cause turbulent blood flow, which is a sign of significant atherosclerosis.

18. Kipanga's claim (gravity pulls large drops harder) is partially correct but incomplete. The gravitational force on a drop is $W = \frac{4}{3} \pi r^3 \rho g$, which is indeed larger for a larger drop. However, this alone does not explain why large drops hit the ground harder, because larger drops also experience greater drag.

Kipute's claim (the difference is about terminal velocity) is the correct and complete explanation. By the terminal velocity formula:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

The terminal velocity is proportional to r^2 . A drop with twice the radius has four times the terminal velocity. This is because the weight grows as r^3 while the Stokes drag grows only as r (since $F = 6\pi\eta r v$). For larger drops, the weight increases faster than the drag, so a higher velocity is needed before the drag catches up. Hence, large drops reach a much higher terminal velocity and hit the ground harder.

Kipute is correct. The key physics is the r^2 dependence of terminal velocity, not simply the gravitational force.

19. When a high-speed train passes, the air between the person and the train is dragged along by the train and moves at high speed. By Bernoulli's equation, this fast-moving air has lower pressure than the still air on the far side of the person (away from the track).

The pressure difference across the person creates a net force directed toward the train:

$$F = (P_{\text{still}} - P_{\text{moving}}) \times A_{\text{body}} \approx \frac{1}{2} \rho v^2 \times A_{\text{body}}$$

The faster the train, the larger the pressure difference and the stronger the force. At very high speeds (such as modern express trains), this force can be large enough to pull a person off balance and into the path of the train.

Safety lines are painted well back from the platform edge so that passengers stand far enough from the train that the high-speed air has slowed significantly before reaching them, reducing the pressure difference and the inward force to a safe level.

20. The saline solution flows from the IV bag through the thin tube into the patient's vein. The flow is driven by the pressure difference between the liquid surface in the bag and the entry point in the vein. This pressure difference is the hydrostatic pressure:

$$\Delta P = \rho gh$$

where h is the vertical height of the liquid surface above the entry point. Raising the bag increases h , which increases ΔP .

The flow through the thin tube is governed by Poiseuille's formula:

$$Q = \frac{\pi \Delta P R^4}{8 \eta L}$$

Since Q is directly proportional to ΔP , a larger pressure difference produces a larger flow rate. Therefore, raising the IV bag increases the flow rate of saline into the patient.

21. A soap bubble has two surfaces (inner and outer), each with surface energy γ per unit area. The total surface energy of the bubble is proportional to its total surface area.

Among all closed shapes enclosing a given volume of air, a sphere has the smallest surface area. Since surface tension drives the film toward the configuration of minimum surface energy, the bubble adopts the shape that minimises its total surface area for the volume of air trapped inside. That shape is a sphere.

Therefore, regardless of how the bubble was blown (through a circular ring, an irregular frame, or by hand), it always relaxes into a spherical shape because this is the state of lowest surface energy.

22. By Poiseuille's formula, the flow rate through a cylindrical tube is:

$$Q = \frac{\pi \Delta P R^4}{8 \eta L}$$

The flow rate depends on the **fourth power** of the radius. If the student switches from a narrow straw of radius R to a wider straw of radius $2R$ (with the same length and the same sucking pressure), the flow rate increases by a factor of:

$$\frac{Q_2}{Q_1} = \left(\frac{2R}{R}\right)^4 = 16$$

The wider straw gives 16 times the flow rate. This dramatic increase comes entirely from the R^4 dependence. Doubling the radius does not merely double the flow; it increases it sixteenfold. This is why even a small increase in straw width makes a thick milkshake much easier to drink.

23. The ocean surface is a water-air interface with high surface tension ($\gamma_{\text{water}} \approx 0.073 \text{ Nm}^{-1}$). When oil is placed on this surface, three interfaces are involved: oil-air, oil-water, and water-air.

The surface energy of the oil-air interface ($\gamma_{\text{oil}} \approx 0.03 \text{ Nm}^{-1}$) plus the oil-water interface is less than the surface energy of the original water-air interface. Consequently, replacing the high-energy water-air surface with the lower-energy combination of oil-air and oil-water surfaces reduces the total surface energy of the system.

Since physical systems tend to minimise their energy, the oil spreads spontaneously to cover as much of the water surface as possible. The spreading continues until the oil forms a very thin film, at which point the energy reduction from further spreading becomes negligible.

24. By Stokes' law, a particle falling through air reaches a terminal velocity:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$$

The terminal velocity is proportional to r^2 . Fine dust particles have very small radii (of the order of micrometres), so their terminal velocity is extremely small. At such low speeds, even the slightest air current keeps the particles suspended. They take hours to settle.

Coarse sand grains have larger radii. Since $v_t \propto r^2$, the terminal velocity of sand is many times larger than that of fine dust. Consequently, sand grains fall rapidly and settle within seconds.

So the r^2 dependence of terminal velocity is the reason why particle size determines settling time so dramatically.

EXERCISE 10C

25. (a) 146Pa (b) 1.99×10^5 Pa

Hint for 25(b): The total pressure inside the bubble is the sum of atmospheric pressure, hydrostatic pressure at depth $h = 10$ m, and excess pressure:

$$P_{\text{total}} = P_{\text{atm}} + \rho gh + \Delta P$$

26. (a) 12cm (b) 1.33Pa

27. (a) 1mm (b) 1.13×10^{-4} J (c) 5.24×10^{-4} K

28. (a) 3.67cm (b) Since the tube is only 3cm long (shorter than 3.67cm), the water rises to the top of the tube but does **not overflow**. Instead, the meniscus at the top adjusts its curvature (becomes flatter, increasing the effective radius of curvature) until the excess pressure across the meniscus is just sufficient to support a liquid column equal to the tube length (3cm). The angle of contact increases from 0° to a value satisfying:

$$\cos\theta' = \frac{\rho g r h_{\text{tube}}}{2\gamma} = \frac{1000 \times 9.8 \times 4 \times 10^{-4} \times 0.03}{2 \times 0.072} = 0.8167$$

$$\theta' = \cos^{-1}(0.8167) \approx 35.2^\circ$$

29. 1.86×10^{-3} m $\approx$ 1.9mm

30. (a) 8ms^{-1} (b) 2.20×10^5 Pa (c) $2.26 \times 10^{-2}\text{m}^3\text{s}^{-1}$

31. (a) 9.90ms^{-1} (b) 1.278s (c) 12.65m

32. 16.0 litres per second

33. Error 1: Bernoulli's equation does not apply at supersonic speeds. Bernoulli's equation is valid only for incompressible flow. At supersonic speeds ($v > 340\text{ms}^{-1}$ in air), the air is compressed significantly by shock waves. The density is no longer constant, and Bernoulli's equation breaks down completely. The actual pressure distribution around a supersonic wing must be calculated using compressible flow theory, not Bernoulli.

Error 2: The student has confused dynamic pressure with total pressure. Even within the valid range of Bernoulli's equation, the pressure around a wing does not drop to zero. The dynamic pressure $\frac{1}{2}\rho v^2$ is the kinetic energy contribution, not the actual static pressure at the wing. The static pressure can decrease but it cannot physically reach zero or become negative in a real fluid. The total pressure (static + dynamic) remains constant along a streamline; the static pressure decreases only by the amount that the dynamic pressure increases.

Therefore, the student's conclusion that the wings have near-zero pressure is wrong on both counts: the equation does not apply, and even if it did, the reasoning misinterprets what the terms mean.

34. (a) $6.28 \times 10^{-3}\text{m}^3\text{s}^{-1}$ (b) $0.1\text{m}^3\text{s}^{-1}$

35. 16

36. (a) 0.0481ms^{-1} (b) 0.703Nsm^{-2} (c) 0.185

Since $Re = 0.185 < 1$, the flow around the sphere is laminar and Stokes' law is valid.

37. Student A (air bubble in water, one surface):

$$\Delta P_A = \frac{2\gamma}{r} = \frac{2 \times 0.073\text{Nm}^{-1}}{2 \times 10^{-3}\text{m}} = 73\text{Pa}$$

Student B (soap bubble in air, two surfaces):

$$\Delta P_B = \frac{4\gamma}{r} = \frac{4 \times 0.025\text{Nm}^{-1}}{2 \times 10^{-3}\text{m}} = 50\text{Pa}$$

Student A gets 73Pa and Student B gets 50Pa. Student A's claim that her bubble has "more surface tension" is **incorrect**. In fact, $\gamma_{\text{water}} = 0.073\text{Nm}^{-1} > \gamma_{\text{soap}} = 0.025\text{Nm}^{-1}$, so water does have higher surface tension. However, the real reason Student A gets a higher excess pressure is a combination of the higher surface tension of water **and** the fact that the air bubble has the formula $\frac{2\gamma}{r}$ while the soap bubble has $\frac{4\gamma}{r}$. Despite the factor of 4 (which should favour the soap bubble), the much higher surface tension of water wins. The student confused the result (higher excess pressure) with a single cause (surface tension) without considering that the two types of bubble use different formulas.

38. 62.6ms^{-1}

39. (a) 1.316 (b) Tsh 158,000

40. (a) $4.8 \times 10^{-3}\text{N}$ (b) 0.49g

41. $2.78 \times 10^5 \text{ Pa}$

42. $7225 \text{ s} \approx 2 \text{ hours}$

43. 71.4 ms^{-1}

44. 56.6 N

45. Under isothermal conditions, total surface area is conserved (for soap bubbles):

$$3 \times 2 \times 4\pi r^2 = 2 \times 4\pi R^2 \Rightarrow R^2 = 3r^2 \Rightarrow R = r\sqrt{3}$$

Both methods give the same result, confirming $R = r\sqrt{3}$.

Ratio of excess pressures:

$$\frac{\Delta P_{\text{original}}}{\Delta P_{\text{new}}} = \frac{4\gamma/r}{4\gamma/R} = \frac{R}{r} = \frac{r\sqrt{3}}{r} = \sqrt{3} \approx 1.73$$

The excess pressure in the original bubbles is $\sqrt{3}$ times that in the new bubble.

46. (a) $1.853 \times 10^{-4} \text{ m}$ (b) 0.0408 Nm^{-1}

47. (a) Sphere density = 1.5ρ , fluid density = ρ :

$$v_t = \frac{2r^2(1.5\rho - \rho)g}{9\eta} = \frac{2r^2 \times 0.5\rho \times g}{9\eta} = \frac{r^2\rho g}{9\eta}$$

(b) Second sphere: radius $2r$, density 3ρ :

$$v_t' = \frac{2(2r)^2(3\rho - \rho)g}{9\eta} = \frac{2 \times 4r^2 \times 2\rho \times g}{9\eta} = \frac{16r^2\rho g}{9\eta}$$

$$\frac{v_t'}{v_t} = \frac{\frac{r^2\rho g}{9\eta}}{\frac{16r^2\rho g}{9\eta}} = \frac{1}{16}$$

48. 14.4 ms^{-1} , $0.0432 \text{ m}^3 = 43.2 \text{ litres per minute}$

49. By Poiseuille's formula, each small pipe (radius $\frac{R}{2}$, length L) carries:

$$Q_{\text{small}} = \frac{\pi \Delta P (R/2)^4}{8\eta L} = \frac{\pi \Delta P R^4}{8\eta L \times 16} = \frac{Q}{16}$$

Four such pipes in parallel carry a total of:

$$Q_{\text{total}} = 4 \times \frac{Q}{16} = \frac{Q}{4}$$

The parallel system carries only **one quarter** of the flow rate of the single pipe. Replacing one pipe with four pipes of half the radius is a very poor trade, because the R^4 law amplifies the effect of the radius reduction far more than the four-fold increase in number compensates.

50. This is the rotating disc problem. Consider an annular ring at radius r , width dr . Area $dA = 2\pi r dr$. Velocity at radius r : $v = \omega r$. Velocity gradient: $dv/dr = \omega$.

Viscous force on ring: $dF = \eta \times 2\pi r dr \times \omega r/d$ Torque from ring: $d\tau = r dF = 2\pi\eta\omega r^3 dr/d$

Total torque:

$$\tau = \frac{2\pi\eta\omega}{d} \int_0^R r^3 dr = \frac{2\pi\eta\omega}{d} \times \frac{R^4}{4} = \frac{\pi\eta\omega R^4}{2d}$$

Given: $R = 0.10 \text{ m}$, $\eta = 0.5 \text{ Nsm}^{-2}$, $d = 0.8 \times 10^{-3} \text{ m}$, $\omega = 2\pi \times 90/60 = 3\pi \text{ rad s}^{-1}$

$$\tau = \frac{\pi \times 0.5 \text{ Nsm}^{-2} \times 3\pi \text{ rad s}^{-1} \times (0.10 \text{ m})^4}{2 \times 8 \times 10^{-4} \text{ m}} = 0.925 \text{ Nm}$$

51. Without air resistance:

$$\text{Time of flight: } t = \sqrt{\frac{2H}{g}} = \sqrt{2 \times \frac{25 \text{ m}}{9.8 \text{ ms}^{-2}}} = 2.26 \text{ s}$$

$$\text{Horizontal distance: } x = v_0 t = 30 \text{ ms}^{-1} \times 2.26 \text{ s} = 67.8 \text{ m}$$

Effect of viscous drag:

(i) Horizontal distance decreases. Air resistance opposes the horizontal motion, decelerating the ball throughout its flight. The horizontal velocity decreases continuously instead of remaining constant. Consequently, the ball lands closer to the launch point.

(ii) Time of flight increases slightly. During the downward fall, air resistance acts upward (opposing the downward motion), reducing the net downward acceleration. The ball takes slightly longer to reach the ground than in the no-drag case.

52. (a) By the equation of continuity:

$$v_2 = v_1 \times \frac{A_1}{A_2} = v_1 \times \frac{r_1^2}{r_2^2} = 0.3\text{ms}^{-1} \times \frac{(2.5\text{mm})^2}{(1.5\text{mm})^2} = 0.3\text{ms}^{-1} \times \frac{6.25}{2.25} = 0.833\text{ms}^{-1}$$

(b) By Bernoulli's equation (horizontal flow):

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) = 1.2 \times 10^4\text{Pa} + \frac{1}{2} \times 1060\text{kgm}^{-3} \times ((0.3\text{ms}^{-1})^2 - (0.833\text{ms}^{-1})^2)$$

$$P_2 = 1.17 \times 10^4\text{Pa}$$

The pressure drops in the narrowed section, which can cause the artery wall to collapse further (a dangerous positive feedback).

(c) By Poiseuille's formula, $Q \propto R^4$:

$$\frac{Q_{\text{narrowed}}}{Q_{\text{healthy}}} = \left(\frac{1.5\text{mm}}{2.5\text{mm}}\right)^4 = (0.6)^4 = 0.1296$$

Percentage reduction:

$$(1 - 0.1296) \times 100\% = 87\%$$

Medical significance: A 40% $\left(\frac{2.5-1.5}{2.5} \times 100\%\right)$ reduction in radius causes an 87% reduction in blood flow through a long narrowed segment. The blood speed nearly triples at the constriction, and the pressure drops. The reduced pressure can cause further collapse of the artery wall. The combination of severely reduced flow and potential wall collapse makes atherosclerosis life-threatening even when the visible narrowing appears moderate on imaging.

SOLUTIONS TO EXAMINATION QUESTIONS

PART ONE

Question 1

(a) Weight is not a fundamental quantity. It is a force: $W = mg$. Force is derived from mass (fundamental), length (fundamental), and time (fundamental) through $F = MLT^{-2}$. The fact that weight can be measured directly on a scale does not make it fundamental. Many derived quantities can be measured directly (velocity with a speedometer, pressure with a barometer), but they remain derived because they can be expressed in terms of simpler quantities.

(b) Work is a scalar defined as the product of force and displacement in the direction of the force: $W = Fd\cos\theta$. Torque is a vector defined as the product of force and the perpendicular distance from the pivot: $\tau = Fr\sin\theta$. Both involve force times distance, giving the same dimensions ML^2T^{-2} , but they describe entirely different physical effects. Work is energy transferred; torque is a turning effect. Dimensional analysis cannot distinguish them because it sees only dimensions, not physical meaning.

Another pair: pressure ($ML^{-1}T^{-2}$) and stress ($ML^{-1}T^{-2}$).

(c) Assume $f = kl^aT^b\mu^c$

Dimensions: $[f] = T^{-1}$, $[l] = L$, $[T] = MLT^{-2}$, $[\mu] = ML^{-1}$

$$T^{-1} = L^a(MLT^{-2})^b(ML^{-1})^c = M^{b+c}L^{a+b-c}T^{-2b}$$

For M: $0 = b + c \dots$ (i)

For T: $-1 = -2b$, giving $b = 1/2 \dots$ (ii)

For L: $0 = a + b - c \dots$ (iii)

From (i): $c = -1/2$. From (iii): $a = c - b = -1/2 - 1/2 = -1$.

$$f = \frac{k}{l} \sqrt{\frac{T}{\mu}}$$

Question 2

(a) A physical quantity is a property that can be measured on a continuous scale and expressed as a number with a unit. The number of students is obtained by counting, not measuring. It is a discrete integer with no unit and no uncertainty. You cannot have 32.7 students. A physical quantity like length can take any value on a continuous scale (2.35 m, 2.351 m, 2.3517 m...) and always requires a unit to have meaning. Counting produces a pure number; measuring produces a physical quantity.

(b) An angle in radians is defined as arc length divided by radius: $\theta = s/r$. Both s and r have dimensions of length, so $[\theta] = L/L = 1$ (dimensionless). In dimensional analysis, angles are invisible: they contribute no powers of M, L, or T. The consequence is that dimensional analysis cannot discover that a formula contains an angle. For example, the formula $x = A\sin(\omega t + \phi)$ cannot be derived by dimensional analysis because $\sin$ and the angle ϕ are dimensionless and undetectable.

(c) First term: $[P] = ML^{-1}T^{-2}$

Second term: $\left[\frac{1}{2}\rho v^2\right] = ML^{-3} \times (LT^{-1})^2 = ML^{-3} \times L^2T^{-2} = ML^{-1}T^{-2}$

Third term: $[\rho gh] = ML^{-3} \times LT^{-2} \times L = ML^{-1}T^{-2}$

All three terms have dimensions $ML^{-1}T^{-2}$. The equation is dimensionally homogeneous.

Question 3

(a) Random errors vary unpredictably in direction: some readings are too high, others too low. When many readings are averaged, the positive deviations tend to cancel the negative ones, and the mean converges toward the true value. Systematic errors shift every reading in the same direction by the same amount. Averaging twenty shifted readings gives a mean that is shifted by exactly the same amount as any single reading. Repetition adds no new information about the systematic shift; only recalibration or a change of technique can remove it.

(b) Parallax occurs when the observer's eye is not perpendicular to the scale. If a pointer sits a small distance above the scale markings, viewing from an angle causes the line of sight to pass through the pointer and hit the scale at the wrong position. The pointer appears to align with a mark that is to the left or right of the true reading. The error is always in the same direction for a given viewing angle, making it systematic. It is eliminated by positioning the eye directly in front of and perpendicular to the scale at the point of reading.

(c)(i) Mean total time for 50 oscillations:

$$\bar{t} = \frac{98.4s + 98.0s + 98.8s}{3} = \frac{295.2s}{3} = 98.4s$$

Individual absolute errors: $|98.4 - 98.4| = 0.0$ s, $|98.0 - 98.4| = 0.4$ s, $|98.8 - 98.4| = 0.4$ s

Mean absolute error in total time: $\Delta \bar{t} = \frac{(0.0+0.4+0.4)s}{3} = 0.27s$

Period: $T = \frac{98.4s}{50} = 1.968s$

Uncertainty in period: $\Delta T = \frac{0.27s}{50} = 0.005s$

Result: $T = (1.968 \pm 0.005) s$

(c)(ii)

$$g = \frac{4\pi^2 \times 0.950m}{(1.968s)^2} = 9.69ms^{-2}$$

$$\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2 \frac{\Delta T}{T} = \frac{0.002m}{0.950m} + 2 \times \frac{0.005s}{1.968s} = 0.21\% + 0.51\% = 0.72\%$$

$$\Delta g = 0.0072 \times 9.69ms^{-2} = 0.07ms^{-2}$$

Result: $g = (9.69 \pm 0.07)ms^{-2}$

(c)(iii) The period T dominates, contributing 0.51% out of the total 0.72%.

Question 4

(a) Different unit systems make communication between scientists, engineers, and trading partners unreliable. A measurement in one system cannot be compared with a measurement in another without conversion, and every conversion is an opportunity for error. A single international system ensures that a kilogram in one place means the same as a kilogram in any other place, enabling science, engineering, and commerce to operate without ambiguity.

(b) The lesson: every number must carry its unit, and every interface between systems must include a unit check.

(c) The pressure exerted by a column of liquid is $P = h\rho g$.

In CGS units: $h = 76cm$, $\rho = 13.6gcm^{-3}$, $g = 980cms^{-2}$

$$P = 76 \times 13.6 \times 980 = 1.013 \times 10^6 \text{ gcm}^{-1}s^{-2}$$

The dimensional formula of pressure is $ML^{-1}T^{-2}$, so $a = 1$, $b = -1$, $c = -2$.

Converting to SI using the method of dimensions:

$$\begin{aligned} n_{SI} &= n_{CGS} \times \left(\frac{M_{CGS}}{M_{SI}}\right)^1 \times \left(\frac{L_{CGS}}{L_{SI}}\right)^{-1} \times \left(\frac{T_{CGS}}{T_{SI}}\right)^{-2} \\ &= 1.013 \times 10^6 \times \left(\frac{1g}{1kg}\right)^1 \times \left(\frac{1cm}{1m}\right)^{-1} \times \left(\frac{1s}{1s}\right)^{-2} \\ &= 1.013 \times 10^6 \times (10^{-3})^1 \times (10^{-2})^{-1} \times 1 \\ &= 1.013 \times 10^5 Nm^{-2} \end{aligned}$$

The pressure is $1.013 \times 10^5 Nm^{-2}$.

Question 5

(a) Random errors vary unpredictably: some readings are too high, others too low. When many readings are averaged, the positive deviations tend to cancel the negative ones, so the mean converges toward the true value. This works on the condition that the errors are truly random (equally likely to be positive or negative) and that no systematic error is present. If a systematic error exists, every reading is shifted in the same direction, and averaging cannot remove the shift.

(b) Six of the seven readings cluster between 3.48 and 3.55 mm, a spread of 0.07 mm. The reading 4.21 mm lies 0.66 mm above the next highest value, far outside the cluster. This is almost certainly a blunder (a severe mistake such as misreading the scale or recording the wrong digit). It is discarded before further analysis. Including it would contaminate the mean and inflate the uncertainty with a single human error rather than reflecting the genuine measurement uncertainty.

(c) Using the six valid readings: 3.52, 3.48, 3.55, 3.50, 3.53, 3.49

Mean: $\bar{t} = (3.52 + 3.48 + 3.55 + 3.50 + 3.53 + 3.49)/6 = 21.07/6 = 3.512 \text{ mm}$

Individual absolute errors: $|3.52 - 3.512| = 0.008$, $|3.48 - 3.512| = 0.032$, $|3.55 - 3.512| = 0.038$, $|3.50 - 3.512| = 0.012$, $|3.53 - 3.512| = 0.018$, $|3.49 - 3.512| = 0.022$

Mean absolute error:

$$\Delta \bar{t} = \frac{(0.008+0.032+0.038+0.012+0.018+0.022)m}{6} = 0.022 \text{ mm} \approx 0.02 \text{ mm}$$

Percentage error: $\left(\frac{0.02}{3.512}\right) \times 100\% = 0.6\%$

Result: $t = (3.51 \pm 0.02) \text{ mm}$

Question 6

(a) If $x = a^n$, then x is the product of n factors of a . By the multiplication rule, the relative error of a product is the sum of the relative errors of each factor. Since each factor is a with relative error $\Delta a/a$, the total relative error is $\Delta a/a$ added n times: $\Delta x/x = n(\Delta a/a)$. The power acts as a multiplier on the relative error.

(b) The question involves error propagation. So it requires all three rules:

Powers:

$k^2 = 0.0400 \text{ m}^2$. Percentage error: $2 \times \frac{0.001}{0.200} \times 100\% = 1.0\%$. Absolute error: $0.01 \times 0.0400 = 0.0004 \text{ m}^2$.

$h^2 = 0.0900 \text{ m}^2$. Percentage error: $2 \times \frac{0.001}{0.300} \times 100\% = 0.67\%$. Absolute error: $0.0067 \times 0.0900 = 0.0006 \text{ m}^2$.

Addition: $k^2 + h^2 = 0.1300 \text{ m}^2$. Absolute error: $0.0004 \text{ m}^2 + 0.0006 \text{ m}^2 = 0.0010 \text{ m}^2$.

Percentage error in *numerator* ($k^2 + h^2$): $\left(\frac{0.0010}{0.1300}\right) \times 100\% = 0.77\%$

Division: The expression inside the square root is $\frac{k^2 + h^2}{gh}$. Since g is exact, the *denominator* contributes only the error from h : $\frac{0.001}{0.300} \times 100\% = 0.33\%$.

Percentage error inside the square root: $0.77\% + 0.33\% = 1.10\%$

Square root: The square root applies a power of $1/2$:

$$\% \text{ error in } T = \frac{1}{2} \times 1.10\% = 0.55\%$$

The percentage error in T is 0.55% .

Question 7

(a) Catastrophic cancellation occurs when two nearly equal measured quantities are subtracted. The absolute errors add (Rule 1), but the result of the subtraction is small, so the relative error of the difference becomes much larger than the relative errors of the individual measurements. Example: if $a = (50.0 \pm 0.5)$ and $b = (49.0 \pm 0.5)$, each has 1% relative error, but the difference $a - b = (1.0 \pm 1.0)$ has 100% percentage error.

(b) The experiment should be redesigned to measure the small difference directly, rather than obtaining it by subtracting two large numbers. Example: to find the temperature rise of water during heating, use the same thermometer to read both temperatures (the change is read directly from the scale) rather than using two separate thermometers and subtracting. Alternatively, use instruments with much higher precision so that the absolute errors are small compared to the expected difference.

(c) First compute $\frac{1}{r_1}$ and $\frac{1}{r_2}$:

$$\frac{1}{r_1} = \frac{1}{6400 \text{ km}} = 1.5625 \times 10^{-4} \text{ km}^{-1}$$

$$\text{Error: } \frac{\Delta\left(\frac{1}{r_1}\right)}{\left(\frac{1}{r_1}\right)} = \frac{\Delta r_1}{r_1} = \frac{10}{6400} = 0.156\%$$

$$\Delta\left(\frac{1}{r_1}\right) = 0.00156 \times 1.5625 \times 10^{-4} = 2.44 \times 10^{-7} \text{ km}^{-1}$$

$$\frac{1}{r_2} = \frac{1}{6800 \text{ km}} = 1.4706 \times 10^{-4} \text{ km}^{-1}$$

$$\text{Error: } \frac{\Delta\left(\frac{1}{r_2}\right)}{\left(\frac{1}{r_2}\right)} = \frac{10}{6800} = 0.147\%$$

$$\Delta\left(\frac{1}{r_2}\right) = 0.00147 \times 1.4706 \times 10^{-4} = 2.16 \times 10^{-7} \text{ km}^{-1}$$

The difference:

$$\frac{1}{r_1} - \frac{1}{r_2} = 1.5625 \times 10^{-4} - 1.4706 \times 10^{-4} = 0.0919 \times 10^{-4} = 9.19 \times 10^{-6} \text{ km}^{-1}$$

Absolute error:

$$\Delta = 2.44 \times 10^{-7} + 2.16 \times 10^{-7} = 4.60 \times 10^{-7} \text{ km}^{-1}$$

Percentage error:

$$\frac{4.60 \times 10^{-7}}{9.19 \times 10^{-6}} \times 100\% = 5.0\%$$

Since GMm is exact, this is also the percentage error in ΔPE : **5.0%**.

Note: The individual measurements of r_1 and r_2 each have only about 0.15% error, but the subtraction of two close reciprocals amplifies this to 5.0%.

Question 8

(a) Resolution (least count) is the smallest change in a quantity that an instrument can detect. It equals the smallest division on the scale. Sensitivity is how much the instrument's output changes per unit change in the measured quantity. A thermometer where the mercury moves 3 cm per $^{\circ}\text{C}$ is more sensitive than one where it moves 0.5 cm per $^{\circ}\text{C}$.

A highly sensitive instrument responds strongly to small changes, but if it has a systematic calibration error, all its readings are shifted from the true value. It detects changes precisely but reports values inaccurately. Sensitivity measures the fineness of the response; accuracy measures the correctness.

(b) First thing: Getting the textbook value does not mean the error is zero. The measurement still carries uncertainty from the instrument resolution, reaction time, and other sources. The uncertainty exists whether or not the mean happens to match the accepted value.

Second thing: The "textbook value" of $g = 9.8\text{ms}^{-2}$ is itself rounded. The actual value varies with location (latitude, altitude, local geology). A more precise accepted value for a specific location might be 9.812ms^{-2} . Comparing with a rounded reference value proves nothing about the quality of the measurement.

$$(c) n = \sin 45.0^{\circ} / \sin 28.0^{\circ} = 0.7071 / 0.4695 = 1.506$$

For the error, using $\Delta(\sin\theta) \approx \cos\theta \times \Delta\theta$ where $\Delta\theta$ is in radians:

$$\Delta\theta = 0.5^{\circ} = 0.5 \times \frac{\pi}{180} = 0.00873 \text{ rad}$$

Error in numerator:

$$\Delta(\sin 45.0^{\circ}) = \cos 45.0^{\circ} \times 0.00873 = 0.7071 \times 0.00873 = 0.00617$$

$$\text{Relative error in numerator: } \frac{0.00617}{0.7071} = 0.872\%$$

Error in denominator:

$$\Delta(\sin 28.0^{\circ}) = \cos 28.0^{\circ} \times 0.00873 = 0.8829 \times 0.00873 = 0.00771$$

$$\text{Relative error in denominator: } \frac{0.00771}{0.4695} = 1.642\%$$

Total percentage error in n (division rule):

$$0.872\% + 1.642\% = \mathbf{2.5\%}$$

$$\Delta n = 0.025 \times 1.506 = 0.038 \approx 0.04$$

Value: $n = 1.51 \pm 0.04$

Question 9

(a) The relationship $T = 2\pi\sqrt{l/g}$ is not linear: T is proportional to $\sqrt{l}$, not to l . Plotting T against l produces a curve, and drawing a straight line through a curve is unreliable and gives an imprecise gradient. Squaring both sides gives $T^2 = (4\pi^2/g)l$, which is a straight line through the origin with gradient $4\pi^2/g$. A straight-line graph is easier to draw accurately, the gradient can be measured precisely using two well-separated points, and deviations from linearity are immediately visible.

(b) First: The best-fit line should be drawn so that data points are distributed as evenly as possible above and below it. Drawing through every point follows the random scatter of individual readings, amplifying noise rather than averaging it out.

Second: The line should not be forced through the origin unless there is a physical reason and experimental confirmation. Even if the theory predicts the line passes through the origin, the data may indicate a small systematic offset (such as a zero error in the length measurement). Forcing the line through the origin hides this information and may distort the gradient.

$$(c) \text{Gradient } m = \frac{4\pi^2}{g}, \text{ so } g = \frac{4\pi^2}{m}.$$

$$\text{Best estimate: } g = \frac{4\pi^2}{4.05\text{s}^2\text{m}^{-1}} = 9.75\text{ms}^{-2}$$

$$\text{From maximum gradient: } g_{\min} = \frac{4\pi^2}{4.32\text{s}^2\text{m}^{-1}} = 9.14\text{ms}^{-2}$$

$$\text{From minimum gradient: } g_{\max} = \frac{4\pi^2}{3.80\text{s}^2\text{m}^{-1}} = 10.39\text{ms}^{-2}$$

$$\Delta g = \frac{g_{\max} - g_{\min}}{2} = \frac{10.39\text{ms}^{-2} - 9.14\text{ms}^{-2}}{2} = 0.6\text{ms}^{-2}$$

Percentage uncertainty: $\left(\frac{0.6}{9.75}\right) \times 100\% = 6.2\%$

Result: $g = (9.8 \pm 0.6)\text{ms}^{-2}$

Question 10

(a) The statement is missing the unit. “ 6×10^{24} ” is a pure number with no physical meaning. It could be 6×10^{24} kg, 6×10^{24} grams, or 6×10^{24} atoms. A physical quantity must always include its unit. Without the unit, the number communicates nothing.

(b) Percentage error in g :

$$\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2 \frac{\Delta T}{T} = \frac{0.002\text{m}}{1.000\text{m}} + 2 \times \frac{0.01\text{s}}{2.01\text{s}} = 0.2\% + 1.0\% = 1.2\%$$

Percentage error in v_e :

Since $v_e = \sqrt{2gR}$ and R is exact, v_e depends on g with a power of $1/2$:

$$\frac{\Delta v_e}{v_e} = \frac{1}{2} \times \frac{\Delta g}{g} = \frac{1}{2} \times 1.2\% = 0.6\%$$

Question 11

(a) Accuracy describes how close measurements are to the true value. Precision describes how closely repeated measurements agree with each other. Example: a student measures the boiling point of water five times using a thermometer and gets 101.2°C , 101.3°C , 101.1°C , 101.2°C , 101.3°C . The readings are precise (spread of only 0.2°C) but inaccurate (the true value at standard pressure is 100°C). A systematic calibration error of about $+1.2^\circ\text{C}$ is present.

(b) Random errors cancel only on average and only approximately. For a finite number of readings, the cancellation is incomplete. The mean of 5 readings may still differ from the true value by a small amount due to chance. As the number of readings increases, the mean approaches the true value more closely, but it reaches it exactly only in the limit of infinitely many readings. Furthermore, the claim assumes no systematic error is present. If a systematic error exists, the mean is displaced from the true value no matter how many readings are taken.

(c)(i) Mean total time for 30 oscillations:

$$\bar{t} = \frac{57.24\text{s} + 57.48\text{s} + 57.12\text{s}}{3} = 57.28\text{s}$$

Individual absolute errors: $|57.24 - 57.28| = 0.04\text{ s}$, $|57.48 - 57.28| = 0.20\text{ s}$, $|57.12 - 57.28| = 0.16\text{ s}$

Mean absolute error: $\Delta \bar{t} = \frac{(0.04+0.20+0.16)\text{s}}{3} = 0.13\text{s}$

Period: $T = \frac{57.28\text{s}}{30} = 1.909\text{s}$

Uncertainty: $\Delta T = \frac{0.13\text{s}}{30} = 0.004\text{s}$

Result: $T = (1.909 \pm 0.004)\text{ s}$

(c)(ii)

$$g = \frac{4\pi^2 \times 0.900\text{m}}{(1.909\text{s})^2} = \frac{35.53\text{m}}{3.644\text{s}^2} = 9.75\text{ms}^{-2}$$

$$\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2 \frac{\Delta T}{T} = \frac{0.002\text{m}}{0.900\text{m}} + 2 \times \frac{0.004\text{s}}{1.909\text{s}} = 0.22\% + 0.42\% = 0.64\%$$

$$\Delta g = 0.0064 \times 9.75\text{ms}^{-2} = 0.06\text{ms}^{-2}$$

Result: $g = (9.75 \pm 0.06)\text{ms}^{-2}$

(c)(iii) The accepted value 9.81ms^{-2} lies at the upper boundary of the student's range $(9.75 \pm 0.06)\text{ms}^{-2}$, which spans from 9.69 to 9.81ms^{-2} . The result is therefore consistent with the accepted value, though only marginally so. The fact that the accepted value falls at the extreme edge rather than near the centre suggests a possible small systematic error, such as measuring the length slightly too long or a slight timing bias.

Question 12

(a) Constant velocity means acceleration is zero. By Newton's second law, zero acceleration implies the resultant force on the vehicle is zero. Fuel is still consumed because the engine must provide a forward driving force to balance resistive forces such as air resistance and rolling friction. The engine sound and fuel consumption therefore indicate that forces are present but balanced, giving zero resultant force.

(b) In dense traffic the vehicle repeatedly accelerates and decelerates, so its velocity changes frequently. Because of inertia, passengers' bodies tend to maintain their previous state of motion while the vehicle's velocity changes. When the vehicle

accelerates forward, passengers tend to lag behind relative to the vehicle; when it brakes, they tend to continue forward. These effects are felt as repeated forward and backward jolts.

- (c) Given: $m = 1400\text{kg}$, $u = 28\text{m/s}$, $v = 0\text{m/s}$, $s = 84\text{m}$.

Using $v^2 = u^2 + 2as$:

$$0 = (28\text{m/s})^2 + 2a(84\text{m})$$

$$a = -4.67\text{m/s}^2$$

- (i) Braking force: $F = ma$

$$F = 1400\text{kg} \left(-\frac{4.67\text{m}}{\text{s}^2} \right) = -6.54 \times 10^3\text{N}$$

The braking force has magnitude $6.54 \times 10^3\text{N}$ and acts opposite the motion.

- (ii) Time to stop: $v = u + at$

$$0 = 28\text{m/s} + (-4.67\text{m/s}^2)t; t = 6.0\text{s}$$

The time to stop is 6.0s.

Question 13

- (a) The difference arises because motion is described relative to a reference frame. The passenger inside the train shares the train's horizontal motion, so relative to the passenger the object has no horizontal velocity and appears to fall straight down under gravity. To the observer beside the track, the object retains the train's horizontal velocity at the moment of release while also accelerating downward due to gravity, so the path is curved. Both descriptions are correct in their own reference frames.
- (b) A headwind increases the resistive force due to air resistance acting opposite to the cyclist's motion. If the cyclist's driving force (from pedalling) remains nearly the same while the opposing resistive force increases, the resultant forward force decreases. With a smaller resultant force, acceleration reduces and sustaining the previous velocity becomes more difficult.
- (c) At $v = 4\text{m/s}$, resistive force: $R = 3v = 3(4) = 12\text{N}$.
Resultant force: $F = 18\text{N} - 12\text{N} = 6\text{N}$ (forward).
Acceleration: $a = \frac{F}{m} = \frac{6\text{N}}{3.0\text{kg}} = 2.0\text{m/s}^2$.
The acceleration is 2.0m/s^2 .

Question 14

- (a) With negligible friction, the horizontal resistive force is very small, so after the push there is approximately zero resultant horizontal force. By Newton's first law, the puck continues moving with constant velocity due to inertia. It is not kept moving by a continuing push; it continues because no significant resultant force acts to change its motion.
- (b) Apparent weight depends on the normal reaction between the lift floor and the passenger. When the lift accelerates upward, the passenger must accelerate upward, requiring a larger upward normal reaction than weight, so the passenger feels heavier. When the lift slows while moving upward, its acceleration is downward; the required normal reaction becomes smaller than weight, so the passenger feels lighter.
- (c) The lift is moving upward but slowing at 1.5m/s^2 , so its acceleration is downward with magnitude 1.5m/s^2 .
So $a = -1.5\text{m/s}^2$.
For the passenger: $N - W = ma$ or $N - mg = ma$
 $N = mg + ma = m(g + a) = 70\text{kg}(9.8\text{m/s}^2 - 1.5\text{m/s}^2) = 581\text{N}$
The normal reaction on the passenger is 581N.

Question 15

- (a) During the push, each skater experiences an interaction force equal in magnitude and opposite in direction. However, acceleration depends on mass through $a = F/m$. The lighter skater has smaller mass, so the same force produces a larger acceleration, causing the lighter skater to gain velocity more rapidly.
- (b) When the runner's foot pushes backward on the ground, the ground exerts an equal and opposite frictional force forward on the foot. That forward force acts on the runner's body and produces forward acceleration. Without this forward force from the ground, forward acceleration would not occur.
- (c) During the push, the interaction forces on the two skaters are equal in magnitude (Newton's third law). But since force is equal to the momentum change divided by time taken and they act for the same time interval (0.40s), the change in momentum on the two skaters are also equal in magnitude.
 $|\Delta p|$ in Kipute = $m(v - u) = 55\text{kg}(2.5\text{m/s} - 0) = 55 \times 2.5\text{kgm/s}$
 $|\Delta p|$ in Kipanga = $m(v - u) = 75\text{kg}(v - 0) = 75\text{kg}(v)$
 $|\Delta p|$ in Kipute = $|\Delta p|$ in Kipanga or $55 \times 2.5\text{kgm/s} = 75\text{kg}(v); v = 1.83\text{m/s}$
(i) The Kipanga's velocity is 1.83m/s.
Kipute changes from 0 to 2.5m/s in 0.40s.
It follows that:

$$a = \frac{\Delta v}{\Delta t} = \frac{2.5\text{m/s}}{0.40\text{s}} = 6.25\text{m/s}^2$$

$$\text{Magnitude of force: } F = ma = 55\text{kg}(6.25\text{m/s}^2) = 344\text{N}$$

- (ii) The magnitude of the force exerted during the push is 344N.

Question 16

- (a) The ball is projected upward with an initial upward velocity. Gravity provides a constant downward acceleration throughout the motion. While the ball moves upward, its upward velocity decreases because the acceleration due to

gravity is opposite to the motion. At the highest point the velocity becomes momentarily zero, but the acceleration remains downward, so the ball then gains downward velocity and returns to its starting level.

- (b) The change in momentum needed to come to rest is fixed, but the average force depends on the stopping time: $F = \Delta p / \Delta t$. Bending the knees increases the time over which the body is brought to rest, reducing the average force during the impact. A smaller force reduces the risk of injury, while a stiff landing produces a shorter stopping time and therefore a larger force.

- (c) Using $s = ut + \frac{1}{2}at^2 = ut + \frac{1}{2} \times (-9.8)t^2 = ut - 4.9t^2$

Ball A (projected upward from ground): $s = 26t - 4.9t^2$

The two balls will be at the same level after ball B travelling a distance of $35 - s$.

So, for ball B (released from rest 35m above ground): $35 - s = \frac{1}{2}at^2 = \frac{1}{2} \times (9.8)t^2 = 4.9t^2$

From which:

$$s = 35 - 4.9t^2$$

It follows that:

$$26t - 4.9t^2 = 35 - 4.9t^2$$

$$26t = 35; t = 1.35s$$

- (i) The time is 1.35s.

Height: substitute into s for ball A:

$$s = 26(1.35) - 4.9(1.35)^2$$

$$s = 35.1 - 8.93 = 26.2m$$

- (ii) The height is 26.2m above the ground.

Question 17

- (a) The ball already has the forward velocity of the moving vehicle at the instant it is released. After release it continues moving forward due to inertia while gravity pulls it downward. Hence, relative to the ground it travels forward as it falls and lands ahead of the release point.
- (b) On a smooth surface the frictional force between the runner's foot and the ground is small, so the ground cannot provide a large forward reaction force on the runner. Since forward acceleration requires a forward resultant force, the runner cannot accelerate effectively. On rough ground, greater friction allows a larger forward reaction force, making acceleration easier.
- (c) Resultant force: $F = 20N - 4N = 16N$
 Acceleration: $a = \frac{F}{m} = \frac{16N}{5.0kg} = 3.2m/s^2$
 Velocity after 5.0s: $v = u + at = 0 + (3.2m/s^2)(5.0s) = 16m/s$
 Displacement in 5.0s: $s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}(3.2m/s^2)(5.0s)^2 = 40m$

Question 18

- (a) Acceleration depends on change of velocity, and velocity changes if either magnitude or direction changes. In circular motion, the direction of velocity changes continuously even when speed is constant, so the object has acceleration directed toward the centre of the circle.
- (b) Passengers tend to continue in their original straight-line motion due to inertia while the car changes direction. Relative to the turning car, this gives the sideways leaning effect. The actual resultant force required for turning is directed toward the centre of the bend, and the passenger's body must be forced sideways to follow the curved path.
- (c) Acceleration: $a = \frac{v - u}{t} = \frac{(0 - 18m/s)}{9.0s} = -2.0m/s^2$
 Braking force: $F = ma = 1200kg(-2.0m/s^2) = -2400N$
 Magnitude of braking force = 2400N (opposite to the motion)
 Stopping distance: $s = ut + \frac{1}{2}at^2 = (18m/s)(9.0s) + \frac{1}{2}(-2.0m/s^2)(9.0s)^2 = 81m$

Question 19

- (a) As velocity increases, resistive forces (especially air resistance and rolling resistance) increase. With roughly constant engine power, the driving force available at higher velocity does not keep increasing, while the opposing forces grow. The resultant force therefore decreases and may reach zero, so acceleration reduces and eventually becomes zero, giving a steady (terminal) velocity on the road.
- (b) Before motion begins, static friction adjusts up to a maximum value and can be large, so the crate strongly resists starting. Once the crate is moving, kinetic friction acts and is typically smaller and more nearly constant. With the same applied push, the resultant forward force becomes larger after slipping begins, so the crate moves more easily.
- (c) Using $v^2 = u^2 + 2as$; where: $u = 12m/s, v = 30m/s, s = 270m$
 $a = (v^2 - u^2)/(2s) = ((30)^2 - (12)^2)/(2 \times 270)$
 $a = (900 - 144)/540 = 756/540 = 1.40m/s^2$
 Resultant force: $F = ma = 1200kg \times 1.40m/s^2 = 1.68 \times 10^3 N$

Question 20

- (a) Velocity is defined relative to a reference frame. Different observers may use different frames (ground, vehicle, train etc), so they can assign different velocities to the same object without contradiction. Each velocity is correct within its own chosen reference frame.
- (b) When the bus changes lanes, it accelerates sideways. Unsecured objects tend to continue in their original straight-line motion due to inertia. Relative to the bus, they appear to shift sideways opposite to the bus's sideways acceleration. The sideways shift is therefore a consequence of inertia during change of direction.
- (c) Initial velocity: $u = +18m/s$, Acceleration: $a = -9.8m/s^2$, Displacement to ground: $s = -40m$

$$\text{Using } s = ut + \frac{1}{2}at^2$$

$$-40 = 18t + \frac{1}{2}(-9.8)t^2$$

$$-40 = 18t - 4.9t^2$$

$$\text{Rearrange: } 4.9t^2 - 18t - 40 = 0; t = 5.23\text{s}$$

The time taken to reach the ground is 5.23s.

Question 21

- (a) Motion does not require a net force; only a change in motion (acceleration) requires a net force. A body can move with constant velocity while the resultant force is zero. Net force is linked to acceleration, not to the existence of velocity.
 (b) Constant velocity means zero acceleration, so resultant horizontal force must be zero. Therefore, the forward push must be equal in magnitude to the frictional (and any other resistive) force acting backward.
 (c) Take initial direction toward the wall as positive: $m = 0.25\text{kg}$, $u = +10\text{m/s}$, $v = -8\text{m/s}$, $t = 0.030\text{s}$

- (i) Change in momentum:

$$\Delta p = m(v - u) = 0.25(-8 - 10) = 0.25(-18) = -4.5\text{kgm/s}$$

The change in momentum of the ball is 4.5kgm/s opposite to the initial motion's direction.

- (ii) Average force:

$$F = \Delta p / \Delta t = (-4.5) / 0.030 = -150\text{N}$$

The average force exerted on the ball by the wall is 150N opposite to the initial motion's direction (away from the wall).

Question 22

- (a) Acceleration depends on change of velocity, and velocity changes if direction changes even when magnitude is constant. Thus, an object moving with constant speed in a curved path can still accelerate because its direction of motion changes continuously.

- (b)

Reason:

A change in direction of motion requires a resultant force toward the centre of the turn.

Explanation:

When the daladala turns, it accelerates sideways toward the centre of the curve. The passenger's body tends to continue moving in a straight line due to inertia. Relative to the turning vehicle, this makes the passenger appear to be pushed sideways outward.

- (c) Using $s = ut + \frac{1}{2}at^2$; where: $u = 24\text{m/s}$, $a = -9.8\text{m/s}^2$, $s = 20\text{m}$

$$20 = 24t - 4.9t^2 \text{ or } 4.9t^2 - 24t + 20 = 0$$

Solving the quadratic equation gives: $t = 1.07\text{s}$ or $t = 3.83\text{s}$

- (i) On the way downward $\rightarrow$ larger time: $t = 3.83\text{s}$

- (ii) Velocity then: $v = u + at = 24 - 9.8(3.83) = -13.5\text{m/s}$ (downward)

Question 23

- (a)

Reason:

Forces cancel only when they act on the same body.

Explanation:

Equal and opposite forces can act on different bodies as part of a Newton's third-law interaction pair. Since these forces act on separate objects, they do not combine to give zero resultant force on any single body. Cancellation occurs only when opposite forces act on the same object.

- (b) Forward motion while walking requires a forward frictional force from the ground. On a slippery surface, friction is too small, so the ground cannot provide a strong forward force on the foot. The foot slips backward and the person cannot generate sufficient forward acceleration.

- (c) Total mass = $3\text{kg} + 5\text{kg} = 8\text{kg}$

$$(i) \text{ Common acceleration: } a = \frac{F}{m_{\text{total}}} = \frac{16\text{N}}{8\text{kg}} = 2.0\text{m/s}^2$$

- (ii) Force exerted by A on B is the one that makes B to accelerate. Thus;

$$F_{AB} = m_B a = 5\text{kg}(2.0\text{m/s}^2) = 10\text{N}$$

Question 24

- (a) In motions such as a vertical throw, the body slows down as it rises because acceleration remains downward. At the highest point the velocity becomes zero only momentarily (instantaneous), but acceleration is still present, so the velocity immediately changes again and the body continues moving in the opposite direction. Hence zero velocity at an instant does not mean the body is at rest for a finite time.

- (b) As the bus goes around the roundabout, its velocity direction changes continuously even if the magnitude stays nearly constant. The passenger's body tends to continue in a straight line due to inertia, so relative to the turning bus the passenger feels pushed sideways. The actual force required is inward (toward the centre) to make the passenger follow the curved path.

- (c) Given: $u = 22\text{m/s}$, $a = -9.8\text{m/s}^2$, height $h = 15\text{m}$.

Solve for times when $s = 15\text{m}$ using: $s = ut + \frac{1}{2}at^2$

$$15 = 22t - 4.9t^2 \text{ or } 4.9t^2 - 22t + 15 = 0$$

Solving the quadratic equation gives two values of t which are:

$$t_1 = 0.84\text{s (on the way upward)}$$

$$t_2 = 3.65\text{s (on the way downward)}$$

$$\text{Time interval above 15m} = t_2 - t_1 = 3.65\text{s} - 0.84\text{s} = 2.81\text{s}$$

So the stone is above 15m for **2.81s**.

Question 25

(a)

Reason:

A constant resultant force does not always guarantee constant acceleration unless the mass of the body remains constant.

Explanation:

Newton's second law in its most general form states that the resultant force equals the rate of change of momentum. If the mass of a body changes with time (for example, systems losing mass), a constant force can produce a non-uniform change in velocity. Therefore, constant force implies constant acceleration only when the mass of the body is constant.

(b)

Reason:

The maximum braking force depends on the friction between the tyres and the road.

Explanation:

On a wet road, the friction between tyres and the road surface is reduced. This limits the maximum braking force that can act on the vehicle. With a smaller braking force, the deceleration is smaller ($a = F/m$), so the vehicle takes more time and travels a longer distance before stopping. Hence the stopping distance increases on wet roads.

(c) Reaction distance: $s = vt$

$$s = (20 \text{ m/s})(0.70 \text{ s}) = 14\text{m}$$

$$\text{Braking acceleration: } a = \frac{F}{m} = -\frac{3600}{1200} = -3\text{m/s}^2$$

(i) Using $v = u + at$

$$0 = 20 - 3t; t = 6.67\text{s}$$

The braking time is 6.67s.

(ii) Using $v^2 = u^2 + 2as$

$$0 = 20^2 + 2(-3)s; s = 66.7\text{m}$$

The braking distance is 66.7m.

(iii) Total distance = Reaction distance + Braking distance = $(14 + 66.7)\text{m}$
The total stopping distance is 80.7m.

Question 26

(a)

Reason:

Acceleration depends on both force and mass.

Explanation:

During a collision, the two bodies exert equal and opposite forces on each other according to Newton's third law. However, acceleration is given by force divided by mass. **If the masses of the two bodies are different**, the same force produces different accelerations. Therefore, equal and opposite forces do not always lead to equal accelerations.

(b)

Reason:

A seatbelt increases the time over which a passenger's momentum changes.

Explanation:

When a vehicle brakes suddenly, the passenger tends to continue moving forward due to inertia. A seatbelt applies a stopping force over a longer time and distance compared with an abrupt stop against the dashboard or seat. Since average force equals change in momentum divided by time, increasing the stopping time reduces the force on the passenger, lowering the risk of injury.

$$(c) \text{ Using } s = h = ut - \frac{1}{2}gt^2 = ut - 4.9t^2$$

If t is the time from the first launch.

$$\text{Height of first ball: } h = 28t - 4.9t^2$$

$$\text{Height of second ball (launched 1.2 s later): } h = 34(t - 1.2) - 4.9(t - 1.2)^2$$

Equating heights and simplifying gives: $t = 2.69\text{s}$

- (i) Time after first launch is 2.69s.
 (ii) Height: $h = 28(2.69) - 4.9(2.69)^2 = 39.9\text{m}$

Question 27

(a)

Reason:

Equal and opposite forces can act on the same body.

Explanation:

A Newton's third-law pair must act on two different bodies. Two forces can be equal and opposite yet both act on the same object (balanced forces). In that case they are not a third-law pair.

(b)

Reason:

Velocity includes direction as well as magnitude.

Explanation:

Two bodies can have the same speedometer reading yet have different velocities if their directions differ. Even if directions are opposite, the magnitudes can match but velocities are not the same

(c) Resistive force: $R = 400 + 50v$

(i) At $v = 12\text{ m/s}$; $R = 400 + 50(12) = 1000\text{ N}$

Resultant force: $F = 2200\text{ N} - 1000\text{ N} = 1200\text{ N}$

Acceleration: $a = \frac{F}{m} = \frac{1200\text{ N}}{1000\text{ kg}} = 1.2\text{ m/s}^2$

The acceleration is 1.2 m/s^2 .

(ii) Acceleration becomes zero when driving force equals resistance:

$$2200 = 400 + 50v; v = 36\text{ m/s}$$

The velocity is 36 m/s .

Question 28

(a) **Choice:** Forces balance

Reason:

On Earth, resistive forces and contact forces are almost always present.

Explanation:

In ordinary motion, forces are rarely absent because bodies interact with the ground, air, and other surfaces. What commonly happens is that several forces act but balance so that the resultant force is zero, giving constant velocity or rest. Therefore "forces balance" matches ordinary motion more often than "forces are absent".

(b)

Reason:

Newton's third law requires interaction forces to be equal and opposite.

Explanation:

In an interaction, the force on A by B equals the force on B by A in magnitude. Therefore, there is no "bigger force" between the two interacting bodies. The body that accelerates more does so because it has smaller mass, not because it experiences a bigger interaction force.

(c) Velocities become equal when; $v_{\text{motorcycle}} = v_{\text{car}}$

For motorcycle: $v = u + at$

$$v = 18\text{ m/s} + (2\text{ m/s}^2)t$$

For car: $v = u + at$

$$v = 30\text{ m/s} + (-3\text{ m/s}^2)t$$

If velocities become equal:

$$18 + 2t = 30 - 3t; t = 2.4\text{ s}$$

(i) Time when velocities become equal is 2.4s.

Motorcycle displacement: $s = ut + \frac{1}{2}at^2$

$$s_m = (18\text{ m/s})(2.4\text{ s}) + \frac{1}{2}(2\text{ m/s}^2)(2.4\text{ s})^2 = 48.96\text{ m}$$

Car displacement: $s = ut + \frac{1}{2}at^2$

$$s_c = (30\text{ m/s})(2.4\text{ s}) + \frac{1}{2}(-3\text{ m/s}^2)(2.4\text{ s})^2 = 63.36\text{ m}$$

$$\text{Separation} = s_c - s_m = 63.36\text{m} - 48.96\text{m} = 14.4\text{m}$$

(ii) The separation between them is 14.4m.

Question 29

(a)

Reason:

Motion does not require a non-zero resultant force.

Explanation:

A body can move with constant velocity when the resultant force is zero. If two forces on the body are equal in magnitude and opposite in direction, the resultant force is zero, so the body does not accelerate but it can still continue moving if it already had velocity. Therefore, movement does not imply unequal forces; unequal forces are required only for changing velocity.

(b)

Reason:

Injury risk depends on the size of the stopping force, which depends on how quickly velocity is reduced.

Explanation:

When the diver enters deep water, the diver is brought to rest over a longer time and distance because the water yields and allows the body to decelerate gradually. Since the change in momentum is spread over a longer time, the average resultant force is smaller. A smaller stopping force reduces the likelihood of serious injury compared with coming to rest in a very short time on a rigid surface.

(c) Use vertical motion equation: $s = ut - \frac{1}{2}gt^2$ (From $s = ut + \frac{1}{2}at^2$; where $a = -g$)

Substituting:

$$18 = 28t - \frac{1}{2}(9.8)t^2 \text{ or } 18 = 28t - 4.9t^2$$

Rearranging:

$$4.9t^2 - 28t + 18 = 0$$

Solving the quadratic equation gives two values of t which are:

$$t_1 = 0.74\text{s (on the way upward)}$$

$$t_2 = 4.98\text{s (on the way downward)}$$

$$\text{Time interval above 15m} = t_2 - t_1 = 4.98\text{s} - 0.74\text{s} = 4.24\text{s}$$

So the stone is above 18m for 4.24s.

$$\text{Total time of flight: } T = \frac{2u}{g} = \frac{2 \times 28\text{m/s}}{9.8\text{m/s}^2} = 5.71\text{s}$$

Time interval below given height = Total time of flight – Time interval above the height

$$\text{Time interval below 18m} = (5.71 - 4.24)\text{s} = 1.47\text{s}$$

The stone is below the height of 18m for **1.47s** in total.

PART TWO

Question 1

(a) Each rope exerts a force in its own direction. Since the two forces act at 90° to each other, their vector sum (the resultant) lies between north and east at an angle determined by the relative magnitudes of the two forces. The crate accelerates in the direction of this resultant, not in the direction of either individual force.

(b) The description is incomplete. While kinetic energy is indeed partially transferred to the goalkeeper, kinetic energy is not conserved in this collision (it is inelastic, since the ball and goalkeeper move together after the catch). The quantity that is conserved is **momentum**, not kinetic energy. Some kinetic energy is converted to thermal energy, sound, and deformation of the ball. Conservation of momentum is the correct principle for analysing such interactions.

$$(c) \text{ Weight: } W = 200\text{kg} \times 9.8\text{m/s}^2 = 1960\text{N}$$

Let T_1 be the tension at 40° and T_2 at 60° .

$$\text{Vertically: } T_1 \sin 40^\circ + T_2 \sin 60^\circ = 1960\text{N}$$

$$\text{Horizontally: } T_1 \cos 40^\circ = T_2 \cos 60^\circ$$

$$\text{From horizontal: } T_1 = T_2 \times \frac{\cos 60^\circ}{\cos 40^\circ} = 0.6527T_2$$

$$\text{Substituting: } 0.6527T_2 \times 0.6428 + T_2 \times 0.8660 = 1960\text{N}$$

$$0.4196T_2 + 0.8660T_2 = 1960\text{N}$$

$$1.2856T_2 = 1960\text{N}$$

$$T_2 = 1524.6\text{N}$$

$$T_1 = 0.6527 \times 1524.6\text{N} = 995.2\text{N}$$

Question 2

- (a) At the top of the loop, both the rider's weight and the normal reaction from the track point downward (toward the centre of the loop). Together they provide the centripetal force needed for circular motion. As long as the rider's speed is high enough that the required centripetal force ($\frac{mv^2}{r}$) exceeds or equals the weight (mg), the track pushes on the rider and the rider stays in contact.
- (b) By Newton's third law, the force on the car equals the force on the lorry during the collision, and both forces act for the same duration. Therefore both vehicles experience the same impulse ($F\Delta t$), and hence the same magnitude of change in momentum. However, because the car has a much smaller mass than the lorry, the same change in momentum produces a much larger change in velocity for the car ($\Delta v = \frac{\Delta p}{m}$). This sudden, large change in velocity is what causes severe deceleration of the occupants inside the car, leading to greater injury. The lorry, having a much larger mass, undergoes a much smaller change in velocity for the same change in momentum, so its occupants are less affected.
- (c) Taking the initial direction as positive:

$$u = +40\text{m/s}, v = -55\text{m/s}$$

$$\Delta p = m(v - u) = 0.06\text{kg} \times (-55 - 40)\text{m/s} = 0.06\text{kg} \times (-95\text{m/s}) = -5.7\text{kgm/s}$$

$$F = \frac{\Delta p}{\Delta t} = \frac{-5.7\text{kgm/s}}{0.008\text{s}} = -712.5\text{N}$$

The average force is 712.5N in the direction of the return.

Question 3

- (a) At the moment of release, the javelin has an upward component of velocity. This upward velocity causes it to rise. However, gravitational acceleration acts downward at all times, progressively reducing the upward velocity. Eventually the upward velocity reaches zero (at the peak), after which gravity accelerates the javelin downward and it descends.
- (b) Gravitational acceleration g varies with altitude: g is slightly smaller at higher altitudes because the object is farther from the Earth's centre. At Iringa (1600m), g is slightly less than at Bagamoyo (sea level). Since the scale measures gravitational force and converts it to mass using its stored value of g , the same mass produces a slightly greater force at sea level, giving a slightly higher reading in Bagamoyo.
- (c) For equilibrium, $F_1 + F_2 + F_3 = 0$, so $F_3 = -(F_1 + F_2)$.

Resultant of F_1 and F_2 :

$$R_x = 10 + 15\cos 120^\circ = 10\text{N} + 15 \times (-0.5)\text{N} = 10\text{N} - 7.5\text{N} = 2.5\text{N}$$

$$R_y = 0 + 15\sin 120^\circ = 12.99\text{N}$$

$$R = \sqrt{(2.5\text{N})^2 + (12.99\text{N})^2} = 13.2\text{N}$$

$$F_3 = 13.2\text{N}, \text{ directed opposite to } R.$$

$$\text{Direction of } R: \alpha = \tan^{-1}\left(\frac{12.99}{2.5}\right) = 79.1^\circ \text{ above the positive x-axis.}$$

Therefore F_3 acts at $79.1^\circ + 180^\circ = 259.1^\circ$ from the positive x-axis (or 79.1° below the negative x-axis).

Question 4

- (a) The satellite is very much affected by gravity. In fact, gravity is the only force acting on it, and it provides the centripetal force that keeps the satellite in orbit. The satellite is in continuous free fall toward the Earth, but because it has a large horizontal velocity, it falls "around" the Earth rather than into it. The curved surface of the Earth drops away as fast as the satellite falls.
- (b) The kinetic energy of the vehicle before impact is converted to thermal energy through friction during skidding: $\frac{1}{2}mv^2 = \mu mgs$, where s is the skid distance. Since mass cancels, $v = \sqrt{2\mu gs}$. By measuring the skid length and knowing the friction coefficient of the road, the initial speed can be calculated. Longer skid marks indicate higher speed before braking.
- (c) Using conservation of mechanical energy:

$$\frac{1}{2}mu^2 + mgh = \frac{1}{2}mv^2$$

$$v^2 = u^2 + 2gh = (20\text{m/s})^2 + 2 \times 9.8\text{m/s}^2 \times 15\text{m} = 694\text{m}^2/\text{s}^2$$

$$v = \sqrt{694\text{m}^2/\text{s}^2} = 26.3\text{m/s}$$

Question 5

- (a) The coin requires centripetal force to move in a circle, which is provided by static friction between the coin and the turntable surface. The required centripetal force is $m\omega^2 r$, which increases with angular velocity ω . The maximum available friction is $\mu_s mg$, which is fixed. When the turntable spins fast enough that the centripetal force demanded exceeds the maximum friction available, friction can no longer hold the coin and it slides off.
- (b) The range is $R = \frac{u^2 \sin 2\theta}{g}$, which depends on the angle θ through the factor $\sin 2\theta$. A steep angle gives great height but short horizontal travel; a shallow angle gives fast horizontal motion but little time of flight. The maximum range occurs when $\sin 2\theta = 1$, which gives $2\theta = 90^\circ$, hence $\theta = 45^\circ$. In this angle both vertical motion and horizontal motion are well optimized.
- (c) Taking the direction of the 0.50kg object as positive:

$$0.50\text{kg} \times 4.0\text{m/s} + 0.30\text{kg} \times (-3.0\text{m/s}) = 0.50\text{kg} \times 1.0\text{m/s} + 0.30\text{kg} \times v$$

$$2.0\text{kgm/s} - 0.9\text{kgm/s} = 0.5\text{kgm/s} + 0.30\text{kg} \times v$$

$$1.1\text{kgm/s} = 0.5\text{kgm/s} + 0.30v$$

$$v = \frac{0.6\text{kgm/s}}{0.30\text{kg}} = 2.0\text{m/s} \text{ (in the original direction of the 0.50kg object)}$$

$$\text{KE before} = \frac{1}{2} \times 0.50\text{kg} \times (4.0\text{m/s})^2 + \frac{1}{2} \times 0.30\text{kg} \times (3.0\text{m/s})^2 = 4.0\text{J} + 1.35\text{J} = 5.35\text{J}$$

$$\text{KE after} = \frac{1}{2} \times 0.50\text{kg} \times (1.0\text{m/s})^2 + \frac{1}{2} \times 0.30\text{kg} \times (2.0\text{m/s})^2 = 0.25\text{J} + 0.60\text{J} = 0.85\text{J}$$

Since $5.35\text{J} \neq 0.85\text{J}$, kinetic energy is not conserved. The collision is inelastic.

Question 6

- (a) While the string is intact, the tension provides a centripetal force directed toward the centre, continuously pulling the stone inward to maintain circular motion. The stone's velocity at any instant is tangent to the circle. When the string breaks, the centripetal force vanishes. By Newton's first law, the stone continues moving in the direction of its velocity at the instant of breaking, which is tangential. There is no outward force; the apparent "outward tendency" during circular motion is simply the stone's inertia resisting the inward change of direction.
- (b) The gravitational constant G is extremely small ($6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$). For two books, the product of masses is too small to produce significant amount of the gravitational force. Gravity becomes significant only when at least one mass is enormous (like a planet), because the large mass compensates for the tiny value of G .
- (c) $u_y = 100\text{m/s} \times \sin 53^\circ = 79.86\text{m/s}$

$$\text{At height 100m: } 100 = 79.86t - \frac{1}{2} \times 9.8 \times t^2$$

$$4.9t^2 - 79.86t + 100 = 0$$

Solve the quadratic equation gives:

$$t = 1.367\text{s} \text{ or } t = 14.93$$

The smaller time corresponds to when the projectile reaches 100 m on its way up (first time), while the larger time corresponds to when it passes 100 m again on its way down (second time).

First time: $t = 1.367\text{s}$

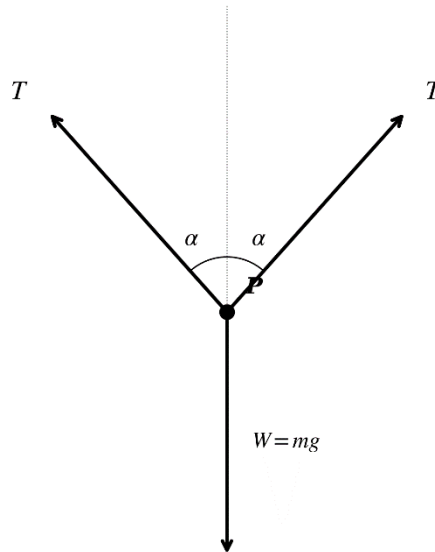
$$\text{Horizontal distance: } x = u_x t = 100\text{m/s} \times \cos 53^\circ \times 1.367\text{s} = 82.3\text{m}$$

Alternative solution: You may use trajectory equation with $s_y = 100\text{m}$, where the smaller value of s_x corresponds to the required horizontal distance.

Question 7

- (a) At the top of the bridge, the car follows a curved (circular) path. The centripetal acceleration is directed downward. Part of the driver's weight provides this centripetal acceleration, so the normal reaction from the seat (which the driver feels as "weight") is reduced: $R = mg - \frac{mv^2}{r}$. Since the sensation of weight comes from the contact force, not from gravity itself, the reduced contact force makes the driver feel lighter.
- (b) The puck and the football both obey Newton's first law: they continue moving until a net force stops them. On ice, friction is extremely small, so the decelerating force on the puck is very small and it takes a long time (and distance) to stop. On grass, friction is much larger, producing a larger decelerating force. By $v^2 = u^2 - 2as$, a larger deceleration a results in a shorter stopping distance s for the same initial speed.

(c) Consider the following Free Body Diagram:



From the geometry of the provided diagram in the question: the two attachment points are 2.0m apart, each rope is 1.5m. The junction hangs below the midpoint. Half-width = 1.0m. Each rope: $\cos\alpha = \frac{1.0}{1.5} = 0.6667$, so $\alpha = 48.2^\circ$ from vertical.

By symmetry, both tensions are equal: $T_1 = T_2 = T$.

Vertically: $2T\cos\alpha = mg$

$$T = \frac{mg}{\cos\alpha} = \frac{15\text{kg} \times 9.8\text{m/s}^2}{0.6667} = \mathbf{110.3\text{N}}$$

Question 8

- (a) The gravitational force on the planet is directed radially toward the Sun. Because this force has no component perpendicular to the radius, it produces zero torque about the Sun. With zero torque, angular momentum $L = mvr$ is conserved. At perihelion, the radius r is smallest, so the speed v must be largest to keep mvr constant. At aphelion, r is largest, so v is smallest.
- (b) A satellite in orbit is in free fall. Gravity provides the centripetal force needed for circular motion, and no work is done (the force is perpendicular to the velocity). The satellite maintains its speed and orbit indefinitely without fuel. An aeroplane, by contrast, is not in free fall. It relies on lift from its wings to support its weight, and this lift is produced only when the aircraft moves through air. Air resistance continuously decelerates the aircraft, so the engines must constantly provide thrust to maintain speed and therefore lift.
- (c) For elastic head-on collision:

$$v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2} = \frac{(3.0 - 1.0)\text{kg} \times 8.0\text{m/s}}{(3.0 + 1.0)\text{kg}} = \frac{2.0\text{kg} \times 8.0\text{m/s}}{4.0\text{kg}} = 4.0\text{m/s (to the right)}$$

$$v_2 = \frac{2m_1u_1}{m_1 + m_2} = \frac{2 \times 3.0\text{kg} \times 8.0\text{m/s}}{4.0\text{kg}} = \frac{48.0\text{kgm/s}}{4.0\text{kg}} = 12.0\text{m/s (to the right)}$$

Question 9

- (a) Escape velocity is derived from energy conservation: $\frac{1}{2}mv_E^2 = \frac{GMm}{r}$. The escaping mass m appears on both sides and cancels, giving $v_E = \sqrt{\frac{2GM}{r}}$. This depends only on M (mass of the planet) and r (radius of the planet). A heavier escaping object needs more energy, but it also has proportionally more kinetic energy at the same speed. These two effects balance exactly, leaving the required speed independent of mass.
- (b) At the outer edge, the radius r of the circular path is larger. The centripetal force needed to maintain circular motion is $F = m\omega^2r$, which increases with r (since ω is the same everywhere on the roundabout). Therefore, the child must hold on with a greater force at the outer edge. If the child's grip cannot provide this increased force, the child will slide off.
- (c) $u_y = 35\text{m/s} \times \sin 50^\circ = 35 \times 0.766 = 26.81\text{m/s}$

$$\text{At height 20m: } 20 = 26.81t - \frac{1}{2} \times 9.8t^2$$

$$4.9t^2 - 26.81t + 20 = 0$$

$$t_1 = 0.891\text{s}, t_2 = 4.58\text{s}$$

Time above 20m: $\Delta t = 4.58\text{s} - 0.891\text{s} = 3.69\text{s}$

Question 10

- (a) For the ring to remain stationary (in equilibrium), the vector sum of all three forces must be zero. This means the third force must be equal in magnitude and opposite in direction to the resultant of the first two forces. This third force is called the **equilibrant**. It can be found by resolving the first two forces into components, finding the resultant, and then reversing its direction.
- (b) The water has mass and is therefore subject to gravitational acceleration. After leaving the nozzle, the water follows a parabolic (projectile) path because gravity pulls it downward while it continues to move forward. The laser beam consists of massless photons that travel at the speed of light. Over the short distances involved, the gravitational deflection of light is completely negligible, so the beam appears perfectly straight.
- (c) The maximum centripetal force equals the maximum static friction:

$$\frac{mv_{\max}^2}{r} = \mu_s mg$$

Mass cancels:

$$v_{\max} = \sqrt{\mu_s gr} = \sqrt{0.45 \times 9.8\text{m/s}^2 \times 50\text{m}} = 14.8\text{m/s}$$

Question 11

- (a) Orbital speed is $v = \sqrt{\frac{GM}{R}}$. A larger R gives a smaller v because gravity is weaker at greater distances and less speed is needed to maintain the orbit. However, total energy is $E = -\frac{GMm}{2R}$, and a larger R makes E less negative (higher total energy). The apparent contradiction dissolves when one realises that the higher satellite has less kinetic energy but much more (less negative) potential energy, and the total energy is indeed greater.
- (b) The follow-through increases the time of contact between bat and ball. By the impulse-momentum theorem ($F\Delta t = \Delta p$), a longer contact time means the bat transfers a greater impulse to the ball for the same average force. A greater impulse produces a greater change in the ball's momentum and hence a higher final speed.
- (c) Weight: $W = 5.0\text{kg} \times 9.8\text{m/s}^2 = 49.0\text{N}$

Let T_1 be at 30° from ceiling (i.e., 60° from horizontal) and T_2 at 45° from ceiling (i.e., 45° from horizontal).

Horizontally: $T_1 \sin 30^\circ = T_2 \sin 45^\circ$

$$T_1 \times 0.5 = T_2 \times 0.7071$$

$$T_1 = 1.414T_2$$

Vertically: $T_1 \cos 30^\circ + T_2 \cos 45^\circ = 49.0\text{N}$

$$1.414T_2 \times 0.866 + T_2 \times 0.7071 = 49.0\text{N}$$

$$1.224T_2 + 0.7071T_2 = 49.0\text{N}$$

$$T_2 = 25.4\text{N}$$

$$T_1 = 1.414 \times 25.4\text{N} = 35.9\text{N}$$

Question 12

- (a) Both bullets experience the same gravitational acceleration ($g = 9.8\text{m/s}^2$) in the vertical direction. The fired bullet has a horizontal velocity, but this does not affect the vertical motion because horizontal and vertical motions are independent. Both bullets start with zero vertical velocity, fall the same height, and accelerate downward at the same rate. Therefore, they reach the ground at the same time. The fired bullet simply lands farther from the firing point.
- (b) At moderate speed, the centripetal force required ($\frac{mv^2}{r}$) is less than the maximum static friction available ($\mu_s mg$). Friction provides the necessary centripetal force and the car follows the curve safely. At excessive speed, the centripetal force required exceeds the maximum friction available. Friction can no longer supply the needed centripetal force, so the car cannot maintain the curved path and skids outward (in accordance with Newton's first law).

$$(c) \quad v = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2} \times 6.0 \times 10^{24} \text{kg}}{3.84 \times 10^8 \text{m}}} = 1021\text{m/s} = 1.02\text{km/s}$$

Question 13

- (a) The conclusion is incorrect. Zero acceleration means zero net force, not zero individual forces. The truck experiences several forces simultaneously: the engine's driving force forward, air resistance backward, friction at the tyres, weight downward, and normal reaction upward. These forces add up to zero (they are in equilibrium), producing zero net force and hence zero acceleration. This is consistent with Newton's first law.

- (b) Once the jumper is airborne, the only external force acting is gravity (air resistance is negligible). By Newton's second law, the centre of mass accelerates at g downward regardless of how the jumper moves their limbs. Internal forces (muscle movements) can change the position of body parts relative to the centre of mass but cannot change the trajectory of the centre of mass itself. Therefore, the centre of mass follows a parabolic path dictated solely by the initial velocity and gravity.
- (c) After impact, the bullet-bob system has velocity v . This velocity sends the system to height 0.10m :

$$\frac{1}{2}(m_b + M)v^2 = (m_b + M)gh$$

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8\text{m/s}^2 \times 0.10\text{m}} = 1.4\text{m/s}$$

By conservation of momentum during the collision:

$$m_b u = (m_b + M)v$$

$$0.025\text{kg} \times u = (0.025 + 4.975)\text{kg} \times 1.4\text{m/s} = 7.0\text{kgm/s}$$

$$u = \frac{7.0\text{kgm/s}}{0.025\text{kg}} = 280\text{m/s}$$

Question 14

- (a) The ball's total mechanical energy is $E = \frac{1}{2}mv^2 - \frac{GMm}{r}$. If the initial speed is less than the escape velocity, E is negative. To reach infinity (where $PE = 0$), the ball would need $E \geq 0$. Since the total energy is negative and conserved, the ball can never reach infinity. It rises until all its kinetic energy is converted to potential energy, then falls back under gravity.
- (b) Both teams exert large forces, but the forces are equal in magnitude and opposite in direction. The net horizontal force on the rope is zero. By Newton's second law, zero net force means zero acceleration, so neither team moves. This is a state of equilibrium. The large individual forces exist (the rope is under high tension), but they cancel each other.
- (c) At the top, the minimum speed occurs when the tension is zero and gravity alone provides the centripetal force:

$$mg = \frac{mv_{\min}^2}{r}$$

$$v_{\min} = \sqrt{gr} = \sqrt{9.8\text{m/s}^2 \times 1.2\text{m}} = 3.43\text{m/s}$$

Question 15

- (a) The orbital speed is derived from $\frac{mv^2}{R} = \frac{GMm}{R^2}$. The satellite mass m cancels from both sides, giving $v = \sqrt{\frac{GM}{R}}$, which depends only on M (Earth's mass) and R (orbital radius). A heavier satellite needs more centripetal force, but gravity provides exactly that extra force because gravitational force is proportional to m . The two effects cancel, leaving the speed independent of satellite mass.
- (b) When a person lands, their momentum must be reduced to zero. The force experienced is $F = \frac{mv}{\Delta t}$, where v is the landing speed and Δt is the stopping time. From a greater height, the landing speed is higher (more KE), so the force is greater. If this force exceeds what bones and joints can withstand, injury results. A soft surface increases the stopping time Δt , reducing the force for the same momentum change.
- (c) $g = \frac{GM}{R^2}$, so $R = \sqrt{\frac{GM}{g}}$

$$R = \sqrt{\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 4.0 \times 10^{24}\text{kg}}{3.2\text{N/kg}}} = 9.131 \times 10^6\text{m}$$

Altitude: $h = R - r = 9.131 \times 10^6\text{m} - 5.0 \times 10^6\text{m} = 4.13 \times 10^6\text{m} = 4130\text{km}$

Question 16

- (a) The gravitational force on a body is $F = mg$. By Newton's second law, the acceleration is $a = \frac{F}{m} = \frac{mg}{m} = g$. The mass cancels because gravity provides a force proportional to mass and inertia resists acceleration proportionally to mass. These two effects balance exactly, giving all objects the same acceleration regardless of mass. Since both balls start from rest and have the same acceleration, they reach the ground at the same time.
- (b) Both children have the same angular velocity ω , but the child at the edge has a larger radius r . The centripetal acceleration is $a = \omega^2 r$, which increases with r . A larger centripetal acceleration requires a larger centripetal force. What the child feels as an "outward pull" is actually the body's tendency to continue in a straight line (inertia). The child at the edge must exert a greater inward force to maintain the circular path, and hence feels a stronger pull.
- (c) At the highest point, the shell moves horizontally at 200m/s with total momentum $= m \times 200\text{m/s}$.

After explosion, one fragment (mass $\frac{m}{2}$) falls vertically, so its horizontal momentum $= 0$.

By conservation of horizontal momentum:

$$m \times 200 \text{ m/s} = \frac{m}{2} \times 0 + \frac{m}{2} \times v$$

$$v = \frac{m \times 200 \text{ m/s}}{\frac{m}{2}} = 400 \text{ m/s (horizontally, in the same direction as the original motion)}$$

Question 17

- (a) On a banked road, the road surface is tilted inward. The normal reaction from the surface therefore has a horizontal component directed toward the centre of the curve. This horizontal component provides the centripetal force needed for circular motion, without relying on friction. At the design speed, the horizontal component of the normal reaction alone equals the required centripetal force: $R \sin \theta = \frac{mv^2}{r}$.
- (b) Negative gravitational potential energy means the asteroid-Earth system is gravitationally bound. Energy must be supplied to the asteroid to separate it to infinity (where PE = 0). The magnitude of the negative PE indicates how much energy would be needed to completely escape the Earth's gravitational influence. As the asteroid approaches Earth, its PE becomes more negative (it falls deeper into the gravitational well), and this lost PE is converted to kinetic energy, causing the asteroid to speed up.
- (c) $u_y = 25 \text{ m/s} \times \sin 37^\circ = 25 \times 0.6018 = 15.05 \text{ m/s (upward)}$

Vertical displacement: -45 m (downward from launch point)

$$\begin{aligned} -45 &= 15.05t - \frac{1}{2} \times 9.8 \times t^2 \\ 4.9t^2 - 15.05t - 45 &= 0 \\ t &= 4.93 \text{ s} \end{aligned}$$

Question 18

- (a) Without a seatbelt, the person would hit the dashboard or windscreen, stopping in a very short time Δt . Since the impulse $F\Delta t = \Delta p$ is fixed (same change in momentum), a very short stopping time means a very large force, causing serious injury. The seatbelt stretches slightly and decelerates the person over a longer time, reducing the peak force. By increasing Δt , the seatbelt decreases F for the same Δp , greatly reducing the risk of injury.
- (b) The follow-through increases the time of contact between the racket and the ball. Since impulse equals force multiplied by time ($F\Delta t = \Delta p$), a longer contact time means a greater impulse is delivered to the ball. A greater impulse produces a greater change in momentum and hence a higher speed.
- (c) On the conical surface, the normal reaction R acts perpendicular to the surface.

Vertically: $R \cos 50^\circ = mg$

$$R = \frac{mg}{\cos 50^\circ} = \frac{2.0 \text{ kg} \times 9.8 \text{ m/s}^2}{\cos 50^\circ} = \frac{19.6 \text{ N}}{0.6428} = 30.49 \text{ N}$$

Horizontally: $R \sin 50^\circ = \frac{mv^2}{r}$

$$\begin{aligned} v^2 &= \frac{R \sin 50^\circ \times r}{m} = \frac{30.49 \text{ N} \times \sin 50^\circ \times 0.30 \text{ m}}{2.0 \text{ kg}} = 3.503 \text{ m}^2/\text{s}^2 \\ v &= \sqrt{3.503 \text{ m}^2/\text{s}^2} = 1.87 \text{ m/s} \end{aligned}$$

Alternatively, you can solve directly by dividing $R \sin 50^\circ$ by $R \cos 50^\circ$ to get $v = \sqrt{rg \tan 50^\circ} = \sqrt{0.3 \times 9.8 \tan 50^\circ} = 1.87 \text{ m/s}$

Question 19

- (a) When a ball just barely completes the vertical circle, the tension at the top is zero and the speed at the top is $v = \sqrt{gr}$. At this instant, the centripetal acceleration equals g , meaning gravity alone provides exactly the centripetal force needed. The ball is effectively in free fall along a circular arc as its acceleration is purely gravitational. There is no contact force, so the ball momentarily behaves as a weightless projectile, giving the appearance of "floating."
- (b) **Advantage:** A high coefficient of friction increases the maximum speed before the car skids ($v_{\text{skid}} = \sqrt{\mu g r}$). The car can negotiate sharper curves at higher speeds without losing grip.

Disadvantage: If the coefficient of friction exceeds $\frac{a}{2h}$, the car reaches the overturning speed before it reaches the skidding speed. The tyres grip so well that instead of sliding safely, the car tips over. In this case, high friction removes the "safety valve" of skidding and exposes the vehicle to the more severe risk of overturning.

- (c) The answer of each part is as follows:

$$(i) \quad v_{\text{skid}} = \sqrt{\mu_s g r} = \sqrt{0.95 \times 9.8 \text{ m/s}^2 \times 50 \text{ m}} = 21.6 \text{ m/s}$$

$$(ii) \quad v_{\text{overturn}} = \sqrt{\frac{arg}{2h}} = \sqrt{\frac{1.8\text{m} \times 50\text{m} \times 9.8\text{m/s}^2}{2 \times 1.0\text{m}}} = 21\text{m/s}$$

(iii) Since $v_{\text{overturn}} (21.0\text{m/s}) < v_{\text{skid}} (21.6\text{m/s})$, the truck **overturns first**.

Explanation:

The high-grip tyres prevent skidding so effectively that the truck reaches its overturning limit first. With ordinary tyres, the truck would skid which is unpleasant but recoverable. With high-grip tyres, it overturns, that is catastrophic.

Question 20

- (a) The satellite orbits in the equatorial plane with a period of exactly 24 hours, the same as the Earth's rotation period. Because the satellite completes one orbit in the same time the Earth completes one rotation, and both move in the same direction (they both move with the same angular velocity), the satellite remains directly above the point on the equator directly below it. From the observer's perspective, the satellite neither falls behind nor moves ahead, so it appears stationary.
- (b) The hose accelerates a large mass of water from rest to a high forward velocity in a short time. As a result, the water gains a large forward momentum. By conservation of momentum, the total momentum of the system (water + hose + firefighter) must remain constant. Since the water gains forward momentum, the hose and firefighter must gain an equal magnitude of backward momentum to keep the total unchanged. Therefore, the firefighter is pushed backward. The faster the water flows and the greater the mass of water ejected per second, the greater the rate of change of momentum, and hence the stronger the backward push on the firefighter.
- (c) At the lowest point, both the tension and the weight act along the radius, with tension directed upward (toward the centre) and weight directed downward (away from the centre):

$$T - mg = \frac{mv^2}{r}$$

$$T = mg + \frac{mv^2}{r} = 0.4\text{kg} \times 9.8\text{m/s}^2 + \frac{0.4\text{kg} \times (6.0\text{m/s})^2}{0.8\text{m}} = 21.9\text{N}$$

Question 21

- (a) Surface gravity is given by $g = \frac{GM}{r^2}$. A larger mass M increases g because there is more matter to exert gravitational pull. A larger radius r decreases g because the surface is farther from the centre. Therefore, g depends on both quantities, and the net effect is determined by which factor dominates.
- (b) By Newton's third law, the force on the bullet equals the force on the gun, and both act for the same duration. Therefore the impulse ($F\Delta t$) is the same on both. Since impulse equals change in momentum, both gain equal magnitudes of momentum. However, since momentum = mass $\times$ velocity, and the gun has a much larger mass than the bullet, the gun's velocity must be much smaller for its momentum to equal the bullet's momentum.
- (c) Horizontal velocity remains constant: $v_x = 12\text{m/s}$

Vertical velocity at impact: $v_y = gt = 9.8\text{m/s}^2 \times 2.4\text{s} = 23.52\text{m/s}$

Speed at impact: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{(12\text{m/s})^2 + (23.52\text{m/s})^2} = 26.4\text{m/s}$

Question 22

- (a) The geostationary orbit requires a period of exactly 24 hours. From Kepler's third law, $T^2 = \frac{4\pi^2}{GM_E} R^3$, the orbital radius R is uniquely determined by the period T . Since T is fixed at 24 hours and GM_E is a constant, there is only one possible value of R , and therefore only one altitude. No other height gives a 24-hour period.
- (b) The gravitational force on each ball is $F = mg$. By Newton's second law, the acceleration is $a = \frac{F}{m} = \frac{mg}{m} = g$. The mass cancels, giving both balls the same acceleration regardless of their mass. Since they start from rest at the same height and have the same acceleration, they reach the ground at the same time.
- (c) First, find the velocity of the block+bullet after impact using conservation of momentum:

$$0.05\text{kg} \times 300\text{m/s} = (0.05 + 2.95)\text{kg} \times v$$

$$v = 5.0\text{m/s}$$

On the incline, the **deceleration** comes from the component of gravity along the slope plus friction:

$$a = -(g\sin 30^\circ + \mu g\cos 30^\circ) = -(9.8\text{m/s}^2 \times 0.5 + 0.25 \times 9.8\text{m/s}^2 \times 0.866) = -7.022\text{m/s}^2$$

Distance up the incline: $v^2 = u^2 + 2as$.

$$0 = (5.0\text{m/s})^2 - 2 \times 7.022\text{m/s}^2 \times s$$

$$s = \frac{25\text{m}^2/\text{s}^2}{14.044\text{m/s}^2} = 1.78\text{m}$$

Question 23

- (a) The kinetic energy is positive because it depends on the square of velocity, which is always positive ($\frac{1}{2}mv^2 > 0$). The gravitational potential energy is negative because the satellite is bound to the Earth. Energy must be supplied to move it from its orbit to infinity, where PE is defined as zero. Since it is currently below infinity in the potential well, its PE is less than zero and therefore negative.
- (b) When the sprinter pushes backward on the blocks, by Newton's third law, the blocks push forward on the sprinter with an equal and opposite force. This forward reaction force from the blocks accelerates the sprinter forward. The sprinter does not accelerate because of the force they exert, but because of the reaction force exerted on them.
- (c) Let T_1 be the tension in the rope at 35° and T_2 in the rope at 55° .

Weight: $W = 40\text{kg} \times 9.8\text{m/s}^2 = 392\text{N}$

Vertically: $T_1 \sin 35^\circ + T_2 \sin 55^\circ = 392\text{N}$

Horizontally: $T_1 \cos 35^\circ = T_2 \cos 55^\circ$

From the horizontal equation: $T_1 = T_2 \times \frac{\cos 55^\circ}{\cos 35^\circ} = T_2 \times \frac{0.5736}{0.8192} = 0.7002T_2$

Substituting into the vertical equation:

$$0.7002T_2 \times 0.5736 + T_2 \times 0.8192 = 392\text{N}$$

$$0.4016T_2 + 0.8192T_2 = 392\text{N}$$

$$T_2 = 321.1\text{N}$$

$$T_1 = 0.7002 \times 321.1\text{N} = 224.8\text{N}$$

Question 24

- (a) Speed is constant, but the direction of motion changes continuously. Since velocity is a vector (having both magnitude and direction), a change in direction means the velocity is changing. By Newton's second law, a changing velocity requires a net force. This force is directed toward the centre of the circle and is called the centripetal force.
- (b) Momentum equals mass times velocity ($p = mv$). At the same speed, the loaded lorry has a greater mass and therefore greater momentum. To stop the lorry, its momentum must be reduced to zero. Since the change in momentum is greater for the loaded lorry, a greater impulse ($F\Delta t$) is needed, making it harder to stop.
- (c) $u_x = 80\text{m/s} \times \cos 50^\circ = 51.42\text{m/s}$

$$u_y = 80\text{m/s} \times \sin 50^\circ = 61.28\text{m/s}$$

Vertical displacement when landing: $s_y = -60\text{m}$ (below launch point)

$$-60 = 61.28t - \frac{1}{2} \times 9.8 \times t^2$$

$$4.9t^2 - 61.28t - 60 = 0$$

$$t = 13.42\text{s}$$

Horizontal distance: $x = u_x t = 51.42\text{m/s} \times 13.42\text{s} = 690\text{m}$

Question 25

- (a) The gravitational force on a planet is directed radially toward the Sun. Because this force has no component perpendicular to the radius vector, it exerts zero torque about the Sun. With zero torque, angular momentum is conserved ($L = mvr = \text{constant}$). The rate of area swept is $\frac{dA}{dt} = \frac{1}{2}rv = \frac{L}{2m}$, which is constant. Therefore, equal areas are swept in equal times.
- (b) Before the bend, both the car and the passengers have momentum in the straight-ahead direction. When the car turns, friction between the tyres and road provides the centripetal force to change the car's direction. However, the passengers' bodies, not being rigidly attached, tend to continue in the original straight-line direction (Newton's first law). Relative to the turning car, the passengers appear to lean outward.
- (c) Momentum is conserved in each direction:

East: $p_x = 3.0\text{kg} \times 5.0\text{m/s} = 15.0\text{kgm/s}$

North: $p_y = 2.0\text{kg} \times 7.5\text{m/s} = 15.0\text{kgm/s}$

Total mass: $3.0 + 2.0 = 5.0\text{kg}$

$$v_x = \frac{15.0 \text{ kgm/s}}{5.0 \text{ kg}} = 3.0 \text{ m/s}$$

$$v_y = \frac{15.0 \text{ kgm/s}}{5.0 \text{ kg}} = 3.0 \text{ m/s}$$

Speed: $v = \sqrt{(3.0 \text{ m/s})^2 + (3.0 \text{ m/s})^2} = 4.24 \text{ m/s}$

Direction: $\theta = \tan^{-1}\left(\frac{3.0}{3.0}\right) = 45^\circ$ north of east.

Question 26

- (a) At the centre, mass is distributed symmetrically in all directions around the point. The gravitational pulls from all directions cancel, giving zero net force and hence zero field strength. However, gravitational potential is a scalar that measures the energy needed per unit mass to escape to infinity. At the centre, an object is at the deepest point of the gravitational potential well, so the potential has its most negative value $\left(-\frac{3GM}{2r_E}\right)$. Zero force does not mean zero energy.
- (b) In a freely falling lift, both the person and the scale accelerate downward at g . There is no relative acceleration between the person and the scale. As a result, the person exerts no force on the scale, so the scale reads zero. The person's weight (gravitational pull) still exists, but it is entirely used to accelerate the person downward, leaving no contact force to register on the scale.
- (c) Thrust: $F_{\text{thrust}} = v_{\text{exhaust}} \times \frac{dm}{dt} = 800 \text{ m/s} \times 20 \text{ kg/s} = 16000 \text{ N}$

Weight: $W = 5000 \text{ kg} \times 9.8 \text{ m/s}^2 = 49000 \text{ N}$

Net upward force: $F_{\text{net}} = 16000 \text{ N} - 49000 \text{ N} = -33000 \text{ N}$

Since the thrust is less than the weight, the rocket does not lift off. The initial acceleration is zero (the rocket remains on the ground).

Question 27

- (a) The range formula is $R = \frac{u^2 \sin 2\theta}{g}$. Since $\sin 2\theta$ has a maximum value of 1 when $2\theta = 90^\circ$, the maximum range occurs at $\theta = 45^\circ$. At this angle, the vertical and horizontal components of the initial velocity are equal, giving the optimum balance between height (time in the air) and horizontal velocity.
- (b) The period is derived from $\frac{mv^2}{R} = \frac{GMm}{R^2}$. The satellite mass m appears on both sides and cancels. The resulting expression $T = 2\pi \sqrt{\frac{R^3}{GM}}$ depends only on the orbital radius R and the mass M of the central body, not on the satellite mass. A heavier satellite needs more centripetal force, but gravity provides proportionally more force, so the motion is the same.
- (c) For the system (both masses accelerate together):

Net force = weight of hanging mass = $1.0 \text{ kg} \times 9.8 \text{ m/s}^2 = 9.8 \text{ N}$

Total mass = $4.0 \text{ kg} + 1.0 \text{ kg} = 5.0 \text{ kg}$

$$a = \frac{9.8 \text{ N}}{5.0 \text{ kg}} = 1.96 \text{ m/s}^2$$

For the hanging mass: $mg - T = ma$

$$T = m(g - a) = 1.0 \text{ kg} \times (9.8 - 1.96) \text{ m/s}^2 = 1.0 \text{ kg} \times 7.84 \text{ m/s}^2 = 7.84 \text{ N}$$

Question 28

- (a) The total energy is $E = K + U = \frac{GMm}{2R} - \frac{GMm}{R} = -\frac{GMm}{2R}$. The potential energy (negative) has twice the magnitude of the kinetic energy (positive). When added, the result is negative. This negative total energy indicates that the satellite is gravitationally bound and cannot escape to infinity without additional energy.
- (b) The maximum height depends only on the vertical component of the initial velocity: $H = \frac{u_y^2}{2g}$. The horizontal component does not affect the vertical motion because horizontal and vertical motions are independent (gravity acts only vertically). As long as u_y is the same, the height is the same regardless of the horizontal component.
- (c) Vertically: $R + F \sin \theta = mg$, so $R = mg - F \sin \theta$

Horizontally (constant velocity, so net force = 0): $F \cos \theta = \mu R$

Substituting R : $F \cos \theta = \mu(mg - F \sin \theta)$

$$F \cos \theta = \mu mg - \mu F \sin \theta$$

$$F \cos \theta + \mu F \sin \theta = \mu mg$$

$$F(\cos\theta + \mu\sin\theta) = \mu mg$$

$$F = \frac{\mu mg}{\cos\theta + \mu\sin\theta}$$

Question 29

- (a) Mass is the quantity of matter in a body, measured in kg. It is a scalar and does not change with location. Weight is the gravitational force on the body ($W = mg$), measured in N. It depends on the local value of g . An astronaut in orbit has the same mass as on Earth, but appears weightless because both the astronaut and the spacecraft are in free fall together. There is no contact force between them, so the astronaut experiences no sensation of weight.
- (b) When a car rounds a curve, the centripetal force (friction at the tyres) acts at road level. This force and the weight of the car (acting at the centre of mass) create a turning effect (torque) about the outer wheels. The torque tending to overturn the car is proportional to the height of the centre of mass. A higher centre of mass produces a larger overturning torque for the same centripetal force, making the car more likely to tip over before it skids.
- (c) Escape velocity: $v_E = \sqrt{2g_{\text{surface}}r} = \sqrt{2 \times 4.5\text{m/s}^2 \times 4.8 \times 10^6\text{m}} = 6573\text{m/s} = 6.57\text{km/s}$

Question 30

- (a) Off the cliff, the ball has a horizontal velocity (from the kick) and simultaneously experiences a vertical gravitational acceleration. These two independent motions combine to produce a curved (parabolic) trajectory. On the flat field, the ball has only horizontal motion. Gravity acts vertically downward and is balanced by the normal reaction from the ground, so there is no vertical acceleration. With motion in only one direction, the path is a straight line.
- (b) Before: total momentum = mv (first stone) + 0 (second stone) = mv . After: total momentum = 0 (first stone) + mv (second stone) = mv . Momentum is conserved. Before: total KE = $\frac{1}{2}mv^2$. After: total KE = $\frac{1}{2}mv^2$. KE is conserved. Both conservation laws are satisfied because the entire momentum and kinetic energy are transferred from the first stone to the second.
- (c) $T = 2\pi\sqrt{\frac{R^3}{GM_E}} = 2 \times 3.14 \times \sqrt{\frac{(3.84 \times 10^8\text{m})^3}{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}} = 2.363 \times 10^6\text{s}$

Converting to days: $\frac{2.363 \times 10^6\text{s}}{86400\text{s/day}} = 27.3$ days

Question 31

- (a) At infinity, both kinetic energy and potential energy are zero, giving total energy of zero. If a body's total energy is negative, it would need to gain energy to reach zero total energy at infinity. Since no external energy source is available (for an unpowered projectile), the body cannot reach infinity. It will rise, slow down, and eventually fall back. Only a body with total energy greater than or equal to zero can escape.
- (b) The rain falls vertically, but the person moves forward horizontally. The velocity of the rain relative to the person is the vector sum of the rain's downward velocity and the person's forward velocity (reversed). This resultant velocity is directed downward and backward relative to the person. Therefore, the rain appears to come from the front at an angle.
- (c) At the design speed, the horizontal component of the normal reaction provides the centripetal force, and friction is zero.

Resolving: $R\sin\theta = \frac{mv^2}{r}$ and $R\cos\theta = mg$

Dividing: $\tan\theta = \frac{v^2}{rg}$

$$v = \sqrt{rg\tan\theta} = \sqrt{80\text{m} \times 9.8\text{m/s}^2 \times \tan 20^\circ} = 16.9\text{m/s}$$

Question 32

- (a) Two effects contribute. First, the Earth is slightly flattened at the poles (oblate), so the equatorial radius is larger than the polar radius. Since $g \propto \frac{1}{r^2}$, the larger equatorial radius gives a smaller g . Second, at the equator, part of the gravitational pull is used to provide the centripetal acceleration for the Earth's rotation. This reduces the effective g . At the poles, no centripetal acceleration is needed (the rotation radius is zero), so the full gravitational pull is felt. Both effects make g smaller at the equator.
- (b) As the parachutist falls, air resistance increases with speed. Initially, gravity exceeds air resistance, producing a net downward force and hence acceleration. As speed increases, air resistance grows until it equals the gravitational force. At this point, the net force is zero, so by Newton's second law, the acceleration is zero. The parachutist continues at constant speed (terminal velocity). Gravity still acts, but it is exactly balanced by air resistance.
- (c) Taking rightward as positive:

$u = +8\text{m/s}$, $v = -6\text{m/s}$

Change in momentum: $\Delta p = mv - mu = 0.2\text{kg} \times (-6\text{m/s}) - 0.2\text{kg} \times 8\text{m/s}$
 $= -1.2\text{kgm/s} - 1.6\text{kgm/s} = -2.8\text{kgm/s}$

Average force: $F = \frac{\Delta p}{\Delta t} = \frac{-2.8 \text{ kg m/s}}{0.05 \text{ s}} = -56 \text{ N}$

The magnitude of the average force is 56 N (directed to the left, toward the ball).

Question 33

- (a) At any distance R from a planet, the orbital velocity is $v_{\text{orb}} = \sqrt{\frac{GM}{R}}$ and the escape velocity is $v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$. Dividing: $\frac{v_{\text{esc}}}{v_{\text{orb}}} = \sqrt{2}$, regardless of the value of R . The factor $\sqrt{2}$ arises from the ratio of the coefficients in the two formulas, and R cancels completely. This relationship is therefore universal.
- (b) The change in momentum of each passenger is the same whether braking is sudden or gradual (the passenger goes from the bus's speed to rest). Since impulse = $F\Delta t = \Delta p$, and Δp is fixed, a longer braking time Δt means a smaller force F on the passengers. Gradual braking increases the stopping time and therefore reduces the force experienced, making it safer and more comfortable.
- (c) Along the slope, net acceleration:

$$a = g \sin 25^\circ - \mu g \cos 25^\circ = 9.8 \text{ m/s}^2 (\sin 25^\circ - 0.10 \cos 25^\circ) = 3.254 \text{ m/s}^2$$

$$v^2 = u^2 + 2as = 0 + 2 \times 3.254 \text{ m/s}^2 \times 200 \text{ m} = 1301.6 \text{ m}^2/\text{s}^2$$

$$v = \sqrt{1301.6 \text{ m}^2/\text{s}^2} = 36.1 \text{ m/s}$$

Question 34

- (a) Throughout the flight, the horizontal component of velocity remains constant (no horizontal force acts). The vertical component decreases as the ball rises (due to gravitational deceleration) and increases as it falls. At the highest point, the vertical component is momentarily zero. Since the speed is $\sqrt{v_x^2 + v_y^2}$, and $v_y = 0$ at the highest point, the speed equals the horizontal component alone, which is its smallest possible value.
- (b) A satellite in circular orbit is in free fall. The gravitational force acts perpendicular to the velocity (directed toward the centre), so it does no work. Since no work is done, the satellite neither gains nor loses energy. Its speed remains constant and its direction changes continuously in exactly the right way to maintain the circular path. No fuel is needed because gravity provides the centripetal force for free.
- (c) Using conservation of momentum (initial momentum = 0):

$$0 = 3.0 \text{ kg} \times (-4.0 \text{ m/s}) + 5.0 \text{ kg} \times v_2$$

$$v_2 = \frac{3.0 \times 4.0}{5.0} = 2.4 \text{ m/s (to the right)}$$

Energy stored in the spring = total KE after release:

$$\begin{aligned} E &= \frac{1}{2} \times 3.0 \text{ kg} \times (4.0 \text{ m/s})^2 + \frac{1}{2} \times 5.0 \text{ kg} \times (2.4 \text{ m/s})^2 \\ &= 24.0 \text{ J} + 14.4 \text{ J} = 38.4 \text{ J} \end{aligned}$$

Question 35

- (a) The approximate formula is derived from the exact formula using the binomial expansion, which drops terms of order $\left(\frac{h}{r_E}\right)^2$ and higher. These dropped terms are negligible only when $\frac{h}{r_E} \ll 1$. As h increases, $\frac{h}{r_E}$ increases, and the dropped terms become increasingly significant. Therefore, the difference between the approximate and exact values grows, producing a rapidly increasing percentage error.
- (b) In the head-on collision of two identical cars at 60 km/h, each car decelerates from 60 km/h to 0 km/h on impact (by symmetry, the wreckage is stationary). The change in velocity for each car is 60 km/h. Hitting a rigid wall at 120 km/h would require the car to decelerate from 120 km/h to 0 km/h, a change of 120 km/h, which is twice as severe. The head-on collision is therefore equivalent to hitting a wall at 60 km/h, not 120 km/h.
- (c) $u_x = 30 \cos 37^\circ = 30 \times 0.7986 = 23.96 \text{ m/s (constant)}$

$$u_y = 30 \sin 37^\circ = 30 \times 0.6018 = 18.05 \text{ m/s (changes: } v_y = u_y - gt)$$

$$\text{Using: } v^2 = v_x^2 + v_y^2$$

$$\text{At time } t, \text{ speed is: } v^2 = u_x^2 + (u_y - gt)^2$$

$$25^2 = 23.96^2 + (18.05 - 9.8t)^2$$

$$(18.05 - 9.8t)^2 = 50.9$$

$$18.05 - 9.8t = \pm 7.13$$

$$t = 1.11\text{s or } t = 2.57\text{s}$$

The speed is 25m/s at $t = 1.11\text{s}$ (on the way up) and $t = 2.57\text{s}$ (on the way down).

Question 36

- (a) Escape velocity is $v_E = \sqrt{\frac{2GM}{r}}$. The Moon has a much smaller mass than the Earth (about $\frac{1}{81}$ of Earth's mass) and a smaller radius (about $\frac{1}{3.7}$ of Earth's radius). The smaller mass dominates, giving a weaker gravitational field. As a result, less kinetic energy is needed to escape from the Moon, and hence a lower speed is sufficient.
- (b) Drawing the hands backward increases the time over which the ball decelerates. The change in momentum is the same regardless of how the ball is caught. Since impulse = $F\Delta t = \Delta p$, increasing Δt decreases the force F for the same Δp . Therefore, the force on the fielder's hands is reduced.
- (c) At the bottom, $KE = \frac{1}{2} \times 1500\text{kg} \times (20\text{m/s})^2 = 300000\text{J}$

At the top (height = $2r = 24\text{m}$), by energy conservation:

$$\begin{aligned}\frac{1}{2}mv_{\text{top}}^2 &= \frac{1}{2}mv_{\text{bottom}}^2 - mg(2r) \\ v_{\text{top}}^2 &= (20\text{m/s})^2 - 2 \times 9.8\text{m/s}^2 \times 24\text{m} = -70.4\text{m}^2/\text{s}^2\end{aligned}$$

Since v_{top}^2 is negative, the car does not have enough speed to reach the top. The car fails to complete the loop.

Question 37

- (a) At the top of the bridge, the car follows a circular arc whose centre of curvature is below the road. The centripetal acceleration is therefore directed downward. Because part of the driver's weight is required to provide this centripetal acceleration, the normal reaction R from the seat is less than the driver's weight: $R = mg - \frac{mv^2}{r}$. Since what the driver feels as "weight" is the contact force R rather than gravity itself, the reduced R creates the sensation of feeling lighter than on the flat road where the normal reaction is equal to the driver's weight.
- (b) The collision between the bullet and block is perfectly inelastic. The kinetic energy that disappears from the motion of the objects is not destroyed but is converted into other forms of energy. Most of it becomes thermal energy (the bullet and the surrounding wood heat up due to friction and deformation). A small portion is converted into sound energy (the impact is audible) and into permanent deformation of the wood and bullet.
- (c) The stone falls through a net vertical height of 70m. Using conservation of energy:

$$\begin{aligned}\frac{1}{2}mu^2 + mgh &= \frac{1}{2}mv^2 \\ v^2 &= u^2 + 2gh \\ v^2 &= (40\text{m/s})^2 + 2 \times 9.8\text{m/s}^2 \times 70\text{m} = 2972\text{m}^2/\text{s}^2 \\ v &= \sqrt{2972\text{m}^2/\text{s}^2} = 54.5\text{m/s}\end{aligned}$$

Question 38

- (a) In the planet's reference frame, the spacecraft approaches and leaves with the same speed (the gravitational interaction is elastic). However, in the Sun's reference frame, the planet is itself moving along its orbit. The spacecraft effectively "borrows" a small amount of kinetic energy from the planet's orbital motion. Because the planet's mass is enormously larger than the spacecraft's, the planet's speed decreases by an insignificant amount, while the spacecraft gains a significant increase in speed.
- (b) This is possible when the two equal forces act in exactly opposite directions. In that case, their vector sum (the resultant) is zero. Since there is no net force, there is no acceleration, and the object remains in equilibrium. This is the principle behind the equilibrant: a force equal in magnitude but opposite in direction to a given force produces equilibrium.
- (c) At the top, both the tension and the weight act toward the centre (downward), so together they provide the centripetal force:

$$\begin{aligned}T + mg &= \frac{mv^2}{r} \\ v^2 &= \frac{(T + mg)r}{m} \\ v^2 &= \frac{(5.0\text{N} + 0.40\text{kg} \times 9.8\text{m/s}^2) \times 0.60\text{m}}{0.40\text{kg}} = 13.38\text{m}^2/\text{s}^2 \\ v &= \sqrt{13.38\text{m}^2/\text{s}^2} = 3.66\text{m/s}\end{aligned}$$

Question 39

- (a) A geostationary satellite must have a period of exactly 24 hours to match the Earth's rotation. From Kepler's third law, $T^2 = \frac{4\pi^2}{GM_E} R^3$, the orbital radius R is uniquely determined by the period T . Since both T and GM_E are fixed constants, only one value of R satisfies the equation. Therefore, only one altitude gives a 24-hour period, and all geostationary satellites must orbit at this height (approximately 36000km).
- (b) When the goalkeeper dives, the body gains significant momentum toward the ground. Upon landing, this momentum must be reduced to zero. By rolling, the goalkeeper extends the time and distance over which the body decelerates after hitting the ground. By the impulse-momentum relationship ($F\Delta t = \Delta p$), since the change in momentum is fixed, a longer deceleration time Δt produces a smaller average impact force F on the body. Rolling also spreads the contact over a larger area of the body rather than concentrating it on a single point such as the elbow or shoulder. As a result, both the force and the pressure on any one part of the body are reduced, greatly lowering the risk of injury from the dive.
- (c) Resolving into components:

$$R_x = 25\text{N} + 18\cos 130^\circ\text{N} + 12\cos 240^\circ\text{N} = 7.43\text{N}$$

$$R_y = 0 + 18\sin 130^\circ\text{N} + 12\sin 240^\circ\text{N} = 3.40\text{N}$$

$$\text{Resultant: } R = \sqrt{(7.43\text{N})^2 + (3.40\text{N})^2} = 8.17\text{N}$$

$$\text{Direction of resultant: } \theta = \tan^{-1}\left(\frac{3.40}{7.43}\right) = 24.6^\circ \text{ above the positive x-axis.}$$

The equilibrant has the same magnitude (8.17N) but acts in the opposite direction: $24.6^\circ + 180^\circ = 204.6^\circ$ from the positive x-axis.

Question 40

- (a) When an object is projected horizontally, it moves forward at constant horizontal velocity while simultaneously falling under gravity. A taller building means the object has a longer time to fall before reaching the ground. Because the horizontal distance equals the horizontal velocity multiplied by the time of flight ($x = u_x \times t$), a longer fall time produces a greater horizontal distance, even though u_x is the same.
- (b) In a freely falling lift, both the person and the scale accelerate downward at g . Because there is no relative acceleration between the person and the scale, the person exerts no force on the scale. As a result, the scale reads zero. The person's weight (mg) still exists as a gravitational force, but it is entirely used to accelerate the person downward, leaving no contact force to register on the scale.
- (c) Momentum is conserved in each direction separately.

$$\text{Eastward: } p_x = 2.0\text{kg} \times 6.0\text{m/s} + 0 = 12.0\text{kgm/s}$$

$$\text{Northward: } p_y = 0 + 3.0\text{kg} \times 4.0\text{m/s} = 12.0\text{kgm/s}$$

$$\text{Total mass: } m = 2.0\text{kg} + 3.0\text{kg} = 5.0\text{kg}$$

Velocity of combined object:

$$v = \frac{\sqrt{p_x^2 + p_y^2}}{m} = \frac{\sqrt{(12.0\text{kgm/s})^2 + (12.0\text{kgm/s})^2}}{5.0\text{kg}} = 3.39\text{m/s}$$

$$\text{Direction: } \theta = \tan^{-1}\left(\frac{12.0}{12.0}\right) = 45^\circ \text{ north of east.}$$

Question 41

- (a) Surface gravity is $g = \frac{GM}{r^2}$. The Moon's smaller radius would, by itself, increase g (because the surface is closer to the centre). However, the Moon's mass is only about $\frac{1}{81}$ of the Earth's mass, which greatly reduces g . Since mass appears in the numerator and radius squared in the denominator, the much smaller mass of the Moon dominates over the slightly smaller radius. As a result, the net effect is a much lower surface gravity.
- (b) The ball was kicked at an angle, so it has both a horizontal and a vertical component of velocity at launch. At the highest point, the vertical component is zero (the ball has momentarily stopped rising). However, the horizontal component remains unchanged throughout the flight because no horizontal force acts on the ball (neglecting air resistance). Therefore, at the highest point, the ball still has a non-zero horizontal velocity, and hence its velocity is not zero.
- (c) For the car to follow the circular path, static friction must provide the centripetal force:

$$\mu_s mg = \frac{mv^2}{r}$$

$$\mu_s = \frac{v^2}{gr}$$

$$\mu_s = \frac{(25\text{m/s})^2}{9.8\text{m/s}^2 \times 60\text{m}} = 1.06$$

Since $\mu_s > 1$ is unrealistic for ordinary tyres on road, the car cannot safely negotiate this bend at 25m/s on a flat road. It would skid.

Question 42

- (a) The total energy of a satellite is $E = K + U = \frac{GMm}{2R} - \frac{GMm}{R} = -\frac{GMm}{2R}$. The potential energy (negative) has twice the magnitude of the kinetic energy (positive). When they are added, the result is negative. This negative total energy signifies that the satellite is gravitationally bound to the Earth. To free the satellite completely (send it to infinity with zero speed), energy equal to $|E|$ must be supplied. A satellite with negative total energy cannot escape without an external energy input.
- (b) The parked car needs only enough fuel to keep the engine running against internal friction (very low energy demand). The moving car, however, must overcome continuous air resistance and road friction. These resistive forces constantly remove kinetic energy from the car, which must be replaced by the engine. Because the driving force must continuously do work against these resistive forces to maintain constant speed, the fuel consumption is much higher. Even though the acceleration is zero in both cases, the moving car dissipates energy at a much higher rate.
- (c) At impact, the horizontal velocity is $v_x = 15\text{m/s}$ (unchanged). The angle below horizontal is 60° , so:

$$\tan 60^\circ = \frac{v_y}{v_x}$$

$$v_y = v_x \tan 60^\circ = 15\text{m/s} \times 1.732 = 25.98\text{m/s}$$

Since the ball starts with zero vertical velocity, using $v_y^2 = 2gh$:

$$h = \frac{v_y^2}{2g} = \frac{(25.98\text{m/s})^2}{2 \times 9.8\text{m/s}^2} = 34.4\text{m}$$

Question 43

- (a) Launching the ball upward allows it to remain in the air longer, increasing the time available for horizontal motion. The longer time of flight allows the ball to travel a greater horizontal distance before landing.
- (b) The coin needs a centripetal force directed toward the centre to move in a circle. On a smooth turntable, there is very little friction to provide this force. As the angular speed increases, the required centripetal force ($m\omega^2 r$) increases. When the required force exceeds the available friction, the friction can no longer maintain the circular path. The coin then continues moving in the direction of its instantaneous velocity (tangentially), which carries it outward and away from the centre. It appears to slide “outward” because the tangential direction at any point on a circle leads away from the centre.
- (c) Using $g = \frac{GM_E}{R^2}$, making R the subject:

$$R = \sqrt{\frac{GM_E}{g}}$$

$$R = \sqrt{\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{4.9\text{N/kg}}} = 9.037 \times 10^6\text{m}$$

Altitude:

$$h = R - r_E = 9.037 \times 10^6\text{m} - 6.4 \times 10^6\text{m} = 2.637 \times 10^6\text{m} = 2637\text{km}$$

Question 44

- (a) Throwing the rope outward gives it a horizontal component of velocity. This allows it to travel away from the cliff while falling. If thrown straight downward, it would simply fall vertically close to the cliff face.
- (b) By Newton’s third law, the force on the boat equals the force on the ship during the collision, and both act for the same duration. Therefore, both experience the same impulse and hence the same magnitude of change in momentum. However, because the boat has a much smaller mass, this momentum change produces a much larger change in velocity ($\Delta v = \frac{\Delta p}{m}$). As a result, the boat decelerates violently, causing severe structural damage, while the ship’s much larger mass means its velocity changes negligibly.
- (c) At the top, for the body to just complete the circle, the minimum condition is $R = 0$ (the string goes slack). Then gravity alone provides the centripetal force:

$$mg = \frac{mv_{\text{top}}^2}{r}$$

$$v_{\text{top}} = \sqrt{gr} = \sqrt{9.8\text{m/s}^2 \times 1.0\text{m}} = 3.13\text{m/s}$$

Using energy conservation from bottom to top (height change = $2r$):

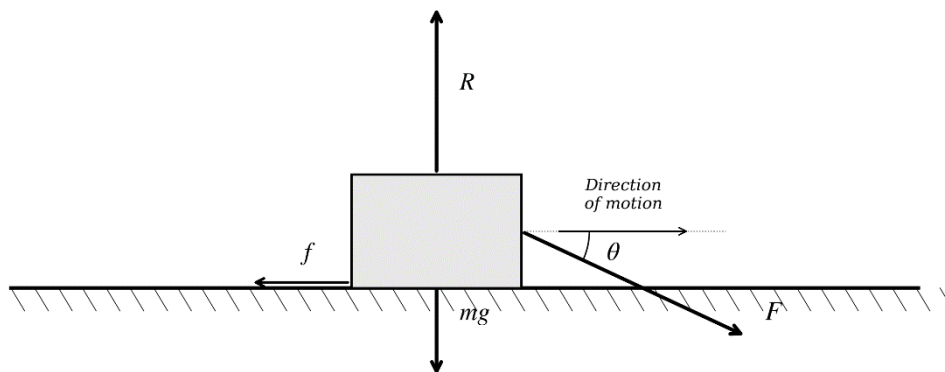
$$\frac{1}{2}mv_{\text{bottom}}^2 = \frac{1}{2}mv_{\text{top}}^2 + mg(2r)$$

$$v_{\text{bottom}} = \sqrt{v_{\text{top}}^2 + 4gr} = \sqrt{gr + 4gr} = \sqrt{5gr}$$

$$v_{\text{bottom}} = \sqrt{5 \times 9.8\text{m/s}^2 \times 1.0\text{m}} = 7.0\text{m/s}$$

Question 45

- (a) The gravitational force on the planet is directed radially toward the Sun and is therefore always perpendicular to the planet's displacement along the orbit only for circular orbits. For an elliptical orbit, the gravitational force has a component along the direction of motion whenever the planet is not at perihelion or aphelion. Between aphelion and perihelion (as the planet moves inward), the radial force has a component in the direction of motion, doing positive work and thus increasing the planet's speed. Between perihelion and aphelion (as the planet moves outward), the component opposes the motion, doing negative work and slowing the planet down. This is consistent with conservation of energy: as the planet falls closer to the Sun, it loses potential energy and gains kinetic energy, and vice versa.
- (b) As sand leaks out, the total mass of the truck decreases over time. If the engine provides the same driving force F and the resistive forces remain approximately constant, the net force stays roughly the same. By Newton's second law, $a = \frac{F_{\text{net}}}{m}$. Because m decreases while F_{net} remains constant, the acceleration increases over time. The truck gradually speeds up faster and faster.
- (c)



Since F acts below the horizontal, it pushes the block into the surface, increasing the normal reaction.

$$\text{Vertically: } R = mg + F\sin\theta = 5.0\text{kg} \times 9.8\text{m/s}^2 + 40\text{N} \times \sin 25^\circ = 65.9\text{N}$$

$$\text{Friction: } f = \mu_k R = 0.30 \times 65.9\text{N} = 19.77\text{N}$$

$$\text{Horizontally: } F\cos\theta - f = ma$$

$$a = \frac{F\cos\theta - f}{m} = \frac{40\text{N} \times \cos 25^\circ - 19.77\text{N}}{5.0\text{kg}} = 3.30\text{m/s}^2$$

Question 46

- (a) Orbital speed is derived from the condition that gravity provides the centripetal force: $\frac{mv^2}{R} = \frac{GM_E m}{R^2}$. The satellite's mass m appears on both sides and cancels, giving $v = \sqrt{\frac{GM_E}{R}}$. This expression depends only on the Earth's mass and the orbital radius, not on the satellite's mass. A heavier satellite requires more centripetal force, but gravity provides exactly that extra force because the gravitational force is proportional to m . Hence the speed is the same for all satellites at the same altitude.
- (b) The change in the ball's momentum when caught is fixed. By the impulse-momentum relationship, $F\Delta t = \Delta p$. By moving the arms backward, the player increases the time Δt over which the ball decelerates. Because Δp is constant, a larger Δt results in a smaller average force F on the arms. Thus the cushioning technique reduces the impact force.
- (c) Components at launch: $v_x = 28\text{m/s} \times \cos 40^\circ = 21.45\text{m/s}$, $v_{y,0} = 28\text{m/s} \times \sin 40^\circ = 17.99\text{m/s}$

At $t = 2.0\text{s}$:

$$v_x = 21.45 \text{ m/s (unchanged)}$$

$$v_y = v_{y,0} - gt = 17.99 \text{ m/s} - 9.8 \text{ m/s}^2 \times 2.0 \text{ s} = -1.61 \text{ m/s}$$

The negative sign means the ball is now moving downward.

Magnitude of velocity (speed):

$$v = \sqrt{(21.45 \text{ m/s})^2 + (-1.61 \text{ m/s})^2} = 21.5 \text{ m/s}$$

$$\text{Direction: } \alpha = \tan^{-1}\left(\frac{1.61}{21.45}\right) = 4.3^\circ \text{ below the horizontal.}$$

Question 47

- (a) **First reason:** The Earth is not a perfect sphere but is slightly flattened at the poles (oblate). The equatorial radius is larger than the polar radius. Since $g \propto \frac{1}{r^2}$, the larger equatorial radius gives a smaller g at the equator.

Second reason: At the equator, the person moves in a circle due to the Earth's rotation. Part of the gravitational pull must provide the centripetal acceleration for this circular motion. As a result, the effective g is reduced by $\omega^2 r_E$. At the poles, the person is on the rotation axis, so no centripetal acceleration is needed and the full gravitational pull is felt.

Both effects make the effective g smaller at the equator, and hence the person weighs less there.

- (b) The ball gains a large momentum in a very short time. By the impulse-momentum theorem, $F = \frac{\Delta p}{\Delta t}$. Because Δt is extremely small (a few milliseconds), even a modest change in momentum produces a very large force.
- (c) First, find the velocity after impact using conservation of momentum:

$$m_b u = (m_b + M)v$$

$$v = \frac{m_b u}{m_b + M} = \frac{0.020 \text{ kg} \times 350 \text{ m/s}}{0.020 \text{ kg} + 2.48 \text{ kg}} = 2.8 \text{ m/s}$$

On the rough patch, friction decelerates the block.

$$F = ma = \mu mg$$

$$a = \mu g \text{ (opposing the motion)}$$

Using $v^2 = u^2 - 2\mu gs$, with the block coming to rest ($v = 0$), and $a = -\mu g$:

$$s = \frac{u^2}{2\mu g} = \frac{(2.8 \text{ m/s})^2}{2 \times 0.40 \times 9.8 \text{ m/s}^2} = 1.0 \text{ m}$$

Alternative solution

Work done by friction = loss of kinetic energy

$$f \times s = \frac{1}{2} mu^2$$

$$\mu mg \times s = \frac{1}{2} mu^2$$

$$s = \frac{u^2}{2\mu g} = \frac{(2.8 \text{ m/s})^2}{2 \times 0.40 \times 9.8 \text{ m/s}^2} = 1.0 \text{ m}$$

Question 48

- (a) Below the Earth's surface, by the Shell Theorem, the spherical shell of mass above the object exerts zero net gravitational force on it. Only the mass within the sphere of radius equal to the object's distance from the centre contributes to gravity. As the object moves deeper, this contributing mass decreases (since less mass lies beneath the object). Therefore, g decreases with depth. At the centre, all the mass is distributed symmetrically around the point, so the gravitational pulls from all directions cancel completely, giving $g = 0$.
- (b) When the car enters the roundabout, all parts of the car must follow a curved path, requiring centripetal force directed toward the centre. The front tyres, being steered, change direction first and friction at the front provides centripetal force. However, the rear of the car, with its greater distance from the turning axis and its inertia, tends to continue in a straight line (in accordance with Newton's first law). If the speed is too high, the rear tyres cannot provide enough friction to keep the back end on the curved path, and the rear swings outward. This is called oversteer.
- (c) For equilibrium, the third force must balance the other two. It must have an upward component of 50 N and a leftward component of 30 N.

Magnitude:

$$F_3 = \sqrt{(50\text{N})^2 + (30\text{N})^2} = 58.3\text{N}$$

Direction:

$$\theta = \tan^{-1}\left(\frac{30}{50}\right) = 31.0^\circ \text{ from the vertical, toward the left.}$$

Equivalently, 59.0° above the negative x-axis (or 121° from the positive x-axis).

Question 49

- (a) When drag removes energy, the total energy $E = -\frac{GMm}{2R}$ becomes more negative. A more negative E corresponds to a smaller R , so the satellite spirals inward to a lower orbit. At this lower orbit, the required orbital speed $v = \sqrt{\frac{GM}{R}}$ is higher because R is smaller. Therefore, the satellite speeds up. This occurs because the gravitational potential energy lost in falling to a lower orbit exceeds the energy lost to drag plus the kinetic energy gained. The satellite trades altitude for speed.
- (b) The ball has large momentum that must be reduced to zero when caught. The change in momentum is fixed. Without pulling the hands back, the stopping time Δt is very short (the ball stops almost instantly). By the impulse-momentum relationship ($F = \frac{\Delta p}{\Delta t}$), a very short Δt produces a very large force F on the hands, causing pain. If the catcher draws the hands back, Δt increases and the force is reduced.
- (c) The design speed for this bank: $v_d = \sqrt{rg \tan 15^\circ} = \sqrt{100 \times 9.8 \times 0.2679} = 16.2\text{m/s}$

Since the car's speed (20m/s) exceeds the design speed (16.2m/s), the banking alone cannot provide enough centripetal force. Friction must act **down the slope** (toward the centre) to supplement the centripetal force.

$$\text{Required centripetal force: } \frac{mv^2}{r} = \frac{1200\text{kg} \times (20\text{m/s})^2}{100\text{m}} = 4800\text{N}$$

$$\text{Force balance toward centre (horizontally): } R \sin 15^\circ + f \cos 15^\circ = 4800\text{N}$$

$$\text{Force balance vertically: } R \cos 15^\circ - f \sin 15^\circ = mg = 1200\text{kg} \times 9.8\text{m/s}^2 = 11760\text{N}$$

$$\text{From vertical: } R = \frac{11760\text{N} + f \sin 15^\circ}{\cos 15^\circ}$$

Substituting into horizontal:

$$\frac{(11760 + f \sin 15^\circ) \sin 15^\circ}{\cos 15^\circ} + f \cos 15^\circ = 4800$$

$$11760 \tan 15^\circ + f \sin 15^\circ \tan 15^\circ + f \cos 15^\circ = 4800$$

$$3151 + f(0.06699 + 0.9659) = 4800$$

$$f = 1596\text{N (acting down the slope, toward the centre)}$$

Question 50

- (a) On a curved road, friction between the tyres and the road surface provides the centripetal force needed to follow the curve. Water on the road acts as a lubricant, significantly reducing the coefficient of friction between the tyres and the surface. As a result, the maximum friction available (μmg) is much smaller on a wet road. Because the centripetal force required ($\frac{mv^2}{r}$) depends on the square of the speed, a speed that was safe on dry road may demand more friction than the wet surface can provide, causing the car to skid.
- (b) The gravitational force on the comet is always directed toward the Sun. In a highly elliptical orbit, the distance from the Sun varies enormously. Near the Sun, the comet is deep in the gravitational potential well. As it falls inward from a great distance, gravitational potential energy is converted to kinetic energy, so it speeds up greatly. When moving away from the Sun, kinetic energy is converted back to potential energy, and the comet slows dramatically. This is consistent with conservation of energy: the total energy remains constant, but the balance between kinetic and potential energy shifts as the distance changes.
- (c) True weight: $W = 2.0\text{kg} \times 9.8\text{m/s}^2 = 19.6\text{N}$

The balance reads 24.0N, which is greater than 19.6N. Since the apparent weight exceeds the true weight, the lift must be accelerating **upward** (the balance must push the mass upward more than gravity alone requires).

$$R - mg = ma$$

$$a = \frac{R - mg}{m} = \frac{24.0\text{N} - 19.6\text{N}}{2.0\text{kg}} = 2.2\text{m/s}^2 \text{ upward}$$

Question 51

- (a) At the bottom of the loop, the centripetal acceleration is directed upward (toward the centre of the loop). The seat must push the pilot upward with a force that both supports the weight and provides the centripetal force: $R = mg + \frac{mv^2}{r}$.

Because R exceeds mg , the pilot feels heavier than normal. At the top of the loop, the centripetal acceleration is directed downward. Gravity assists in providing the centripetal force, so the seat needs to push less: $R = \frac{mv^2}{r} - mg$. If the speed is close to the minimum value for the loop, R is very small, and the pilot feels almost weightless.

- (b) The loaded minibus has a much greater momentum ($p = mv$) than the empty one at the same speed. Since the braking force is similar, the impulse delivered per second is similar. Because the loaded minibus has more momentum to remove, it takes a longer time to stop.

- (c) For the planet: $r_p = 2r_E$ and $\rho_p = \frac{1}{2}\rho_E$

Using $g = \frac{4}{3}\pi G\rho r$:

$$\frac{g_p}{g_E} = \frac{\rho_p \times r_p}{\rho_E \times r_E} = \frac{\frac{1}{2}\rho_E \times 2r_E}{\rho_E \times r_E} = 1$$

$$g_p = g$$

The surface gravity is equal to g (the same as Earth's).

Question 52

- (a) When two forces act on an object, the resultant depends on the angle between them. When both forces act in the same direction (0°), their magnitudes add directly: $R = 10 + 10 = 20\text{N}$. When they act in opposite directions (180°), their magnitudes subtract: $R = 10 - 10 = 0\text{N}$. At any intermediate angle, the resultant lies between these extremes, as given by the parallelogram law. Therefore the resultant ranges from 0 to 20N.
- (b) Throwing distance depends on the range formula $R = \frac{v^2 \sin 2\theta}{g}$. On the Moon, g is smaller, so the range is greater than on the Earth for the same launch speed. However, running speed depends on how fast the legs can push the body forward, which is not determined by gravity. The Moon's lower gravity makes each stride longer (the astronaut stays airborne longer between steps), but the maximum leg-push speed is essentially the same. Therefore, throwing range improves dramatically but running speed does not.
- (c) For an elastic head-on collision between equal masses, the velocities are simply exchanged. The moving ball stops completely, and the stationary ball moves off at the original velocity.

$v_1 = 0\text{m/s}$, $v_2 = 10\text{m/s}$ (in the original direction)

Question 53

- (a) A lower orbit has a smaller radius R . From $T = 2\pi\sqrt{\frac{R^3}{GM}}$ a smaller R gives a shorter period T . With a shorter period, the satellite completes each orbit faster, and therefore fits more orbits into a 24-hour day. Additionally, the orbital speed $v = \sqrt{\frac{GM}{R}}$ is higher at smaller R , so the satellite moves faster along a shorter path, both effects reducing the period.
- (b) The helicopter must continuously push air downward to generate an upward thrust (lift) equal to its weight. Each parcel of air pushed downward quickly moves away and must be replaced by new air. The engines power the rotor blades, which continuously accelerate air downward, giving it momentum. By Newton's third law, this produces an equal upward force on the helicopter. If the engines stopped, no air would be pushed downward, the upward force would vanish, and the helicopter would fall.
- (c) For the 3.0kg block on the incline, the driving force down the incline is the component of its weight along the slope: $3.0\text{kg} \times 9.8\text{m/s}^2 \times \sin 30^\circ = 14.7\text{N}$

For the 2.0kg hanging block, the driving force is its weight: $2.0\text{kg} \times 9.8\text{m/s}^2 = 19.6\text{N} > 14.7\text{N}$

The hanging block accelerates downward, pulling the block on the incline upward along the slope. The net driving force is:

$$F_{\text{net}} = 19.6\text{N} - 14.7\text{N} = 4.9\text{N}$$

$$a = \frac{4.9\text{N}}{(3.0 + 2.0)\text{kg}} = 0.98\text{m/s}^2$$

Question 54

- (a) When pushing at a downward angle, the applied force has a downward component that presses the box harder into the floor. This increases the normal reaction R , which in turn increases the friction force ($f = \mu R$). As a result, more of the applied force is wasted overcoming the increased friction. When pushing horizontally, there is no downward component, so the normal reaction equals just the weight, and friction is at its minimum for the given conditions. Therefore, horizontal pushing requires less total effort to move the box.
- (b) Crumple zones are designed to deform (crush) during a collision. This deformation increases the distance over which the car decelerates. By the work-energy theorem, $F \times d = \Delta KE$. Since the change in kinetic energy is fixed (determined

by the initial speed), a larger deformation distance d results in a smaller average force F . Equivalently, the crumpling increases the stopping time, which by the impulse-momentum theorem ($F\Delta t = \Delta p$) reduces the force. The smaller force is transmitted to the occupants, reducing the severity of injuries.

(c) At the crest: $mg - R = \frac{mv^2}{r}$

$$R = mg - \frac{mv^2}{r} = 1400\text{kg} \times 9.8\text{m/s}^2 - \frac{1400\text{kg} \times (18\text{m/s})^2}{40\text{m}} = 2380\text{N}$$

Since $R > 0$, the car maintains contact with the road.

Question 55

(a) Gravitational potential energy is $U = -\frac{GMm}{R}$. As R increases (higher orbit), the denominator becomes larger, making $\frac{GMm}{R}$ smaller. Since U is the negative of this quantity, U becomes less negative (closer to zero). Physically, the satellite moves farther from the Earth and is less tightly bound, so its potential energy is higher (less negative). Energy must be supplied to move the satellite to a higher orbit, and this energy appears as an increase in potential energy.

(b) Both the ball and the satellite are acted upon by gravity alone. The key difference is the horizontal velocity. The ball has negligible horizontal velocity, so it falls in a nearly straight vertical path toward the ground. The satellite has a very large horizontal velocity (about 7.7km/s), so although it falls toward the Earth at the same rate, its forward motion carries it far enough that the curved surface of the Earth drops away beneath it at the same rate. As a result, the satellite continually falls but never gets closer to the surface. It “falls around” the Earth.

(c) Net driving force: $(5.0 - 3.0)\text{kg} \times 9.8\text{m/s}^2 = 19.6\text{N}$

Total mass: $(5.0 + 3.0)\text{kg} = 8.0\text{kg}$

$$a = \frac{19.6\text{N}}{8.0\text{kg}} = 2.45\text{m/s}^2$$

$$s = \frac{1}{2}at^2 = \frac{1}{2} \times 2.45\text{m/s}^2 \times (2.0\text{s})^2 = \frac{1}{2} \times 2.45 \times 4.0 = 4.9\text{m}$$

Each mass travels 4.9m in 2.0s.

Question 56

(a) Both the bullet and the ball follow parabolic paths under gravity. The bullet’s horizontal velocity (hundreds of metres per second) is vastly greater than the ball’s. Over short distances, the bullet’s time of flight is extremely small, so it falls only a tiny distance vertically before reaching the target. This tiny vertical drop is unnoticeable, making the path appear flat. The ball’s much lower velocity means a much longer flight time, during which gravity pulls it through a clearly visible vertical distance, producing an obviously curved arc.

(b) At the design speed, the horizontal component of the normal reaction from the banked surface alone provides exactly the centripetal force needed for the circular path: $R\sin\theta = \frac{mv^2}{r}$. No friction is needed at all. The vertical component of R supports the weight: $R\cos\theta = mg$. This is the ideal condition for which the road was designed.

(c) Let T_1 be the tension in the ceiling string (at 50° to horizontal) and T_2 the tension in the horizontal string.

Vertically: $T_1\sin 50^\circ = mg = 0.80\text{kg} \times 9.8\text{m/s}^2 = 7.84\text{N}$

$$T_1 = \frac{7.84\text{N}}{\sin 50^\circ} = 10.23\text{N}$$

Horizontally: $T_2 = T_1\cos 50^\circ = 10.23\text{N} \times 0.6428 = 6.58\text{N}$

Question 57

(a) Both trucks have similar braking forces (similar brakes). The loaded truck has greater mass and therefore greater momentum at the same speed ($p = mv$). Since the braking force is similar, the rate of momentum removal ($F = \frac{\Delta p}{\Delta t}$) is similar for both. Because the loaded truck has more momentum to remove, it requires a longer time and a greater distance to stop. Therefore, it must begin braking earlier.

(b) Near the surface, the variation of distance from the Earth’s centre is very small compared to the Earth’s radius. Because the change in r is negligible, the change in $g = \frac{GM}{r^2}$ is also negligible, and the field appears uniform. At the scale of satellite orbits, the distances involved are comparable to or larger than the Earth’s radius. The variation in r is significant, so g changes noticeably with position, and the radial (non-uniform) nature of the field becomes apparent.

(c) Maximum height: $H = \frac{u_y^2}{2g}$, so $u_y = \sqrt{2gH} = \sqrt{2 \times 9.8\text{m/s}^2 \times 20\text{m}} = 19.80\text{m/s}$

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin \theta \cos \theta}{g} = \frac{2 \times (u \cos \theta)(u \sin \theta)}{g} = \frac{2u_x u_y}{g}$$

$$\text{So } u_x = \frac{Rg}{2u_y} = \frac{60 \text{ m} \times 9.8 \text{ m/s}^2}{2 \times 19.80 \text{ m/s}} = 14.85 \text{ m/s}$$

$$u = \sqrt{u_x^2 + u_y^2} = \sqrt{(14.85 \text{ m/s})^2 + (19.80 \text{ m/s})^2} = 24.7 \text{ m/s}$$

$$\theta = \tan^{-1} \left(\frac{19.80}{14.85} \right) = 53.1^\circ$$

Question 58

- (a) The period of a pendulum is $T = 2\pi \sqrt{\frac{L}{g}}$. On the Moon, g is smaller (about 6 times smaller) than on Earth. Because T is proportional to $\frac{1}{\sqrt{g}}$, a smaller g gives a longer period. With a longer period, each swing takes more time, so the pendulum appears to swing more slowly. The length L is the same, but the weaker gravitational restoring force means the pendulum accelerates more slowly toward the equilibrium position.
- (b) After the burn, the satellite has more kinetic energy than required for its current circular orbit. The orbit becomes elliptical, with the burn point as the perigee (lowest point) of the new ellipse. The satellite climbs to a higher point (apogee) on the opposite side of the orbit. The period increases because the semi-major axis of the elliptical orbit is larger than the original circular radius.
- (c) By conservation of momentum:

$$0.030 \text{ kg} \times 400 \text{ m/s} = 0.030 \text{ kg} \times 100 \text{ m/s} + 1.5 \text{ kg} \times v$$

$$v = \frac{9.0 \text{ kg m/s}}{1.5 \text{ kg}} = 6.0 \text{ m/s}$$

$$\text{KE before: } \frac{1}{2} \times 0.030 \text{ kg} \times (400 \text{ m/s})^2 = 2400 \text{ J}$$

$$\text{KE after: } \frac{1}{2} \times 0.030 \text{ kg} \times (100 \text{ m/s})^2 + \frac{1}{2} \times 1.5 \text{ kg} \times (6.0 \text{ m/s})^2 = 177 \text{ J}$$

$$\text{KE lost: } 2400 \text{ J} - 177 \text{ J} = 2223 \text{ J}$$

Question 59

- (a) The range of a jump is $R = \frac{u^2 \sin 2\theta}{g}$, where u is the takeoff speed. Running builds up a large horizontal velocity before the jump. At takeoff, this horizontal velocity combines with the vertical push from the legs to produce a higher overall launch velocity u . Since range is proportional to u^2 , even a modest increase in launch velocity produces a significant increase in horizontal distance. From a standing position, the launch velocity comes only from the legs, giving a much smaller u and hence a much shorter range.
- (b) Above the surface, the entire mass of the Earth attracts the object. As r increases, the force decreases as $\frac{1}{r^2}$ (inverse square law) because the same mass pulls from a greater distance. Below the surface, the Shell Theorem means that only the mass within radius r contributes. This contributing mass is proportional to r^3 (for uniform density). The field strength is $g = \frac{GM'}{r^2} = \frac{G \times \frac{4}{3}\pi \rho r^3}{r^2} = \frac{4}{3}\pi G \rho r$, which is proportional to r . Therefore g decreases linearly with decreasing r inside, rather than as $\frac{1}{r^2}$.
- (c) $T = 12 \text{ h} = 43200 \text{ s}$

$$R^3 = \frac{GMT^2}{4\pi^2} = \frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 5.0 \times 10^{24} \text{ kg} \times (43200 \text{ s})^2}{4 \times (3.14)^2} = 1.577 \times 10^{22} \text{ m}^3$$

$$R = (1.577 \times 10^{22})^{1/3} = 2.51 \times 10^7 \text{ m} = 25100 \text{ km}$$

Question 60

- (a) The escape velocity is derived from energy conservation: $\frac{1}{2}mv_E^2 = \frac{GMm}{r}$. The mass m of the escaping object appears on both sides and cancels, giving $v_E = \sqrt{\frac{2GM}{r}}$. Because m cancels completely, the escape velocity depends only on M (the planet's mass) and r (the planet's radius). A heavier object needs more energy to escape, but it also has proportionally more kinetic energy at the same speed. These effects balance exactly.
- (b) At the bottom of the dip, the car follows a circular arc whose centre of curvature is above the road. The centripetal acceleration is directed upward. The road must push the car upward with a force greater than the weight to provide both the support against gravity and the centripetal acceleration: $R = mg + \frac{mv^2}{r}$. Because R exceeds mg , the driver's body is pushed into the seat with greater force than normal, creating the sensation of feeling heavier.

- (c) Speed at the bottom of the ramp (energy conservation, smooth ramp):

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ m/s}^2 \times 3.0 \text{ m}} = 7.67 \text{ m/s} = \text{initial velocity on horizontal floor}$$

On the rough floor, friction decelerates the body:

$$u^2 = 2\mu_k g s \text{ (from } v^2 = u^2 - 2\mu_k g s; \text{ with } v = 0)$$

$$s = \frac{u^2}{2\mu_k g} = \frac{58.8 \text{ m}^2/\text{s}^2}{2 \times 0.50 \times 9.8 \text{ m/s}^2} = 6.0 \text{ m}$$

Question 61

- (a) At the top of the circle, the water requires centripetal force directed downward (toward the centre of the circle, which is below the bucket). Gravity provides part or all of this centripetal force. If the speed is high enough that $\frac{mv^2}{r} \geq mg$, gravity alone (or gravity plus the bucket pushing down) is sufficient. The water presses against the bottom of the bucket and stays inside. If the speed is too low ($\frac{mv^2}{r} < mg$), gravity exceeds the centripetal force needed, and the water falls out of the bucket.
- (b) Each spin increases the tangential speed of the hammer. The hammer moves in a circle, and with each revolution the thrower's muscles add kinetic energy to the system. After several spins, the hammer has a much higher tangential speed than after just one spin. Since the range of the hammer (once released as a projectile) is proportional to the square of the release speed ($R = \frac{u^2 \sin 2\theta}{g}$), even a modest increase in speed produces a significant increase in distance.
- (c) Height after each bounce: $h_n = e^{2n} h_0$

After 3rd bounce: $h_3 = (0.80)^6 \times 5 \text{ m} = 0.2621 \times 5 \text{ m} = 1.31 \text{ m}$

Total distance before 4th bounce:

Drop: 5m

After 1st bounce: up $e^2 \times 5 = 0.64 \times 5 = 3.2 \text{ m}$, down 3.2m

After 2nd bounce: up $e^4 \times 5 = 0.4096 \times 5 = 2.048 \text{ m}$, down 2.048m

After 3rd bounce: up 1.31m, down 1.31m

Total = $5 + 2(3.2 + 2.048 + 1.31) = 18.1 \text{ m}$

Question 62

- (a) Momentum is $p = mv$. A lorry with large m and small v can have the same product mv as a car with small m and large v . In a collision, however, the damage depends on the kinetic energy ($\frac{1}{2}mv^2$), not just momentum. For the same momentum, the faster-moving car has more kinetic energy (because KE increases with v^2 or $K.E \propto \frac{1}{\text{mass}}$ for constant momentum). Therefore, the car is more dangerous in terms of the energy released during impact, even though both have the same momentum.
- (b) Surface gravity is $g = \frac{GM}{r^2}$. If the radius halves while mass stays the same, the new g becomes $\frac{GM}{(r/2)^2} = \frac{4GM}{r^2} = 4g$. The person's weight would become $4 \times 700 \text{ N} = 2800 \text{ N}$, four times heavier. The person's mass remains unchanged, but the stronger surface gravity quadruples the weight.
- (c) Ball's horizontal velocity: $u_x = 16 \text{ m/s} \times \cos 30^\circ = 13.86 \text{ m/s}$

Ball's vertical velocity: $u_y = 16 \text{ m/s} \times \sin 30^\circ = 8 \text{ m/s}$ (upward)

Time for ball to reach ground (falling 25m below launch):

$$-25 = 8t - \frac{1}{2} \times 9.8t^2$$

$$4.9t^2 - 8t - 25 = 0$$

$$t = 3.22 \text{ s}$$

Horizontal distance from building base: $x_{\text{ball}} = 13.86 \times 3.22 = 44.6 \text{ m}$

Person is 40m from the base, so must cover $44.6 - 40 = 4.6 \text{ m}$ (away from building) in 3.22s:

$$v_{\text{person}} = \frac{4.6 \text{ m}}{3.22 \text{ s}} = 1.43 \text{ m/s}$$

Question 63

- (a) Equilibrium requires zero net force. A single force has a definite magnitude and direction and therefore produces a non-zero net force. There is no way to make a single force sum to zero by itself. At least two forces are needed (equal and opposite) to achieve equilibrium.
- (b) The parabolic path assumes that the only force acting is constant gravity directed vertically downward, and that the Earth's surface is flat. In reality, air resistance acts on the projectile, opposing its motion. This reduces the horizontal velocity (which the parabolic model assumes is constant) and alters the vertical motion, making the trajectory asymmetric: the descending branch is steeper than the ascending branch.
- (c) By conservation of momentum:

$$3\text{kg} \times 4\text{m/s} = 3\text{kg} \times 1.6\text{m/s} + 2 \times v_2$$

$$12\text{kgm/s} = 4.8\text{kgm/s} + 2v_2$$

$$v_2 = 3.6\text{m/s}$$

Coefficient of restitution:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{v_2 - v_1}{u_1 - u_2} = \frac{3.6\text{m/s} - 1.6\text{m/s}}{4\text{m/s} - 0} = 0.50$$

Question 64

- (a) At the moment of release, the package has the same horizontal velocity as the helicopter. After release, no horizontal force acts on it (ignoring air resistance), so it continues forward at the helicopter's speed. Since both the helicopter and the package move forward at the same horizontal velocity, the package remains directly below the helicopter throughout the fall and lands at a point directly below the helicopter's position at impact time, which is ahead of the release point as seen from the ground.
- (b) On a curved track, the train requires a centripetal force directed toward the centre of the curve. By tilting (banking) the track inward, the normal reaction from the track has a horizontal component directed toward the centre. This component provides the centripetal force without relying on sideways friction between the wheels and rails. Without banking, the rails would need to push the wheels sideways, causing rapid wear on both the wheels and the track. Banking also improves passenger comfort, because the resultant of the normal reaction and gravity points more nearly through the passengers' seats rather than sideways.
- (c) The ball must satisfy $y \geq 15\text{m}$ at $x = 30\text{m}$:

By using trajectory equation:

$$15 = 30\tan\theta - \frac{9.8 \times 900}{2 \times 625\cos^2\theta}$$

$$15 = 30\tan\theta - \frac{7.056}{\cos^2\theta}$$

Let $p = \tan\theta$, so $\frac{1}{\cos^2\theta} = 1 + p^2$:

$$15 = 30p - 7.056(1 + p^2)$$

$$7.056p^2 - 30p + 22.056 = 0$$

$$p = 3.307 \text{ or } p = 0.946$$

$$\theta = 73.2^\circ \text{ or } \theta = 43.4^\circ$$

The minimum angle to clear the wall is 43.4° (the lower of the two solutions).

Question 65

- (a) A polar orbit passes over the North and South Poles. As the Earth rotates beneath the satellite, different strips of longitude pass under the orbital plane on each revolution. After enough orbits (12 to 14 per day for a low-orbit satellite), the entire surface has passed beneath the satellite.

An equatorial orbit stays above the equator at all times. No matter how many orbits the satellite completes, it never passes over regions at high latitudes. Therefore, it can only photograph a narrow band around the equator.

- (b) The rotating station creates a centripetal acceleration directed inward (toward the rotation axis). For an astronaut standing on the outer wall, the wall pushes inward on the astronaut's feet, providing the centripetal force. The astronaut's body resists this inward push (due to inertia), pressing outward against the wall. This outward press on the feet mimics the sensation of standing on a floor under gravity. The outer wall must be used because that is where the reaction force pushes inward. On the inner wall, the geometry is reversed and no outward-pressing sensation exists.
- (c) The 3kg mass pulls one way, the 2kg mass pulls the opposite way. The 5kg mass sits between them on the table.

Net driving force: $(3 - 2)\text{kg} \times 9.8\text{m/s}^2 = 9.8\text{N}$

Total mass: $(5 + 3 + 2)\text{kg} = 10\text{kg}$

$$a = \frac{9.8\text{N}}{10\text{kg}} = 0.98\text{m/s}^2$$

For the 3kg mass (accelerating downward): $3g - T_1 = 3a$

$$T_1 = 3(9.8 - 0.98) = 26.5\text{N}$$

For the 2kg mass (accelerating upward): $T_2 - 2g = 2a$

$$T_2 = 2(9.8 + 0.98) = 21.6\text{N}$$

Question 66

(a) In level flight, the lift force on the wings acts vertically upward, balancing the weight. When the pilot banks the wings, the lift tilts inward. The vertical component of the tilted lift still supports the weight, while the horizontal component provides a centripetal force directed toward the centre of the turn. This centripetal force changes the aircraft's direction, allowing it to curve toward the runway.

(b) The ball arrives with large momentum. To stop it, the fielder must reduce its momentum to zero, which requires an impulse ($J = \Delta p$). By the impulse-momentum theorem, $F\Delta t = \Delta p$. Drawing the hands backward increases the time Δt over which the ball decelerates. For the same momentum change, a longer time means a smaller average force on the hands, reducing the chance of injury.

(c) By conservation of momentum:

$$m_1 v_1 = (m_1 + m_2) V$$

$$V = \frac{0.012\text{kg} \times 150\text{ms}^{-1}}{0.012\text{kg} + 1.188\text{kg}} = \frac{1.8\text{kgms}^{-1}}{1.2\text{kg}} = 1.5\text{ms}^{-1}$$

$$\text{KE before: } KE_i = \frac{1}{2} \times 0.012\text{kg} \times (150\text{ms}^{-1})^2 = 135\text{J}$$

$$\text{KE after: } KE_f = \frac{1}{2} \times 1.2\text{kg} \times (1.5\text{ms}^{-1})^2 = 1.35\text{J}$$

$$\text{KE lost: } \Delta KE = 135\text{J} - 1.35\text{J} = 133.65\text{J}$$

Question 67

(a) The horizontal range of a projectile is $R = u^2 \sin 2\theta / g$. The range depends on the square of the initial velocity. If the initial velocity is doubled ($u \rightarrow 2u$), the range becomes $R' = (2u)^2 \sin 2\theta / g = 4u^2 \sin 2\theta / g = 4R$. The range quadruples.

(b) In an elliptical orbit, the satellite moves under gravity alone. As the satellite moves closer to the Earth, it falls deeper into the gravitational potential well, losing gravitational PE. By conservation of energy, this lost PE is converted to KE, so the satellite speeds up. As it moves away, KE converts back to PE and the satellite slows down.

(c) At the maximum height (top of barrier), vertical velocity $v_y = 0$:

$$v_y^2 = u_y^2 - 2gh$$

$$u_y = \sqrt{2gh} = \sqrt{2 \times 9.8\text{ms}^{-2} \times 2.2\text{m}} = 6.57\text{ms}^{-1}$$

$$\text{Horizontal velocity: } u_x = \sqrt{u^2 - u_y^2} = \sqrt{(8.0\text{ms}^{-1})^2 - (6.57\text{ms}^{-1})^2} = 4.57\text{ms}^{-1}$$

$$\text{Time to reach max height: } t_{\text{up}} = \frac{u_y}{g} = \frac{6.57\text{ms}^{-1}}{9.8\text{ms}^{-2}} = 0.670\text{s}$$

$$\text{Total time of flight: } T = 2t_{\text{up}} = 2 \times 0.670\text{s} = 1.34\text{s}$$

Question 68

(a) Gravitational PE is defined as the work done against gravity to bring a mass from infinity (where PE is zero by convention) to the point in question. Since gravity is always attractive, a mass moving from infinity toward the attracting body moves in the direction of the gravitational force, meaning gravity does positive work. The mass therefore loses PE. Starting from zero at infinity, the PE becomes negative as the mass approaches. It can never be zero at any finite distance because gravity, though weakening with distance, never vanishes entirely.

(b) Conservation of momentum is not violated. The person pushes backward on the Earth with their foot, and by Newton's third law, the Earth pushes the person forward. The person gains forward momentum $+p$ while the Earth gains an equal backward momentum $-p$. The total momentum of the person-Earth system remains zero. The Earth's velocity change is imperceptibly small because its mass is extremely large.

(c) At the knot, three forces act: horizontal tension T_H (from the block string), tension T_1 at 30° above horizontal (to ceiling), and weight Q vertically downward.

Horizontal equilibrium at the knot:

$$T_H = T_1 \cos 30^\circ$$

Vertical equilibrium at the knot:

$$T_1 \sin 30^\circ = Q$$

For the block on the table, the horizontal string pulls it with force T_H , resisted by friction:

$$T_H \leq f_{\max} = 12\text{N}$$

At maximum: $T_H = 12\text{N}$

$$T_1 = \frac{T_H}{\cos 30^\circ} = \frac{12\text{N}}{\cos 30^\circ} = 13.86\text{N}$$

$$Q_{\max} = T_1 \sin 30^\circ = 13.86\text{N} \times 0.5 = 6.93\text{N}$$

Question 69

(a) The rocket engines expel exhaust gases backward at high speed. By Newton's third law, the gases exert an equal and opposite force (thrust) on the rocket, pushing it forward. No external medium is required. The thrust depends only on the rate at which mass is ejected and the exhaust velocity: $F = v_e \times dm/dt$. In a vacuum, there is no air resistance, so the thrust is entirely available for acceleration.

(b) The Earth rotates about its axis once every 24 hours. Every part of the Earth undergoes circular motion and requires a centripetal acceleration directed toward the axis. At the equator, this centripetal acceleration is greatest (largest radius of rotation). The material at the equator tends to move outward (away from the axis), bulging the equator. At the poles, the radius of rotation is zero and there is no outward tendency. Over geological time, the Earth's material has deformed to accommodate this, producing the oblate spheroid shape.

(c) Here $\theta = 65^\circ$ (from horizontal) and $\beta = 28^\circ$ (uphill, positive).

Range along the incline:

$$L = \frac{u^2}{g \cos^2 \beta} (\sin(2\theta - \beta) - \sin \beta)$$

$$L = \frac{(33\text{ms}^{-1})^2}{9.8\text{ms}^{-2} \times \cos^2 28^\circ} (\sin(2 \times 65^\circ - 28^\circ) - \sin 28^\circ)$$

$$L = 72.5\text{m}$$

Time of flight:

$$T = \frac{2u \sin(\theta - \beta)}{g \cos \beta} = \frac{2 \times 33\text{ms}^{-1} \times \sin(65^\circ - 28^\circ)}{9.8\text{ms}^{-2} \times \cos 28^\circ}$$

$$T = 4.59\text{s}$$

Question 70

(a) In a perfectly inelastic collision, momentum is conserved: $m_1 v_1 + m_2 v_2 = (m_1 + m_2)V$. However, kinetic energy is not conserved because the collision involves permanent deformation, heat generation, and sound. The "lost" kinetic energy is converted into internal energy (deformation of the bodies, heating of materials, sound waves). The amount lost can be calculated from the difference between the initial and final kinetic energies.

(b) The combined wreckage moves in the direction of the total momentum vector of the system before impact. If vehicle 1 has momentum p_1 (eastward, say) and vehicle 2 has momentum p_2 (northward), the wreckage moves in the direction of $\vec{p}_1 + \vec{p}_2$. The angle of the skid marks from the east direction is $\tan^{-1} \left(\frac{p_2}{p_1} \right)$. Thus the angle depends on the ratio of the momenta (mv for each).

(c) Speed: $v = 72\text{kmh}^{-1} = 20\text{ms}^{-1}$

Centripetal acceleration: $a_c = \frac{v^2}{r} = \frac{(20\text{ms}^{-1})^2}{100\text{m}} = 4\text{ms}^{-2}$ (directed toward centre)

Tangential deceleration: $a_t = 5\text{ms}^{-2}$ (directed backward along path)

Net acceleration:

$$a = \sqrt{a_c^2 + a_t^2} = \sqrt{(4\text{ms}^{-2})^2 + (5\text{ms}^{-2})^2} = 6.4\text{ms}^{-2}$$

Angle from the velocity direction (toward centre):

$$\theta = \tan^{-1} \left(\frac{a_c}{a_t} \right) = \tan^{-1} \left(\frac{4}{5} \right) = 38.7^\circ$$

The net acceleration makes 38.7° with the backward tangential direction, toward the centre of the circular path.

Question 71

(a) The rocket exerts a backward force on the exhaust gases, accelerating them rearward. By Newton's third law, the gases exert an equal forward force on the rocket. By conservation of momentum, the backward momentum of the exhaust equals the forward momentum gained by the rocket. No external medium is needed because the interaction is between the rocket and its own exhaust, not between the rocket and anything external.

(b) The total energy of a satellite in orbit is $E = KE + PE$, where PE is negative. When the engines fire, they add energy to the satellite. As the satellite moves to a higher orbit, it moves further from the Earth and its gravitational PE increases (becomes less negative). For a circular orbit, $KE = -E$ and $PE = 2E$, so the PE increase is twice the total energy increase while the KE actually decreases. The extra energy comes from the fuel burned by the engines.

(c) By conservation of momentum (cannon + ball initially at rest):

$$0 = m_{\text{ball}}v_{\text{ball}} + m_{\text{cannon}}v_{\text{cannon}}$$

The muzzle speed 55ms^{-1} is relative to the cannon. Let the cannon's recoil velocity be v_c (backward) and the ball's velocity relative to Earth be v_b (forward):

$$v_b - (-v_c) = 55\text{ms}^{-1}$$

$$\text{so } v_b + v_c = 55\text{ms}^{-1} \dots (1)$$

$$72v_b = 1300v_c \dots (2)$$

From (2): $v_b = 18.06v_c$. Substituting into (1):

$$18.06v_c + v_c = 55$$

$$\rightarrow v_c = \frac{55}{19.06} = 2.89\text{ms}^{-1}$$

$$v_b = 52.1\text{ms}^{-1}$$

Question 72

(a) As the insect slides down the curved surface, it gains speed. At any point, Newton's second law in the radial direction gives: $mg\cos\theta - R = \frac{mv^2}{r}$, where θ is measured from the vertical and R is the normal reaction. As v increases and $\cos\theta$ decreases, R must decrease. The insect loses contact when $R = 0$, which occurs when $mg\cos\theta = \frac{mv^2}{r}$. Using energy conservation to relate v to θ , this gives $\cos\theta = \frac{2}{3}$, or $\theta \approx 48^\circ$.

(b) At the neutral point, the gravitational pulls from the two bodies are equal and opposite. If d is the distance between centres and x is the distance from the larger body to the neutral point: $\frac{GM}{x^2} = \frac{Gm}{(d-x)^2}$, giving $\frac{x}{d-x} = \sqrt{\frac{M}{m}}$. Since $M > m$, $x > d - x$, meaning the neutral point is closer to the less massive body. The lighter body needs less distance for its weaker field to match the stronger field of the heavier body.

(c) Let x be the distance from the Earth's centre:

$$\frac{M_E}{x^2} = \frac{M_M}{(d-x)^2}$$

$$\frac{x}{d-x} = \sqrt{\frac{M_E}{M_M}} = \sqrt{81} = 9$$

$$x = 9(d-x) = 9d - 9x$$

$$\rightarrow 10x = 9d \rightarrow x = 0.9d = 0.9 \times 4 \times 10^5 \text{km} = 3.6 \times 10^5 \text{km}$$

Question 73

(a) Water leaving the nozzle is a projectile: gravity immediately pulls it downward. If aimed directly at the window, the water would follow a parabolic path that curves below the target and miss. By aiming above the window, the fireman gives the water an upward component of velocity that compensates for the downward pull of gravity during the flight. The water rises initially, then curves downward through the window at the correct height.

(b) The bullet is a projectile. From the moment it leaves the barrel, gravity pulls it downward. Over 200m, the bullet drops by $h = \frac{1}{2}gt^2$. Since the bullet is moving with very high velocity (say 1000ms^{-1}), the flight time is over 200m is too small ($t = \frac{200}{1000} = 0.2\text{s}$), giving very small height drop ($h = \frac{1}{2} \times 9.8 \times 0.04 = 0.196\text{m} \approx 20\text{cm}$). This height drop (20cm) over 200m is too small to notice with the naked eye, which is why the path appears straight. But it is genuinely curved.

(c) The water must reach a point 40m horizontally ($38\text{m} + 2\text{m}$) and 20m vertically. Using the trajectory equation:

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$20 = 40 \tan 60^\circ - \frac{9.8 \times (40)^2}{2u^2 \cos^2 60^\circ}$$

$$u^2 = 636.4 \text{ m}^2 \text{ s}^{-2}$$

$$u = 25.2 \text{ m s}^{-1}$$

Question 74

(a) A projectile's vertical position is $y = u_y t - \frac{1}{2}gt^2$. Setting $y = 7\text{m}$ gives a quadratic equation in t with two solutions: one on the way up and one on the way down. These two times correspond to two horizontal positions ($x_1 = u_x t_1$ and $x_2 = u_x t_2$). The walls must be placed at precisely these two positions. For any height below the maximum, there are always exactly two such points (the trajectory is symmetric about its peak for level-ground launch).

(b) Each board removes the same fraction ($1/20$) of the current speed, not the same absolute amount. After the first board, the speed is $19v/20$. After the second, $(19/20)^2 v$. After n boards, the speed is $(19/20)^n v$. The kinetic energy is proportional to v^2 , so it decreases as $(19/20)^{2n}$. Each successive board encounters a slower bullet with less KE. However, each board absorbs the same fraction of the remaining speed, meaning the absolute KE removed per board decreases. The bullet is stopped when its KE is fully absorbed. Because the percentage loss compounds geometrically, far fewer than 20 boards are needed.

(c) $u_x = 100 \cos 60^\circ = 50 \text{ m s}^{-1}$, $u_y = 100 \sin 60^\circ = 86.6 \text{ m s}^{-1}$

At height $y = 7\text{m}$: $7 = 86.6t - 4.9t^2$

$$4.9t^2 - 86.6t + 7 = 0$$

$t_1 = 0.0816\text{s}$ and $t_2 = 17.59\text{s}$

Distance between walls:

$$d = u_x(t_2 - t_1) = 50 \text{ m s}^{-1} \times (17.59 - 0.0816)\text{s} = 875\text{m}$$

Alternatively, you can solve it by using trajectory equation to get x_1 and x_2 and then calculate $x_2 - x_1$ to get the final answer.

Question 75

(a) In static equilibrium, every part of the system is at rest. Since $p = mv$ and $v = 0$ for every component, the total momentum is zero and remains zero. If a connection suddenly breaks, the internal forces that maintained equilibrium vanish. Each freed part now has a net unbalanced external force (typically its weight) acting on it. By Newton's second law, each part accelerates and gains momentum. However, the total momentum of the entire system remains zero because no new external force acts on the system as a whole.

(b) Escape velocity is derived from energy conservation: $\frac{1}{2}mv_{\text{esc}}^2 = GMm/R$. The mass m of the escaping object cancels on both sides, giving $v_{\text{esc}} = \sqrt{2GM/R}$. This depends only on the planet's mass and radius, not on the direction. The direction of launch does not matter because gravity is a conservative force: the work done against gravity depends only on the initial and final positions (surface to infinity), not on the path taken.

(c)(i) Taking the initial direction of the ball as positive:

$$J = \Delta p = m(v_f - v_i) = 0.060\text{kg} \times (-45 \text{ m s}^{-1} - 35 \text{ m s}^{-1}) = -4.8 \text{ kg m s}^{-1}$$

The impulse has magnitude 4.8 kg m s^{-1} , directed opposite to the initial motion (toward the racket side).

(ii) Average force:

$$F_{\text{avg}} = \frac{J}{\Delta t} = \frac{4.8 \text{ kg m s}^{-1}}{0.006\text{s}} = 800\text{N}$$

Question 76

(a) From Newton's second law in its general form, thrust $= v_e \times \frac{dm}{dt}$, where v_e is the exhaust velocity and $\frac{dm}{dt}$ is the rate of mass ejection. Higher exhaust speed or faster mass ejection produces greater thrust. As fuel is consumed, the total mass M of the rocket decreases. Since $a = \frac{F_{\text{net}}}{M}$ and the thrust is constant, a decreasing M produces an increasing acceleration.

(b) At the equilibrium point, $\frac{GM_E}{x^2} = \frac{GM_S}{(d-x)^2}$, giving $\frac{x}{(d-x)} = \sqrt{\frac{M_E}{M_S}}$. Since $M_S \gg M_E$ (the Sun is about 333,000 times more massive), $\sqrt{\frac{M_E}{M_S}} \ll 1$, so $x \ll d - x$. The equilibrium point lies very close to the Earth because the Earth's much weaker field can only match the Sun's field at a point where the Earth is nearby and the Sun is very far away.

(c)(i) For liftoff, thrust $\geq$ total weight:

$$v_e \times \frac{dm}{dt} \geq (M + m_{\text{fuel}})g$$

$$\frac{dm}{dt} \geq \frac{(20 + 180)\text{kg} \times 9.8\text{ms}^{-2}}{1600\text{ms}^{-1}} = 1.225\text{kgs}^{-1}$$

The minimum rate of fuel consumption is 1.225kgs^{-1} .

(c)(ii) Burn time: $t = \frac{180\text{kg}}{2.0\text{kgs}^{-1}} = 90\text{s}$

Using the rocket equation:

$$v = v_e \ln \frac{M_0}{M_f} - gt = 1600\text{ms}^{-1} \times \ln \frac{200\text{kg}}{20\text{kg}} - 9.8\text{ms}^{-2} \times 90\text{s} = 2803\text{ms}^{-1} \approx 2.8\text{kms}^{-1}$$

Question 77

(a) Static friction is a self-adjusting force: it takes whatever value is needed to prevent motion, up to its maximum $f_{\max} = \mu_s R$. At a small angle, the component of gravity along the plane ($mg \sin \theta$) is small, and static friction matches it exactly, keeping the block stationary. As the angle increases, $mg \sin \theta$ grows and friction increases to match. The block remains stationary as long as $mg \sin \theta \leq \mu_s mg \cos \theta$. At the critical angle ($\tan \theta_c = \mu_s$), friction reaches its maximum and can no longer match the gravitational component. Beyond this angle, the block slides.

(b) The horizontal range is $R = u^2 \sin 2\theta / g$. The range depends on $\sin 2\theta$, not on θ alone. The sine function has the property that $\sin 2\theta = \sin(180^\circ - 2\theta)$, which means $\sin 2\theta_1 = \sin 2\theta_2$ whenever $\theta_1 + \theta_2 = 90^\circ$. For example, $\sin 60^\circ = \sin 120^\circ$, so 30° and 60° give the same range. Physically, the steep angle (60°) gives a high trajectory with long flight time but small horizontal speed, while the shallow angle (30°) gives a low trajectory with short flight time but large horizontal speed. The two effects compensate exactly for complementary angles.

(c)(i) At projection, the ball has: $v_x = 8.5\text{ms}^{-1}$ (horizontal) and $v_y = +14.7\text{ms}^{-1}$ (upward, inherited from helicopter).

Vertical displacement to ground: -196m (downward).

$$-196 = 14.7t - \frac{1}{2}(9.8)t^2$$

$$4.9t^2 - 14.7t - 196 = 0$$

$$t = 8\text{s}$$

(c)(ii) Horizontal distance: $x = v_x \times t = 8.5\text{ms}^{-1} \times 8\text{s} = 68\text{m}$

(c)(iii) At ground: $v_x = 8.5\text{ms}^{-1}$, $v_y = 14.7\text{ms}^{-1} - 9.8\text{ms}^{-2} \times 8\text{s} = -63.7\text{ms}^{-1}$

$$v = \sqrt{v_x^2 + v_y^2} = 64.3\text{ms}^{-1}$$

Question 78

(a) After a perfectly inelastic collision, the wreckage moves in the direction of the total momentum. If two vehicles of equal mass approach at right angles, one with speed v_1 and the other with speed v_2 , the total momentum has components mv_1 and mv_2 along the two original directions. The skid marks point along the resultant momentum, at an angle $\tan \theta = \frac{mv_2}{mv_1} = \frac{v_2}{v_1}$ from the first vehicle's direction. Since the masses are equal, they cancel, and hence the angle directly reveals the speed ratio.

(b) The elevator is moving downward and decelerating (slowing down), which means its acceleration is directed upward. By Newton's second law applied to a passenger: $R - mg = ma$ (upward acceleration), so $R = m(g + a)$. The scale reading R exceeds the true weight mg . The passengers feel heavier during this phase than when the elevator moves at constant speed (where $R = mg$).

(c)(i) The skid marks are at 55° north of east, meaning the angle is measured from the eastward direction. The eastward momentum ($m \times 13$) is the adjacent component and the northward momentum ($m \times v$) is the opposite component:

$$\tan 55^\circ = \frac{mv}{m \times 13} = \frac{v}{13}$$

$$v = 13\text{ms}^{-1} \times \tan 55^\circ = 18.6\text{ms}^{-1}$$

(c)(ii) The speed limit is 15.6ms^{-1} . Since $v = 18.6\text{ms}^{-1} > 15.6\text{ms}^{-1}$, the northward-moving driver was **exceeding the speed limit** by 3ms^{-1} .

Question 79

(a) The variation of g with altitude is $g' = g(R_E / (R_E + h))^2$. For typical projectile heights h are very small compared to Earth's radius, making the change in g negligible. So the constant- g approximation introduces an error far smaller than other neglected effects (air resistance, wind), and hence it is entirely justified for near-surface problems.

(b) Using $g' = g \left(\frac{R_E}{R_E + h} \right)^2$: $\frac{R_E + h}{R_E} = \sqrt{\frac{g}{g'}}$, giving $h = R_E \left(\sqrt{\frac{g}{g'}} - 1 \right)$ from which the height of the mountain can be estimated. For example, with $g = 9.81$ and $g' = 9.70$: $h = 6400\text{km} \times \left(\sqrt{\frac{9.81}{9.70}} - 1 \right) = 36.3\text{km}$. One reason the actual value might differ: the density of rock beneath the mountain differs from the average Earth density, so the local gravitational field is affected by the mass distribution of nearby geological structures.

(c)(i) The ball starts at 1000m and must reach 400m: vertical drop = 600m.

$$\begin{aligned} -600 &= u_y t - \frac{1}{2} g t^2 \\ u_y &= 80 \sin 60^\circ = 69.3 \text{ms}^{-1} \\ 4.9 t^2 - 69.3 t - 600 &= 0 \\ t &= 20.2 \text{s} \end{aligned}$$

(c)(ii) $u_x = 80 \cos 60^\circ = 40 \text{ms}^{-1}$

$$x = u_x \times t = 40 \text{ms}^{-1} \times 20.2 \text{s} = 808 \text{m}$$

Question 80

(a) Gravity acts on the satellite directed toward the Earth's centre, which is perpendicular to the satellite's velocity (tangent to the orbit). Work done is $W = F \cdot d \cos \theta$. Since $\theta = 90^\circ$, $\cos 90^\circ = 0$, and gravity does zero work on the satellite. By the work-energy theorem, zero work means zero change in kinetic energy. Therefore, the speed remains constant even though a force acts continuously.

(b) Speed is the magnitude of velocity (a scalar), while average velocity is displacement divided by time (a vector). Over a semicircle of radius r , the distance travelled is πr but the displacement is $2r$ (a straight line from start to end). Average speed = $\frac{\pi r}{t}$ while average velocity = $\frac{2r}{t}$. Since $\pi > 2$, average speed exceeds the magnitude of average velocity. In circular motion, the speed may be constant while the velocity changes direction continuously.

(c)(i) Time for semicircle: $t = \frac{\pi r}{v} = \pi \times \frac{6 \text{m}}{12 \text{ms}^{-1}} = \frac{\pi}{2} = 1.571 \text{s}$

Displacement: $2r = 12 \text{m}$ (diameter, from A to B)

Average velocity: $v_{\text{avg}} = \frac{12 \text{m}}{1.571 \text{s}} = 7.64 \text{ms}^{-1}$ (directed from A to B)

(c)(ii) For circular orbit: $\frac{GM_E M_s}{r^2} = \frac{M_s v^2}{r}$, so $v = \sqrt{\frac{GM_E}{r}}$.

Angular momentum: $L = M_s v r = M_s r \sqrt{\frac{GM_E}{r}} = M_s \sqrt{GM_E r}$

Question 81

(a) A vertical launch gives the satellite only radial velocity (directed away from Earth). For a circular orbit, the satellite needs a velocity entirely tangential to the orbit (perpendicular to the radius). A vertically launched object will rise, slow down, stop, and fall back along the same radial line. To enter a circular orbit, the satellite must be turned so that its velocity is horizontal at the orbital altitude, which requires a separate horizontal impulse.

(b) Orbital speed is $v_{\text{orb}} = \sqrt{\frac{GM}{r}}$ and escape speed is $v_{\text{esc}} = \sqrt{\frac{2GM}{r}} = \sqrt{2} \times v_{\text{orb}}$. The additional speed needed is $v_{\text{esc}} - v_{\text{orb}} = (\sqrt{2} - 1)v_{\text{orb}} = 0.414 v_{\text{orb}}$. This is only 41.4% of the orbital speed, which is less than the orbital speed itself. The satellite already has most of the energy needed to escape.

(c)(i) Orbital radius: $r = R_E + h = 6.4 \times 10^6 \text{m} + 5 \times 10^5 \text{m} = 6.9 \times 10^6 \text{m}$

$$v_{\text{orb}} = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.9 \times 10^6}} = 7597 \text{ms}^{-1} \approx 7.6 \text{kms}^{-1}$$

(c)(ii) The satellite has vertical momentum $p_v = 50 \text{kg} \times 2000 \text{ms}^{-1} = 1 \times 10^5 \text{kgms}^{-1}$ and needs horizontal momentum $p_h = 50 \text{kg} \times 7597 \text{ms}^{-1} = 3.799 \times 10^5 \text{kgms}^{-1}$ with zero vertical momentum.

The impulse must cancel the vertical momentum and provide the horizontal momentum:

$$J = \sqrt{p_v^2 + p_h^2} = \sqrt{(10^5 \text{kgms}^{-1})^2 + (3.799 \times 10^5 \text{kgms}^{-1})^2} = 3.93 \times 10^5 \text{kgms}^{-1}$$

Question 82

(a) In the reference frame of one projectile, the other approaches in a straight line (because both experience the same gravitational acceleration g , which cancels in the relative motion). The relative velocity is constant (a straight line), so the

collision condition depends only on whether the initial relative velocity points from one launch point to the other. This condition involves only the initial velocities and positions, not g . Hence, the collision condition is independent of g .

(b) The 10m diver hits the water with greater speed ($v = \sqrt{2gh}$) than the 3m diver. Greater speed means greater momentum at entry. The water must reduce this momentum to zero. By the impulse-momentum theorem ($F_{\text{avg}} \times \Delta t = \Delta p$), a larger momentum change requires a larger impulse. If the stopping time Δt is similar in both cases (comparable depth of submersion), the average force on the 10m diver must be larger.

(c) Let the velocity of A be u_A at 30° from height 2m. At collision time t :

$$\text{Position of A: } x_A = u_A \cos 30^\circ \times t, y_A = 2 + u_A \sin 30^\circ \times t - \frac{1}{2}gt^2$$

$$\text{Position of B (from ground, directly below A): } x_B = 30 \cos 60^\circ \times t = 15t, y_B = 30 \sin 60^\circ \times t - \frac{1}{2}gt^2 = 25.98t - 4.9t^2$$

For collision: $y_A = y_B$ and $x_A = x_B$ (but A moves horizontally, B moves at angle — they must meet at same point).

Since both experience the same g , the $\frac{1}{2}gt^2$ terms cancel in the vertical equation:

$$\begin{aligned} 2 + u_A \sin 30^\circ \times t &= 30 \sin 60^\circ \times t \\ 2 &= (30 \sin 60^\circ - u_A \sin 30^\circ)t = (25.98 - 0.5u_A)t \end{aligned}$$

$$\text{From horizontal: } u_A \cos 30^\circ \times t = 30 \cos 60^\circ \times t \rightarrow u_A = \frac{30 \cos 60^\circ}{\cos 30^\circ} = 17.32 \text{ms}^{-1}$$

$$\text{Substituting back: } 2 = (25.98 - 0.5 \times 17.32)t = (25.98 - 8.66)t = 17.32t$$

$$t = \frac{2}{17.32} = 0.115 \text{s}$$

Question 83

(a) The bullet is a projectile: gravity pulls it downward throughout its flight. If aimed directly at the target, the bullet follows a parabolic path that curves below the line of aim. Over the flight time $t = \frac{x}{v}$, the bullet drops $h = \frac{1}{2}gt^2$. To compensate, the marksman aims slightly above the target so the bullet's parabolic path intersects the target at the correct height.

(b) Yes, the bullet hits the monkey. At the instant of firing, the bullet is aimed directly at the monkey. Both the bullet and the monkey begin falling under gravity at the same instant. In time t , both fall the same vertical distance $\frac{1}{2}gt^2$ below their initial positions. The bullet's path would have pointed directly at the monkey's original position if gravity were absent. However, since both the bullet and the monkey fall by the same amount under gravity, their relative positions remain aligned. Therefore, the bullet and the monkey meet at the same point below the original line of aim.

$$(c)(i) \text{ Time of flight: } t = \frac{x}{v} = \frac{200\text{m}}{1000\text{ms}^{-1}} = 0.2\text{s}$$

$$\text{Drop: } h = \frac{1}{2}gt^2 = \frac{1}{2} \times 9.8\text{ms}^{-2} \times (0.2\text{s})^2 = 0.196\text{m} = 19.6\text{cm}$$

(c)(ii) The marksman must aim 19.6cm above the target. At this small angle, the upward component compensates for the gravitational drop during the flight time. The bullet rises 19.6cm above the direct line while simultaneously dropping 19.6cm due to gravity, arriving at target height.

Question 84

(a) At the equator, a person undergoes circular motion with the Earth's rotation. The centripetal acceleration required is $a_c = \omega^2 R_E$, directed toward the Earth's axis. The normal reaction R (apparent weight) and the true weight mg must provide this centripetal acceleration: $mg - R = m\omega^2 R_E$, so $R = m(g - \omega^2 R_E) < mg$. At the North Pole, the person is on the rotation axis ($r = 0$), requires no centripetal acceleration, and $R = mg$.

(b) Escape speed is $v_{\text{esc}} = \sqrt{2}v_{\text{orb}}$. An orbiting spaceship at speed v_{orb} needs to reach $\sqrt{2}v_{\text{orb}} = 1.414v_{\text{orb}}$. The additional speed is $0.414v_{\text{orb}}$, which is 41.4% of the orbital speed. This is because the spaceship already possesses significant kinetic energy in orbit. Escape requires only the additional energy to bring the total mechanical energy from negative (bound orbit) to zero (just escaping).

$$(c)(i) \text{ For zero apparent weight: } mg = m\omega^2 R_E, \text{ so } \omega = \sqrt{\frac{g}{R_E}}$$

$$\omega = \sqrt{\frac{9.8\text{ms}^{-2}}{6.4 \times 10^6\text{m}}} = 1.238 \times 10^{-3} \text{rads}^{-1}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1.238 \times 10^{-3} \text{rads}^{-1}} = 5075\text{s} = 1.41 \text{hours}$$

$$(c)(ii) v_{\text{esc}} = \sqrt{2} \times v_{\text{orb}} = 1.414 \times 7.9\text{kms}^{-1} = 11.17\text{kms}^{-1}$$

$$\text{Additional speed: } \Delta v = (11.17 - 7.9)\text{kms}^{-1} = 3.27\text{kms}^{-1}$$

Question 85

(a) Gravitational PE is zero at infinity (by convention) and negative at all finite distances because work must be done against gravity to move from any point to infinity. The negative value represents a bound state: the body is trapped in the gravitational well. To escape, the body must be given enough kinetic energy to climb out of the well entirely, reaching zero total energy at infinity. The minimum KE needed equals the magnitude of the PE at the surface: $KE_{\min} = \frac{GMm}{R}$.

(b) At launch, the rocket has kinetic energy $KE_0 = \frac{1}{2}mv^2$. As the rocket rises, this energy is converted into two forms: gravitational potential energy (mgh) gained by climbing to height h , and thermal energy lost to atmospheric drag (E_{drag}). By conservation of energy:

$$\frac{1}{2}mv^2 = mgh_{\max} + E_{\text{drag}}$$

The rocket reaches its maximum height when all the initial KE has been used up. Rearranging:

$$h_{\max} = \frac{v^2}{2g} - \frac{E_{\text{drag}}}{mg}$$

The first term is the height the rocket would reach without drag. The second term is the height reduction caused by drag. The greater the fraction of KE lost to drag, the lower the maximum height.

(c)(i) Work to move to infinity:

$$W = \frac{GM_E m}{R_E} = \frac{6.67 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2} \times 6 \times 10^{24} \text{kg} \times 10 \text{kg}}{6.4 \times 10^6 \text{m}} = 6.25 \times 10^8 \text{J}$$

(c)(ii) By energy conservation: $\frac{1}{2}mv^2 = W$

$$v = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2 \times 6.25 \times 10^8 \text{J}}{10 \text{kg}}} = 11.2 \text{kms}^{-1}$$

(c)(iii) Initial KE: $\frac{1}{2}mv^2 = \frac{1}{2}m(2000 \text{ms}^{-1})^2 = 2 \times 10^6 \text{m J}$

After drag loss (20%): $KE_{\text{remaining}} = \frac{80}{100} \times 2 \times 10^6 \text{m J} = 1.6 \times 10^6 \text{m J}$

$$mgh = 1.6 \times 10^6 \text{m}$$

$$h = \frac{1.6 \times 10^6 \text{m}}{g_{\text{Mars}}} = \frac{1.6 \times 10^6 \text{m}}{3.7 \text{ms}^{-2}} = 4.32 \times 10^5 \text{m} = 432 \text{km}$$

Question 86

(a) By Newton's first law, an object in motion continues in a straight line unless acted upon by a force. During circular motion, the centripetal force continuously deflects the object inward toward the centre. At the instant the string breaks, the centripetal force vanishes. No inward force acts, so the object continues with whatever velocity it had at that instant. That velocity is tangent to the circle (perpendicular to the radius).

Note: There is no outward force and never was; the sensation of being "thrown outward" is simply the object continuing in a straight line while the circular constraint is removed.

(b) Braking: the friction force $f = \mu mg$ decelerates the truck. Using $v^2 = 2as$, the stopping distance is $s = \frac{v^2}{2\mu g}$.

Swerving: the same friction provides centripetal force $\mu mg = \frac{mv^2}{r}$, giving the minimum radius $r_{\min} = \frac{v^2}{\mu g}$.

Comparing: $s = \frac{v^2}{2\mu g} = \frac{r_{\min}}{2}$. The stopping distance is half the minimum turning radius. Since the wall is at distance r , braking requires $r > \frac{v^2}{2\mu g}$ while swerving requires $r > \frac{v^2}{\mu g}$.

Hence, **braking is the better option** as it needs only half the distance.

(c) After the string breaks, the stone is a horizontal projectile from height $h = 2 \text{m}$.

Time to hit ground:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 2 \text{m}}{9.8 \text{ms}^{-2}}} = \sqrt{0.4082 \text{s}^2} = 0.639 \text{s}$$

Horizontal speed at break:

$$v = \frac{x}{t} = \frac{10 \text{m}}{0.639 \text{s}} = 15.65 \text{ms}^{-1}$$

Centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{(15.65 \text{ ms}^{-1})^2}{1.5 \text{ m}} = \frac{244.9 \text{ m}^2 \text{ s}^{-2}}{1.5 \text{ m}} = 163 \text{ ms}^{-2}$$

Question 87

(a) Each star moves in a circle about the common centre of mass. Circular motion requires a centripetal force directed toward the centre of the circle, which is the centre of mass. This force is provided by the gravitational attraction of the other star. By the definition of centre of mass, $m_1 r_1 = m_2 r_2$, where r_1 and r_2 are the orbital radii. The less massive star orbits at a larger radius, and the more massive star orbits at a smaller radius.

(b) The centre of mass lies at $r_1 = \frac{m_2 d}{m_1 + m_2}$ from the heavier star. If $m_1 \gg m_2$, the centre of mass lies very close to the heavier star. The heavier star barely moves (tiny orbit) while the lighter star traces a large orbit. In the Earth-Sun system, the Sun is about 333,000 times the Earth's mass. The centre of mass lies inside the Sun, so the Sun wobbles imperceptibly while the Earth traces a large orbit. This is why we say the Earth orbits the Sun.

(c)(i) At height h : $g' = g \left(\frac{R_E}{R_E + h} \right)^2$

For $g' = \frac{g}{2}$:

$$\frac{1}{2} = \frac{R_E^2}{(R_E + h)^2}$$

$$(R_E + h)^2 = 2R_E^2$$

$$R_E + h = R_E \sqrt{2}$$

$$h = R_E(\sqrt{2} - 1) = 6400 \text{ km} \times 0.4142 = 2651 \text{ km}$$

(c)(ii) For geostationary orbit: $T = 9.84 \times 3600 \text{ s} = 35424 \text{ s}$

$$R^3 = \frac{GMT^2}{4\pi^2} = \frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 1.9 \times 10^{27} \text{ kg} \times (35424 \text{ s})^2}{4\pi^2}$$

$$R^3 = 4.028 \times 10^{24} \text{ m}^3$$

$$R = (4.028 \times 10^{24} \text{ m}^3)^{1/3} = 1.591 \times 10^8 \text{ m}$$

Altitude: $h = R - R_J = 1.591 \times 10^8 \text{ m} - 6.99 \times 10^7 \text{ m} = 8.92 \times 10^7 \text{ m} = 89200 \text{ km}$

Question 88

(a) Momentum is a vector quantity. The total momentum is the vector sum of all individual momenta. For the total to be zero, the momentum vectors must cancel when added. This allows individual fragments to have large momenta provided they point in directions that sum to zero. For example, in a two-fragment explosion, the fragments must fly in exactly opposite directions with equal magnitudes; whereas in a three-fragment explosion, the three momentum vectors must form a closed triangle.

(b) Both the person and the briefcase are inside the lift. If the briefcase does not fall relative to the person, it means the briefcase has the same acceleration as the person and the lift. The only way a released object can remain stationary relative to its surroundings is if everything is in free fall together: the lift, the person, and the briefcase all accelerate downward at g . The lift cable must have snapped or the lift is in free fall with $a = g$.

(c) At the instant of explosion, the bomb is at rest (just cut from cable, before gravity acts appreciably). Total initial momentum is zero.

$$\text{x-direction: } 0 = \frac{M}{2}V_0 + \frac{M}{3}(2V_0)\cos 37^\circ + \frac{M}{6}v_x$$

$$0 = \frac{V_0}{2} + \frac{2V_0 \times 0.7986}{3} + \frac{v_x}{6} = 0.5V_0 + 0.5324V_0 + \frac{v_x}{6}$$

$$v_x = -6 \times 1.0324V_0 = -6.195V_0$$

$$\text{y-direction: } 0 = 0 + \frac{M}{3}(2V_0)\sin 37^\circ + \frac{M}{6}v_y$$

$$0 = \frac{2V_0 \times 0.6018}{3} + \frac{v_y}{6} = 0.4012V_0 + \frac{v_y}{6}$$

$$v_y = -6 \times 0.4012V_0 = -2.407V_0$$

$$\text{Speed: } \mathbf{v}_3 = \sqrt{v_x^2 + v_y^2} = \sqrt{(6.195V_0)^2 + (2.407V_0)^2} = 6.65V_0$$

$$\text{Direction: } \theta = \tan^{-1} \left(\frac{-2.407}{-6.195} \right) = \tan^{-1}(0.3885) = 21.2^\circ \text{ below the negative x-axis.}$$

The third piece moves at $6.65V_0$ at 21.2° below the negative x-direction (into the third quadrant).

Question 89

(a) Maximum height depends on the vertical component of initial velocity: $H = \frac{u^2 \sin^2 \theta}{2g}$. An initial velocity at steeper angle gives a larger vertical component $u \sin \theta$, and hence greater height. The horizontal range is $R = \frac{u^2 \sin 2\theta}{g}$. For a given speed u , the range depends on $\sin 2\theta$, which has a maximum value of 1 when $2\theta = 90^\circ$ (i.e. $\theta = 45^\circ$). No angle can make $\sin 2\theta > 1$, so the range cannot exceed $\frac{u^2}{g}$ regardless of the angle chosen.

(b) The rotor blades push a mass of air downward each second, giving it momentum. By Newton's third law, the air pushes upward on the rotor with an equal force. This upward force is the thrust. The thrust equals the rate of momentum given to the air: $F = \left(\frac{dm}{dt}\right) \times v$, where $\frac{dm}{dt}$ is the mass of air pushed down per second and v is its downward speed. For hovering, this thrust must equal the helicopter's weight.

(c)(i) The ball must not exceed height 25m. Maximum height: $H = \frac{u^2 \sin^2 \theta}{2g}$.

$$\text{Setting } H = 25\text{m: } \sin^2 \theta = \frac{2gH}{u^2} = 2 \times 9.8\text{ms}^{-2} \times \frac{25\text{m}}{(40\text{ms}^{-1})^2} = 0.30625$$

$$\sin \theta = 0.5534$$

$$\theta = 33.6^\circ$$

Range at this angle:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(40\text{ms}^{-1})^2 \times \sin 67.2^\circ}{9.8\text{ms}^{-2}} = 150.5\text{m}$$

(c)(ii) Volume of air pushed down per second:

$$V = A \times v = \pi(2\text{m})^2 \times 8\text{ms}^{-1} = 32\pi \text{ m}^3\text{s}^{-1}$$

$$\text{Mass per second: } \frac{dm}{dt} = \rho V = 1.3\text{kgm}^{-3} \times 32\pi \text{ m}^3\text{s}^{-1} = 130.7\text{kgs}^{-1}$$

$$\text{Thrust: } F = \left(\frac{dm}{dt}\right) \times v = 130.7\text{kgs}^{-1} \times 8\text{ms}^{-1} = 1045.6\text{N}$$

For hovering: $mg = F$

$$m = \frac{F}{g} = \frac{1045.6\text{N}}{9.8\text{ms}^{-2}} = 106.7\text{kg}$$

Question 90

(a) The rope has a vertical component pulling upward on the object: $F \sin \theta$ (where θ is the angle above horizontal). This partially supports the weight. The surface needs to provide less normal reaction: $R = mg - F \sin \theta$. Since kinetic friction is $f = \mu R$, a smaller R means a smaller friction force. Pulling at an angle therefore reduces friction, making the object easier to move than if pulled horizontally.

(b) On a banked curve, the road surface tilts inward. The normal reaction R acts perpendicular to the road surface, which is no longer vertical. Its horizontal component $R \sin \theta$ (where θ is the banking angle) points toward the centre of the curve, providing the centripetal force. Its vertical component $R \cos \theta$ supports the weight. No friction is needed when $\tan \theta = \frac{v^2}{rg}$; the geometry of the bank naturally provides exactly the right centripetal force.

(c)(i) At constant velocity, net force is zero. Horizontal: $F \cos 30^\circ = \mu R$. Vertical: $R = mg - F \sin 30^\circ$.

Substituting:

$$F \cos 30^\circ = \mu(mg - F \sin 30^\circ)$$

$$F \cos 30^\circ + \mu F \sin 30^\circ = \mu mg$$

$$F(\cos 30^\circ + \mu \sin 30^\circ) = \mu mg$$

$$F = \frac{\mu mg}{(\cos 30^\circ + \mu \sin 30^\circ)} = \frac{0.57 \times 20\text{kg} \times 9.8\text{ms}^{-2}}{0.866 + 0.57 \times 0.5} = 97.1\text{N}$$

(c)(ii) Speed: $v = 72\text{kmh}^{-1} = 20\text{ms}^{-1}$

For banking without friction: $\tan \theta = \frac{v^2}{rg}$

$$\tan \theta = \frac{(20\text{ms}^{-1})^2}{50\text{m} \times 9.8\text{ms}^{-2}} = 0.8163$$

$$\theta = \tan^{-1}(0.8163) = 39.2^\circ$$

Question 91

(a) The range is $R = u^2 \sin 2\theta / g$. For a given u , the range depends on $\sin 2\theta$, which varies with θ . Different angles give different values of $\sin 2\theta$ and hence different ranges. However, $\sin 2\theta = \sin(180^\circ - 2\theta)$, so two angles θ_1 and θ_2 give the same range whenever $\theta_1 + \theta_2 = 90^\circ$. These are called complementary angles (e.g. 30° and 60° , or 20° and 70°).

(b) The gravel lands on the belt with zero horizontal velocity. The belt moves at constant speed v . To bring each piece of gravel from rest to speed v , its horizontal momentum must increase by mv . By the impulse-momentum theorem, a horizontal force must act on the gravel over some time interval. This force comes from kinetic friction between the belt surface and the gravel. The belt's motor must supply a driving force to compensate for this friction and maintain constant speed.

(c)(i) Equal vertical components: $u_P \sin \alpha = u_Q \sin \beta$.

$$R_P = \frac{2u_P^2 \sin \alpha \cos \alpha}{g}, \quad R_Q = \frac{2u_Q^2 \sin \beta \cos \beta}{g}$$

$$\frac{R_P}{R_Q} = \frac{u_P^2 \sin \alpha \cos \alpha}{u_Q^2 \sin \beta \cos \beta} = \frac{u_P (u_P \sin \alpha) \cos \alpha}{u_Q (u_Q \sin \beta) \cos \beta} = \frac{u_P \cos \alpha}{u_Q \cos \beta}$$

From: $u_P \sin \alpha = u_Q \sin \beta$; $\frac{u_P}{u_Q} = \frac{\sin \beta}{\sin \alpha}$:

$$\frac{R_P}{R_Q} = \frac{\sin \beta \cos \alpha}{\sin \alpha \cos \beta} = \frac{\sin \beta}{\cos \beta} \times \frac{1}{\frac{\sin \alpha}{\cos \alpha}} = \frac{\tan \beta}{\tan \alpha}$$

(c)(ii) Force to maintain belt speed:

$$F = v \times \frac{dm}{dt} = 2 \text{ ms}^{-1} \times 75 \text{ kgs}^{-1} = 150 \text{ N}$$

Rate of kinetic energy given to gravel:

$$P_{KE} = \frac{1}{2} \times \frac{dm}{dt} \times v^2 = \frac{1}{2} \times 75 \text{ kgs}^{-1} \times (2 \text{ ms}^{-1})^2 = 150 \text{ Js}^{-1} = 150 \text{ W}$$

Question 92

(a) In an elliptical orbit, the distance between the satellite and Earth changes continuously. When the satellite moves closer to Earth, gravity does positive work (the force has a component along the direction of motion). Positive work increases kinetic energy, so the satellite speeds up. When moving away, gravity does negative work, and the satellite slows down. The total mechanical energy remains constant throughout.

(b) Escape velocity from a body's surface is $v_{esc} = \sqrt{2GM/R}$. The Moon has both a smaller mass and a smaller radius than Earth, but the mass effect dominates, giving the Moon a much lower escape velocity ($\approx 2.4 \text{ kms}^{-1}$) compared to Earth ($\approx 11.2 \text{ kms}^{-1}$). Gas molecules at a given temperature have a distribution of speeds. On the Moon, many molecules (especially lighter ones) exceed the low escape velocity and leave permanently. Over billions of years, the Moon has lost virtually all its atmosphere. On Earth, only the lightest molecules occasionally exceed the higher escape velocity, so the atmosphere is retained.

(c) By conservation of energy. Taking the initial separation as $d = 10^9 \text{ km} = 10^{12} \text{ m}$ and the final separation (surfaces touching) as $2R = 2 \times 10^4 \text{ km} = 2 \times 10^7 \text{ m}$:

$$-\frac{Gm^2}{d} + 0 = -\frac{Gm^2}{2R} + 2 \times \frac{1}{2} mv^2$$

$$mv^2 = Gm^2 \left(\frac{1}{2R} - \frac{1}{d} \right)$$

Since $d \gg 2R$, $\frac{1}{d}$ is negligible:

$$v^2 \approx \frac{Gm}{2R} = \frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 2 \times 10^{30} \text{ kg}}{2 \times 10^7 \text{ m}} = 6.67 \times 10^{12} \text{ m}^2 \text{ s}^{-2}$$

$$v = \sqrt{6.67 \times 10^{12} \text{ m}^2 \text{ s}^{-2}} = 2.58 \times 10^6 \text{ ms}^{-1} \approx 2580 \text{ kms}^{-1}$$

Question 93

(a) The car has two accelerations: centripetal acceleration $a_c = \frac{v^2}{r}$ directed toward the centre (perpendicular to velocity) and tangential acceleration a_t along the velocity direction (from the speed increase). Since these are perpendicular, the resultant is $a = \sqrt{a_c^2 + a_t^2}$. The angle between the resultant and the velocity is $\theta = \tan^{-1} \left(\frac{a_c}{a_t} \right)$, directed toward the centre. As the speed increases, a_c grows and θ increases.

(c)(i) Optimal angle: $\theta_{\text{opt}} = 45^\circ + \frac{\beta}{2} = 45^\circ + \frac{30^\circ}{2} = 45^\circ + 15^\circ = 60^\circ$

(c)(ii) Centripetal acceleration: $a_c = \frac{v^2}{r} = \frac{(30\text{ms}^{-1})^2}{500\text{m}} = 1.8\text{ms}^{-2}$

Tangential acceleration: $a_t = 2\text{ms}^{-2}$

Resultant: $a = \sqrt{(1.8\text{ms}^{-2})^2 + (2\text{ms}^{-2})^2} = 2.69\text{ms}^{-2}$

Angle from velocity: $\theta = \tan^{-1}\left(\frac{a_c}{a_t}\right) = \tan^{-1}\left(\frac{1.8}{2}\right) = 42^\circ$ toward the centre.

Question 94

(a) At the highest point, the vertical velocity is zero but the horizontal velocity $v_x = u\cos 45^\circ = \frac{u}{\sqrt{2}}$ is unchanged (no air resistance, no horizontal force) leading to non-zero KE.

$$\text{KE}_{\text{top}} = \frac{1}{2}m\left(\frac{u}{\sqrt{2}}\right)^2 = \frac{1}{2}\frac{mu^2}{2} = \frac{1}{2} \times \text{KE}_{\text{launch}}$$

Hence kinetic energy at the top is exactly half the launch kinetic energy.

(b) As more passengers step onto the belt, the total friction force on the belt increases (more mass to accelerate from rest to belt speed). The driving mechanism must supply a force equal to the total friction acting backward on the belt. For constant speed, the drive force must continuously match the friction force, which increases with the number of passengers. The motor must therefore increase its output automatically in response to the increased load.

(c) At 45° , $\text{KE}_{\text{top}} = \frac{1}{2} \times \text{KE}_{\text{launch}} = \frac{1}{2} \times 500\text{J} = 250\text{J}$

Question 95

(a) By conservation of energy, the PE lost equals the KE gained: $mgh = \frac{1}{2}mv^2$. The mass cancels, giving $v = \sqrt{2gh}$. This depends only on h , not on the angle or length of the slope. A steeper slope is shorter but has a larger component of gravity along it; a gentler slope is longer but has a smaller component. Both effects compensate exactly, producing the same final speed.

(b) Projectile motion is defined as motion under the influence of gravity alone (no other forces), with a constant horizontal velocity and a constant vertical acceleration g . A stone thrown vertically upward satisfies these conditions: the only force acting is gravity, and the acceleration is constant at g downward throughout the flight. It is a special case of projectile motion where the horizontal velocity is zero. The student's claim is correct.

(c) $u_x = 20\cos 30^\circ = 17.32\text{ms}^{-1}$, $u_y = 20\sin 30^\circ = 10\text{ms}^{-1}$

Time to reach wall (3m away): $t_1 = \frac{3\text{m}}{17.32\text{ms}^{-1}} = 0.1732\text{s}$

Height at wall: $y_1 = 2\text{m} + 10\text{ms}^{-1} \times 0.1732\text{s} - \frac{1}{2} \times 9.8\text{ms}^{-2} \times (0.1732\text{s})^2 = 3.585\text{m}$

Vertical velocity at wall: $v_y = 10\text{ms}^{-1} - 9.8\text{ms}^{-2} \times 0.1732\text{s} = 8.303\text{ms}^{-1}$ (upward)

After elastic bounce: $v_x = -17.32\text{ms}^{-1}$ (reversed), $v_y = 8.303\text{ms}^{-1}$ (unchanged)

The ball now moves away from the wall at height 3.585m with $v_x = 17.32\text{ms}^{-1}$ (away) and $v_y = 8.303\text{ms}^{-1}$ (upward).

Time to hit ground from this point ($y = 0$):

$$0 = 3.585 + 8.303t - 4.9t^2$$

$$4.9t^2 - 8.303t - 3.585 = 0$$

$$t = 2.051\text{s}$$

Horizontal distance from wall: $x = 17.32\text{ms}^{-1} \times 2.051\text{s} = 35.5\text{m}$

Question 96

(a) The plates have inertia: they resist changes in motion. When the cloth is pulled, friction between the cloth and plates acts on the plates. A fast pull means the cloth is in contact with the plates for a very short time. By the impulse-momentum theorem, $J = F\Delta t$: even though friction is large, the extremely short contact time produces a tiny impulse, and the plates barely move. A slow pull keeps friction acting for a long time, producing a large impulse that drags the plates along with the cloth.

(b) Thrust is $F = v_e \times \frac{dm}{dt}$. If both the exhaust velocity and the burn rate are constant, the thrust is constant. However, acceleration is $a = \frac{(F - Mg)}{M}$ (for vertical flight) or $a = \frac{F}{M}$ (in space). As fuel is consumed, the total mass M decreases. With constant F and decreasing M , the acceleration increases.

(c) At launch ($t = 0$): mass = 100kg

$$a_0 = \frac{F - mg}{m} = \frac{1500\text{N} - 100\text{kg} \times 9.8\text{ms}^{-2}}{100\text{kg}} = 5.2\text{ms}^{-2}$$

At time t : mass = $(100 - 20t)\text{kg}$

$$a(t) = \frac{F - mg}{m} = \frac{F}{m} - g = \frac{1500\text{N}}{(100 - 20t)\text{kg}} - 9.8\text{ms}^{-2}$$

Question 97

(a) The planet has an enormous mass compared to the spacecraft. In the planet's reference frame, the spacecraft swings around and leaves at the same speed it arrived (elastic gravitational interaction). But the planet is moving relative to the Sun. In the Sun's frame, the spacecraft gains speed from the planet's motion. By conservation of momentum for the planet-spacecraft system, the planet loses a tiny amount of momentum. Because the planet's mass is so large, its speed change is negligible, while the spacecraft gains a significant velocity boost.

(b) In a circular orbit, gravity acts perpendicular to the velocity (radially inward while velocity is tangential). The work done by a force perpendicular to displacement is zero: $W = Fd\cos 90^\circ = 0$. By the work-energy theorem, zero work means no change in KE, so speed is constant. In an elliptical orbit, gravity is not always perpendicular to the velocity. When the satellite moves toward Earth, the component of gravity along the motion does positive work, increasing KE and speed. When moving away, gravity does negative work, decreasing speed.

(c) Orbital radius: $R = R_E + h = 6.4 \times 10^6\text{m} + 4 \times 10^5\text{m} = 6.8 \times 10^6\text{m}$

Orbital speed:

$$v = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 5.97 \times 10^{24}\text{kg}}{6.8 \times 10^6\text{m}}} = 7653\text{ms}^{-1}$$

Head-on collision (equal masses, equal speeds, opposite directions):

$$p_{\text{total}} = mv - mv = 0$$

After sticking together: $2m \times V = 0$, so $V = 0$.

The wreckage has zero velocity. With no orbital speed, it cannot maintain orbit and falls directly toward the Earth under gravity. It crashes into the Earth (or burns up in the atmosphere).

Question 98

(a) Two conditions: (1) The orbital period must equal 24 hours (one sidereal day) so that the satellite completes one orbit in the same time the Earth completes one rotation. If the periods differ, the satellite drifts east or west relative to the ground. (2) The orbit must lie in the equatorial plane and the satellite must orbit in the same direction as the Earth's rotation (west to east). If the orbit is inclined, the satellite oscillates north and south of the equator, appearing to move in a figure-eight pattern as seen from the ground.

(b) At the hanging point, three forces act: tension T_L in the left segment at 10° below horizontal, tension T_R in the right segment at 5° below horizontal, and the boy's weight 90N downward.

Horizontal: $T_L \cos 10^\circ = T_R \cos 5^\circ$

$$T_L = T_R \times \frac{\cos 5^\circ}{\cos 10^\circ} = T_R \times \frac{0.9962}{0.9848} = 1.0116 T_R$$

Vertical: $T_L \sin 10^\circ + T_R \sin 5^\circ = 90\text{N}$

$$1.0116 T_R \times 0.1736 + T_R \times 0.08716 = 90\text{N}$$

$$T_R(0.1756 + 0.08716) = 90\text{N}$$

$$T_R \times 0.2628 = 90\text{N}$$

$$T_R = 342.5\text{N}$$

$$T_L = 1.0116 \times 342.5\text{N} = 346.5\text{N}$$

Question 99

(a) Both stones fall the same vertical distance h and take the same time $t = \sqrt{2h/g}$ to reach the ground (the horizontal velocity does not affect the fall time). The horizontal distance depends on two factors: the initial horizontal velocity u and the fall time t . Since the fall time is fixed by the height, the horizontal distance is $x = ut$. A greater horizontal velocity or a greater height (which gives more fall time) increases the distance.

(b) The player must estimate: (1) the range of the ball; that is how far from the kicker it will land, which depends on the kick speed and angle; (2) the time of flight; that is how long the ball stays in the air, which determines how much time the player

has to run; and (3) the direction of the kick. From these, the player calculates the distance between their starting position and the landing point, then divides by the time of flight to find the minimum running speed.

(c) Range of the ball:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(19.5 \text{ ms}^{-1})^2 \times \sin 90^\circ}{9.8 \text{ ms}^{-2}} = 38.8 \text{ m}$$

Time of flight:

$$T = \frac{2u \sin \theta}{g} = \frac{2 \times 19.5 \text{ ms}^{-1} \times \sin 45^\circ}{9.8 \text{ ms}^{-2}} = 2.81 \text{ s}$$

The player is 55m away from the kicker. The ball lands 38.8m from the kicker. The player must cover $55 - 38.8 = 16.2 \text{ m}$ in 2.81s:

$$v_{\min} = \frac{16.2 \text{ m}}{2.81 \text{ s}} = 5.77 \text{ ms}^{-1}$$

Question 100

(a) At the instant the ball is thrown, it already has the platform's horizontal velocity (by inertia). The throw adds an additional velocity. To a stationary ground observer, the ball's velocity is the vector sum of the platform velocity and the throw velocity. For example, if the platform moves east and the ball is thrown north relative to the platform, the ground observer sees the ball move northeast. The trajectory shape (parabolic) is the same, but it is tilted relative to what the thrower sees.

(b) The shell follows a parabolic trajectory with a definite range and flight time. The car moves at constant speed along the same road. For the shell to hit the car, the shell must land at the exact position where the car is at that instant. This requires: (1) the shell's range must equal the car's initial distance plus the distance the car travels during the shell's flight time, and (2) the shell and car must arrive at the same point at the same time.

(c) Speed of car: $v_c = 72 \text{ kmh}^{-1} = 20 \text{ ms}^{-1}$

At 45° , the shell's range is $R = \frac{u^2}{g}$ and flight time is $T = \frac{2u \sin 45^\circ}{g} = \frac{u\sqrt{2}}{g}$.

During time T , the car moves an additional $v_c \times T$ from the gun.

The shell must reach: $500 \text{ m} + v_c T$

$$\begin{aligned} \frac{u^2}{g} &= 500 \text{ m} + 20 \text{ ms}^{-1} \times \frac{u\sqrt{2}}{g} \\ u^2 &= 500 \times 9.8 + 20\sqrt{2} \times u = 4900 + 28.28u \\ u^2 - 28.28u - 4900 &= 0 \\ u &= 85.6 \text{ ms}^{-1} \end{aligned}$$

Question 101

(a) On each slope, the forces along the plane are: the gravity component down the slope ($mg \sin \theta$), friction (which can act either up or down the slope), and the string tension. Friction can take any value up to μR , and can reverse direction. This creates a range of tensions for which neither mass slides. A higher μ widens this range; more friction means a larger band of weight ratios and angles that maintain equilibrium. Without friction ($\mu = 0$), equilibrium requires a single precise condition.

(b) The bob has weight mg acting vertically downward. The only other force is the string tension, which acts along the string toward the pivot. If the string were horizontal, the tension would be entirely horizontal and there would be no upward force to balance the weight. The bob would accelerate downward. For equilibrium in the vertical direction, the string must have an upward component: $T \cos \theta = mg$. This requires $\theta < 90^\circ$, so the string can never be horizontal.

(c) For constant velocity (zero acceleration), the net force along each slope is zero.

On the 60° slope (mass slides down, friction acts up):

$$mg \sin 60^\circ - T - \mu mg \cos 60^\circ = 0 \quad \dots (1)$$

On the 30° slope (mass is pulled up, friction acts down):

$$T - mg \sin 30^\circ - \mu mg \cos 30^\circ = 0 \quad \dots (2)$$

Adding (1) and (2) to eliminate T :

$$\begin{aligned} mg \sin 60^\circ - mg \sin 30^\circ &= \mu mg (\cos 60^\circ + \cos 30^\circ) \\ \mu &= \frac{\sin 60^\circ - \sin 30^\circ}{\cos 60^\circ + \cos 30^\circ} = 0.27 \end{aligned}$$

Question 102

(a) When only gravity acts, the normal reaction balances the perpendicular component of weight: $R = mg\cos\theta$. A horizontal force F has a component perpendicular to the slope: $F\sin\theta$ (pushing into the surface). This adds to the normal reaction: $R = mg\cos\theta + F\sin\theta$.

(b) The total energy of a satellite in circular orbit is $E = -\frac{GMm}{2r}$. When engines fire, they add energy, making E less negative. The satellite moves to a larger radius r . But orbital speed is $v = \sqrt{\frac{GM}{r}}$, which decreases with increasing r . The added energy goes primarily into gravitational PE (which increases as $-\frac{GMm}{r}$, becoming less negative). The KE ($= \frac{GMm}{2r}$) actually decreases. The satellite trades kinetic energy for potential energy as it climbs to the higher orbit.

(c) Forces on the block perpendicular to slope:

$$R = mg\cos\theta + F\sin\theta$$

Forces along the slope (about to slide up, friction acts downward):

$$F\cos\theta = mg\sin\theta + \mu_s R = mg\sin\theta + \mu_s(mg\cos\theta + F\sin\theta)$$

$$F\cos\theta - \mu_s F\sin\theta = mg\sin\theta + \mu_s mg\cos\theta$$

$$F(\cos\theta - \mu_s\sin\theta) = mg(\sin\theta + \mu_s\cos\theta)$$

$$F_{\max} = \frac{mg(\sin\theta + \mu_s\cos\theta)}{\cos\theta - \mu_s\sin\theta}$$

Question 103

(a) The bob moves in a horizontal circle. In the vertical direction there is no acceleration. By Newton's second law in the vertical direction: the net vertical force must be zero. The only upward force is $T\cos\theta$ and the only downward force is mg . Therefore $T\cos\theta = mg$.

(b) At the design speed, the horizontal component of the normal reaction exactly provides the centripetal force. At a higher speed, the required centripetal force ($\frac{mv^2}{r}$) exceeds the horizontal component of R . The deficit must be supplied by friction acting down the bank (inward), and the truck tends to slide outward. At a lower speed, the centripetal requirement is less than the horizontal component of R . The excess pushes the truck inward, and friction must act up the bank (outward) to prevent sliding.

(c) Vertical: $T\cos 25^\circ = mg$

$$T = \frac{mg}{\cos 25^\circ} = \frac{0.2\text{kg} \times 9.8\text{ms}^{-2}}{0.9063} = 2.16\text{N}$$

Horizontal (centripetal): $T\sin 25^\circ = \frac{mv^2}{r}$

Radius of circle: $r = L\sin 25^\circ = 1.5\text{m} \times 0.4226 = 0.634\text{m}$

$$v^2 = \frac{T\sin 25^\circ \times r}{m} = \frac{2.16\text{N} \times 0.4226 \times 0.634\text{m}}{0.2\text{kg}} = 2.894\text{m}^2\text{s}^{-2}$$

$$v = 1.70\text{ms}^{-1}$$

Period: $T_{\text{period}} = \frac{2\pi r}{v} = 2\pi \times \frac{0.634\text{m}}{1.70\text{ms}^{-1}} = 2.34\text{s}$

Question 104

(a) At the highest point, only gravity acts on the projectile (the same force that acted before the explosion). The explosion is an internal event: internal forces cannot change the motion of the centre of mass. By Newton's second law applied to the centre of mass, $F_{\text{ext}} = Ma_{\text{cm}}$, and the only external force is gravity. Therefore, $a_{\text{cm}} = g$ downward, the same as before the explosion. The centre of mass continues along the original parabola.

(b) Before firing, the total momentum of the cannon-shell system is zero (both at rest). After firing, momentum is conserved: $0 = m_s v_s - m_c v_c$, so $v_c = \left(\frac{m_s}{m_c}\right)v_s$. The recoil speed is proportional to the shell-to-cannon mass ratio. A heavier cannon recoils more slowly; a heavier shell increases the recoil.

(c) At the highest point: $v_x = u\cos 60^\circ = 50\text{ms}^{-1} \times 0.5 = 25\text{ms}^{-1}$ (horizontal), $v_y = 0$

Momentum before split: $p = 3\text{kg} \times 25\text{ms}^{-1} = 75\text{kgms}^{-1}$ (horizontal)

After split: the 1kg fragment has zero horizontal velocity.

$$75\text{kgms}^{-1} = 1\text{kg} \times 0 + 2\text{kg} \times v_2$$

$v_2 = 37.5\text{ms}^{-1}$ (horizontal)

$$\text{Height at highest point: } H = \frac{u^2 \sin^2 60^\circ}{2g} = \frac{(50 \text{ ms}^{-1})^2 \times 0.75}{2 \times 9.8 \text{ ms}^{-2}} = \frac{1875 \text{ m}^2 \text{ s}^{-2}}{19.6 \text{ ms}^{-2}} = 95.66 \text{ m}$$

$$\text{Time from peak to ground: } H = \frac{1}{2} g t_2^2, \text{ so } t_2 = \sqrt{\frac{2H}{g}} = \sqrt{2 \times \frac{95.66 \text{ m}}{9.8 \text{ ms}^{-2}}} = 4.42 \text{ s}$$

$$\text{Time from launch to peak: } t_1 = \frac{u \sin 60^\circ}{g} = 50 \times \frac{0.866}{9.8} = 4.42 \text{ s}$$

$$\text{Horizontal distance of launch to peak: } x_1 = 25 \text{ ms}^{-1} \times 4.42 \text{ s} = 110.5 \text{ m}$$

Horizontal distance of peak to landing (2kg fragment at 37.5 ms^{-1}):

$$x_2 = 37.5 \text{ ms}^{-1} \times 4.42 \text{ s} = 165.8 \text{ m}$$

$$\text{Total distance from launch: } x_1 + x_2 = 110.5 \text{ m} + 165.8 \text{ m} = 276.3 \text{ m}$$

Question 105

(a) The stone moves slowly and stays in contact with the glass for a long time. The impulse spreads across the entire pane, stressing it everywhere and causing it to shatter. The bullet passes through in a tiny fraction of a second. The contact time is so short that only the small area in the bullet's path receives the impulse. The rest of the glass has no time to respond, leaving a clean hole.

(b) Objects on the surface: with gravity gone, no downward force acts. Objects continue with the rotational velocity they had and fly off tangentially.

The Moon: gravity provides the centripetal force. Without it, by Newton's first law, the Moon continues in a straight line tangent to its orbit.

The atmosphere: gas molecules are held near the surface by gravity. Without it, they disperse into space following their thermal velocities.

(c) The water must reach $x = 20 \text{ m}$, $y = 18 \text{ m} - 1 \text{ m} = 17 \text{ m}$ above the nozzle:

By using trajectory equation:

$$17 \text{ m} = 20 \text{ m} \times \tan 55^\circ - \frac{9.8 \text{ ms}^{-2} \times (20 \text{ m})^2}{2u^2 \cos^2 55^\circ}$$

$$u^2 = 515 \text{ m}^2 \text{ s}^{-2}$$

$$u = 22.7 \text{ ms}^{-1}$$

Question 106

(a) In a horizontal circle, speed is constant because gravity has no component along the motion. The tension is constant: $T = \frac{mv^2}{r}$.

In a vertical circle, gravity has a component along the motion, so speed varies; fastest at the bottom, slowest at the top. The tension is $T_{\text{bottom}} = \frac{mv_{\text{bottom}}^2}{r} + mg$ at the bottom and $T_{\text{top}} = \frac{mv_{\text{top}}^2}{r} - mg$ at the top. Tension is maximum at the bottom and minimum at the top.

(b) By conservation of momentum: $m_1 v_1 = (m_1 + m_2) V$, so $V = \frac{m_1 v_1}{m_1 + m_2} < v_1$. The momentum is shared among a larger mass, reducing the speed.

(c) Let a_1 = downward acceleration of the 4kg mass (= upward acceleration of pulley B). Let a_2 = acceleration of the 3kg mass downward relative to pulley B. Let T_1 = tension over pulley A, T_2 = tension over pulley B.

Constraint (massless pulley B): $T_1 = 2T_2$

Absolute accelerations (downward positive): 4kg: a_1 ; 3kg: $(-a_1 + a_2)$; 1kg: $(-a_1 - a_2)$.

Equations of motion:

$$4g - T_1 = 4a_1 \quad \dots (1)$$

$$3g - T_2 = 3(-a_1 + a_2) \quad \dots (2)$$

$$g - T_2 = (-a_1 - a_2) \quad \dots (3)$$

From (2): $T_2 = 3g + 3a_1 - 3a_2$. From (3): $T_2 = g + a_1 + a_2$.

Equating: $3g + 3a_1 - 3a_2 = g + a_1 + a_2 \rightarrow g + a_1 = 2a_2 \dots (4)$

From (1) and (2): $T_1 = 4g - 4a_1$, so $T_2 = 2g - 2a_1$.

Setting equal to (3) expression: $2g - 2a_1 = g + a_1 + a_2 \rightarrow g = 3a_1 + a_2 \dots (5)$

From (4): $a_2 = (g + a_1)/2$. Substituting into (5):

$$g = 3a_1 + \frac{g + a_1}{2} = 3a_1 + \frac{g}{2} + \frac{a_1}{2} = \frac{7a_1}{2} + \frac{g}{2}$$

$$\frac{g}{2} = \frac{7a_1}{2}$$

$$\rightarrow a_1 = \frac{g}{7} = 1.4\text{ms}^{-2}$$

$$a_2 = \frac{\left(g + \frac{g}{7}\right)}{2} = \frac{4g}{7} = 5.6\text{ms}^{-2}$$

Accelerations:

$$4\text{kg: } \frac{g}{7} = 1.4\text{ms}^{-2} \text{ downward}$$

$$3\text{kg: } -\frac{g}{7} + \frac{4g}{7} = \frac{3g}{7} = 4.2\text{ms}^{-2} \text{ downward}$$

$$1\text{kg: } -\frac{g}{7} - \frac{4g}{7} = -\frac{5g}{7} = 7\text{ms}^{-2} \text{ upward}$$

Tensions:

$$T_2 = g + a_1 + a_2 = 9.8 + 1.4 + 5.6 = 16.8\text{N}$$

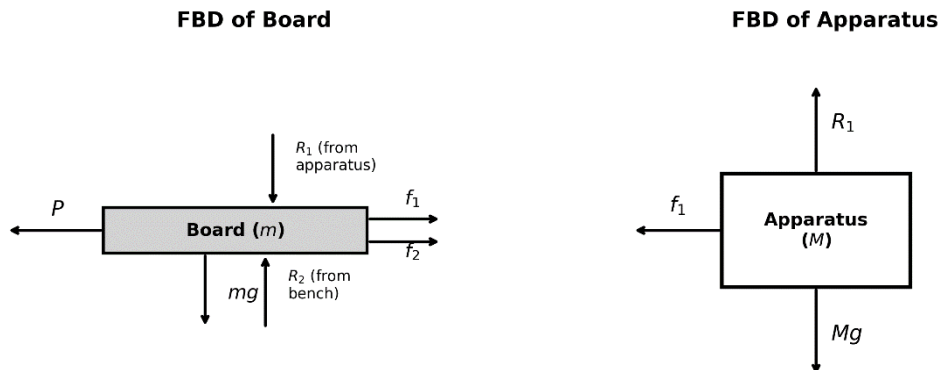
$$T_1 = 2T_2 = 33.6\text{N}$$

Question 107

(a) For a circular orbit, total energy is $E = -GMm/(2r)$ and orbital speed is $v = \sqrt{GM/r}$. As the satellite loses energy to drag, E becomes more negative, so r decreases. A smaller r means a larger $v = \sqrt{GM/r}$. The satellite spirals inward and speeds up. The energy lost to drag exceeds the KE gained because the PE decreases by twice the KE increase.

(b) The board is sandwiched between the apparatus above and the bench below. The friction force between the apparatus and the board is $f_1 = \mu_s Mg$ (due to the weight of the apparatus pressing on the board). The friction force between the board and the bench is $f_2 = \mu_s (M + m)g = \mu_s (M + 0.1M)g = 1.1\mu_s Mg$ because the bench supports both the apparatus and the board. Since $1.1\mu_s Mg > \mu_s Mg$, the friction at the lower surface is greater even though μ_s is the same. This is because the bench carries more weight.

(c)(i) Free body diagram:



(c)(ii) For the system to start moving as a whole, P must overcome the friction between the board and the bench (the only external friction on the system):

$$P_{\min} = f_2 = \mu_s (M + m)g = 0.1 \times (M + 0.1M)g = 0.1 \times 1.1Mg = \mathbf{0.11Mg}$$

(c)(iii) For the board to slide out while the apparatus remains stationary, P must overcome both friction surfaces acting on the board:

$$\text{Friction from apparatus on board (top): } f_1 = \mu_s Mg = 0.1Mg$$

$$\text{Friction from bench on board (bottom): } f_2 = \mu_s (M + m)g = 0.11Mg$$

$$P = f_1 + f_2 = 0.1Mg + 0.11Mg = \mathbf{0.21Mg}$$

(c)(iv) While Kipanga pulls the board, kinetic friction from the sliding board acts on the apparatus in the direction of the pull: $f_{\text{board} \rightarrow \text{app}} = \mu_k Mg = 0.1Mg$ (kinetic, because the board slides under the apparatus).

The bench resists this with static friction on the apparatus: $f_{\text{bench} \rightarrow \text{app}} \leq \mu_s Mg = 0.1Mg$.

Since the driving force ($0.1Mg$) exactly equals the maximum resisting force ($0.1Mg$), the apparatus is on the verge of motion but never actually accelerates. It acquires no velocity during the pull.

After the board is removed, the apparatus sits stationary on the bench, 5cm from the edge, with zero velocity. No horizontal force acts to start it moving.

The apparatus does not slide off the edge.

PART THREE

Question 1

(a) Both cylinders have the same mass and radius, but their moments of inertia differ. The solid cylinder has $I = \frac{1}{2}MR^2$ while the hollow cylinder has $I = MR^2$. When rolling without slipping, gravitational potential energy is converted into both translational and rotational kinetic energy. The hollow cylinder, with its larger moment of inertia, diverts a greater fraction of its energy into rotation, leaving less for translation. Consequently, it moves more slowly down the incline. The solid cylinder, with a smaller moment of inertia, retains more energy as translational kinetic energy and reaches the bottom first.

(b) While the umbrella spins, each water droplet undergoes circular motion. The surface of the umbrella provides the centripetal force required. When the droplet reaches the edge, the surface can no longer provide sufficient centripetal force, and the droplet leaves the umbrella. After leaving, no centripetal force acts on the droplet. By Newton's first law, it continues in a straight line tangent to the circular path at the point of release.

(c) Moment of inertia of the disc:

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 4.0\text{kg} \times (0.30\text{m})^2 = 0.18\text{kgm}^2$$

Torque: $\tau = FR = 12\text{N} \times 0.30\text{m} = 3.6\text{Nm}$

Angular acceleration:

$$\alpha = \frac{\tau}{I} = \frac{3.6\text{Nm}}{0.18\text{kgm}^2} = 20\text{rad s}^{-2}$$

Angular velocity after 5.0s from rest:

$$\omega = \omega_0 + \alpha t = 0 + 20\text{rad s}^{-2} \times 5.0\text{s} = 100\text{rad s}^{-1}$$

Question 2

(a) With no external torque, angular momentum is conserved: $I\omega = \text{constant}$. When the skater pulls her arms inward, her mass moves closer to the axis of rotation. This decreases her moment of inertia I . For the product $I\omega$ to remain constant, the angular velocity ω must increase. Hence, she spins faster.

(b) By the capillary rise formula, $h = 2\gamma \cos\theta / (\rho g r)$, the height of rise is inversely proportional to the tube radius. A narrower tube has a smaller r , producing a larger h . Physically, surface tension acts along the circumference of the meniscus ($2\pi r$) while supporting a column of weight proportional to $\pi r^2 h$. In a narrower tube, the weight per unit contact length is smaller, so surface tension can support a taller column.

(c) A soap bubble has two surfaces. Excess pressure:

$$\Delta P = \frac{4\gamma}{r} = \frac{4 \times 0.025\text{Nm}^{-1}}{0.03\text{m}} = 3.33\text{Pa}$$

Total pressure inside the bubble:

$$P = P_{\text{atm}} + \Delta P = 1.01 \times 10^5\text{Pa} + 3.33\text{Pa} = 1.01003 \times 10^5\text{Pa}$$

Question 3

(a) The flywheel has a larger radius than the bicycle wheel. Since both have the same mass, the flywheel has a larger moment of inertia (more mass is distributed farther from the axis). Rotational kinetic energy is $KE = \frac{1}{2}I\omega^2$. At the same angular velocity, the flywheel stores more kinetic energy. Under similar frictional torque, the flywheel takes longer to dissipate this larger energy store. The bicycle wheel, with smaller moment of inertia, stores less energy and stops sooner.

(b) When wind blows horizontally across the top of the roof, the air above the roof moves at high speed. The air inside the house below the roof is stationary. By Bernoulli's equation, the fast-moving air above has lower pressure than the still air below. This pressure difference produces a net upward force on the roof equal to $F = \frac{1}{2}\rho v^2 \times A_{\text{roof}}$. If this upward force exceeds the weight of the roof and the strength of its fastenings, the roof is lifted off.

(c) Terminal velocity:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta} = \frac{2 \times (2.0 \times 10^{-3}\text{m})^2 \times (7800 - 1260)\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 1.5\text{Nsm}^{-2}}$$

$$v_t = 0.038\text{ms}^{-1}$$

Question 4

(a) When no external torque acts, angular momentum is conserved: $L = I\omega = \text{constant}$. When the person walks from the centre to the edge, they move mass farther from the axis, increasing the total moment of inertia I of the system. For $I\omega$ to remain constant, ω must decrease. When the person walks back to the centre, I decreases and ω increases.

(b) The pond skater's legs are hydrophobic and distribute its tiny weight over a large contact perimeter. The surface tension force acting upward along the contact line around each leg is sufficient to support the insect's weight without breaking the surface. A coin has much greater weight concentrated over a short contact perimeter. The surface tension force along this short perimeter is far too small to support the coin's weight, so the surface breaks and the coin sinks.

(c) Using conservation of energy. The rod's centre of mass falls through $h = L/2 = 0.60\text{m}$ from horizontal to vertical:

$$\begin{aligned} Mgh &= \frac{1}{2} I\omega^2 \\ Mg \times \frac{L}{2} &= \frac{1}{2} \times \frac{1}{3} ML^2 \times \omega^2 \\ g \times \frac{L}{2} &= \frac{L^2 \omega^2}{6} \\ \omega &= \sqrt{\frac{3g}{L}} = \sqrt{\frac{3 \times 9.8\text{ms}^{-2}}{1.2\text{m}}} = 4.95\text{rad s}^{-1} \end{aligned}$$

Question 5

(a) While on the table, both translational and rotational motion are maintained by the rolling condition. When the ball leaves the edge, gravity acts on its centre of mass (producing translational acceleration downward) but exerts zero torque about the centre of mass (because the gravitational force passes through the centre). With no torque to change the angular velocity, the ball continues to spin at the same rate throughout the fall, by the rotational analogue of Newton's first law.

(b) In a healthy artery, blood flows in a smooth, laminar pattern. When an artery is partially blocked by plaque (atherosclerosis), the internal radius decreases. By the continuity equation, blood speed increases at the narrowed section. This raises the Reynolds number. If it exceeds approximately 2000–3000, the flow transitions from laminar to turbulent. Turbulent flow produces chaotic pressure fluctuations that vibrate the artery wall, creating the audible murmur.

(c) By Poiseuille's formula:

$$\begin{aligned} Q &= \frac{\pi \Delta P R^4}{8\eta L} = \frac{\pi \times 600\text{Pa} \times (4 \times 10^{-3}\text{m})^4}{8 \times 1.0 \times 10^{-3}\text{Nsm}^{-2} \times 1.5\text{m}} \\ Q &= \frac{\pi \times 600\text{Pa} \times 2.56 \times 10^{-10}\text{m}^4}{1.2 \times 10^{-2}\text{Nsm}^{-1}} = 4.02 \times 10^{-5}\text{m}^3\text{s}^{-1} \end{aligned}$$

Maximum velocity at the centre:

$$v_{\text{max}} = \frac{\Delta P R^2}{4\eta L} = \frac{600\text{Pa} \times (4 \times 10^{-3}\text{m})^2}{4 \times 1.0 \times 10^{-3}\text{Nsm}^{-2} \times 1.5\text{m}} = 1.6\text{ms}^{-1}$$

Question 6

(a) When two gears are meshed, the tangential velocity at the contact point is the same for both. For the smaller gear (radius r_1), $v = \omega_1 r_1$. For the larger gear (radius r_2), $v = \omega_2 r_2$. Since $r_2 > r_1$, $\omega_2 < \omega_1$: the larger gear rotates more slowly. However, assuming negligible friction, power is conserved: $\tau_1 \omega_1 = \tau_2 \omega_2$. Since $\omega_2 < \omega_1$, $\tau_2 > \tau_1$: the larger gear delivers greater torque. The gear system trades angular velocity for torque.

(b) The ocean surface is a water-air interface with high surface energy ($\gamma \approx 0.073\text{Nm}^{-1}$). When oil is placed on this surface, it creates oil-air and oil-water interfaces whose combined surface energy is less than the original water-air surface energy. Since physical systems tend toward minimum energy, the oil spreads spontaneously to replace the high-energy water-air surface with the lower-energy combination, continuing until it forms a thin film.

(c) For a solid sphere rolling without slipping, conservation of energy gives:

$$\begin{aligned} Mgh &= \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} Mv^2 + \frac{1}{2} \times \frac{2}{5} MR^2 \times \frac{v^2}{R^2} = \frac{1}{2} Mv^2 + \frac{1}{5} Mv^2 = \frac{7}{10} Mv^2 \\ v &= \sqrt{\frac{10gh}{7}} = \sqrt{\frac{10 \times 9.8\text{ms}^{-2} \times 3.0\text{m}}{7}} = 6.48\text{ms}^{-1} \end{aligned}$$

Question 7

(a) A spinning top has angular momentum $L = I\omega$ directed along its spin axis. While spinning fast, L is large. Any small disturbance (such as a slight tilt) produces a gravitational torque that attempts to change the direction of L . However, a large angular momentum resists rapid changes in direction, so the top remains stable. As friction slows the spin, ω and hence L

decrease. The same gravitational torque now produces a larger rate of change of the direction of L , the top wobbles increasingly, and eventually falls over.

(b) In a glass tube, the cohesive forces between mercury atoms are much stronger than the adhesive forces between mercury and glass. Consequently, mercury does not wet the glass surface. The angle of contact exceeds 90° , producing a convex meniscus. The surface tension force acts downward along the contact line (because the meniscus curves downward), pulling the mercury surface below the level of mercury outside the tube. Water, by contrast, has stronger adhesion to glass than cohesion, producing an angle of contact less than 90° and a capillary rise.

(c) Angular velocity:

$$\omega = 2\pi \times \frac{120}{60} = 4\pi \text{ rad s}^{-1}$$

Using the rotating disc torque formula:

$$\tau = \frac{\pi\eta\omega R^4}{2d} = \frac{\pi \times 0.80 \text{ Nsm}^{-2} \times 4\pi \text{ rad s}^{-1} \times (0.12 \text{ m})^4}{2 \times 1.0 \times 10^{-3} \text{ m}}$$

$$\tau = 3.27 \text{ Nm}$$

Question 8

(a) The rotational kinetic energy of a body is $\text{KE} = \frac{1}{2}I\omega^2$. The solid sphere has $I = \frac{2}{5}MR^2$ and the hollow shell has $I = \frac{2}{3}MR^2$. At the same angular velocity, the shell has a larger moment of inertia and therefore greater rotational kinetic energy. The work needed to stop a rotating body equals its kinetic energy (by the work-energy theorem). Since the shell has more kinetic energy, more work is needed to stop it.

(b) When air is blown between the two sheets, the air between them moves at high speed. The air on the outer sides of the sheets is stationary. By Bernoulli's equation, the fast-moving air between the sheets has lower pressure than the still air outside. The pressure difference across each sheet produces a net inward force, pushing the sheets toward each other.

(c) By Torricelli's theorem:

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ ms}^{-2} \times 4.0 \text{ m}} = 8.85 \text{ ms}^{-1}$$

Initial volume flow rate:

$$Q = av = 2.0 \times 10^{-4} \text{ m}^2 \times 8.85 \text{ ms}^{-1} = 1.77 \times 10^{-3} \text{ m}^3 \text{ s}^{-1}$$

Question 9

(a) Torque is the product of force and the perpendicular distance from the line of action of the force to the pivot: $\tau = F \times d$. When the same force F is applied near the handle (large d), the torque is large. Near the hinge (small d), the torque is small. Since the angular acceleration of the door is $\alpha = \tau/I$, a larger torque produces a greater angular acceleration. Therefore, the door opens more easily when pushed near the handle.

(b) On a waxed surface, the wax is hydrophobic. The cohesive forces within the water are stronger than the adhesive forces between water and wax. The angle of contact exceeds 90° , so water beads up into near-spherical drops. On an unwaxed surface, the adhesive forces between water and the paint are stronger than the cohesive forces. The angle of contact is small, so the water spreads and wets the surface.

(c) Terminal velocity: $v_t \propto r^2$ (when density and fluid are the same).

$$\frac{v_Q}{v_P} = \frac{(3r)^2}{r^2} = 9$$

The terminal velocity of Q is 9 times that of P .

Question 10

(a) When the main rotor spins, it exerts a torque on the air (pushing air downward and around). By Newton's third law, the air exerts an equal and opposite torque on the rotor. This reaction torque is transmitted through the rotor shaft to the helicopter body, tending to rotate the body in the opposite direction to the rotor. The tail rotor produces a sideways thrust that creates a counter-torque about the vertical axis, preventing the body from spinning.

(b) By Stokes' law, the terminal velocity of a particle in a fluid is $v_t \propto r^2$. Fine clay particles have very small radii, so their terminal velocities are extremely small (fractions of a millimetre per second). Even slight air currents keep them suspended. Coarse sand grains have much larger radii, so their terminal velocities are much higher (proportional to r^2), and they settle rapidly under gravity.

(c) Moment of inertia:

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 5.0 \text{ kg} \times (0.20 \text{ m})^2 = 0.10 \text{ kgm}^2$$

Angular deceleration:

$$\alpha = \frac{\tau}{I} = \frac{0.50 \text{ Nm}}{0.10 \text{ kg m}^2} = 5.0 \text{ rad s}^{-2}$$

Time to stop:

$$t = \frac{\omega_0}{\alpha} = \frac{10 \text{ rad s}^{-1}}{5.0 \text{ rad s}^{-2}} = 2.0 \text{ s}$$

Angular displacement:

$$\theta = \omega_0 t - \frac{1}{2} \alpha t^2 = 10 \times 2.0 - \frac{1}{2} \times 5.0 \times (2.0)^2 = 20 - 10 = 10 \text{ rad}$$

Number of revolutions:

$$n = \frac{\theta}{2\pi} = \frac{10 \text{ rad}}{2\pi} = 1.59 \approx 1.6 \text{ revolutions}$$

Question 11

(a) The moment of inertia of a disc about its centre is $I_c = \frac{1}{2} MR^2$. By the parallel axis theorem, the moment about an axis through the edge (distance R from the centre) is $I_e = I_c + MR^2 = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2$. Since $I_e = 3I_c$, the disc has three times the moment of inertia about the edge. By $\tau = I\alpha$, achieving the same angular acceleration requires three times the torque.

(b) When the river passes through the narrow arches, the continuity equation ($A_1 v_1 = A_2 v_2$) requires the water speed to increase. By Bernoulli's equation, the increased speed corresponds to a decrease in pressure. Since the water surface is exposed to the atmosphere, a lower pressure within the flowing water means the water surface drops to maintain equilibrium. Hence, the water level decreases at the narrow section.

(c) Given: $A = \pi(0.075 \text{ m})^2 = 1.767 \times 10^{-2} \text{ m}^2$, $a = \pi(0.025 \text{ m})^2 = 1.963 \times 10^{-3} \text{ m}^2$

Pressure difference from manometer:

$$\Delta P = (\rho' - \rho)gh = (13,600 - 1000) \text{ kg m}^{-3} \times 9.8 \text{ m s}^{-2} \times 0.03 \text{ m} = 3704 \text{ Pa}$$

$$A^2 - a^2 = (1.767 \times 10^{-2})^2 - (1.963 \times 10^{-3})^2 = 3.084 \times 10^{-4} \text{ m}^4$$

$$v_1 = \frac{a}{\sqrt{A^2 - a^2}} \times \sqrt{\frac{2\Delta P}{\rho}} = \frac{1.963 \times 10^{-3} \text{ m}^2}{1.756 \times 10^{-2} \text{ m}^2} \times \sqrt{\frac{2 \times 3704 \text{ Pa}}{1000 \text{ kg m}^{-3}}} = 0.304 \text{ m s}^{-1}$$

$$Q = Av_1 = 1.767 \times 10^{-2} \text{ m}^2 \times 0.304 \text{ m s}^{-1} = 5.37 \times 10^{-3} \text{ m}^3 \text{ s}^{-1}$$

Question 12

(a) Both balls have the same mass and speed, so their translational kinetic energies are equal. However, a rolling ball also has rotational kinetic energy $\frac{1}{2} I \omega^2$. The hollow ball has $I = \frac{2}{3} MR^2$ while the solid ball has $I = \frac{2}{5} MR^2$. At the same speed (and therefore same $\omega = v/R$), the hollow ball has more rotational kinetic energy. The work needed to stop a body equals its total kinetic energy. Since the hollow ball has more total energy, it is harder to stop.

(b) By the equation of continuity ($A_1 v_1 = A_2 v_2$), the volume of water passing through any cross-section per unit time must be the same. The narrow end has a much smaller cross-sectional area than the wide end. For the same volume to pass through per unit time, the water must move much faster through the narrow end. No additional force is needed; the speed increase is a direct consequence of mass conservation.

(c) By conservation of volume: $r = R/n^{1/3} = 2 \times 10^{-3} \text{ m}/8^{1/3} = 1 \times 10^{-3} \text{ m}$

Work done = increase in surface energy:

$$W = \gamma(8 \times 4\pi r^2 - 4\pi R^2) = 4\pi \times 0.5 \text{ N m}^{-1} \times (8 \times (10^{-3} \text{ m})^2 - (2 \times 10^{-3} \text{ m})^2)$$

$$W = 2\pi \times (8 \times 10^{-6} \text{ m}^2 - 4 \times 10^{-6} \text{ m}^2) = 2.51 \times 10^{-5} \text{ J}$$

Mass of original drop:

$$m = \frac{4}{3} \pi R^3 \rho = \frac{4}{3} \pi \times (2 \times 10^{-3} \text{ m})^3 \times 13,600 \text{ kg m}^{-3} = 4.571 \times 10^{-4} \text{ kg}$$

Temperature rise:

$$\Delta T = \frac{W}{mc} = \frac{2.51 \times 10^{-5} \text{ J}}{4.571 \times 10^{-4} \text{ kg} \times 140 \text{ J kg}^{-1} \text{ K}^{-1}} = 3.92 \times 10^{-4} \text{ K}$$

Question 13

(a) A rolling ball has both translational KE ($\frac{1}{2} Mv^2$) and rotational KE ($\frac{1}{2} I \omega^2$). The total kinetic energy is the sum of both. A sliding ball at the same speed has only translational KE ($\frac{1}{2} Mv^2$) and no rotational KE (since it does not rotate). The extra

energy in the rolling ball comes from the work done by friction at the contact point during the initial phase when rolling was established. Once rolling without slipping is achieved, friction does no further work, but the rotational energy it created remains.

(b) Viscosity in liquids arises from intermolecular attractive forces that resist the sliding of adjacent layers. At low temperature, molecules have low kinetic energy and intermolecular forces are strong, producing high resistance to flow. When heated, molecules gain kinetic energy and overcome these forces more easily. The resistance between layers decreases, the viscosity drops, and the honey flows more freely.

(c) For level flight, lift = weight:

$$\begin{aligned}\frac{1}{2}\rho A(v_1^2 - v_2^2) &= Mg \\ v_1^2 - v_2^2 &= \frac{2Mg}{\rho A} = \frac{2 \times 8000\text{kg} \times 9.8\text{ms}^{-2}}{1.2\text{kgm}^{-3} \times 30\text{m}^2} = 4356\text{m}^2\text{s}^{-2} \\ v_2^2 &= (90\text{ms}^{-1})^2 - 4356\text{m}^2\text{s}^{-2} = 3744\text{m}^2\text{s}^{-2} \\ v_2 &= \sqrt{3744\text{m}^2\text{s}^{-2}} = 61.2\text{ms}^{-1}\end{aligned}$$

Question 14

(a) A heavier flywheel has a larger moment of inertia I . Starting it from rest requires kinetic energy $\frac{1}{2}I\omega^2$, which is larger for larger I . Hence more energy (and time) is needed to bring it to operating speed. However, once running, the large I stores more kinetic energy. When the engine encounters a sudden load (such as a compression stroke), the flywheel releases stored energy to maintain the angular velocity nearly constant. A lighter flywheel stores less energy and allows greater speed fluctuations. Thus, a heavier flywheel smooths out variations in angular velocity.

(b) The fast-moving train drags nearby air along with it. By Bernoulli's equation, this fast-moving air has lower pressure than the still air on the far side of the person. The pressure difference across the person creates a net force directed toward the train.

(c) Height of rise:

$$h = \frac{2\gamma\cos\theta}{\rho g r} = \frac{2 \times 0.073\text{Nm}^{-1} \times 1}{1000\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 5 \times 10^{-4}\text{m}} = 0.0298\text{m} = 2.98\text{cm}$$

Since the tube is only 2cm long (shorter than 2.98cm), the water rises to the top but does not overflow. The meniscus at the top adjusts its curvature (becomes flatter), increasing the effective radius of curvature until the reduced excess pressure is just sufficient to support a 2cm column.

Question 15

(a) Before the child jumps, the total angular momentum is $L = I_{\text{total}}\omega$. When the child jumps off tangentially, she carries angular momentum with her ($L_{\text{child}} = mvr$, where r is the radius). By conservation of angular momentum, the remaining angular momentum of the merry-go-round must increase to compensate: $L_{\text{merry}} = L - L_{\text{child}}$. Since the merry-go-round now has a smaller moment of inertia (the child's mass is gone) but must retain its share of angular momentum, its angular velocity increases.

(b) A soap bubble has two surfaces (inner and outer), each with surface energy proportional to the surface area. Among all closed shapes enclosing a given volume, a sphere has the smallest surface area. Surface tension drives the film toward the configuration of minimum total surface energy. Therefore, regardless of the initial shape, the bubble relaxes into a sphere.

(c) Initial angular momentum:

$$L_i = I_{\text{disc}}\omega_i = \frac{1}{2} \times 8.0\text{kg} \times (0.40\text{m})^2 \times 6.0\text{rad s}^{-1} = 3.84\text{kgm}^2\text{s}^{-1}$$

After clay sticks at the rim, the new moment of inertia:

$$I_f = I_{\text{disc}} + m_{\text{clay}}R^2 = 0.64\text{kgm}^2 + 2.0\text{kg} \times (0.40\text{m})^2 = 0.96\text{kgm}^2$$

By conservation of angular momentum:

$$\omega_f = \frac{L_i}{I_f} = \frac{3.84\text{kgm}^2\text{s}^{-1}}{0.96\text{kgm}^2} = 4.0\text{rad s}^{-1}$$

Question 16

(a) The long pole has a large moment of inertia about the tightrope (because its mass is distributed far from the axis). By $\tau = I\alpha$, a large I means that any gravitational torque caused by a slight tilt produces only a small angular acceleration. This gives the walker more time to shift body weight and correct the tilt before it becomes dangerous. Without the pole, the smaller moment of inertia allows faster tilting, making balance harder.

(b) By Bernoulli's equation for horizontal flow: $P + \frac{1}{2}\rho v^2 = \text{constant}$. The total of pressure energy and kinetic energy per unit volume is constant along a streamline. At the wider section, the speed is lower (by continuity), so the kinetic energy term $\frac{1}{2}\rho v^2$ is smaller. For the sum to remain constant, the pressure P must be larger. At the narrow section, the speed is higher, so the pressure must be lower.

(c) Velocity gradient:

$$\frac{dv}{dy} = \frac{v}{d} = \frac{0.6\text{ms}^{-1}}{2.0 \times 10^{-3}\text{m}} = 300\text{s}^{-1}$$

By Newton's law of viscosity:

$$F = \eta A \frac{dv}{dy} = 0.30\text{Nsm}^{-2} \times 0.04\text{m}^2 \times 300\text{s}^{-1} = 3.6\text{N}$$

Question 17

(a) The point is the centre of mass of the hammer. External forces (gravity and air resistance) act on the body as a whole. The motion of the centre of mass depends only on these external forces, regardless of how the body rotates. Gravity produces a uniform downward acceleration, so the centre of mass follows a parabolic trajectory (just like a point particle). All other points rotate around the centre of mass while it follows this parabola, (which is why the hammer appears to tumble).

(b) By Poiseuille's formula, $Q \propto \Delta P \times R^4$. The flow rate depends on the fourth power of the radius but only the first power of the pressure. To triple Q by pressure alone (keeping R constant), ΔP must triple, which requires a 200% increase. To triple Q by increasing R alone (keeping ΔP constant), R^4 must triple, so R must increase by a factor of $3^{1/4} = 1.32$, only a 32% increase. The R^4 dependence makes radius changes far more effective than pressure changes.

(c) For a solid sphere rolling without slipping:

$$\begin{aligned} \frac{1}{2}Mv^2 + \frac{1}{2} \times \frac{2}{5}MR^2 \times \frac{v^2}{R^2} &= Mgh \\ \frac{7}{10}Mv^2 &= Mgh \\ h = \frac{7v^2}{10g} &= \frac{7 \times (4.0\text{ms}^{-1})^2}{10 \times 9.8\text{ms}^{-2}} = 1.14\text{m} \end{aligned}$$

Question 18

(a) Angular momentum is conserved: $I\omega = (I + I)\omega_f$, giving $\omega_f = \omega/2$. Initial KE: $\frac{1}{2}I\omega^2$. Final KE: $\frac{1}{2}(2I)(\omega/2)^2 = \frac{1}{4}I\omega^2$. Half the kinetic energy is lost. The lost energy is converted to heat by friction between the disc surfaces as they slide against each other before reaching the common angular velocity. Angular momentum is conserved because no external torque acts, but kinetic energy is not conserved because internal friction is an inelastic process.

(b) When the rubber bulb is squeezed, air rushes through the horizontal tube at high speed. At the junction with the vertical tube, the fast-moving air creates low pressure (by Bernoulli's principle). The atmospheric pressure on the liquid surface in the reservoir is greater than the pressure at the junction. This pressure difference pushes the liquid up the vertical tube. At the junction, the fast airstream breaks the liquid into fine droplets, producing a spray.

(c) Terminal velocity:

$$v_t = \frac{d}{t} = \frac{0.30\text{m}}{6.5\text{s}} = 0.0462\text{ms}^{-1}$$

Rearranging the terminal velocity formula:

$$\eta = \frac{2r^2(\rho_s - \rho_f)g}{9v_t} = \frac{2 \times (1.0 \times 10^{-3}\text{m})^2 \times (2500 - 800)\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 0.0462\text{ms}^{-1}} = 0.0801\text{Nsm}^{-2}$$

Question 19

(a) When the ball with backspin hits the ground, the point of contact has two velocity components: a forward translational velocity and a backward velocity due to the spin. If the backward spin velocity exceeds the forward translational velocity at the contact point, the friction force from the ground acts forward on the ball. This forward friction decelerates the forward motion, momentarily stops it, and then accelerates the ball backward. The ball reverses direction. The angular momentum about the contact point is partially transferred to linear momentum in the reverse direction.

(b) When pivoted at one end, the moment of inertia is $I_{\text{end}} = \frac{1}{3}ML^2$. When pivoted at the centre, $I_{\text{centre}} = \frac{1}{12}ML^2$. The restoring torque due to gravity depends on the displacement of the centre of mass from below the pivot. For the rod pivoted at one end, the centre of mass is $\frac{L}{2}$ from the pivot, giving torque $\tau = \frac{MgL\sin\theta}{2}$. The angular acceleration is $\alpha = \frac{\tau}{I} = \frac{\frac{MgL\sin\theta}{2}}{\frac{1}{3}ML^2} = \frac{3g\sin\theta}{2L}$. For the rod pivoted at the centre, there is no restoring torque (the centre of mass is at the pivot). The rod pivoted at the end swings; the one at the centre does not swing at all under gravity. Hence, the end-pivoted rod swings faster.

(c) Moment of inertia of the ring: $I_{\text{ring}} = mr^2 = 2.0\text{kg} \times (0.25\text{m})^2 = 0.125\text{kgm}^2$

Total initial angular momentum:

$$L = I_{\text{turntable}}\omega_i = 0.80\text{kgm}^2 \times 3.0\text{rad s}^{-1} = 2.4\text{kgm}^2\text{s}^{-1}$$

After the ring is placed on the turntable:

$$I_f = 0.80\text{kgm}^2 + 0.125\text{kgm}^2 = 0.925\text{kgm}^2$$

$$\omega_f = \frac{L}{I_f} = \frac{2.4\text{kgm}^2\text{s}^{-1}}{0.925\text{kgm}^2} = 2.59\text{rad s}^{-1}$$

Kinetic energy lost:

$$KE_i = \frac{1}{2} \times 0.80\text{kgm}^2 \times (3.0\text{rad s}^{-1})^2 = 3.6\text{J}$$

$$KE_f = \frac{1}{2} \times 0.925\text{kgm}^2 \times (2.59\text{rad s}^{-1})^2 = 3.10\text{J}$$

$$\Delta KE = 3.6\text{J} - 3.10\text{J} = 0.50\text{J}$$

Question 20

(a) When the cylinder rolls without slipping, gravitational PE is converted into both translational and rotational KE: $Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$. Some energy goes into spinning the cylinder. When the cylinder slides without rolling, all the PE converts to translational KE only: $Mgh = \frac{1}{2}Mv^2$. The sliding cylinder has more translational speed at the bottom because no energy is “lost” to rotation. Therefore, it reaches the bottom faster.

(b) By the equation of continuity ($Av = \text{constant}$), the product of cross-sectional area and flow speed is constant throughout the circulatory system. The aorta has a small total cross-sectional area, so blood flows quickly. Capillaries are individually very narrow, but there are billions of them. Their combined total cross-sectional area is vastly larger than the aorta's. For the same volume of blood to pass through per unit time, the speed in the capillaries must be correspondingly lower.

(c) The smaller bubble has greater excess pressure ($\Delta P = 4\gamma/r$). When connected, air flows from the smaller (higher pressure) bubble into the larger (lower pressure) bubble. The smaller bubble shrinks and the larger one grows.

Radius of curvature of the common interface:

$$\frac{1}{R} = \frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{0.02\text{m}} - \frac{1}{0.05\text{m}} = 50\text{m}^{-1} - 20\text{m}^{-1} = 30\text{m}^{-1}$$

$$R = \frac{1}{30}\text{m} = 0.0333\text{m} = 3.33\text{cm}$$

Question 21

(a) The spinning gyroscope has angular momentum $L = I\omega$ directed along its spin axis. Gravity exerts a torque about the support point, trying to pull the axle downward. This torque is perpendicular to L , so it changes the direction of L without changing its magnitude. The result is that the angular momentum vector rotates horizontally in a slow circle. This is precession. The gyroscope does not fall because the gravitational torque is entirely used up in changing the direction of the large angular momentum, not in creating a new downward rotation.

(b) When the capillary tube is lifted, the water column remains supported by surface tension at the meniscus inside the tube. The excess pressure across the curved meniscus ($\Delta P = \frac{2\gamma\cos\theta}{r}$) supports the weight of the water column. As long as the bottom of the tube remains submerged and the meniscus is intact, the water does not drain out because the surface tension force at the top meniscus balances the hydrostatic pressure of the column.

(b) Let the acceleration of the falling mass be a and the tension in the string be T .

For the hanging mass: $mg - T = ma \rightarrow 2 \times 9.8 - T = 2a \rightarrow 19.6 - T = 2a \dots (1)$

For the cylinder: $\tau = I\alpha \rightarrow TR = \frac{1}{2}MR^2 \times \frac{a}{R} \rightarrow T = \frac{1}{2}Ma = \frac{1}{2} \times 4\text{kg} \times a = 2a \dots (2)$

Substituting (2) into (1): $19.6\text{N} - 2a = 2a \rightarrow 4a = 19.6\text{N} \rightarrow a = 4.9\text{ms}^{-2}$

$$T = 2 \times 4.9 = 9.8\text{N}$$

Question 22

(a) The moment of inertia of a body about a given axis depends on how far its mass is distributed from that axis ($I = \sum m_i r_i^2$). For the T-shape and L-shape, the same total mass is distributed differently relative to the pivot at one end. The shape that has more mass farther from the pivot has a larger moment of inertia. By $\tau = I\alpha$, a larger I requires more torque for the same angular acceleration, making that shape harder to rotate.

(b) Paper towel fibres are hydrophilic (water-attracting). The adhesive forces between water and the fibres are stronger than the cohesive forces within the water. The angle of contact is small (less than 90°), so capillary rise occurs in the narrow spaces between fibres. Waxed paper has hydrophobic fibres. The cohesive forces in water are stronger than the adhesive forces with wax. The angle of contact exceeds 90° , so capillary rise does not occur.

(c) By continuity: $v_2 = v_1 \times r_1^2 / r_2^2 = 6\text{ms}^{-1} \times (3\text{cm})^2 / (6\text{cm})^2 = 1.5\text{ms}^{-1}$

By Bernoulli's equation:

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) - \rho g(h_2 - h_1)$$

$$P_2 = 2.5 \times 10^5 \text{Pa} + \frac{1}{2} \times 1000 \text{kgm}^{-3} \times ((6\text{ms}^{-1})^2 - (1.5\text{ms}^{-1})^2) - 1000 \text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 4\text{m}$$

$$P_2 = 2.28 \times 10^5 \text{Pa}$$

Question 23

(a) When the heavy mass is at one end, it creates a large gravitational torque about the pivot (the centre). Any slight displacement is amplified because the torque increases as the ruler tilts. The ruler is in unstable equilibrium and tips over. When the mass is at the centre (at the pivot), it contributes zero torque regardless of the tilt angle, because its distance from the pivot is zero. The ruler remains balanced because only the ruler's own symmetrically distributed weight acts, and this produces a restoring effect for small displacements.

(b) When a cricket ball spins while moving forward, it drags the surrounding air in the direction of spin. On one side, the spin adds to the oncoming airflow (faster air, lower pressure). On the opposite side, the spin opposes the airflow (slower air, higher pressure). The pressure difference produces a net sideways force (Magnus effect) that curves the ball's path.

(c) Total KE = translational KE + rotational KE:

$$\text{KE} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2} \times \frac{2}{3}MR^2 \times \frac{v^2}{R^2}$$

$$\text{KE} = \frac{1}{2}Mv^2 + \frac{1}{3}Mv^2 = \frac{3}{6}Mv^2 + \frac{2}{6}Mv^2 = \frac{5}{6}Mv^2$$

Question 24

(a) In an Atwood machine, the net torque on the pulley is $\tau = (m_1 - m_2)gR$. The angular acceleration is $\alpha = \tau / (I_{\text{pulley}} + (m_1 + m_2)R^2)$. A massive pulley has a large I_{pulley} , increasing the denominator and reducing α . The system accelerates more slowly because the net force must accelerate not only the hanging masses but also spin up the pulley. A light pulley contributes negligible I and the system accelerates as if the pulley were not there.

(b) The terminal velocity of a raindrop is $v_t \propto r^2$ (by Stokes' law). A large raindrop has a larger radius and therefore a much higher terminal velocity. The weight of a drop grows as r^3 while the viscous drag grows only as r . A larger drop must reach a higher speed before drag balances its weight. Hence, large drops fall faster.

(c) The soap film has two surfaces. Force on the wire:

$$F = 2\gamma l = 2 \times 0.03 \text{Nm}^{-1} \times 0.06\text{m} = 3.6 \times 10^{-3} \text{N}$$

Work done:

$$W = F \times d = 3.6 \times 10^{-3} \text{N} \times 0.04\text{m} = 1.44 \times 10^{-4} \text{J}$$

Question 25

(a) The sliding block has only translational kinetic energy: $\text{KE} = \frac{1}{2}Mv^2$. The rolling sphere has both translational and rotational kinetic energy: $\text{KE} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{5}Mv^2 = \frac{7}{10}Mv^2$. The extra $\frac{1}{5}Mv^2$ is rotational energy, which exists because the sphere is spinning. This extra energy was originally supplied by friction at the contact point when the rolling motion was first established.

(b) The excess pressure inside a soap bubble is $\Delta P = 4\gamma/r$, which is inversely proportional to the radius. A smaller bubble has a smaller r , giving a larger ΔP . Physically, the smaller bubble has a more tightly curved surface. The surface tension forces along this tightly curved surface have a larger inward component per unit area, requiring a greater internal pressure to maintain equilibrium.

(c) Using conservation of energy. Let the speed of the hanging mass be v after falling $h = 2.0\text{m}$:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2} \times \frac{1}{2}MR^2 \times \frac{v^2}{R^2} = \frac{1}{2}mv^2 + \frac{1}{4}Mv^2$$

$$5.0\text{kg} \times 9.8\text{ms}^{-2} \times 2.0\text{m} = \frac{1}{2} \times 5.0\text{kg} \times v^2 + \frac{1}{4} \times 10\text{kg} \times v^2$$

$$98\text{J} = 2.5v^2 + 2.5v^2 = 5.0v^2$$

$$v = \sqrt{\frac{98}{5.0}} \text{ms}^{-1} = 4.43 \text{ms}^{-1}$$

Question 26

(a) A gearbox conserves power: $P = \tau\omega$. In low gear, a small gear drives a large gear. The output angular velocity ω is low, but since power is conserved, the output torque $\tau = P/\omega$ is high. In high gear, a large gear drives a small gear. The output ω is high but τ is low. The gearbox trades angular velocity for torque (or vice versa) while conserving power (minus friction losses).

(b) The pitot tube has an opening facing directly into the airflow, bringing the air to rest at the stagnation point. By Bernoulli's equation, the stagnation pressure exceeds the static pressure by $\frac{1}{2}\rho v^2$. A manometer measures this pressure difference ΔP . The airspeed is then $v = \sqrt{\frac{2\Delta P}{\rho}}$.

(c) By Poiseuille's formula:

$$Q = \frac{\pi \Delta P R^4}{8 \eta L} = \frac{\pi \times 5 \times 10^4 \text{Pa} \times (3 \times 10^{-3} \text{m})^4}{8 \times 1.5 \text{Nsm}^{-2} \times 0.8 \text{m}} = 1.325 \times 10^{-6} \text{m}^3 \text{s}^{-1}$$

$$\text{Average velocity: } v_{\text{avg}} = \frac{Q}{\pi R^2} = 1.325 \times \frac{10^{-6} \text{m}^3 \text{s}^{-1}}{\pi \times 9 \times 10^{-6} \text{m}^2} = 0.0469 \text{ms}^{-1}$$

Reynolds number:

$$R_e = \frac{\rho v_{\text{avg}} \times 2R}{\eta} = \frac{1260 \text{kgm}^{-3} \times 0.0469 \text{ms}^{-1} \times 6 \times 10^{-3} \text{m}}{1.5 \text{Nsm}^{-2}} = 0.237$$

Since $R_e = 0.237 \ll 2000$, the flow is **laminar**.

Question 27

(a) In a rigid body, all particles are locked in position relative to each other. When the body rotates, every particle moves in a circle about the axis. All particles complete one full revolution in the same time, so they all have the same angular velocity ω . However, each particle's linear speed is $v = \omega r$, where r is its distance from the axis. Particles farther from the axis travel larger circles in the same time, so they move at higher linear speeds.

(b) By Torricelli's theorem, the efflux speed from a hole at depth h below the surface is $v = \sqrt{2gh}$. The lower hole is at greater depth, so its efflux speed is higher. After exiting horizontally, the water follows a projectile path. Although the lower hole has less fall height (less time in the air), its much greater horizontal speed more than compensates, and the jet strikes the ground further from the tank.

$$(c) \text{Moment of inertia: } I = \frac{1}{2}MR^2 = \frac{1}{2} \times 6.0 \text{kg} \times (0.30 \text{m})^2 = 0.27 \text{kgm}^2$$

$$\text{Rotational KE: } KE = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.27 \text{kgm}^2 \times (8.0 \text{rad s}^{-1})^2 = 8.64 \text{J}$$

$$\text{Angular deceleration: } \alpha = \frac{\omega}{t} = \frac{8.0 \text{rad s}^{-1}}{4.0 \text{s}} = 2.0 \text{rad s}^{-2}$$

$$\text{Torque: } \tau = I\alpha = 0.27 \text{kgm}^2 \times 2.0 \text{rad s}^{-2} = 0.54 \text{Nm}$$

Question 28

(a) When two hands push in opposite directions on opposite sides of the wheel, they form a couple: two equal, opposite, parallel forces. A couple produces pure rotation with no net translational force. The total torque of the couple is $\tau = F \times d$, where d is the diameter of the wheel (the distance between the forces). With one hand, the torque is $\tau = F \times r$ (only the radius). Since the diameter is twice the radius, the couple provides twice the torque for the same force, making it more effective.

(b) The soap film has two surfaces, each with surface tension γ . Surface tension acts to minimise the total surface area. The sliding wire is free to move, so the surface tension forces pull it inward (toward the frame), reducing the film area and hence the total surface energy. The force on the wire is $F = 2\gamma l$, where l is the wire length.

(c) Sphere density = 2ρ , fluid density = ρ :

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta} = \frac{2r^2(2\rho - \rho)g}{9\eta} = \frac{2r^2\rho g}{9\eta}$$

Question 29

(a) At the instant the ball is momentarily at rest on the incline, both its translational and rotational kinetic energies are zero ($v = 0$ and $\omega = 0$, since $v = R\omega$ for rolling without slipping). All the initial kinetic energy (translational + rotational) has been converted into gravitational potential energy Mgh , where h is the height gained on the incline.

(b) Both have the same mass M and the same angular velocity ω . The rotational kinetic energy is $KE = \frac{1}{2}I\omega^2$. The solid disc has $I = \frac{1}{2}MR^2$ while the thin ring has $I = MR^2$. Since $I_{\text{ring}} > I_{\text{disc}}$, the ring has greater kinetic energy at the same ω . The ring stores more energy because all its mass is concentrated at the maximum distance from the axis, maximising the mr^2 contribution.

(c) For rolling without slipping, $Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ with $\omega = \frac{v}{R}$:

$$\text{Cylinder } (I = \frac{1}{2}MR^2): Mgh = \frac{1}{2}Mv_c^2 + \frac{1}{4}Mv_c^2 = \frac{3}{4}Mv_c^2 \rightarrow v_c = \sqrt{\frac{4gh}{3}}$$

$$\text{Sphere } (I = \frac{2}{5}MR^2): Mgh = \frac{1}{2}Mv_s^2 + \frac{1}{5}Mv_s^2 = \frac{7}{10}Mv_s^2 \rightarrow v_s = \sqrt{\frac{10gh}{7}}$$

$$\frac{v_s}{v_c} = \frac{\sqrt{\frac{10gh}{7}}}{\sqrt{\frac{4gh}{3}}} = \sqrt{\frac{10}{7} \cdot \frac{3}{4}} = \sqrt{\frac{10}{4}} = 1.035$$

The sphere is about 3.5% faster than the cylinder.

Question 30

(a) Static friction at the contact point is essential for rolling without slipping. It provides the torque that maintains the rotational motion in step with the translational motion ($v = R\omega$). Without friction (on a smooth surface), the ball would slide without rotating. However, rolling friction (a separate, smaller effect caused by deformation at the contact point) dissipates energy and gradually slows both the translation and rotation. Eventually, all kinetic energy is converted to heat and the ball stops.

(b) Repeated heating at high temperatures causes the oil molecules to undergo chemical changes: polymerisation (molecules joining into longer chains) and oxidation. Longer-chain molecules have stronger intermolecular forces and resist sliding past each other more strongly. This increases the viscosity of the oil, making it thicker and harder to pour.

(c) Let a be the acceleration, T_1 and T_2 the tensions on the 5.0kg and 3.0kg sides.

$$\text{Pulley: } I = \frac{1}{2} \times 3.0\text{kg} \times (0.20\text{m})^2 = 0.06\text{kgm}^2$$

$$\text{For 5.0kg mass: } 5 \times 9.8 - T_1 = 5a \rightarrow 49 - T_1 = 5a \dots (1)$$

$$\text{For 3.0kg mass: } T_2 - 3.0 \times 9.8 = 3a \rightarrow T_2 - 29.4 = 3a \dots (2)$$

$$\text{For pulley: } (T_1 - T_2)R = I\alpha = \frac{Ia}{R} \rightarrow T_1 - T_2 = \frac{Ia}{R^2} = 0.06 \times \frac{a}{0.04} = 1.5a \dots (3)$$

$$\text{Adding (1) + (2): } 49 - 29.4 - (T_1 - T_2) = 8.0a$$

$$\text{Substituting (3): } 19.6 - 1.5a = 8.0a \rightarrow 9.5a = 19.6 \rightarrow a = 2.06\text{ms}^{-2}$$

$$T_1 = 49 - 5.0 \times 2.06 = 49 - 10.3 = 38.7\text{N}$$

$$T_2 = 29.4 + 3.0 \times 2.06 = 29.4 + 6.18 = 35.6\text{N}$$

Question 31

(a) The rotational analogue of Newton's second law is $\tau = I\alpha$, where $\alpha = \frac{\Delta\omega}{\Delta t}$. Rearranging: $\tau\Delta t = I\Delta\omega$. The product $\tau\Delta t$ is the angular impulse. For a constant torque, doubling the time doubles the angular impulse and therefore doubles the change in angular velocity. This is the rotational equivalent of the linear impulse-momentum theorem ($F\Delta t = m\Delta v$).

(b) Bernoulli's principle: The nozzle narrows the exit area. By continuity, the water speed increases at the nozzle. By Bernoulli's equation, the pressure drops, and the water emerges as a fast jet. Newton's third law: The hose pushes water forward at high momentum. By Newton's third law, the water pushes back on the hose with an equal and opposite force. This reaction force causes the hose to recoil backward.

$$\text{(c) Excess pressure (one surface): } \Delta P = \frac{2\gamma}{r} = 2 \times \frac{0.073\text{Nm}^{-1}}{5 \times 10^{-4}\text{m}} = 292\text{Pa}$$

$$\text{Hydrostatic pressure at depth 20m: } P_h = \rho gh = 1000\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 20\text{m} = 196,000\text{Pa}$$

Total pressure:

$$P = P_{\text{atm}} + P_h + \Delta P = 1.01 \times 10^5\text{Pa} + 1.96 \times 10^5\text{Pa} + 292\text{Pa} = 2.97 \times 10^5\text{Pa}$$

Question 32

(a) A long rod has a larger moment of inertia about its base than a short rod of the same mass. By $\tau = I\alpha$, the gravitational torque (which causes tilting) produces a smaller angular acceleration in the long rod. The long rod tilts more slowly, giving the person more time to make corrective adjustments with the finger. A short rod tilts quickly and is harder to correct in time.

(b) By Poiseuille's formula, $Q = \pi \Delta P R^4 / (8 \eta L)$. The flow rate depends on the fourth power of the straw radius. Doubling the radius increases the flow by a factor of $2^4 = 16$. A thick milkshake has high viscosity η , which already reduces Q . Through a narrow straw (small R), the R^4 factor makes the flow rate extremely small. A wider straw dramatically increases R^4 and restores a usable flow rate.

(c) Moment of inertia: $I = \frac{2}{5} MR^2 = \frac{2}{5} \times 3.0 \text{ kg} \times (0.10 \text{ m})^2 = 0.012 \text{ kg m}^2$

Angular momentum: $L = I\omega = 0.012 \text{ kg m}^2 \times 20 \text{ rad s}^{-1} = 0.24 \text{ kg m}^2 \text{ s}^{-1}$

Torque to stop in 5.0s: $\tau = \frac{L}{t} = \frac{0.24 \text{ kg m}^2 \text{ s}^{-1}}{5.0 \text{ s}} = 0.048 \text{ Nm}$

Question 33

(a) On a smooth incline, there is no friction at the contact point. The only force along the incline is the component of gravity, which accelerates the ball's centre of mass downward. With no friction, there is no torque about the centre of mass, so the ball does not rotate. On a rough incline, static friction acts at the contact point, directed up the incline. This friction provides a torque about the centre of mass that initiates and maintains rotation. The friction does not dissipate energy (it is static, with no sliding), but it converts some translational energy into rotational energy, producing rolling without slipping.

(b) In free fall, no external contact forces act on the drop. The only force shaping it is surface tension, which minimises the surface area. For a given volume, a sphere has the smallest surface area. Hence, the drop is perfectly spherical. On a table, the drop's weight presses its base flat against the surface, and the normal reaction from the table distorts the shape. The drop is flattened because gravity competes with surface tension.

(c) The hole is at depth 2m below the surface, so the efflux speed is:

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ ms}^{-2} \times 2 \text{ m}} = \sqrt{39.2 \text{ m}^2 \text{ s}^{-2}} = 6.26 \text{ ms}^{-1}$$

The hole is at height $6 \text{ m} - 2 \text{ m} = 4 \text{ m}$ above the ground. Time to fall:

$$t = \sqrt{\frac{2H}{g}} = \sqrt{2 \times \frac{4 \text{ m}}{9.8 \text{ ms}^{-2}}} = 0.903 \text{ s}$$

Horizontal distance: $x = vt = 6.26 \text{ ms}^{-1} \times 0.903 \text{ s} = 5.65 \text{ m}$

Question 34

(a) When the disc rotates about an axis perpendicular to its face (through the centre), every particle of the disc is at its maximum possible distance from the axis (the full radial distance r). When the disc rotates about a diameter, particles near the diameter are very close to the axis and contribute little to I , while only particles near the rim contribute significantly. On average, less mass is far from the axis when rotating about a diameter. Since $I = \sum m_i r_i^2$, the perpendicular-face axis gives a larger I and greater resistance to rotation.

(b) In untilled soil, narrow spaces between soil particles act as capillary tubes, drawing water upward from moist subsoil to the dry surface, where it evaporates. Tilling breaks the top layer into larger, irregular clumps with much wider gaps. Capillary rise is inversely proportional to gap width ($h \propto 1/r$). The wide gaps cannot sustain capillary rise, so the connection between the moist subsoil and the surface is broken. The moisture stays underground.

(c) Velocity gradient: $\frac{dv}{dy} = \frac{v}{d} = \frac{0.50 \text{ ms}^{-1}}{5 \times 10^{-4} \text{ m}} = 1000 \text{ s}^{-1}$

Force: $F = \eta A \left(\frac{dv}{dy} \right) = 0.40 \text{ N s m}^{-2} \times 0.04 \text{ m}^2 \times 1000 \text{ s}^{-1} = 16 \text{ N}$

Question 35

(a) Linear momentum is conserved because no external horizontal force acts on the system (the cart is on a smooth surface). The forward momentum of the ball equals the backward momentum of the cart-cannon system. Angular momentum about the barrel axis is not conserved because gravity exerts an external torque on the system about that axis (the ball follows a projectile path after leaving the barrel, and gravity acts on it throughout the flight).

(b) The flow rate from the tap is governed by Torricelli's theorem: $v = \sqrt{2gh}$, where h is the height of water above the tap. As water drains out, the water level drops, h decreases, and the efflux speed v decreases. Since the volume flow rate is $Q = av$ (where a is the tap area), a lower v means a lower Q . The flow rate decreases continuously as the tank empties.

(c) For rolling without slipping: $a = R\alpha$ and $v = R\omega$.

Forces along the incline: $Mg \sin \theta - f = Ma \dots (1)$

Torque about centre: $fR = I\alpha = \frac{1}{2} MR^2 \times a/R \rightarrow f = \frac{1}{2} Ma \dots (2)$

Substituting (2) into (1): $Mg \sin \theta - \frac{1}{2} Ma = Ma \rightarrow Mg \sin \theta = \frac{3}{2} Ma$

$$a = \frac{2g \sin \theta}{3}$$

Question 36

(a) Torque equals force times perpendicular distance from the pivot: $\tau = Fd$. A long-handled spanner has a larger distance d from the hand to the bolt. For the same applied force F , a larger d produces a larger torque. Since the bolt resists turning with a fixed frictional torque, a larger applied torque is more likely to overcome this resistance.

(b) Surface tension pulls every part of the liquid surface inward, minimising the total surface area. Among all closed shapes enclosing a given volume, a sphere has the smallest surface area. Consequently, surface tension moulds the drop into a sphere. The “teardrop” shape is incorrect for free-falling drops; it occurs only when a drop is attached to a surface and being elongated by gravity just before detaching.

(c) From mass and density: $m = \frac{4}{3}\pi r^3 \rho_s$

$$r = \left(\frac{3m}{4\pi\rho_s}\right)^{1/3} = \left(\frac{3 \times 2.0\text{kg}}{4\pi \times 7800\text{kgm}^{-3}}\right)^{1/3} = 0.0396\text{m}$$

Terminal velocity:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta} = \frac{2 \times (0.0396\text{m})^2 \times (7800 - 900)\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 0.50\text{Nsm}^{-2}}$$

$$v_t = 47.1\text{ms}^{-1}$$

Note: This very high terminal velocity confirms that Stokes' law would not be valid for this large, fast sphere (the Reynolds number would be far above 1). In practice, the actual terminal velocity would be much lower due to turbulent drag.

Question 37

(a) The child's push is applied as a force at the rim for a certain time. By the angular impulse-momentum theorem ($\tau\Delta t = \Delta L$), the push delivers an angular impulse that increases the angular momentum of the roundabout. Even after the child stops pushing, the roundabout has gained angular momentum (and hence angular velocity) from the impulse already delivered. It does not decelerate immediately because the only remaining torque is friction, which is typically much smaller than the child's push. The roundabout continues to accelerate briefly if the angular velocity has not yet caught up with the impulse already received, then friction gradually decelerates it.

(b) Water between the two ships is confined to a narrow channel. By the equation of continuity, the water in this narrow space flows faster than the water on the outer sides of the ships. By Bernoulli's equation, faster-moving water has lower pressure. The pressure between the ships is therefore lower than the pressure on the outer sides. This pressure difference pushes both ships inward, toward each other.

(c) Angular deceleration:

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{4.0 - 10}{8.0}\text{rad s}^{-2} = -0.75\text{rad s}^{-2}$$

Frictional torque: $\tau = I|\alpha| = 0.60\text{kgm}^2 \times 0.75\text{rad s}^{-2} = 0.45\text{Nm}$

Angular displacement:

$$\theta = \omega_i t + \frac{1}{2}\alpha t^2 = 10 \times 8.0 + \frac{1}{2}(-0.75)(8.0)^2 = 56\text{rad}$$

Revolutions: $n = \frac{56}{2\pi} = 8.91 \approx 8.9$ revolutions

Energy dissipated:

$$\Delta KE = \frac{1}{2}I\omega_i^2 - \frac{1}{2}I\omega_f^2 = \frac{1}{2} \times 0.60\text{kgm}^2 \times ((10)^2 - (4)^2)\text{rad}^2\text{s}^{-2} = 0.30 \times 84 = 25.2\text{J}$$

Question 38

(a) Friction between the coin's edge and the table surface dissipates kinetic energy. Air resistance also removes energy, though more slowly. As the rotational kinetic energy decreases, the angular momentum decreases, and the gyroscopic stability that kept the coin upright weakens. Eventually, the angular momentum is too small to resist the gravitational torque, the coin begins to wobble (precession with increasing amplitude), and it falls flat.

(b) The excess pressure inside the soap bubble is $\Delta P = 4\gamma/r$. This pressure is higher than atmospheric pressure outside. When the blowing tube is left open, the air inside the bubble has a direct path to the atmosphere. The higher internal pressure pushes air out through the tube. As air escapes, the bubble's volume decreases and it contracts. The bubble shrinks until the internal pressure equalises with the atmosphere (which, for a soap film, means the bubble disappears entirely).

(c) In series, the flow rate Q is the same through both tubes. By Poiseuille's formula:

$$\Delta P = \frac{8\eta LQ}{\pi R^4}$$

For tube 1 (radius R): $\Delta P_1 = \frac{8\eta LQ}{\pi R^4}$

For tube 2 (radius 2R): $\Delta P_2 = \frac{8\eta LQ}{\pi(2R)^4} = \frac{8\eta LQ}{16\pi R^4}$

$$\frac{\Delta P_1}{\Delta P_2} = \frac{16\pi R^4}{\pi R^4} = 16$$

The pressure drop across the narrow tube is **16 times** that across the wide tube.

Question 39

(a) At the same translational speed v , both bodies also rotate at $\omega = v/R$ (rolling condition). The total kinetic energy of a rolling body is $KE = \frac{1}{2}Mv^2(1 + I/(MR^2))$. For the sphere ($I = \frac{2}{5}MR^2$): $KE = \frac{7}{10}Mv^2$. For the cylinder ($I = \frac{1}{2}MR^2$): $KE = \frac{3}{4}Mv^2$. The cylinder has more total kinetic energy because its moment of inertia is larger. Under the same rolling friction (which dissipates energy at the same rate for both at any given speed), the cylinder has more energy to lose before stopping. Therefore, the cylinder takes longer to stop.

(b) The viscosity of oil increases as temperature drops (viscosity of liquids is inversely related to temperature). In winter, the oil is colder and more viscous. By Poiseuille's formula ($Q = \pi\Delta PR^4/(8\eta L)$), the flow rate is inversely proportional to viscosity. Higher η in winter means lower Q , even though ΔP , R , and L are unchanged.

(c) Angular acceleration: $\alpha = \frac{(\omega - \omega_0)}{t} = \frac{(50 - 0)\text{rad s}^{-1}}{10\text{s}} = 5.0\text{rad s}^{-2}$

Torque: $\tau = I\alpha = 0.40\text{kg m}^2 \times 5.0\text{rad s}^{-2} = 2.0\text{Nm}$

Power at end: $P = \tau\omega = 2.0\text{Nm} \times 50\text{rad s}^{-1} = 100\text{W}$

Question 40

(a) By conservation of angular momentum (no external torque): $I\omega = (I + I)\omega_f$, giving $\omega_f = \omega/2$. The angular velocity halves. Initial KE: $\frac{1}{2}I\omega^2$. Final KE: $\frac{1}{2}(2I)(\omega/2)^2 = \frac{1}{4}I\omega^2$. Half the kinetic energy is lost, converted to heat by friction between the disc surfaces as they slide against each other before reaching the common velocity.

(b) By the equation of continuity ($A_1v_1 = A_2v_2$), the same volume of water must pass through the narrow gap per unit time as through the wide river. Since the gap has a much smaller cross-sectional area, the water must move much faster to maintain the same volume flow rate.

(c) $\frac{1}{2}\rho_{\text{air}}v^2 = \rho_{\text{water}}gh$

$$v = \sqrt{\frac{2\rho_{\text{water}}gh}{\rho_{\text{air}}}} = \sqrt{\frac{2 \times 1000\text{kg m}^{-3} \times 9.8\text{ms}^{-2} \times 0.08\text{m}}{1.2\text{kg m}^{-3}}} = 36.1\text{ms}^{-1}$$

Question 41

(a) When a ball with topspin hits the ground, the spinning surface at the contact point moves forward (in the same direction as the ball's translational motion). Friction at the contact point acts backward on the ball (opposing the relative motion of the contact surface). This backward friction has two effects: it slightly reduces the ball's forward speed, and it increases the forward rotation. The net result is that after the bounce, the ball has more forward spin and a flatter (lower) trajectory angle than a non-spinning ball, because friction redirected some of the vertical bounce energy into forward motion.

(b) By Poiseuille's formula, the flow rate through a channel is proportional to R^4 (the fourth power of the effective radius). Replacing the wide channel with a narrower one drastically reduces R . Since the flow depends on R^4 , even a modest reduction in width causes a dramatic reduction in flow rate.

(c) Moment of inertia of sphere about its centre: $I_{\text{cm}} = \frac{2}{5}MR^2 = \frac{2}{5} \times 2.5\text{kg} \times (0.08\text{m})^2 = 6.4 \times 10^{-3}\text{kg m}^2$

Distance from centre of sphere to pivot: $d = 0.60\text{m}$

By the parallel axis theorem:

$$I_{\text{pivot}} = I_{\text{cm}} + Md^2 = 6.4 \times 10^{-3}\text{kg m}^2 + 2.5\text{kg} \times (0.60\text{m})^2 = 0.906\text{kg m}^2$$

Question 42

(a) The excess pressure inside a soap bubble is $\Delta P = 4\gamma/r$. The smaller bubble has a larger excess pressure (smaller r , larger ΔP). When connected, air flows from the region of higher pressure (small bubble) to the region of lower pressure (large bubble). The small bubble shrinks (increasing its pressure further) and the large bubble grows (decreasing its pressure). This is a positive feedback process that continues until the small bubble disappears entirely.

(b) Consider a small cube of fluid at rest at depth h . If the pressure on one face were greater than on the opposite face, there would be a net force on the cube, and it would accelerate. Since the fluid is at rest (no acceleration), the pressure on all faces must be equal. This argument holds for any orientation of the cube, proving that pressure at a point in a static fluid is the same in all directions (Pascal's result).

(c) Total KE on the horizontal surface:

$$KE = \frac{7}{10} Mv^2 = \frac{7}{10} \times 1.5\text{kg} \times (5.0\text{ms}^{-1})^2 = 26.25\text{J}$$

On the incline, all KE converts to PE: $KE = Mgh = Mg \times d \sin 30^\circ$

$$d = \frac{KE}{Mg \sin 30^\circ} = \frac{26.25\text{J}}{1.5\text{kg} \times 9.8\text{ms}^{-2} \times 0.5} = 3.57\text{m}$$

Question 43

(a) The sand falls vertically and has zero angular momentum about the platform's axis before landing. When it lands on the rotating platform, friction accelerates it to the platform's angular velocity, giving it angular momentum. By conservation of angular momentum for the system, the angular momentum gained by the sand must come from the platform. The platform's angular momentum decreases. Since $L = I\omega$ and the total I has increased (added mass), ω decreases.

(b) Inside the pipe, the water is under pressure (greater than atmospheric). At the leak, the pressurised water pushes outward against atmospheric pressure. Since the internal pressure exceeds atmospheric pressure, water is forced out through the leak. Air would only be drawn in if the internal pressure were lower than atmospheric, which does not occur in a pressurised water pipe.

(c) Before impact, the bullet has linear momentum $p = mv = 0.05\text{kg} \times 200\text{ms}^{-1} = 10\text{kgms}^{-1}$.

Its angular momentum about the pivot (centre of rod) at distance $L/2 = 0.40\text{m}$:

$$L_i = mv \times \frac{L}{2} = 10\text{kgms}^{-1} \times 0.40\text{m} = 4.0\text{kgm}^2\text{s}^{-1}$$

After impact, the bullet is embedded at the end. Total moment of inertia:

$$I = I_{\text{rod}} + m_{\text{bullet}} \left(\frac{L}{2}\right)^2 = \frac{1}{12} ML^2 + m \left(\frac{L}{2}\right)^2$$

$$I = \frac{1}{12} \times 2.0\text{kg} \times (0.80\text{m})^2 + 0.05\text{kg} \times (0.40\text{m})^2 = 0.1147\text{kgm}^2$$

By conservation of angular momentum:

$$\omega = \frac{L_i}{I} = \frac{4.0\text{kgm}^2\text{s}^{-1}}{0.1147\text{kgm}^2} = 34.9\text{rad s}^{-1}$$

Question 44

(a) The spinning gyroscope has a large angular momentum $L = I\omega$ directed along its spin axis. By the law of conservation of angular momentum, L remains constant in direction unless an external torque acts. When the aircraft turns or tilts, no torque is applied to the gyroscope (it is mounted on gimbals). The gyroscope's spin axis continues to point in the original direction, providing a fixed reference against which the aircraft's orientation can be measured.

(b) During high winds, the fast-moving air outside the building has lower pressure than the still air inside (by Bernoulli's equation). The internal air pressure exceeds the external air pressure on the windward side. This pressure difference pushes windows outward from inside. If the pressure difference exceeds the strength of the glass, the window shatters outward.

(c) Terminal velocity: $v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta}$. Since r , g , and η are the same for all three:

$$v_t \propto (\rho_s - \rho_f)$$

$$v_A \propto (8000 - 1000) = 7000, v_B \propto (4000 - 1000) = 3000, v_C \propto (2000 - 1000) = 1000$$

Ranking: $v_A > v_B > v_C$

$$v_A : v_B : v_C = 7000 : 3000 : 1000 = 7 : 3 : 1$$

Question 45

(a) The oil film lies between the rotating disc and the fixed surface below. The oil layer in contact with the disc moves with the disc, while the oil layer in contact with the fixed surface remains stationary. By Newton's law of viscosity, the velocity difference between these layers produces a viscous drag force that opposes the disc's rotation. This drag force creates a retarding torque on the disc. The torque does negative work on the disc, steadily converting its rotational kinetic energy into heat energy in the oil. As the kinetic energy decreases, the angular velocity decreases. The disc eventually stops when all its rotational kinetic energy has been transferred to the oil as heat.

(b) The pressure of water at a depth h below the surface is $P = \rho gh$. The pressure increases linearly with depth. At the base of the dam, the water pressure is maximum. The dam wall must withstand this pressure. A thicker base provides greater structural strength to resist the larger force per unit area at greater depth. Near the top, the water pressure is small and a thinner wall suffices.

(c) On the table, the ball has translational and rotational KE:

$$KE_{\text{table}} = \frac{7}{10}Mv^2 = \frac{7}{10} \times 0.30\text{kg} \times (3.0\text{ms}^{-1})^2 = 1.89\text{J}$$

During the fall, the ball gains translational KE from gravity (rotational KE is unchanged since gravity exerts no torque about the centre of mass):

$$KE_{\text{fall}} = Mgh = 0.30\text{kg} \times 9.8\text{ms}^{-2} \times 1.5\text{m} = 4.41\text{J}$$

Total KE just before hitting the ground:

$$KE_{\text{total}} = KE_{\text{table}} + KE_{\text{fall}} = 1.89\text{J} + 4.41\text{J} = 6.30\text{J}$$

This total KE is the sum of translational and rotational:

$$KE_{\text{total}} = \frac{1}{2}Mv_{\text{final}}^2 + \frac{1}{2}I\omega^2$$

The rotational KE is unchanged from the table: $\frac{1}{2}I\omega^2 = \frac{1}{5}Mv_{\text{table}}^2 = \frac{1}{5} \times 0.30 \times 9 = 0.54\text{J}$

$$\frac{1}{2}Mv_{\text{final}}^2 = KE_{\text{total}} - \frac{1}{2}I\omega^2 = 6.30\text{J} - 0.54\text{J} = 5.76\text{J}$$

$$v_{\text{final}} = \sqrt{\frac{2 \times 5.76\text{J}}{0.30\text{kg}}} = 6.2\text{ms}^{-1}$$

Note: This is the translational speed. The total speed (including the tangential speed due to rotation at the surface) involves the spin, but in examination context, the “speed of the ball” refers to the speed of the centre of mass, which is 6.2ms^{-1} .

Question 46

(a) The hard-boiled egg is a rigid body. When stopped, every particle inside it stops simultaneously. When released, there is no internal motion to restart the spin.

The raw egg has a liquid interior. When the shell is stopped, the liquid inside continues to rotate (it was not in direct contact with the finger). When the finger is released, the still-rotating liquid drags the shell back into motion through viscous friction between the liquid and the inner wall of the shell.

(b) At terminal velocity, the net force on the ball is zero: the weight equals the sum of the upthrust and viscous drag. By Stokes' law, the viscous drag is $F = 6\pi\eta rv_t$. A higher viscosity produces a larger drag force at any given speed, so the ball reaches terminal velocity at a lower speed and takes longer to fall the same distance. The liquid in which the ball takes longer has the higher viscosity.

(c) Given: $h = 1.3\text{m}$, $r = 0.18\text{mm} = 1.8 \times 10^{-4}\text{m}$, $L = 0.03\text{m}$, $Q = 4.5\text{cm}^3/\text{min} = 7.5 \times 10^{-8}\text{m}^3\text{s}^{-1}$

Pressure difference driving the flow:

$$\Delta P = \rho gh = 1060\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 1.3\text{m} = 13504\text{Pa}$$

By Poiseuille's formula:

$$\eta = \frac{\pi\Delta Pr^4}{8QL} = \frac{\pi \times 13504\text{Pa} \times (1.8 \times 10^{-4}\text{m})^4}{8 \times 7.5 \times 10^{-8}\text{m}^3\text{s}^{-1} \times 0.03\text{m}} = 2.33 \times 10^{-3}\text{Nsm}^{-2}$$

Question 47

(a) When rolling without slipping, gravitational PE converts into both translational and rotational KE. The fraction going to rotation depends on the moment of inertia. The solid sphere has $I = \frac{2}{5}MR^2$ and the solid cylinder has $I = \frac{1}{2}MR^2$. The cylinder has a larger moment of inertia, so it diverts a greater fraction of its energy into rotation, leaving less for translation. The sphere retains more translational KE and reaches the bottom with the greater translational speed.

(b) Gas flows at high speed through the narrow jet at the base of the burner. By Bernoulli's equation, the fast-moving gas creates a region of low pressure at the jet. The atmospheric pressure outside the side hole is greater than the pressure at the jet. This pressure difference pushes air through the side hole into the barrel, where it mixes with the gas before combustion.

(c) By Torricelli's theorem: $v = \sqrt{2gh} = \sqrt{2 \times 9.8\text{ms}^{-2} \times 10\text{m}} = 14\text{ms}^{-1}$

Required flow rate: $Q = 26.4\text{m}^3/\text{min} = 26.4/60 = 0.44\text{m}^3\text{s}^{-1}$

$$Q = Av = \frac{\pi d^2}{4} \times v$$

$$d = \sqrt{\frac{4Q}{\pi v}} = \sqrt{\frac{4 \times 0.44\text{m}^3\text{s}^{-1}}{\pi \times 14\text{ms}^{-1}}} = 0.20\text{m}$$

The outlet diameter is 0.20m (20 cm).

Question 48

(a) When the ball strikes the step, the step exerts a reaction force at the contact point (the edge of the step). The torque of this reaction force about the ball's centre determines whether the ball rotates over the step or is pushed back.

If the ball's centre is above the step edge, the reaction force at the edge passes below the centre. This produces a torque that rotates the ball forward over the step. If the ball's centre is below the step edge, the reaction force passes above the centre. This produces a torque that rotates the ball backward, and the ball bounces back.

(b) At low pouring speed, the Reynolds number of the flow is small and the flow is laminar: smooth, orderly, and quiet. When the bottle is tilted steeply, the flow speed increases, raising the Reynolds number. When it exceeds the critical value (approximately 2000), the flow transitions from laminar to turbulent. Turbulent flow is chaotic, with eddies and irregular pressure fluctuations that produce the gurgling noise.

(c) At terminal velocity, the net force on the bubble is zero. The buoyant force (upward) equals the weight of the bubble plus the viscous drag (downward on a rising bubble):

$$\text{Upthrust} = \frac{4}{3}\pi r^3 \rho_f g, \text{ Weight} = \frac{4}{3}\pi r^3 \rho_{\text{air}} g, \text{ Drag} = 6\pi\eta r v_t$$

Since $\rho_{\text{air}} \ll \rho_f$, the weight of the bubble is negligible:

$$6\pi\eta r v_t = \frac{4}{3}\pi r^3 (\rho_f - \rho_{\text{air}})g$$

$$\eta = \frac{2r^2(\rho_f - \rho_{\text{air}})g}{9v_t} = \frac{2 \times (0.01\text{m})^2 \times (1470 - 1.2)\text{kgm}^{-3} \times 9.8\text{ms}^{-2}}{9 \times 0.002\text{ms}^{-1}}$$

$$\eta = 16.0\text{Nsm}^{-2}$$

Question 49

(a) The moment of inertia of a body is $I = \sum m_i r_i^2$, where r_i is each particle's distance from the axis. In the wheel with a heavy rim, most of the mass sits at the maximum distance (the rim radius R), contributing $m_{\text{rim}} R^2$ to I . In the wheel with heavy spokes, most of the mass is distributed between $r = 0$ (the hub) and $r = R$ (the rim), at an average distance much less than R . Since the r^2 weighting penalises mass near the centre, the heavy-spoke wheel has a smaller I despite the same total mass.

(b) At the ankle, the column of blood between the heart and the ankle adds hydrostatic pressure $\Delta P = \rho gh$, where h is the vertical distance from the heart to the ankle. This additional pressure raises the reading above the value produced by the heart's pumping alone. At the upper arm (level with the heart), $h \approx 0$ and no hydrostatic correction is needed. The arm reading therefore reflects the heart's pumping pressure directly.

(c) By Poiseuille's formula, $Q = \pi \Delta P R^4 / (8\eta L)$. For constant Q : $\Delta P \propto 1/R^4$.

New radius: $R' = R - 0.08R = 0.92R$

$$\frac{\Delta P'}{\Delta P} = \frac{R^4}{R'^4} = \frac{R^4}{(0.92R)^4} = \frac{1}{(0.92)^4} = \frac{1}{0.7164} = 1.396$$

The pressure difference must increase by a factor of approximately 1.40 (a 40% increase) to maintain the same flow rate through a pipe whose radius has decreased by only 8%.

Question 50

(a) On the smooth incline, no friction acts. There is no tangential force along the surface, so no torque acts about the disc's centre of mass. By the rotational analogue of Newton's first law, the angular velocity ω remains constant throughout the climb. However, the component of gravity along the incline decelerates the translational motion, so the translational speed v decreases. The rolling condition $v = R\omega$ breaks immediately because v changes while ω does not. The disc slides and spins simultaneously as it climbs. Eventually $v = 0$ and the disc stops translating, but it is still spinning at the original ω .

(b) As the stream falls, it accelerates under gravity. By the equation of continuity ($Av = \text{constant}$), the stream must narrow as it speeds up. The narrowing increases the surface area to volume ratio ($A - V \text{ ratio} \propto \frac{1}{r}$). Surface tension, which always acts to minimise surface area, eventually breaks the thin stream into separate droplets, because a series of spherical drops has less total surface energy than a long, thin cylinder of the same volume.

(c) Mass of the raindrop:

$$m = \frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi \times (2.0 \times 10^{-3}\text{m})^3 \times 1000\text{kgm}^{-3} = 3.351 \times 10^{-5}\text{kg}$$

The work done by gravity depends only on the force (mg) and the vertical displacement, regardless of the speed at which the drop travels.

First half (500m to 250m, distance = 250m):

$$W_1 = mgh_1 = 3.351 \times 10^{-5} \text{kg} \times 9.8 \text{ms}^{-2} \times 250 \text{m} = 0.082 \text{J}$$

Second half (250m to ground, distance = 250m):

$$W_2 = mgh_2 = 3.351 \times 10^{-5} \text{kg} \times 9.8 \text{ms}^{-2} \times 250 \text{m} = 0.082 \text{J}$$

The work done by gravity is the **same** in both halves (0.082J).

In the first half, this work goes partly into increasing the drop's kinetic energy and partly into overcoming air resistance. In the second half (constant speed, zero net acceleration), all of gravity's work goes entirely into overcoming air resistance.

Question 51

(a) By Torricelli's theorem, the efflux speed is $v = \sqrt{2gh}$, where h is the depth of the hole below the water surface. This depends only on h and g , not on the size of the hole. Enlarging the hole does not change v . However, the volume flow rate is $Q = Av$. A larger hole has a larger cross-sectional area A , so Q increases even though v stays the same.

(b) The wet canvas has water filling the tiny gaps between the fibres. Surface tension at each gap supports the water and prevents it from passing through, because the water surface curves into a meniscus at each gap and the excess pressure ($\Delta P = 2\gamma/r$) holds the water in place. When the inside is touched, the finger pushes the water surface through the gaps, breaking the menisci. Once broken, the surface tension barrier is destroyed and water drips through freely.

(c) Total pressure = static pressure + dynamic pressure:

$$\frac{1}{2}\rho v^2 = P_{\text{total}} - P_{\text{static}} = 1.4 \times 10^5 \text{Pa} - 1.3 \times 10^5 \text{Pa} = 1.0 \times 10^4 \text{Pa}$$

$$v = \sqrt{\frac{2 \times 1.0 \times 10^4 \text{Pa}}{1000 \text{kgm}^{-3}}} = 4.47 \text{ms}^{-1}$$

Mass flow rate:

$$\dot{m} = \rho Av = 1000 \text{kgm}^{-3} \times 25 \times 10^{-4} \text{m}^2 \times 4.47 \text{ms}^{-1} = 11.2 \text{kgs}^{-1}$$

Question 52

(a) When the rod is released, gravity acts at its centre of mass (distance $L/2$ from the pivot). The gravitational torque about the pivot is $\tau = Mg(L/2)$. The angular acceleration is $\alpha = \tau/I = Mg(L/2)/(\frac{1}{3}ML^2) = 3g/(2L)$. The linear acceleration of the free end (at distance L from the pivot) is $a_{\text{tip}} = \alpha L = 3g/2$. This is 1.5 times g , which exceeds the acceleration of a freely falling object. The explanation is that the pivot reaction force prevents the upper end from falling freely, so the gravitational torque concentrates the rotational effect at the far end, where the lever arm is greatest.

(b) Before the parachute opens, the skydiver's cross-sectional area is small. The drag force ($F = 6\pi\eta rv$ for Stokes drag, or more generally proportional to Av^2 at high speeds) is small, so a high speed is needed before drag equals the weight. Opening the parachute dramatically increases the effective cross-sectional area. The drag force at any given speed is now much larger. Equilibrium between weight and drag is reached at a much lower speed, giving a much lower terminal velocity.

(c) At terminal velocity, weight = drag (buoyancy neglected):

$$mg = 6\pi\eta rv_t$$

$$\frac{4}{3}\pi r^3 \rho g = 6\pi\eta rv_t$$

$$v_t = \frac{2r^2 \rho g}{9\eta} = \frac{2 \times (2.0 \times 10^{-5} \text{m})^2 \times 1200 \text{kgm}^{-3} \times 9.8 \text{ms}^{-2}}{9 \times 1.8 \times 10^{-5} \text{Nsm}^{-2}} = 0.058 \text{ms}^{-1}$$

Viscous force at terminal velocity:

$$F = 6\pi\eta rv_t = 6\pi \times 1.8 \times 10^{-5} \text{Nsm}^{-2} \times 2.0 \times 10^{-5} \text{m} \times 0.058 \text{ms}^{-1} = 3.94 \times 10^{-10} \text{N}$$

Question 53

(a) Rotational work is $W = \tau\theta$, where θ is the angular displacement through which the torque acts. If the bolt does not move, $\theta = 0$, and therefore $W = 0$ regardless of how large the torque is. A force that produces no displacement does no work. The mechanic's effort is absorbed by elastic deformation in the wrench and the bolt, but no rotation occurs and no energy is transferred to the bolt.

(b) Through the same siphon (same radius R , same length L , same pressure difference ΔP), Poiseuille's formula gives $Q \propto 1/\eta$. A liquid that flows faster (empties the bottle in less time) has a lower viscosity. Since petrol empties 2.5 times faster, its flow rate is 2.5 times larger. Therefore, the viscosity of petrol is 2.5 times smaller than that of water: $\eta_{\text{petrol}} = \eta_{\text{water}}/2.5$.

(c) Pressure difference: $\Delta P = \rho gh = 1000 \text{kgm}^{-3} \times 9.8 \text{ms}^{-2} \times 1.0 \text{m} = 9800 \text{Pa}$

Areas: $A_A = \pi(0.30 \text{m})^2 = 0.2827 \text{m}^2$, $A_B = \pi(0.10 \text{m})^2 = 0.03142 \text{m}^2$

By continuity: $v_A = v_B \times \frac{A_B}{A_A} = \frac{v_B}{9}$

By Bernoulli (horizontal):

$$P_A - P_B = \frac{1}{2} \rho (v_B^2 - v_A^2) = \frac{1}{2} \rho v_B^2 \left(1 - \frac{1}{81}\right) = \frac{1}{2} \rho v_B^2 \times \frac{80}{81}$$

$$9800 \text{ Pa} = \frac{1}{2} \times 1000 \text{ kg m}^{-3} \times v_B^2 \times 0.9877$$

$$v_B^2 = 19.84 \text{ m}^2 \text{ s}^{-2}$$

$$v_B = 4.45 \text{ ms}^{-1}$$

$$Q = A_B v_B = 0.03142 \text{ m}^2 \times 4.45 \text{ ms}^{-1} = 0.140 \text{ m}^3 \text{ s}^{-1}$$

Question 54

(a) The rolling condition requires $v = R\omega$ at all times during rolling without slipping. As the ball decelerates on the rough incline, friction simultaneously reduces both v (through the translational equation of motion) and ω (through the rotational equation). The constraint $v = R\omega$ ensures that both quantities decrease in proportion. When v reaches zero, $\omega = v/R = 0$ simultaneously. One cannot reach zero without the other, because that would violate the rolling condition, which is maintained as long as the surface is rough enough to prevent slipping.

(b) Water wets glass because the adhesive forces between water molecules and glass are stronger than the cohesive forces within the water. The angle of contact is small (less than 90°), so water clings to the glass surface and forms a thin film when withdrawn. Mercury does not wet glass because the cohesive forces between mercury atoms are much stronger than the adhesive forces between mercury and glass. The angle of contact exceeds 90° , so mercury pulls away from the glass surface and the rod emerges clean.

(c) At the critical point: $R_e = \frac{\rho v_c r}{\eta}$

$$v_c = \frac{R_e \times \eta}{\rho r} = \frac{2000 \times 1.0 \times 10^{-3} \text{ N s m}^{-2}}{1000 \text{ kg m}^{-3} \times 0.01 \text{ m}} = \frac{2.0 \text{ N s m}^{-1}}{10 \text{ kg m}^{-2}} = 0.20 \text{ ms}^{-1}$$

Maximum volume flow rate:

$$Q = A v_c = \pi r^2 v_c = \pi \times (0.01 \text{ m})^2 \times 0.20 \text{ ms}^{-1} = 6.28 \times 10^{-5} \text{ m}^3 \text{ s}^{-1}$$

Question 55

(a) The work done by a torque is $W = \tau\theta$, where θ is the angular displacement. Both children push with the same force at the same distance from the axis, producing the same torque τ . But the first child pushes through $\theta = 2 \times 2\pi = 4\pi$ radians while the second pushes through $\theta = \pi$ radians. Since $W = \tau\theta$, the first child does four times as much work. More energy is transferred to the roundabout because the torque acts through a greater angular displacement.

(b) The height of capillary rise is $h = 2\gamma \cos\theta / (\rho g r)$. For the same tube (same r), the rise depends on three liquid-specific properties: the surface tension γ , the angle of contact θ , and the density ρ . Paraffin has a lower surface tension, a non-zero contact angle, and lower density than water. The combined effect of the lower $\gamma \cos\theta$ relative to ρ produces a smaller rise. Different liquids rise to different heights because these three properties differ from liquid to liquid.

(c) By the capillary rise formula: $h = \frac{2\gamma \cos\theta}{\rho g r}$

$$r = \frac{2\gamma \cos\theta}{\rho g h} = \frac{2 \times 0.026 \text{ N m}^{-1} \times \cos 26^\circ}{800 \text{ kg m}^{-3} \times 9.8 \text{ ms}^{-2} \times 0.045 \text{ m}} = 1.325 \times 10^{-4} \text{ m}$$

Internal diameter: $d = 2r = 2.65 \times 10^{-4} \text{ m} = 0.265 \text{ mm}$

Question 56

(a) Bernoulli's theorem applies to an ideal (non-viscous) fluid. Real water has viscosity, which causes internal friction between adjacent layers. This friction dissipates energy along the pipe, converting kinetic energy into heat. To maintain the same flow speed (as required by continuity in a uniform pipe), the pressure must drop along the length to compensate for the energy lost to viscous friction. The steady pressure decrease from inlet to outlet reflects this continuous energy loss.

(b) When wind blows past the flag, the air on both sides moves at varying speeds due to the flag's flexible shape. Any slight ripple creates regions where air moves faster on one side (lower pressure) and slower on the other (higher pressure). The resulting pressure difference pushes the flag sideways. Once displaced, the flag creates new ripples with new pressure differences, sustaining a continuous fluttering motion. On a calm day, with no airflow, no pressure differences develop and the flag hangs limp.

(c) When the valve is closed, the water is stationary and the gauge reads the static pressure. When water flows, Bernoulli's equation gives:

$$P_{\text{static}} + \frac{1}{2} \rho v^2 = P_{\text{closed}}$$

$$\frac{1}{2}\rho v^2 = P_{\text{closed}} - P_{\text{flowing}} = 2.5 \times 10^5 \text{ Pa} - 2.0 \times 10^5 \text{ Pa} = 5.0 \times 10^4 \text{ Pa}$$

$$v = \sqrt{\frac{2 \times 5.0 \times 10^4 \text{ Pa}}{1000 \text{ kg m}^{-3}}} = 10 \text{ ms}^{-1}$$

Question 57

(a) Both objects experience the same gravitational acceleration initially. As they fall, air resistance builds up. The stone is compact and dense, so its terminal velocity is very high and it accelerates nearly freely for most of the fall. The parachutist deploys a large canopy that presents an enormous surface area to the air. The drag force on the parachute equals the person's weight at a much lower speed. The parachutist reaches a low terminal velocity and descends gently, while the stone reaches a high terminal velocity and strikes the ground with great force.

(b) After emerging horizontally from the hole, the water has a horizontal velocity but no vertical velocity. Gravity immediately begins to accelerate it downward. The horizontal velocity remains constant (no horizontal force acts) while the vertical velocity increases steadily. The combination of constant horizontal motion and uniformly accelerated vertical motion produces a parabolic curve, exactly as in projectile motion.

(c) By conservation of volume: $\frac{4}{3}\pi R^3 = 2 \times \frac{4}{3}\pi r^3$, giving $R = r \times 2^{1/3}$.

Terminal velocity: $v_t \propto r^2$.

$$\frac{V_{\text{combined}}}{V} = \frac{R^2}{r^2} = \frac{(r \times 2^{1/3})^2}{r^2} = 2^{2/3} = \sqrt[3]{4}$$

$$V_{\text{combined}} = 2^{2/3}V = \sqrt[3]{4}V \approx 1.587V$$

Question 58

(a) The student is incorrect. By Torricelli's theorem, $v = \sqrt{2gh}$, the efflux velocity depends only on the depth h of the hole below the water surface and not on the width or shape of the tank. A wider tank holds more water (so it takes longer to drain), but the speed at which water exits is determined solely by the height of water above the hole.

(b) By the equation of continuity ($A_1v_1 = A_2v_2$), the same volume of water must pass through every cross-section per unit time. At the narrow section, the cross-sectional area is smaller. For the same volume to pass through per unit time, the water must move faster. The extra kinetic energy comes from a drop in pressure: by Bernoulli's equation, the pressure in the narrow section is lower than in the wide section.

(c) The hole is 3.0m below the surface. Efflux speed:

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ ms}^{-2} \times 3.0 \text{ m}} = 7.67 \text{ ms}^{-1}$$

The hole is $10 \text{ m} - 3.0 \text{ m} = 7.0 \text{ m}$ above the ground. Time to fall:

$$t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 7.0 \text{ m}}{9.8 \text{ ms}^{-2}}} = 1.20 \text{ s}$$

Horizontal distance:

$$x = vt = 7.67 \text{ ms}^{-1} \times 1.20 \text{ s} = 9.2 \text{ m}$$

Question 59

(a) Plain water has a high surface tension ($\gamma \approx 0.073 \text{ Nm}^{-1}$), which prevents it from spreading into narrow gaps and pores in dirty fabric or skin. Soap molecules reduce the surface tension of water dramatically. The low-surface-tension soap solution penetrates small gaps, loosens dirt particles, and surrounds them. The soap molecules also have hydrophilic (water-attracting) and hydrophobic (water-repelling) ends, which attach to grease and dirt and suspend them in the water for removal.

(b) A soap bubble has an inner and an outer liquid surface. Each surface contributes an excess pressure of $2\gamma/R$. The total excess pressure is the sum: $\Delta P_{\text{soap}} = 2 \times 2\gamma/R = 4\gamma/R$. An air bubble in water has only one liquid surface (the water-air interface on the outside). It contributes a single excess pressure: $\Delta P_{\text{air}} = 2\gamma/R$. Therefore, $\Delta P_{\text{soap}} = 2 \times \Delta P_{\text{air}}$.

(c) The excess pressure inside the soap bubble:

$$\Delta P = \frac{4\gamma}{R}$$

This is balanced by the oil column pressure:

$$\frac{4\gamma}{R} = \rho_{\text{oil}}gh = 0.8 \times 1000 \text{ kg m}^{-3} \times 9.8 \text{ ms}^{-2} \times 2.0 \times 10^{-3} \text{ m} = 15.68 \text{ Pa} = \Delta P$$

$$\gamma = \frac{\Delta P \times R}{4} = \frac{15.68 \text{ Pa} \times 0.01 \text{ m}}{4} = 0.0392 \text{ Nm}^{-1}$$

Question 60

(a) By Torricelli's theorem, the efflux speed is $v = \sqrt{2gh}$, where h is the height of water above the hole. When the tank is full, h is large and the jet emerges at high speed. As the tank drains, h decreases, and the efflux speed decreases proportionally to $\sqrt{h}$. The volume flow rate $Q = av$ also decreases. Consequently, the jet becomes progressively weaker.

(b) Before the hole is unplugged, the liquid pushes equally on all internal walls. When the hole is opened, the liquid at the hole escapes, so there is no longer any outward force on the wall at the hole. The opposite wall, however, still has liquid pressing against it. This unbalanced force pushes the container away from the hole. By Newton's third law, the escaping liquid carries momentum in one direction and the container gains equal momentum in the opposite direction.

(c) Initial efflux speed:

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ ms}^{-2} \times 0.30 \text{ m}} = 2.42 \text{ ms}^{-1}$$

Initial drainage rate:

$$Q = av = 6.5 \times 10^{-4} \text{ m}^2 \times 2.42 \text{ ms}^{-1} = 1.57 \times 10^{-3} \text{ m}^3 \text{ s}^{-1}$$

At 30cm below the hole, the water has fallen freely through 0.30m. Its speed there:

$$v_2 = \sqrt{v^2 + 2g \times 0.30 \text{ m}} = \sqrt{5.88 + 5.88} = 3.43 \text{ ms}^{-1}$$

By continuity ($a_1 v_1 = a_2 v_2$):

$$a_2 = \frac{a_1 v_1}{v_2} = \frac{6.5 \times 10^{-4} \text{ m}^2 \times 2.42 \text{ ms}^{-1}}{3.43 \text{ ms}^{-1}} = 4.59 \times 10^{-4} \text{ m}^2 = 4.6 \text{ cm}^2$$

Question 61

(a) The oil exits horizontally and then follows a projectile path. The horizontal distance depends on two factors: the horizontal speed (which is the same in both cases, since the can produces the same exit speed) and the time the oil is in the air. At waist height, the oil falls a greater vertical distance, taking more time to reach the ground. More time in the air means more horizontal distance before landing. At ground level, the fall height is small, the air time is short, and the oil lands close to the can.

(b) At the wide section, the water moves slowly (by continuity). At the constriction, the water moves fast. By Bernoulli's equation, faster-moving fluid has lower pressure. The vertical tubes act as pressure indicators: the water rises to a height proportional to the local pressure. Since the pressure at the constriction is lower, the water level in its vertical tube is lower than at the wide section.

(c) By continuity, the nozzle speed:

$$v_{\text{nozzle}} = v_{\text{piston}} \times \frac{A_{\text{piston}}}{A_{\text{nozzle}}} = 0.20 \text{ ms}^{-1} \times \frac{(5d)^2}{d^2} = 0.20 \text{ ms}^{-1} \times 25 = 5.0 \text{ ms}^{-1}$$

The water exits horizontally at 5.0 ms^{-1} and falls 1.25m. Vertical velocity on hitting the ground:

$$v_y = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ ms}^{-2} \times 1.25 \text{ m}} = 4.95 \text{ ms}^{-1}$$

Speed on hitting the ground:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(5.0 \text{ ms}^{-1})^2 + (4.95 \text{ ms}^{-1})^2} = 7.04 \text{ ms}^{-1}$$

Question 62

(a) By Poiseuille's formula, the flow rate through the needle is $Q = \pi \Delta P R^4 / (8\eta L)$. The flow depends on the fourth power of the radius. Even a small decrease in R causes R^4 to drop dramatically. For example, reducing the radius by 20% reduces R^4 to $(0.8)^4 = 0.41$, cutting the flow by nearly 60%. This extreme sensitivity to radius explains why the drip rate falls so sharply with a slightly smaller needle.

(b) In series, the same volume flow rate Q passes through both tubes. By Poiseuille's formula, the pressure drop across each tube is $\Delta P = 8\eta LQ / (\pi R^4)$. For the same Q and L , $\Delta P \propto 1/R^4$. The narrower tube has a smaller R and therefore a much larger ΔP , carrying the larger share of the total pressure drop.

(c) In series, Q is constant. $\Delta P \propto 1/R^4$ for each tube.

$$\frac{\Delta P_1}{\Delta P_3} = \frac{R_3^4}{R_1^4} = \frac{(5r)^4}{(3r)^4} = \frac{625}{81} = 7.716$$

$$\Delta P_1 = 7.716 \times 8.1 \text{ mm} = 62.5 \text{ mm of liquid}$$

Question 63

(a) By Stokes' law, the terminal velocity of a spherical drop is $v_t \propto r^2$. Tiny fog droplets have very small radii, so their terminal velocities are extremely low (fractions of a millimetre per second). Even the slightest air currents keep them suspended. Large

raindrops have much greater radii, so their terminal velocities are proportionally much higher (r^2 dependence), and they fall rapidly.

(b) At terminal velocity, the viscous drag equals the net downward force (weight minus upthrust). The drag force is $F = 6\pi\eta r v_t$, so $v_t \propto 1/\eta$. When glycerine is warmed, its viscosity η decreases (intermolecular forces weaken at higher temperature). A lower η means a smaller drag force at any given speed, so the sphere must reach a higher speed before drag balances its weight. Hence, the terminal velocity increases.

(c) Finding the radius:

$$v_t = \frac{2r^2(\rho_s - \rho_{\text{air}})g}{9\eta}$$

$$r^2 = \frac{9\eta v_t}{2(\rho_s - \rho_{\text{air}})g} = \frac{9 \times 1.8 \times 10^{-5} \text{Nsm}^{-2} \times 3.2 \times 10^{-2} \text{ms}^{-1}}{2 \times (900 - 1.2) \text{kgm}^{-3} \times 9.8 \text{ms}^{-2}} = 2.944 \times 10^{-10} \text{m}^2$$

$$r = 1.72 \times 10^{-5} \text{m}$$

For a drop of half the radius: $v_t \propto r^2$

$$v_t' = v_t \times \left(\frac{r}{2}\right)^2 = 3.2 \times 10^{-2} \text{ms}^{-1} \times \frac{1}{4} = 8.0 \times 10^{-3} \text{ms}^{-1}$$

Question 64

(a) Snow or ice accumulation roughens the normally smooth wing surface and alters its aerodynamic shape. The rough surface disrupts the smooth flow of air over the wing, causing the boundary layer to separate earlier. This reduces the speed difference between air flowing over the top and bottom of the wing. By Bernoulli's equation, a smaller speed difference means a smaller pressure difference, which reduces the lift. In severe cases, the disrupted airflow causes a stall.

(b) By Bernoulli's equation, the total energy per unit volume along a streamline is constant: $P + \frac{1}{2}\rho v^2 = \text{constant}$ (for horizontal flow). When the fluid enters the constriction and speeds up, its kinetic energy per unit volume increases. This increase comes at the expense of pressure energy. The pressure in the constriction drops, and it is this pressure drop that provides the energy for the increased speed.

(c) By Poiseuille's formula, for tubes in series (same Q): $\Delta P \propto \frac{L}{R^4}$.

$$\frac{\Delta P_1}{\Delta P_2} = \frac{\frac{L_1}{R_1^4}}{\frac{L_2}{R_2^4}} = \frac{L_1 R_2^4}{L_2 R_1^4} = \frac{21 \text{cm} \times (0.0375 \text{cm})^4}{7 \text{cm} \times (0.1125 \text{cm})^4} = 0.03704$$

$$\Delta P_1 + \Delta P_2 = 14 \text{cm}$$

$$0.03704 \Delta P_2 + \Delta P_2 = 14 \text{cm}$$

$$\Delta P_2 = \frac{14}{1.03704} = 13.5 \text{cm of water}$$

$$\Delta P_1 = 14 - 13.5 = 0.5 \text{cm of water}$$

Question 65

(a) Engine oil must lubricate moving parts by maintaining a film between metal surfaces. In hot weather, oil becomes thinner (lower viscosity) and may fail to maintain an adequate film, leading to increased wear. In cold weather, oil becomes thicker (higher viscosity) and resists flow, making it harder to start the engine and increasing energy loss to internal friction. Different viscosity grades are designed so that the oil remains within the optimal viscosity range for the prevailing temperature.

(b) The engine contains oil that lubricates moving parts. Cold oil has high viscosity, producing large frictional forces between the oil layers and the moving surfaces. As the engine runs, it generates heat that warms the oil. The viscosity of the oil decreases with increasing temperature. Lower viscosity means less internal friction between the oil layers, so the engine encounters less resistance and runs more smoothly.

(c) First, find the density of the sphere:

$$\rho_s = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi r^3} = \frac{1.2 \times 10^{-4} \text{kg}}{\frac{4}{3}\pi \times (2.0 \times 10^{-3} \text{m})^3} = 3580 \text{kgm}^{-3}$$

Terminal velocity formula rearranged for viscosity:

$$\eta = \frac{2r^2(\rho_s - \rho_f)g}{9v_t} = \frac{2 \times (2.0 \times 10^{-3} \text{m})^2 \times (3580 - 900) \text{kgm}^{-3} \times 9.8 \text{ms}^{-2}}{9 \times 5.4 \times 10^{-2} \text{ms}^{-1}}$$

$$\eta = 0.432 \text{Nsm}^{-2}$$

Question 66

(a) The tank is continuously fed at a constant rate Q_{in} . The outflow rate depends on the water height: $Q_{out} = a\sqrt{2gh}$. Initially, the water level is high and $Q_{out} > Q_{in}$, so the level drops. As h decreases, Q_{out} decreases. At a particular height, $Q_{out} = Q_{in}$: water leaves at the same rate it enters. Below this height, $Q_{out} < Q_{in}$ and the level would rise back up. Hence, the system self-corrects to the equilibrium height where inflow equals outflow.

(b) By Poiseuille's formula, $Q = \pi \Delta P R^4 / (8\eta L)$. Doubling L halves the flow. Halving the diameter halves R , so R^4 drops by a factor of $2^4 = 16$. The combined effect reduces the flow by a factor of $2 \times 16 = 32$. The R^4 dependence is devastating: a small reduction in radius produces a dramatic reduction in flow.

(c) By Poiseuille's formula: $Q \propto \frac{R^4}{L}$. New length = $2L$, new radius = $\frac{R}{2}$:

$$Q' = Q \times \frac{\left(\frac{R}{2}\right)^4}{R^4} \times \frac{L}{2L} = Q \times \frac{1}{16} \times \frac{1}{2} = \frac{Q}{32}$$

Question 67

(a) In normal flight, the curved upper surface and angled wing cause air to move faster over the top than the bottom, creating lower pressure above and hence lift. When inverted, the wing's curved surface faces downward and would normally push the aircraft down. However, the pilot tilts the nose upward, increasing the angle of attack. This forces air to deflect downward off the underside of the wing (now the top surface), producing an upward reaction force. The lift from this high angle of attack overcomes the unfavourable wing curvature.

(b) Lift is proportional to $\frac{1}{2} \rho A v^2 \times C_L$, where A is the wing area and C_L is the lift coefficient (which increases with angle of attack). Extending flaps increases both A and C_L simultaneously. A larger product $A \times C_L$ means the same lift can be generated at a lower speed v . This allows the aircraft to take off and land at safely reduced speeds without stalling.

(c) By continuity: $v_2 = v_1 \times \left(\frac{d_1}{d_2}\right)^2 = 5\text{ms}^{-1} \times \left(\frac{1.5\text{cm}}{0.75\text{cm}}\right)^2 = 20\text{ms}^{-1}$

By Bernoulli (with height difference $h = 8\text{m}$):

$$P_2 = P_1 + \frac{1}{2} \rho (v_1^2 - v_2^2) - \rho gh$$

$$P_2 = 6.5 \times 10^5 \text{Pa} + \frac{1}{2} \times 1000 \text{kgm}^{-3} \times ((5)^2 - (20)^2) \text{m}^2 \text{s}^{-2} - 1000 \text{kgm}^{-3} \times 9.8 \text{ms}^{-2} \times 8 \text{m}$$

$$P_2 = 3.84 \times 10^5 \text{Pa}$$

Question 68

(a) The student's prediction reverses the causality. By the equation of continuity, water slows down in the wider section (larger area, lower speed). By Bernoulli's equation, slower-moving fluid has higher pressure. The gauge at the wider section reads higher, not lower. The "more space means less pressure" reasoning confuses gas behaviour (where expansion reduces pressure) with incompressible fluid flow (where pressure and speed trade off along a streamline).

(b) By Torricelli: $v = \sqrt{2gh}$. Volume outflow: $-A \frac{dh}{dt} = a\sqrt{2gh}$

Separating variables: $\frac{dh}{\sqrt{h}} = -\frac{a}{A} \sqrt{2g} dt$

Integrating from H_1 to H_2 and 0 to t :

$$\int_{H_1}^{H_2} \frac{dh}{\sqrt{h}} = \int_0^t -\frac{a}{A} \sqrt{2g} dt$$

$$[2\sqrt{h}]_{H_1}^{H_2} = -\frac{a}{A} \sqrt{2g} \times t$$

$$2(\sqrt{H_2} - \sqrt{H_1}) = -\frac{a}{A} \sqrt{2g} \times t$$

$$t = \frac{A}{a} \sqrt{\frac{2}{g}} (\sqrt{H_1} - \sqrt{H_2})$$

(c) Areas: $A_1 = \pi(0.04\text{m})^2 = 5.027 \times 10^{-3} \text{m}^2$, $A_2 = \pi(0.08\text{m})^2 = 2.011 \times 10^{-2} \text{m}^2$

By continuity: $v_1 = v_2 \times \frac{A_2}{A_1}$. Since $A_2 = 4A_1$: $v_1 = 4v_2$

Flow rate: $Q = 96 \times 10^{-3} \text{m}^3 \text{s}^{-1} = 0.096 \text{m}^3 \text{s}^{-1}$

$$v_1 = \frac{Q}{A_1} = \frac{0.096}{5.027} \times 10^{-3} = 19.1 \text{ ms}^{-1}$$

$$v_2 = \frac{Q}{A_2} = \frac{0.096}{2.011} \times 10^{-2} = 4.77 \text{ ms}^{-1}$$

By Bernoulli:

$$P_2 = P_1 + \frac{1}{2} \rho (v_1^2 - v_2^2) = 3.5 \times 10^5 \text{ Pa} + 500 \text{ kg m}^{-3} \times ((19.1)^2 - (4.77)^2) \text{ m}^2 \text{ s}^{-2}$$

$$P_2 = 5.21 \times 10^5 \text{ Pa}$$

Question 69

(a) Water flows in at a constant rate Q_{in} . The outflow rate is $Q_{\text{out}} = a\sqrt{2gh}$, which increases with height. Initially h is small and $Q_{\text{out}} < Q_{\text{in}}$, so water accumulates and h rises. As h increases, Q_{out} increases. The level stops rising when $Q_{\text{out}} = Q_{\text{in}}$: the outflow through the hole exactly matches the inflow. Any further rise would make $Q_{\text{out}} > Q_{\text{in}}$, causing the level to drop back. The system is self-regulating.

(b) Terminal velocity is $v_t \propto r^2$. The larger sphere (diameter $3d$, radius $3r/2$) has a terminal velocity 9 times that of the smaller sphere (radius $r/2$). Since both fall the same distance and the larger sphere moves 9 times faster, the smaller sphere takes 9 times as long.

(c) At equilibrium: $Q_{\text{in}} = Q_{\text{out}} = a\sqrt{2gh}$

$$h = \frac{Q_{\text{in}}^2}{2ga^2} = \frac{(2 \times 10^{-4} \text{ m}^3 \text{ s}^{-1})^2}{2 \times 9.8 \text{ ms}^{-2} \times (2 \times 10^{-4} \text{ m}^2)^2} = 0.051 \text{ m} = 5.1 \text{ cm}$$

Question 70

(a) A tennis ball is lighter and has a rough, fuzzy surface. When spinning, the rough surface drags more air with it, creating a larger asymmetry in air speed between the two sides. By the Magnus effect, this larger speed difference produces a larger pressure difference and hence a greater sideways force. A cricket ball is heavier (requiring more force to deflect) and smoother (less surface drag), so the same spin produces less curve.

(b) Let the piston area be A_p and the orifice area be a . The piston moves at speed $\frac{V}{A_p t}$ (to push volume V in time t , the piston distance is $\frac{V}{A_p}$). By continuity, the efflux speed is $v = \frac{V}{at}$.

By Bernoulli (piston to orifice, horizontal, $a \ll A_p$ so piston speed is negligible):

$$P_{\text{piston}} = \frac{1}{2} \rho v^2 = \frac{1}{2} \rho \frac{V^2}{a^2 t^2}$$

Work done by force $F = P_{\text{piston}} \times A_p$ through distance $d = \frac{V}{A_p}$:

$$W = F \times d = P_{\text{piston}} \times A_p \times \frac{V}{A_p} = P_{\text{piston}} \times V = \frac{1}{2} \rho \frac{V^3}{a^2 t^2} \times V$$

$$W = \frac{\rho V^3}{2a^2 t^2}$$

(c) Terminal velocity:

$$v_t = \frac{2r^2(\rho_s - \rho_f)g}{9\eta} = \frac{2 \times (2.0 \times 10^{-3} \text{ m})^2 \times (7800 - 1260) \text{ kg m}^{-3} \times 9.8 \text{ ms}^{-2}}{9 \times 0.83 \text{ N s m}^{-2}}$$

$$v_t = 0.069 \text{ ms}^{-1}$$

Question 71

(a) By Torricelli's theorem, a lower hole has a greater depth h below the surface, giving a higher efflux speed $v = \sqrt{2gh}$. However, the lower hole is also closer to the ground, so the jet has less fall height H and less air time $t = \sqrt{\frac{2H}{g}}$. The horizontal distance is $x = vt = \sqrt{2gh} \times \sqrt{\frac{2H}{g}} = 2\sqrt{hH}$. For a tank of total height L : $h + H = L$, and $x = 2\sqrt{h(L-h)}$, which is maximised when $h = \frac{L}{2}$. A hole below the midpoint has high speed but very little fall height, so x decreases. Hence, a faster jet can land closer.

(b) Terminal velocity is $v_t = 2r^2(\rho_s - \rho_f)g/(9\eta)$, so $v_t \propto r^2$. A drop of half the radius has $v_t' = v_t \times (r/2)^2/r^2 = v_t/4$. The terminal velocity is exactly one quarter because the weight (which drives the fall) decreases as r^3 while the drag force (which opposes it) decreases only as r . The net effect is that smaller drops require much less drag to reach equilibrium, which occurs at a much lower speed.

(c) Efflux speed: $v = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ms}^{-2} \times 0.80 \text{m}} = 3.96 \text{ms}^{-1}$

Time to fall $H = 1.20 \text{m}$: $t = \sqrt{\frac{2H}{g}} = \sqrt{2 \times 1.20 \text{m} / 9.8 \text{ms}^{-2}} = 0.495 \text{s}$

Horizontal distance: $x = vt = 3.96 \text{ms}^{-1} \times 0.495 \text{s} = 1.96 \text{m}$

Question 72

(a) When the diver leaves the board, her angular momentum is fixed (no external torque acts during the fall). Angular momentum is $L = I\omega$. Tucking reduces the moment of inertia I by bringing her mass closer to the axis of rotation. For L to remain constant, ω must increase. A higher angular velocity means more revolutions completed in the same fall time.

(b) The moment of inertia depends on the axis of rotation chosen. It is defined as $I = \sum m_i r_i^2$, where r_i is each particle's perpendicular distance from the axis. A different axis gives different distances r_i for every particle, producing a different value of I . Therefore, a rigid body has infinitely many possible moments of inertia, one for each possible axis.

(c) Initial angular velocity: $\omega_1 = 20 \times 2\pi/60 = 2.094 \text{rad s}^{-1}$

After the wax is added:

$$I_2 = I_1 + mr^2 = 2.5 \times 10^{-4} \text{kgm}^2 + 0.04 \text{kg} \times (0.05 \text{m})^2 = 3.5 \times 10^{-4} \text{kgm}^2$$

By conservation of angular momentum:

$$\omega_2 = \frac{I_1 \omega_1}{I_2} = \frac{2.5 \times 10^{-4} \times 2.094}{3.5 \times 10^{-4}} = 1.496 \text{rad s}^{-1}$$

$$n_2 = \frac{1.496 \times 60}{2\pi} = 14.3 \text{ rev/min}$$

Question 73

(a) The rotational kinetic energy stored is $KE = \frac{1}{2} I \omega^2$. For a given mass and angular velocity, concentrating the mass at the rim maximises I (since $I = \sum m_i r_i^2$ and rim mass sits at the maximum r). A larger I stores more energy at the same speed. When the engine encounters a sudden load, the flywheel releases this stored energy to maintain ω nearly constant. Conversely, when the engine surges, the large I resists the speed increase. Both effects smooth out speed fluctuations.

(b) A uniform solid sphere has perfect symmetry: every axis through its centre is identical to every other (by rotational symmetry of the sphere). Since $I = \frac{2}{5} MR^2$ for any such axis, the value is the same for all axes through the centre. A cylinder does not have this property because it has different symmetry about its central axis ($I = \frac{1}{2} MR^2$) compared to a diameter through the centre ($I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$). The cylinder's geometry distinguishes between these axes.

(c) Treating the wheel as a disc: $I = \frac{1}{2} MR^2 = \frac{1}{2} \times 3.0 \text{kg} \times (0.40 \text{m})^2 = 0.24 \text{kgm}^2$

$$\omega_0 = 3600 \times 2\pi/60 = 120\pi \text{rad s}^{-1}$$

$$\alpha = \frac{\omega_0}{t} = \frac{120\pi}{10} = 12\pi \text{rad s}^{-2}$$

$$\tau = I\alpha = 0.24 \text{kgm}^2 \times 12\pi \text{rad s}^{-2} = 9.05 \text{Nm}$$

Question 74

(a) Moment of inertia is $I = \sum m_i r_i^2$. Every real object consists of particles with non-zero mass. For $I = 0$, every particle would need $r_i = 0$, meaning all the mass lies exactly on the axis. A real three-dimensional object always has mass distributed at some distance from any chosen axis. Therefore, $I > 0$ for any real rotating object.

(b) On a frictionless incline, the acceleration would be $g \sin \theta$ (all PE converts to translational KE). When rolling without slipping, part of the PE converts to rotational KE. Less energy is available for translation, so the translational acceleration is reduced. The fraction diverted to rotation depends on $I/(MR^2)$: for a solid cylinder ($I = \frac{1}{2} MR^2$), one-third of the gravitational component goes to rotation, giving $a = \frac{2}{3} g \sin \theta$.

(c) By conservation of energy: $Mg\ell \sin \alpha = \frac{1}{2} Mv^2 + \frac{1}{2} I \omega^2$

With $I = Mk^2$ and $\omega = v/R$:

$$Mg\ell \sin \alpha = \frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \times \frac{v^2}{R^2} = \frac{1}{2} Mv^2 \left(1 + \frac{k^2}{R^2} \right)$$

$$v = \sqrt{\frac{2gl\sin\alpha}{1 + \frac{k^2}{R^2}}}$$

Question 75

(a) The hollow cylinder has all its mass at the maximum distance from the axis ($I = MR^2$), while the solid cylinder has mass distributed from the axis to the rim ($I = \frac{1}{2}MR^2$). At the same angular velocity, $KE = \frac{1}{2}I\omega^2$, so the hollow cylinder has twice the kinetic energy. By the work-energy theorem, the work required to bring it to that angular velocity is twice as large.

(b) Place the diameter along the x-axis. A small element at angle θ from the axis of symmetry subtends $d\theta$ and has mass $dm = \frac{M}{\pi R} R d\theta = \frac{M}{\pi} d\theta$. Its y-coordinate (along the axis of symmetry) is $y = R\sin\theta$.

$$y_{cm} = \frac{1}{M} \int_0^\pi y dm = \frac{1}{M} \int_0^\pi R \sin\theta \times \frac{M}{\pi} d\theta = \frac{R}{\pi} \int_0^\pi \sin\theta d\theta$$

$$y_{cm} = \frac{R}{\pi} [-\cos\theta]_0^\pi = \frac{R}{\pi} (-\cos\pi + \cos 0) = \frac{R}{\pi} (1 + 1)$$

$$y_{cm} = \frac{2R}{\pi}$$

(c) Initial angular momentum: $L = I_1\omega_1 = MR^2\omega_1$ (ring about its axis: $I = MR^2$ since $r = R$... using the variable in the question, $I = Mr^2$).

After adding two masses at the ring: $I_2 = Mr^2 + 2mr^2 = (M + 2m)r^2$

By conservation of angular momentum:

$$\omega_2 = \frac{Mr^2\omega_1}{(M + 2m)r^2} = \frac{M\omega_1}{M + 2m}$$

Question 76

(a) An object rotating at constant angular velocity has $\alpha = 0$, so by $\tau_{net} = I\alpha$, the net torque is zero. A torque is needed to change the angular velocity, not to maintain it. Example: a frictionless flywheel spinning at constant ω requires no torque to keep spinning. This is the rotational analogue of Newton's first law.

(b) The pipe extension increases the effective handle length d . By $\tau = Fd$, the same hand force F produces a much larger torque. This larger torque is more likely to exceed the frictional resistance of the seized bolt. However, the bolt head and threads are designed to withstand only a certain maximum torque. The extended handle can easily produce a torque that exceeds the bolt's structural strength, stripping the threads or shearing the bolt head.

(c) Using conservation of energy. The heavier mass falls $\frac{h}{2} = 0.15\text{m}$ while the lighter rises 0.15m :

$$(m_2 - m_1)g \times 0.15\text{m} = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}I\omega^2$$

With $v = \omega R$: $\omega = \frac{v}{R} = \frac{v}{0.15\text{m}}$

$$(0.25 - 0.20)\text{kg} \times 9.8\text{ms}^{-2} \times 0.15\text{m} = \frac{1}{2}(0.45\text{kg})v^2 + \frac{1}{2}(0.12\text{kgm}^2)\frac{v^2}{(0.15\text{m})^2}$$

$$v = 0.159\text{ms}^{-1}$$

$$\omega = \frac{v}{R} = \frac{0.159\text{ms}^{-1}}{0.15\text{m}} = 1.06\text{rad s}^{-1}$$

Question 77

(a) When the sliding ball moves onto the rough surface, friction at the contact point acts backward (opposing the translational motion) and simultaneously creates a torque that initiates rotation. The translational speed decreases while the rotational speed increases until the rolling condition $v = R\omega$ is satisfied. The "lost" translational kinetic energy has been converted into rotational kinetic energy. The total kinetic energy has decreased slightly because friction dissipated some energy as heat during the transition.

(b) No external torque acts on the Earth about the Sun (gravity passes through the Sun, producing zero torque about it). Angular momentum is conserved: $L = mr^2\omega = \text{constant}$. When the Earth is closer to the Sun, r is smaller. For $r^2\omega$ to remain constant, ω must increase. When farther away, r is larger and ω decreases.

(c) For a uniform solid sphere: $I = \frac{2}{5}MR^2$. If $R' = \frac{R}{2}$:

$$I' = \frac{2}{5}M\left(\frac{R}{2}\right)^2 = \frac{1}{4} \times \frac{2}{5}MR^2 = \frac{I}{4}$$

By conservation of angular momentum: $I\omega = I'\omega'$

$$\omega' = \frac{I\omega}{I'} = \frac{I\omega}{\frac{I}{4}} = 4\omega$$

Since $\omega \propto \frac{1}{T}$,

$$\text{Period: } T' = \frac{T}{4} = \frac{24 \text{ hours}}{4} = 6 \text{ hours}$$

The day would shrink from 24 hours to **6 hours**.

Question 78

(a) Opening a door requires a single torque about the hinge axis. A single force applied far from the hinge (large lever arm) produces this torque efficiently. Twisting a bottle cap requires a couple: two equal, opposite, parallel forces that produce pure rotation without translation. A single finger cannot create a couple. Thumb and finger on opposite sides produce equal forces in opposite directions, forming a couple whose torque equals $F \times d$ (where d is the cap diameter).

(b) At the instant a torque is first applied to a stationary object, $\omega = 0$ but $\alpha = \tau/I \neq 0$. Angular acceleration is the rate of change of angular velocity. The object has not yet acquired any angular velocity, but it is about to. (This is analogous to a ball at the instant of release from rest: its velocity is zero, but its acceleration is g .)

(c) Mass of Earth: $M = \rho \times \frac{4}{3}\pi R^3 = 5450 \text{ kg m}^{-3} \times \frac{4}{3}\pi \times (6.37 \times 10^6 \text{ m})^3 = 5.90 \times 10^{24} \text{ kg}$

$$I = \frac{2}{5}MR^2 = \frac{2}{5} \times 5.90 \times 10^{24} \text{ kg} \times (6.37 \times 10^6 \text{ m})^2 = 9.58 \times 10^{37} \text{ kg m}^2$$

Question 79

(a) Brick walls have porous surfaces with tiny gaps and cracks. Without plastering, water can enter these gaps by capillary action and penetrate the wall. Cement plaster fills the pores and creates a smooth, continuous surface. The adhesive forces between cement and brick bond them together, strengthening the wall structurally. The smooth plaster surface eliminates the narrow capillary channels, preventing water from being drawn into the wall by surface tension.

(b) A molecule inside the liquid is surrounded by neighbours on all sides. The attractive forces from all directions cancel out, and the molecule is in its lowest energy state. A molecule at the surface has neighbours only on the sides and below, but none above (the air above exerts negligible attraction). There is a net inward pull on the surface molecule. To bring a molecule from the interior to the surface, work must be done against this inward pull. This work increases the molecule's potential energy. Hence, surface molecules are in a higher energy state.

(c) By volume conservation: $64 \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$, giving $R = 4r$.

Total surface energy of 64 small drops: $E_1 = 64 \times 4\pi r^2 \gamma = 256\pi r^2 \gamma$

Surface energy of one large drop: $E_2 = 4\pi(4r)^2 \gamma = 64\pi r^2 \gamma$

$$\frac{E_1}{E_2} = \frac{256\pi r^2 \gamma}{64\pi r^2 \gamma} = 4$$

Question 80

(a) In a narrow capillary tube, the liquid rises by an amount $h_c = \frac{2\gamma \cos \theta}{\rho g r}$, which adds to the true height. If two tubes of radii r_1 and r_2 are used, the observed heights are $H_1 = H_{\text{true}} + h_{c1}$ and $H_2 = H_{\text{true}} + h_{c2}$. The difference $H_1 - H_2 = h_{c1} - h_{c2} = \frac{2\gamma \cos \theta}{\rho g \left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$, which is known. This allows h_{c1} and h_{c2} to be calculated and subtracted from each reading, yielding the true pressure.

(b) Surface tension acts along the contact line (the perimeter where the liquid, glass, and air meet). The force is $F = \gamma \times \text{perimeter} = \gamma \times 2\pi r$ for a circular plate (factor of 2 for wetting on both sides). This force pulls the plate downward into the liquid. The extra force needed to detach the plate equals this surface tension force. It depends on the perimeter because surface tension is a force per unit length, not a force per unit area.

(c) The difference in readings is due to the difference in capillary rises:

$$\begin{aligned} H_1 - H_2 &= \frac{2\gamma \cos \theta}{\rho g} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \\ &= \frac{2 \times 0.073 \text{ N m}^{-1} \times 1}{1000 \text{ kg m}^{-3} \times 9.8 \text{ m s}^{-2}} \times \left(\frac{1}{6.5 \times 10^{-4} \text{ m}} - \frac{1}{1.24 \times 10^{-3} \text{ m}} \right) = 0.01091 \text{ m} \end{aligned}$$

Observed difference = 0.20 m. True pressure difference (in metres of water):

$$H_{\text{true}} = H_{\text{narrow}} - h_{c,\text{narrow}}$$

The capillary rise in the narrow tube:

$$h_{c1} = \frac{2\gamma\cos\theta}{\rho g r_1} = \frac{2 \times 0.073 \times 1}{1000 \times 9.8 \times 6.5 \times 10^{-4}} = 0.0229\text{m}$$

The capillary rise in the wide tube:

$$h_{c2} = \frac{2\gamma\cos\theta}{\rho g r_2} = 0.01202\text{m}$$

The true height difference (correcting the narrow tube): the 0.20m excess reading is partly due to the capillary difference $h_{c1} - h_{c2} = 0.0229 - 0.0120 = 0.0109\text{m}$.

True pressure difference:

$$\Delta P = \rho g(0.20\text{m} - 0.0109\text{m}) = 1000\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 0.189\text{m} = 1852\text{Pa}$$

Question 81

(a) The excess pressure inside a soap bubble is $\Delta P = 4\gamma/r$. The smaller bubble has a smaller radius and therefore a higher internal pressure. When connected, air flows from the higher-pressure (smaller) bubble to the lower-pressure (larger) bubble. The smaller bubble shrinks while the larger one grows. This continues until the smaller bubble collapses and only the larger bubble remains.

(b) Consider a rectangular film on a wire frame with a sliding wire of length l . The film has two surfaces, each exerting force γl , for a total force $F = 2\gamma l$. If the wire is pulled outward by distance dx , the increase in area is $dA = 2l dx$ (two surfaces). Work done: $dW = F dx = 2\gamma l dx = \gamma \times 2l dx = \gamma dA$.

Therefore: $\gamma = dW/dA$ (energy per unit area) and $\gamma = F/(2l)$ (force per unit length of one-sided contact). Both give the same γ with dimensions $\text{Nm}^{-1} = \text{Jm}^{-2}$.

(c) By volume conservation: $1000 \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$, so $R = 10r = 10 \times 5 \times 10^{-3}\text{m} = 0.05\text{m}$.

Work done = increase in surface area $\times \gamma$:

$$W = \gamma(1000 \times 4\pi r^2 - 4\pi R^2) = 4\pi \times 0.50\text{Nm}^{-1} \times (1000 \times (5 \times 10^{-3}\text{m})^2 - (0.05\text{m})^2)$$

$$W = 0.1414\text{J}$$

Question 82

(a) At higher temperature, the surface tension of the soup decreases. A lower surface tension allows the soup surface to break more easily, releasing vapour bubbles that carry volatile flavour molecules into the air above the bowl. These molecules reach the nose (smell) and tongue (taste) more readily. Cold soup has higher surface tension, which keeps the surface intact and traps flavour molecules, reducing the aroma and perceived flavour.

(b) The nib of a fountain pen contains narrow channels (slits) between the tines. These channels act as capillary tubes. The adhesive forces between ink and the nib material (usually metal or plastic) are stronger than the cohesive forces within the ink. This produces a small angle of contact, enabling capillary rise. The ink is drawn through the narrow channels by capillary action from the reservoir toward the paper, even against gravity, because the surface tension force along the contact line exceeds the weight of the ink column in the channel.

(c) A soap bubble has two surfaces. Surface area: $A = 2 \times 4\pi r^2$ with $r = 0.5\text{cm} = 5 \times 10^{-3}\text{m}$:

$$W = \gamma \times 2 \times 4\pi r^2 = 0.030\text{Nm}^{-1} \times 8\pi \times (5 \times 10^{-3}\text{m})^2 = 1.88 \times 10^{-5}\text{J}$$

Question 83

(a) Glue is a liquid when applied and flows into the microscopic pores, cracks, and irregularities on both surfaces. As it sets, strong adhesive bonds form between the glue molecules and the molecules of each surface. The large contact area (including all the pores filled by the glue) means that an enormous number of adhesive bonds form across the joint. To pull the pieces apart, all these bonds must be broken simultaneously. The total force required is proportional to the true contact area, which is much larger than the apparent surface area due to the penetration into pores.

(b) Surface tension is a property of the liquid-air (or liquid-liquid) interface arising from intermolecular forces. Each molecule at the surface experiences the same imbalance of forces regardless of how large the surface is. Doubling the surface area doubles the number of surface molecules but does not change the force per unit length experienced by each molecule. The coefficient γ characterises the strength of the intermolecular interaction at the interface, not the total force on the entire surface.

(c) By volume conservation: $\frac{4}{3}\pi r_1^3 + \frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi R^3$

$$R^3 = r_1^3 + r_2^3 = (0.001\text{m})^3 + (0.002\text{m})^3 = 10^{-9} + 8 \times 10^{-9} = 9 \times 10^{-9}\text{m}^3$$

$$R = (9 \times 10^{-9})^{1/3} = 2.080 \times 10^{-3}\text{m}$$

Energy released = decrease in surface energy:

$$\Delta E = \gamma(4\pi r_1^2 + 4\pi r_2^2 - 4\pi R^2) = 4\pi \times 0.50\text{Nm}^{-1} \times ((0.001)^2 + (0.002)^2 - (2.080 \times 10^{-3})^2)\text{m}^2$$

$$\Delta E = 4.23 \times 10^{-6}\text{J}$$

Question 84

(a) The angle of contact depends on the relative strengths of cohesive forces (within the liquid) and adhesive forces (between the liquid and the solid). For water on glass, adhesive forces are stronger than cohesive forces. Water molecules are attracted to the glass more than to each other, so the liquid spreads and the meniscus is concave (acute angle). For mercury on glass, cohesive forces within mercury are much stronger than adhesive forces to glass. Mercury molecules pull away from the glass, forming a convex meniscus (obtuse angle).

(b) The excess pressure needed to inflate a bubble is $\Delta P = 4\gamma/r$. When r is very small (just starting), ΔP is very large. The person must blow hard to generate this high pressure. As the bubble grows, r increases and ΔP decreases. Less effort is required to continue inflating. This is why the hardest part of blowing a bubble is the initial formation.

(c) By the capillary rise formula, $h \propto \frac{1}{r}$. If $r_2 = \frac{r_1}{3}$:

$$h_2 = h_1 \times \frac{r_1}{r_2} = 2.0\text{cm} \times 3 = 6.0\text{cm}$$

Question 85

(a) Cotton is made of hydrophilic (water-attracting) cellulose fibres. The adhesive forces between water and cotton exceed the cohesive forces within water. The angle of contact is small, and water is drawn into the narrow spaces between fibres by capillary action, soaking the towel. A plastic tarpaulin is hydrophobic. The cohesive forces within water are stronger than the adhesive forces to the plastic surface. The angle of contact exceeds 90° , so water beads up into nearly spherical droplets and rolls off.

(b) The excess pressure inside a bubble is $\Delta P = 2\gamma/r$. For a very small bubble (say $r \sim 0.1\text{mm}$), this gives $\Delta P \sim 1460\text{Pa}$, which is about 1.4% of atmospheric pressure and measurable. For a large bubble (say $r \sim 1\text{cm}$), $\Delta P \sim 14.6\text{Pa}$, which is 0.014% of atmospheric pressure and negligible. The inverse dependence on r means excess pressure becomes significant only when the radius is very small.

(c) Volume conservation: $1000 \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$, so $R = 10r$.

With $r = 0.5 \times 10^{-8}\text{m} = 5 \times 10^{-9}\text{m}$, $R = 5 \times 10^{-8}\text{m}$.

Energy liberated = decrease in surface area $\times \gamma$:

$$\Delta E = \gamma(1000 \times 4\pi r^2 - 4\pi R^2) = 4\pi \times 0.073\text{Nm}^{-1} \times (1000 \times (5 \times 10^{-9}\text{m})^2 - (5 \times 10^{-8}\text{m})^2)$$

$$\Delta E = 2.06 \times 10^{-14}\text{J}$$

Question 86

(a) In untilled soil, narrow spaces between soil particles act as capillary tubes. Water is drawn upward from the moist subsoil to the dry surface by capillary action ($h = 2\gamma\cos\theta/(\rho g r)$). At the surface, it evaporates. Tilling breaks the top layer into larger clumps with much wider gaps. Capillary rise is inversely proportional to the gap width. The wide gaps cannot sustain capillary rise, so the connection between the moist subsoil and the surface is broken. The water stays underground.

(b) Let n small drops of radius r form one large drop of radius R .

Volume conservation: $n \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$, so $n = \frac{R^3}{r^3}$

Surface energy released:

$$\Delta E = \gamma(n \times 4\pi r^2 - 4\pi R^2) = 4\pi\gamma(nr^2 - R^2) = 4\pi\gamma\left(\frac{R^3}{r^3} \times r^2 - R^2\right) = 4\pi\gamma R^3\left(\frac{1}{r} - \frac{1}{R}\right)$$

This energy heats the combined drop of mass $m = \frac{4}{3}\pi R^3\rho$:

$$\Delta E = mc\Delta\theta = \frac{4}{3}\pi R^3\rho c\Delta\theta$$

Equating:

$$4\pi\gamma R^3\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{4}{3}\pi R^3\rho c\Delta\theta$$

$$\Delta\theta = \frac{3\gamma}{\rho c}\left(\frac{1}{r} - \frac{1}{R}\right)$$

(c) By the capillary rise formula:

$$r = \frac{2\gamma\cos\theta}{\rho gh} = \frac{2 \times 5 \times 10^{-2}\text{Nm}^{-1} \times \cos 45^\circ}{1000\text{kgm}^{-3} \times 9.8\text{ms}^{-2} \times 20\text{m}} = 3.61 \times 10^{-7}\text{m} \approx 0.36\mu\text{m}$$

Question 87

(a) Initial state: bubble of radius r , chamber pressure P_0 , pressure inside bubble $P_0 + \frac{4\gamma}{r}$.

Final state: bubble of radius $2r$, chamber pressure P , pressure inside bubble $P + \frac{4\gamma}{2r} = P + \frac{2\gamma}{r}$.

By Boyle's law (isothermal): $P_i V_i = P_f V_f$

$$\left(P_0 + \frac{4\gamma}{r}\right) \times \frac{4}{3}\pi r^3 = \left(P + \frac{2\gamma}{r}\right) \times \frac{4}{3}\pi (2r)^3$$

$$\left(P_0 + \frac{4\gamma}{r}\right) r^3 = \left(P + \frac{2\gamma}{r}\right) \times 8r^3$$

$$P_0 + \frac{4\gamma}{r} = 8P + \frac{16\gamma}{r}$$

$$8P = P_0 + \frac{4\gamma}{r} - \frac{16\gamma}{r} = P_0 - \frac{12\gamma}{r}$$

$$P = \frac{P_0}{8} - \frac{3\gamma}{2r}$$

(b) The expected rise height is $h_e = 2\gamma \cos\theta / (\rho g r)$. If the tube length $L < h_e$, the water rises to the top but cannot rise further. The meniscus adjusts: it flattens, increasing its radius of curvature from $r/\cos\theta$ to a new larger value. This means the angle of contact increases from θ to a new value θ' such that $L = 2\gamma \cos\theta' / (\rho g r)$. The water does not overflow because the meniscus adjusts its curvature to support exactly the column L .

(c) Excess pressure in the soap bubble (two surfaces):

$$\Delta P = \frac{4\gamma}{r} = \frac{4 \times 2.5 \times 10^{-2} \text{ Nm}^{-1}}{0.01 \text{ m}} = 10 \text{ Pa}$$

For the air bubble at depth 40cm in soap solution (one surface):

$$\Delta P_{\text{bubble}} = \frac{2\gamma}{r} = \frac{2 \times 2.5 \times 10^{-2} \text{ Nm}^{-1}}{0.01 \text{ m}} = 5 \text{ Pa}$$

Hydrostatic pressure at 40cm depth:

$$P_h = \rho g h = 1200 \text{ kg m}^{-3} \times 9.8 \text{ m s}^{-2} \times 0.40 \text{ m} = 4704 \text{ Pa}$$

Total pressure inside the air bubble:

$$P = P_{\text{atm}} + P_h + \Delta P_{\text{bubble}} = 1.013 \times 10^5 \text{ Pa} + 4704 \text{ Pa} + 5 \text{ Pa} = 1.060 \times 10^5 \text{ Pa}$$

Question 88

(a) The remaining soap film outside the loop has surface tension pulling equally in all directions. Before puncturing, the tension on both sides of the thread cancels out. After the film inside the loop is destroyed, there is no longer any inward pull from inside the loop. The surface tension from the intact film outside pulls the thread outward from every point along its length. The shape that maximises the enclosed area for a given perimeter is a circle. The thread springs into a circle because surface tension minimises the remaining film area, which is equivalent to maximising the area enclosed by the thread.

(b) Water wets glass (adhesive forces stronger than cohesive forces, acute contact angle), so it is pulled upward in the tube by surface tension, forming a concave meniscus. Mercury does not wet glass (cohesive forces stronger than adhesive forces, obtuse contact angle), so the surface tension pulls the mercury surface downward inside the tube, forming a convex meniscus. The same tube produces opposite behaviour because the two liquids have opposite relationships between cohesive and adhesive forces with glass.

$$(c) \text{ For water: } h_w = \frac{2\gamma_w \cos\theta_w}{\rho_w g r} \dots (1)$$

$$\text{For mercury: } h_{\text{Hg}} = \frac{2\gamma_{\text{Hg}} \cos\theta_{\text{Hg}}}{\rho_{\text{Hg}} g r} \dots (2)$$

Dividing (2) by (1):

$$\frac{h_{\text{Hg}}}{h_w} = \frac{\gamma_{\text{Hg}} \cos\theta_{\text{Hg}} \times \rho_w}{\gamma_w \cos\theta_w \times \rho_{\text{Hg}}}$$

$$\frac{\gamma_{\text{Hg}}}{\gamma_w} = \frac{h_{\text{Hg}}}{h_w} \times \frac{\cos\theta_w}{\cos\theta_{\text{Hg}}} \times \frac{\rho_{\text{Hg}}}{\rho_w}$$

Using $h_{\text{Hg}} = -3.4 \text{ cm}$, $h_w = 9.0 \text{ cm}$:

$$\frac{\gamma_{\text{Hg}}}{\gamma_w} = \frac{-3.4}{9.0} \times \frac{\cos 0^\circ}{\cos 135^\circ} \times \frac{13600}{1000}$$

$$\frac{\gamma_{\text{Hg}}}{\gamma_{\text{w}}} = 7.27$$

Question 89

(a) The spinning wheels have angular momentum directed along their axles. A large angular momentum resists changes in direction (gyroscopic stability). Any small tilt of the bicycle produces a gravitational torque, but the angular momentum vector resists this change, keeping the bike upright. On a stationary bicycle, the wheels have no angular momentum. Any small tilt is immediately amplified by gravity with no gyroscopic resistance, making balance very difficult.

(b) The moment of inertia of the system about the pivot is $I = I_{\text{ruler}} + 2md^2$, where d is the distance of each mass from the pivot. Moving the masses further from the pivot increases d and therefore increases I . The angular acceleration for a given torque is $\alpha = \tau/I$. A larger I means a smaller α for the same tilt (which produces the same gravitational torque). The system resists angular acceleration more strongly.

(c) For a thin rectangular sheet of mass M , length a , and width b :

Axis parallel to the length (through centre): $I_l = \frac{1}{12} Mb^2$

$$I_l = \frac{1}{12} \times 0.032\text{kg} \times (0.10\text{m})^2 = \frac{1}{12} \times 0.032 \times 0.01 = 2.67 \times 10^{-5}\text{kgm}^2$$

Axis parallel to the width (through centre): $I_w = \frac{1}{12} Ma^2$

$$I_w = \frac{1}{12} \times 0.032\text{kg} \times (0.25\text{m})^2 = \frac{1}{12} \times 0.032 \times 0.0625 = 1.67 \times 10^{-4}\text{kgm}^2$$

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