GRAVITATIONAL FIELD AND POTENTIAL

So far, we have described gravity as a force between two masses. But there is another way to think about it, one that becomes essential in more advanced physics. Instead of asking *"what force does mass A exert on mass B?"*, we ask *"what has mass A done to the space around it?"*

The answer: mass A has created a **gravitational field** which may be defined as *a region of space where any other mass placed in it will experience a gravitational force*. The field exists whether or not a second mass is present to feel it. The Earth's gravitational field fills the space around it; the Moon, satellites, and falling apples merely respond to the field that is already there.

Gravitational Field Strength

Imagine placing a small test mass **m** at some point in space near a large mass **M**. The test mass experiences a gravitational force **F** directed toward M. The **gravitational field strength** at that point is defined as *the gravitational force per unit mass experienced by a small test mass placed (at that point):*

$$\mathbf{g} = \frac{\mathbf{F}}{\mathbf{m}}$$

The SI unit of gravitational field strength is N/kg.

Since $\frac{F}{m}$ also equals acceleration (by Newton's second law), gravitational field strength is numerically equal to the acceleration due to gravity at that point. The units N/kg and m/s^2 are therefore equivalent. The two quantities: field strength (N/kg) and acceleration (m/s^2) have the same value, but they represent different physical ideas. *Field strength describes a property of the field while acceleration describes the response (resulting motion) of a mass placed in the field.*

For a point at distance **r** from the centre of a uniform sphere of mass **M**, the gravitational force on the test mass is $F = \frac{GMm}{r^2}$ (by Newton's law). Dividing by m:

$$\mathbf{g} = \frac{F}{m} = \frac{\mathbf{GM}}{\mathbf{r^2}}$$

This is the same expression we derived in the previous subtopic for the acceleration due to gravity. But now we interpret it differently: it describes a property of the field created by M, independent of whatever test mass we choose to place in it.

The gravitational field strength is a **vector** quantity. It points in the direction of the force that a test mass would experience, that is, toward the mass creating the field.

Gravitational field lines

Gravitational field lines are a visual tool for representing the direction and relative strength of a gravitational field.

Rules for field lines:

Rule 1: The direction of a field line at any point gives the direction of the gravitational field (and therefore the direction of the force on a test mass) at that point. Field lines for gravity always point **toward** the mass creating the field, because gravity is always attractive.

Rule 2: The spacing between field lines indicates the strength of the field: closer lines mean a stronger field; wider spacing means a weaker field.

How these features appear depends on the situation, as shown in the two cases below:

Radial field: Far from a spherical mass (like the Earth seen from space), the field lines point radially inward toward the centre. They spread out with distance, reflecting the inverse square decrease in field strength. This is called a **radial field**.

Uniform field: Very close to the Earth's surface (over a small area), the field lines are approximately parallel, equally spaced, and pointing vertically downward. The field is approximately **uniform** (the same strength and direction everywhere in the region). This is why we treat g as a constant in most problems near the surface.

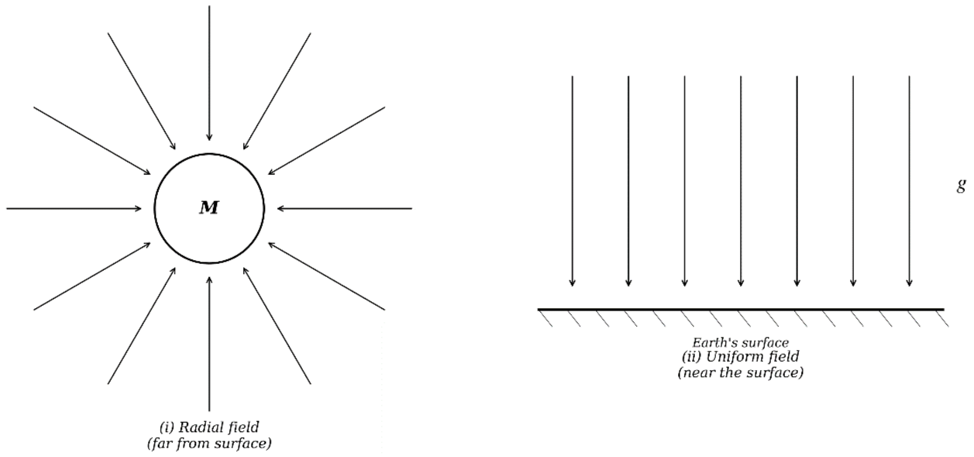

Figure: ***Gravitational field lines.*** *(i) Radial field around a spherical mass M: field lines point radially inward from all directions, spreading out with distance, reflecting the inverse square decrease in field strength. (ii) Uniform field near the Earth's surface: over a small area, the field lines are parallel, equally spaced, and point vertically downward, giving an approximately constant field strength g.*

Gravitational Potential

Gravitational field strength tells us about forces. But in many problems especially those involving energy, orbits, and escape velocity, we need a quantity that describes **energy** rather than force. That quantity is **gravitational potential**.

The **gravitational potential V** at a point in a gravitational field is defined as *the work done per unit mass in bringing a small test mass from infinity to that point:*

$$V = \frac{W_{\infty \to \text{point}}}{m}$$

To understand gravitational potential, consider the work required to move a mass through a gravitational field. Suppose we want to move a small mass m from a point A to a point B in the gravitational field of a large mass M. The work done against gravity depends on the positions of A and B, not on the path taken (because gravity is a **conservative force**). This means we can define a quantity at each point in the field that determines the work done in moving between any two points.

To derive the expression for **V**, consider bringing a test mass **m** from infinity to a point at distance **r** from the centre of mass **M**. At some intermediate position, at distance **x** from the centre, the gravitational force on the test mass is:

$$F = \frac{GMm}{x^2}$$

The work done by gravity in moving the test mass through a small displacement dx (inward, toward M) is $dW = Fdx$. The total work done by gravity in bringing the mass from infinity to distance r is:

$$W = \int_{\infty}^{r} \frac{GMm}{x^2} dx = GMm \left[-\frac{1}{x} \right]_{\infty}^{r} = GMm \left(-\frac{1}{r} + \frac{1}{\infty} \right) = -\frac{GMm}{r}$$

The work done per unit mass is:

$$V = \frac{W}{m} = -\frac{GM}{r}$$

Hence, the gravitational potential at distance r from the centre of a uniform sphere of mass M is given by:

$$\mathbf{V = -\frac{GM}{r}}$$

The SI unit of gravitational potential is J/kg (joules per kilogram).

Why is the potential negative?

The negative sign deserves careful thought, because it confuses many students.

The reference point is infinity, where the potential is defined as zero (when $r = \infty$, gravitational potential formula gives V=0). Now consider what happens as a mass moves from infinity toward a planet. Gravity pulls the mass inward, doing positive work on it. The mass speeds up, gaining kinetic energy. But this kinetic energy comes at the expense of potential energy, which decreases. Since the potential energy at infinity was zero, and the potential energy has decreased, it must now be negative.

A more negative potential means the point is deeper inside the gravitational field. The surface of the Earth has a more negative potential than a point high above it, and the centre of the Earth has the most negative potential of all.

To move a mass upward (away from the planet), you must do work against gravity. This work increases the potential energy and makes the potential less negative. At infinity, the potential returns to zero and the mass is completely free of the gravitational field.

Think of it this way: the gravitational potential tells you how much energy per kilogram you would need to supply to completely free an object from the gravitational field and send it to infinity. At the Earth's surface, this value is about 6.25×10^7 J/kg, an enormous amount of energy per kilogram. This is why escaping the Earth's gravity requires such powerful rockets.

To move a mass **away** from a planet (upward), work must be done **against** gravity. This increases the potential (makes it less negative). At infinity, the potential returns to zero.

Gravitational potential energy

The gravitational potential V is a property of the field at a point (energy per unit mass). When a mass m is actually placed at that point, the gravitational potential energy of the system results.

Gravitational potential energy U *is the energy of a* ***system*** *of two masses due to their gravitational interaction*. For a mass **m** at distance **r** from the centre of mass **M**:

$$U = mV = -\frac{GMm}{r}$$

The gravitational potential energy is negative for the same reason the gravitational potential is negative: the system is bound, and energy must be supplied to separate the masses to infinity.

It is important to note that this formula replaces the familiar $U = mgh$ from earlier chapters. The formula mgh is an approximation that assumes g is constant, which is valid only near the Earth's surface over **small** height changes. The general formula $U = -\frac{GMm}{r}$ works at all distances and accounts for the variation of g with position.

The connection between the two can be seen as follows. For a given height h above the surface, the change in potential energy is:

$$\Delta U = \text{PE at h} - \text{PE at earth surface } (h = 0)$$

But:

$$\text{PE at h} = -\frac{GMm}{r_E + h}$$

$$\text{PE at earth surface} = -\frac{GMm}{r_E}$$

It follows that:

$$\Delta U = -\frac{GMm}{r_E + h} - \left(-\frac{GMm}{r_E}\right) = GMm\left(\frac{1}{r_E} - \frac{1}{r_E + h}\right) = GMm\left(\frac{h}{r_E(r_E + h)}\right)$$

For $h \ll r_E$, the denominator becomes approximately r_E^2. Thus:

$$\Delta U \approx \frac{GMmh}{r_E^2}$$

But:

$$\frac{GM}{r_E^2} = g$$

Hence, over small height changes:

$$\boldsymbol{\Delta U \approx mgh}$$

So mgh is indeed a special case of the general formula, valid when the height is small compared to the Earth's radius.

Relationship Between Field Strength and Potential

Gravitational field strength and gravitational potential are two descriptions of the same field, one based on force and the other on energy. They are connected by a precise mathematical relationship.

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Recall that the potential is defined through work: $V = -\frac{GM}{r}$. If we differentiate this with respect to r:

$$\frac{dV}{dr} = \frac{d}{dr}\left(-\frac{GM}{r}\right) = \frac{GM}{r^2}$$

But we know that $\frac{GM}{r^2} = g$. Therefore:

$$\frac{dV}{dr} = g$$

However, there is a sign to consider. The field strength points in the direction of *decreasing* potential (toward the mass, where V becomes more negative). Since r increases **outward** (away from the mass), and g points **inward** (opposite to displacement, r), the correct relationship including direction is:

$$\mathbf{g = -\frac{dV}{dr}}$$

In words: the gravitational field strength at any point is equal to the negative of the rate of change of potential with distance at that point. The field points in the direction of steepest decrease of potential, and its magnitude equals the rate at which the potential decreases.

This is analogous to a ball on a hill. The ball accelerates in the direction of steepest descent (where height decreases fastest), and the steeper the slope, the greater the acceleration. In gravitation, the "height" is the potential, the "acceleration" is the field strength ,and the "steepness" is $\frac{dV}{dr}$.

On a graph of V against r, the gradient (slope) at any point gives the field strength at that distance. At the Earth's surface, the slope is steep (strong field). Far from the Earth, the curve flattens (weak field). At infinity, the slope is zero and the field vanishes.

Variation of potential with distance

Outside the Earth ($r > r_E$), the potential follows:

$$V = -\frac{GM_E}{r}$$

This implies that the potential is increasing (becoming less negative) as r increases and approaching zero as $r \to \infty$.

At the surface ($r = r_E$):

$$V_{\text{surface}} = -\frac{GM_E}{r_E}$$

Inside the Earth (assuming uniform density), the potential continues to decrease toward the centre. It can be shown that the potential inside a uniform sphere is:

$$V = -\frac{GM_E}{2r_E}\left(3 - \frac{r^2}{r_E^2}\right)$$

At the centre ($r = 0$):

$$V_{\text{centre}} = -\frac{3GM_E}{2r_E}$$

This is 1.5 times more negative than at the surface. The centre of the Earth is the deepest point in the gravitational potential "well."

At the surface ($r = r_E$):

$$V_{\text{surface}} = -\frac{GM_E}{r_E}$$

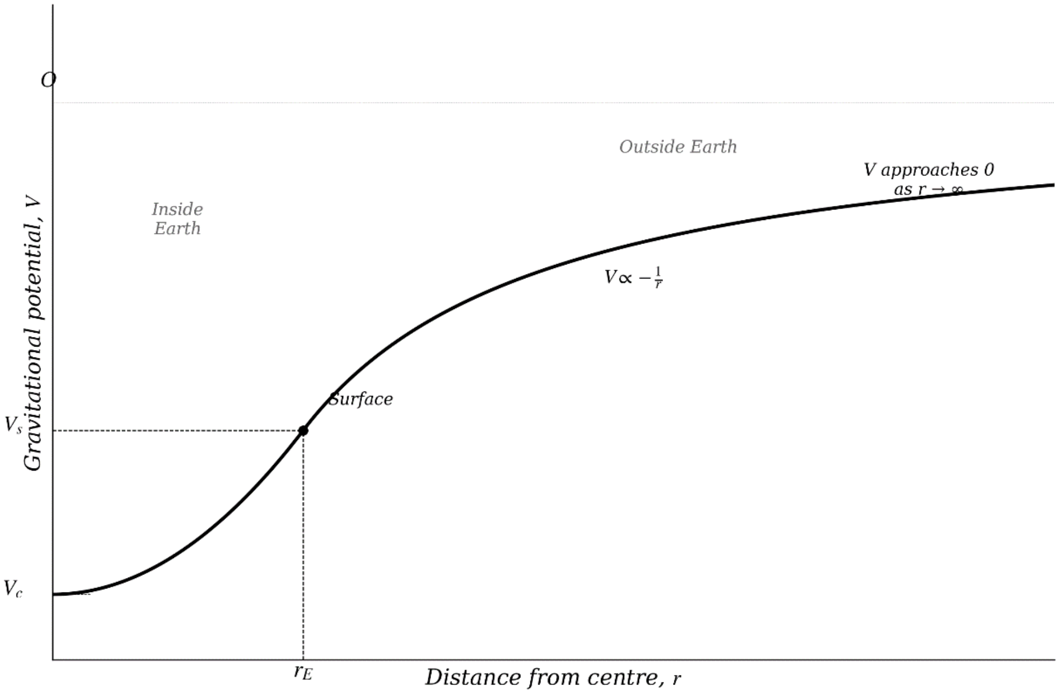

Figure: ***Variation of gravitational potential V with distance r from the centre of the Earth.*** *Inside the Earth, V decreases from V_s at the surface to $V_c = -\frac{3GM_E}{2r_E}$ at the centre. Outside the Earth, V increases (becomes less* *negative) as 1/r, approaching zero as r → ∞. The potential is negative everywhere, reflecting the work needed to escape the gravitational field.*

With this solid theoretical foundation in place, we are now ready to see these ideas come alive through carefully chosen worked examples.

BINDER Example 14

(a) Calculate the gravitational field strength at a height of 200km above the Earth's surface. Take $\mathrm{M_E} = 6.0 \times 10^{24}\text{kg}$, $\mathrm{r_E} = 6.4 \times 10^6\text{m}$, and $\mathrm{G} = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) Compare this with the value at the surface.

Solution

(a) The distance from the centre of the Earth is:

$$\mathrm{R} = \mathrm{r_E} + \mathrm{h} = 6.4 \times 10^6\text{m} + 200 \times 10^3\text{m} = 6.6 \times 10^6\text{m}$$

$$\mathrm{g} = \frac{\mathrm{GM_E}}{\mathrm{R}^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{(6.6 \times 10^6\text{m})^2} = \frac{4.0 \times 10^{14}\text{Nm/kg}}{4.356 \times 10^{13}\text{m}^2} = 9.19\text{N/kg}$$

(b) At the surface: $\mathrm{g_{surface}} = \frac{4.0\times10^{14}}{(6.4\times10^6)^2} = \frac{4.0\times10^{14}}{4.096\times10^{13}} = 9.77\text{N/kg} > 9.19\text{N/kg}$

The field strength at 200km altitude is smaller than the value at the surface (about 94% of the surface value).

Making Sense of the Answer: 200km *seems like a great height, but it is only about* 3% *of the Earth's radius. Since* g *decreases as* $\frac{1}{\mathrm{r}^2}$ *from the centre, a* 3% *increase in* r *produces only about a* 6% *decrease in* g.

Think Like a Physicist: *This result explains why the "weightlessness" of astronauts in orbit is not due to being far from Earth. They are in a gravitational field almost as strong as the one you are sitting in right now. Their apparent weightlessness comes from the absence of a contact force, not the absence of gravity.*

REAL Example 15

A well is dug to a depth of 3km below the Earth's surface. Explain whether the gravitational potential at the bottom of the well is higher or lower (more negative or less negative) than at the surface.

Solution

The gravitational potential at the bottom of the well is **lower** (more negative) than at the surface. Moving from the surface toward the centre of the Earth means moving deeper into the gravitational field. Gravity does positive work as the object descends, meaning the object loses potential energy. Since potential energy decreases, and $\mathrm{U} = \mathrm{mV}$, the potential V also decreases (becomes more negative).

To climb back out of the well, work must be done against gravity, which increases the potential back to its surface value.

Making Sense of the Answer: *Think of the gravitational field as a "well" or valley. The surface is already deep in the well (large negative potential). Going deeper makes the potential even more negative. Going upward (increasing altitude) makes it less negative, approaching zero at infinity.*

Think Like a Physicist: *The sign convention is consistent: potential decreases in the direction of the gravitational force (downward), and increases in the opposite direction (upward). This matches* $\mathrm{g} = -\frac{\mathrm{dV}}{\mathrm{dr}}$; *the field points in the direction of decreasing potential.*

HOT Example 16

(a) Calculate the gravitational potential at the surface of the Earth. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) Calculate the gravitational potential at a height of 36000km above the surface (geostationary orbit altitude).

(c) Find the work done per unit mass in moving an object from the Earth's surface to the geostationary orbit.

Solution

(a) The gravitational potential is given by:

$$V_{\text{surface}} = -\frac{GM_E}{r_E} = -\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{6.4 \times 10^6\text{m}} = -6.25 \times 10^7\,\text{J/kg}$$

The gravitational potential at the surface is -6.25×10^7 J/kg.

(b) The distance from the centre to the geostationary orbit:

$$R = r_E + h = 6.4 \times 10^6\text{m} + 36.0 \times 10^6\text{m} = 42.4 \times 10^6\text{m}$$

$$V_{\text{geo}} = -\frac{GM_E}{R} = -\frac{4.0 \times 10^{14}\text{Nm/kg}}{42.4 \times 10^6\text{m}} = -9.43 \times 10^6\,\text{J/kg}$$

The gravitational potential at the geostationary orbit is -9.43×10^6 J/kg.

(c) The **work done per unit mass** *equals the change in gravitational potential:*

$$\mathbf{\frac{W}{m}} = \mathbf{V_{geo}} - \mathbf{V_{surface}} = (-9.43 \times 10^6\,\text{J/kg}) - (-6.25 \times 10^7\,\text{J/kg})$$

$$\frac{W}{m} = -9.43 \times 10^6\,\text{J/kg} + 6.25 \times 10^7\,\text{J/kg} = 5.31 \times 10^7\,\text{J/kg}$$

The work done is 5.31×10^7 J/kg.

Making Sense of the Answer: *The positive answer confirms that work must be done to move the object upward (against gravity). For a 1000kg satellite, this is $5.31 \times 10^{10} J$ ($5.31 \times 10^7 \times 1000$). This enormous energy requirement is why rocket launches are so expensive and why most of a rocket's mass is fuel.*

Think Like a Physicist: *The work done equals the difference in potential, not the potential itself. This is why the reference point (infinity) does not matter for practical calculations; only the difference between two points matters, and the reference cancels out.*

HOT Example 17

There is a point along the line joining the centres of the Earth and the Moon where the gravitational field strength due to both bodies is zero. Calculate the distance of this point from the centre of the Earth. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $M_M = 7.35 \times 10^{22}\text{kg}$, and the Earth-Moon distance $D = 3.84 \times 10^8\text{m}$.

Solution

Let the point be at distance x from the centre of the Earth. Then it is at distance $(D - x)$ from the centre of the Moon.

At this point, the gravitational field strength due to the Earth (pointing toward Earth) equals the field strength due to the Moon (pointing toward Moon):

$$\frac{GM_E}{x^2} = \frac{GM_M}{(D - x)^2}$$

G cancels:

$$\frac{M_E}{x^2} = \frac{M_M}{(D - x)^2}$$

Taking square roots of both sides:

$$\frac{\sqrt{M_E}}{x} = \frac{\sqrt{M_M}}{D - x}$$

Cross-multiplying:

$$(D - x)\sqrt{M_E} = x\sqrt{M_M}$$

$$D\sqrt{M_E} = x\sqrt{M_M} + x\sqrt{M_E} = x\left(\sqrt{M_M} + \sqrt{M_E}\right)$$

$$x = \frac{D\sqrt{M_E}}{\sqrt{M_E} + \sqrt{M_M}}$$

Substituting:

$$x = \frac{3.84 \times 10^8 \mathrm{m}\sqrt{6.0 \times 10^{24}\mathrm{kg}}}{\sqrt{6.0 \times 10^{24}\mathrm{kg}} + \sqrt{7.35 \times 10^{22}}} = 3.46 \times 10^8 \mathrm{m}$$

The point of zero gravitational field is 3.46×10^8m from the centre of the Earth (about 90% of the way to the Moon).

Making Sense of the Answer: *The neutral point is much closer to the Moon than to the Earth, because the Earth is far more massive. The Moon's weaker field can only match the Earth's strong field at a point very close to the Moon, where the Moon's inverse-square field is large enough to compensate for its smaller mass.*

Think Like a Physicist: *This is a point of zero field, not zero potential. The potential at this point is not zero, it is still negative (both the Earth and Moon contribute negative potential). A spacecraft at this point feels no net gravitational force, but it is not free from the gravitational influence of either body.*

With the concepts of gravitational field strength and potential now in place, we have two complementary descriptions of gravity: one based on force (field strength) and one based on energy (potential). Both will be essential in the next section, where we apply them to the physics of orbits and escape velocity.