ACCELERATION DUE TO GRAVITY

In Chapter 2, we introduced $g = 9.8\text{m/s}^2$ as a constant, which is the acceleration of any object falling freely near the Earth's surface. We used it in equation after equation without asking where it comes from. Now, with Newton's law of gravitation in hand, we can finally answer that question and discover that g is not truly constant at all.

The Origin of g

Consider an object of mass **m** on the surface of the Earth (mass $\mathbf{M_E}$, radius $\mathbf{r_E}$). The gravitational force on the object is its weight:

$$F = \frac{GM_E m}{r_E^2}$$

But we also know that weight equals mg:

$$mg = \frac{GM_E m}{r_E^2}$$

The mass m cancels:

$$\mathbf{g = \frac{GM_E}{r_E^2}}$$

This is a profound result. The acceleration due to gravity at the surface of a planet depends only on the **mass** and **radius** of the planet, not on the mass of the falling object. A feather and a boulder experience the same gravitational acceleration. This is a fact Galileo demonstrated centuries ago, and which Newton's law now explains mathematically.

Mass and Density of the Earth

Since $g = \frac{GM_E}{r_E^2}$, we can rearrange to find the mass of the Earth:

$$\mathbf{M_E = \frac{gr_E^2}{G}} = \frac{9.8\text{m/s}^2 \times (6.4 \times 10^6\text{m})^2}{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}} = 6.02 \times 10^{24}\text{kg}$$

The density of the Earth follows. Modelling the Earth as a sphere:

$$\boldsymbol{\rho} = \frac{\mathbf{M_E}}{\mathbf{\frac{4}{3}\pi r_E^3}} = \frac{6.02 \times 10^{24}\text{kg}}{\frac{4}{3} \times 3.14 \times (6.4 \times 10^6\text{m})^3} = 5483\text{kg/m}^3$$

This is about 5500kg/m^3, roughly five and a half times the density of water. Since surface rocks have a density of only about $2500 - 3000\text{kg/m}^3$, the interior of the Earth must be much denser than the surface; evidence for a heavy iron-nickel core, deduced from nothing more than g, G, and r_E.

Variation of g with Altitude

Consider the following diagram:

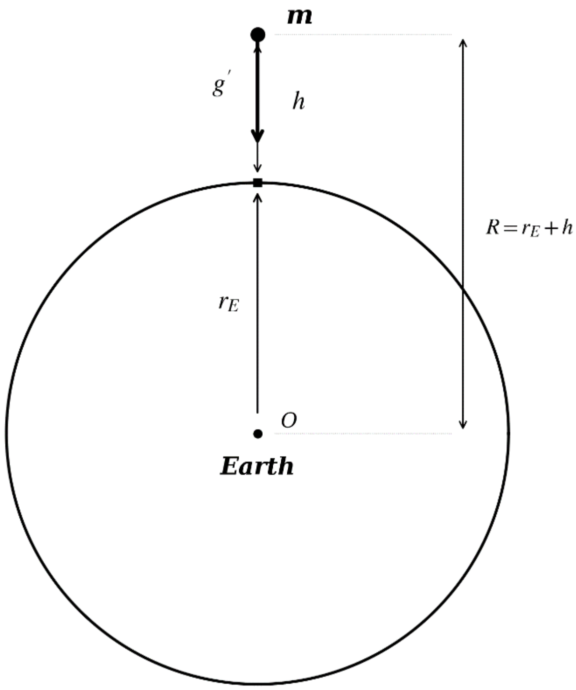

Figure: ***An object of mass m at height h above the Earth's surface.*** *The distance from the centre of the Earth is* $R = r_E + h$*. The acceleration due to gravity g' at this height is directed toward the centre and is less than g at the surface.*

At a height h above the Earth's surface, the distance from the centre of the Earth becomes:

$$R = r_E + h.$$

The acceleration due to gravity at this height is:

$$g' = \frac{GM_E}{R^2} = \frac{GM_E}{(r_E + h)^2}$$

Dividing by $g = \frac{GM_E}{r_E^2}$:

$$\frac{g'}{g} = \frac{r_E^2}{(r_E + h)^2}$$

$$\mathbf{g' = g\left(\frac{r_E}{r_E + h}\right)^2}$$

This is the **exact** expression, valid for any height. It shows that g decreases with altitude, following an inverse square relationship with the distance from the centre (not from the surface).

Approximate formula for small heights ($h \ll r_E$**):**

By multiplying both the numerator and the denominator inside the bracket by $\frac{1}{r_E}$, the equation can be rewritten as:

$$g' = g\left(\frac{1}{1 + \frac{h}{r_E}}\right)^2 = g\left(1 + \frac{h}{r_E}\right)^{-2}$$

If $h \ll r_E$:

$$\frac{h}{r_E} \approx 0(\text{too small})$$

Using the binomial approximation $(1 + x)^{-2} \approx 1 - 2x$ for small x:

$$\mathbf{g' \approx g\left(1 - \frac{2h}{r_E}\right)}$$

This approximation is useful for heights up to a few hundred kilometres, where $\frac{h}{r_E}$ is still small. For larger heights, the exact formula must be used.

Variation of g with Depth

What happens below the Earth's surface? This is where Part 2 of the Shell Theorem becomes essential.

Consider an object at depth d below the surface. Its distance from the centre is $R = r_E - d$. According to the Shell Theorem, only the mass of the Earth **below** the object (within radius R) contributes to the gravitational pull. All the mass in the shell above the object exerts zero net force on it.

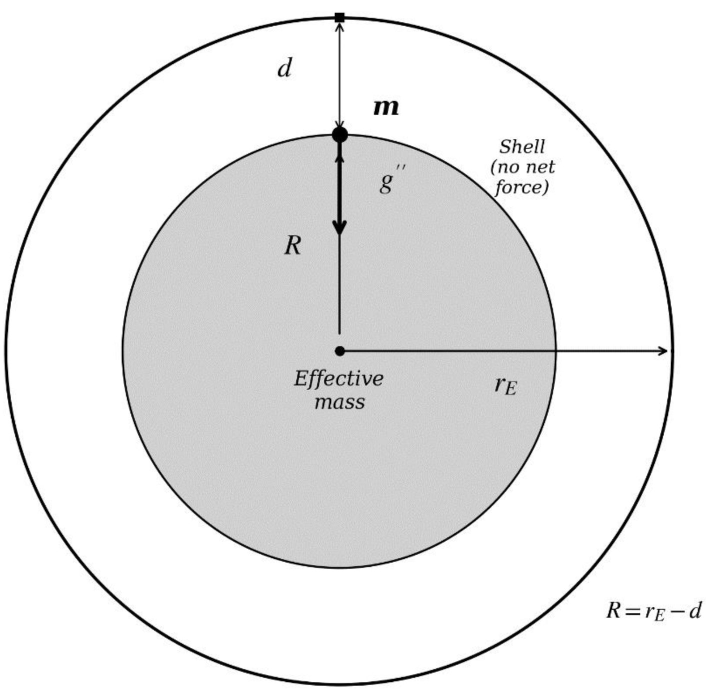

Figure: ***An object of mass m at depth d below the Earth's surface,*** *at distance $R = r_E - d$ from the centre. The shaded region represents the effective mass that exerts gravitational force on the object. The outer shell (unshaded) exerts zero net force, in accordance with the Shell Theorem.*

Assuming the Earth has uniform density ρ (an approximation, but one that reveals the essential physics):

The mass within radius R is:

$$M' = \rho \times \frac{4}{3}\pi R^3$$

The total mass of the Earth is:

$$M_E = \rho \times \frac{4}{3}\pi r_E^3$$

Dividing:

$$\frac{M'}{M_E} = \frac{R^3}{r_E^3}$$

The acceleration due to gravity at depth d is:

$$g'' = \frac{GM'}{R^2} = \frac{G}{R^2} \times \frac{M_E R^3}{r_E^3} = \frac{GM_E}{r_E^3} R = \frac{g}{r_E} R$$

Since $R = r_E - d$:

$$g'' = \frac{g}{r_E}(r_E - d)$$

Hence:

$$\mathbf{g'' = g\left(1 - \frac{d}{r_E}\right)}$$

This tells us that g decreases **linearly** with depth. At the surface ($d = 0$), $g'' = g$. At the centre of the Earth ($d = r_E$), $g'' = 0$.

An object at the centre of the Earth would be pulled equally in all directions by the surrounding mass, so the net gravitational force would be zero. This is true weightlessness with genuine zero gravitational force, not the "weightlessness" of orbit, where gravity still acts but produces no contact force.

Important assumption: The derivation above assumes uniform density throughout the Earth. In reality, the Earth's core is much denser than its mantle and crust, so the actual variation of g with depth is more complex. In fact, g initially *increases* slightly as you descend from the surface (because you move closer to the dense core), reaching a maximum at the core-mantle boundary, before decreasing to zero at the centre. However, for examination purposes, the uniform density approximation is standard.

Variation of g with Latitude

The value of g at the Earth's surface also varies with latitude, for two reasons:

Reason 1: The Earth is not a perfect sphere. The Earth is slightly flattened at the poles and bulges at the equator (an oblate spheroid). The polar radius is about 21km shorter than the equatorial radius. Since $g \propto \frac{1}{r^2}$, the smaller radius at the poles means g is slightly larger there.

Reason 2: The Earth rotates. A body on the surface of the Earth moves in a circle (due to the Earth's rotation) and therefore requires centripetal acceleration. Part of the gravitational pull goes toward providing this centripetal acceleration, leaving less to be felt as "weight."

At the equator, the centripetal acceleration is maximum because the radius of the circular path (the Earth's equatorial radius r_E) is largest. At the poles, the body is on the axis of rotation, the radius of the circular path is zero, and no centripetal acceleration is needed.

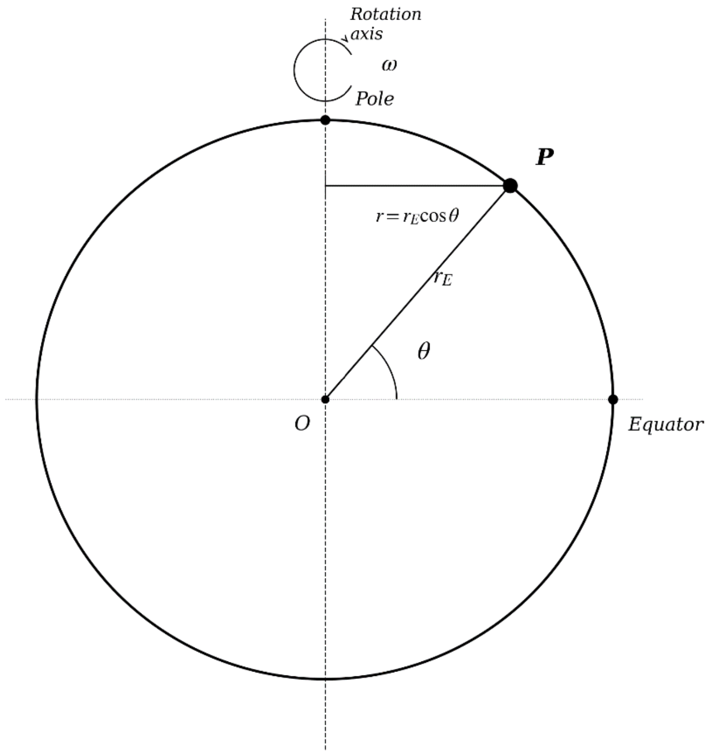

Figure: ***A point P on the Earth's surface at latitude θ.*** *The Earth rotates about its polar axis with angular velocity ω. The distance from P to the rotation axis is* $r = r_E cos\theta$*. As the Earth rotates, P moves in a horizontal circle of this radius, requiring centripetal acceleration* $\omega^2 r_E cos\theta$ *directed toward the axis.*

At latitude θ, the radius of the circular path is $r = r_E cos\theta$, and the centripetal acceleration is $\omega^2 r_E \cos\theta$. The effective (measured) gravitational acceleration is:

$$\mathbf{g' = g - \omega^2 r_E \cos^2\theta}$$

At the **equator** ($\theta = 0°$): $g' = g - \omega^2 r_E$ (minimum)

At the **poles** ($\theta = 90°$): $g' = g$ (maximum)

The correction $\omega^2 r_E = (7.27 \times 10^{-5} \text{rad/s})^2 \times 6.4 \times 10^6 \text{m} = 0.034 \text{m/s}^2$. This is small compared to 9.8m/s^2 (about 0.3%), but it is measurable. Combined with the shape effect, the total variation in g from equator to pole is about 0.053m/s^2, giving $g \approx 9.78 \text{m/s}^2$ at the equator and $g \approx 9.83 \text{m/s}^2$ at the poles.

Summary on variation of g

As distance from the centre of the Earth **increases from zero to** r_E (moving from centre to surface): g increases linearly from 0 to g.

As distance from the centre **increases beyond** r_E (moving above the surface): g decreases as $\frac{1}{r^2}$.

The maximum value of g (under the uniform density assumption) occurs at the surface. A graph of g versus distance from the centre rises linearly to a peak at $r = r_E$ and then falls off as an inverse square curve.

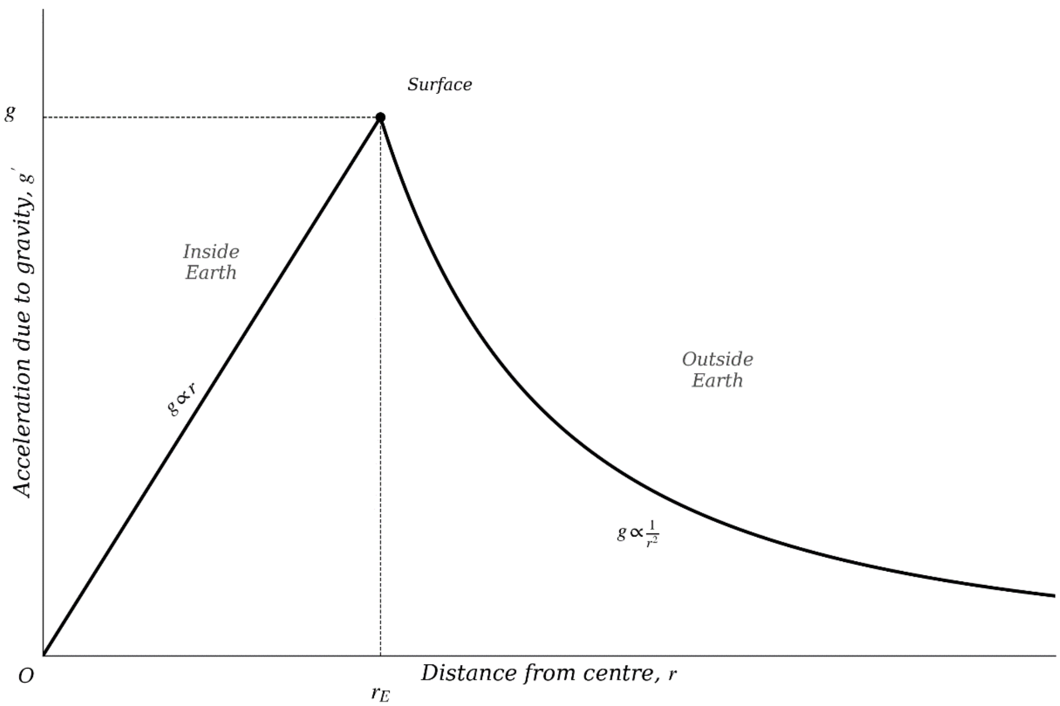

Figure: ***Variation of the acceleration due to gravity with distance from the centre of the Earth (assuming uniform density).*** *Inside the Earth, g increases linearly with distance (g ∝ r). Outside the Earth, g decreases with the inverse square of distance (g ∝ 1/r²). The maximum value occurs at the surface ($r = r_E$).*

Let us now apply these ideas through worked examples.

BINDER Example 7

(a) Calculate the acceleration due to gravity on the surface of Mars. Take the mass of Mars as 6.4×10^{23}kg, the radius of Mars as 3.4×10^{6}m, and $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) A person weighs 700N on Earth. What would this person weigh on Mars?

Solution

(a) Using:

$$g_{Mars} = \frac{GM_{Mars}}{r_{Mars}^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.4 \times 10^{23}\text{kg}}{(3.4 \times 10^{6}\text{m})^2} = 3.69\text{m/s}^2$$

The acceleration due to gravity on Mars is 3.69m/s^2.

(b) The person's mass: $m = \frac{W_{Earth}}{g_{Earth}} = \frac{700\text{N}}{9.8\text{m/s}^2} = 71.4\text{kg}$

Weight on Mars: $W_{Mars} = mg_{Mars} = 71.4\text{kg} \times 3.69\text{m/s}^2 = 263\text{N}$

The person would weigh 263N on Mars.

Making Sense of the Answer: *Mars has about 38% of Earth's surface gravity. The person feels roughly a third of their Earth weight. This is why Mars rovers can make soft landings with less braking than would be needed on Earth, but enough gravity exists to keep objects firmly on the surface.*

Think Like a Physicist: *Notice that Mars is both less massive* ***and*** *smaller than Earth. Less mass reduces g, but a smaller radius increases g (because you are closer to the centre). The net effect depends on which factor wins. For Mars, the mass effect dominates, giving a lower g.*

BINDER Example 8

A body weighing 72N on the surface of the Earth is taken to a height equal to half the radius of the Earth above the surface. Calculate the new weight. Take $r_E = 6400\text{km}$.

Solution

At height $h = \frac{r_E}{2}$, the distance from the centre of the Earth becomes:

$$R = r_E + h = r_E + \frac{r_E}{2} = \frac{3}{2}r_E$$

Using the exact formula:

$$g' = g\left(\frac{r_E}{r_E + h}\right)^2 \text{ or } \frac{g'}{g} = \left(\frac{r_E}{r_E + h}\right)^2 = \left(\frac{r_E}{R}\right)^2$$

The above equation is the same as writing as:

$$\frac{mg'}{mg} = \left(\frac{r_E}{r_E + h}\right)^2 = \left(\frac{r_E}{R}\right)^2$$

Where: $mg' = W'$, $mg = W$. Thus:

$$\frac{W'}{W} = \left(\frac{r_E}{R}\right)^2 = \left(\frac{r_E}{\frac{3}{2}r_E}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Therefore:

$$W' = \frac{4}{9} \times W = \frac{4}{9} \times 72\text{N} = 32\text{N}$$

The new weight is 32N.

Making Sense of the Answer: *Moving to $1.5r_E$ from the centre reduces the weight to $\frac{4}{9}$ of the surface value which is less than half. The inverse square law is powerful: even a modest increase in distance produces a significant drop in gravitational force.*

Think Like a Physicist: *The approximate formula $g' \approx g\left(1 - \frac{2h}{r_E}\right)$ would give $g' = g(1 - 1) = 0$, which is clearly wrong. The approximation is only valid when $h \ll r_E$. Here $h = 0.5r_E$, which is not small, so the exact formula is essential.*

REAL Example 9

A gold dealer in Dar es Salaam (near sea level) uses a precision digital scale to weigh a gold bar, and reads 500.00g. The same gold bar is taken to a dealer in Mbeya (altitude approximately 1700m) and placed on an identical scale. Explain whether the reading in Mbeya will be the same, and if not, which reading will be higher.

Solution

The reading will be slightly different. A precision scale measures the gravitational force (weight) on the object and converts it to a mass reading using a calibrated value of g.

At higher altitude, g is smaller because the object is farther from the centre of the Earth. With a lower g, the gravitational force on the gold bar is slightly less, so the scale in Mbeya reads a slightly lower value than the scale in Dar es Salaam. **The Dar es Salaam reading will be higher.**

(Using the approximate formula: $\frac{\Delta g}{g} \approx \frac{2h}{r_E} = \frac{2\times1700\text{m}}{6.4\times10^6\text{m}} = 0.00053 = 0.053\%$. The difference is tiny (about 0.27g on a 500g bar), but it is real and measurable with a precision scale.)

Making Sense of the Answer: *For everyday trade this difference is negligible. But for high-precision scientific work, for gold trading in large quantities, or for pharmaceutical measurements, the variation of* g *with altitude matters. A scale calibrated in Dar es Salaam is not perfectly accurate in Mbeya.*

Think Like a Physicist: *A balance (two-pan) scale would give the same reading at both locations, because it compares masses, not weights. Both sides are affected equally by the change in* g*. A spring scale or digital scale, which measures force, is the one affected by altitude.*

HOT Example 10

At what height above the Earth's surface is the acceleration due to gravity reduced by 36% from its surface value? Take $r_E = 6400\text{km}$.

Solution

If g is reduced by 36%, the remaining value is 64% of g:

$$g' = 0.64g$$

Using the exact formula:

$$g' = g\left(\frac{r_E}{r_E + h}\right)^2$$

Substituting $g' = 0.64g$:

$$0.64g = g\left(\frac{r_E}{r_E + h}\right)^2$$

Dividing both sides by g:

$$0.64 = \left(\frac{r_E}{r_E + h}\right)^2$$

Taking the square root of both sides:

$$\sqrt{0.64} = \frac{r_E}{r_E + h}$$

$$0.8 = \frac{r_E}{r_E + h}$$

Making $(r_E + h)$ the subject:

$$r_E + h = \frac{r_E}{0.8} = 1.25r_E$$

$$h = 1.25r_E - r_E = 0.25r_E = 0.25 \times 6400\text{km} = 1600\text{km}$$

The height is 1600km above the surface.

Making Sense of the Answer: *Commercial aircraft fly at about 10km which is far too low for any noticeable change in g. Even the International Space Station at 400km experiences only about an 11% reduction. Gravity does not "switch off" in space; it simply weakens with distance.*

Think Like a Physicist: *Notice that* $0.64 = 0.8^2$*, so the distance from the centre need only increase by a factor of* $\frac{1}{0.8} = 1.25$*. Recognising perfect squares speeds up problem solving enormously.*

HOT Example 11

(a) A planet has the same mass as the Earth but half the radius. Calculate the acceleration due to gravity on the surface of this planet in terms of g.

(b) A planet has the same density as the Earth but twice the radius. Show that the acceleration due to gravity on its surface is 2g.

Solution

(a) Using $g_p = \frac{GM}{r_p^2}$, with $M = M_E$ and $r_p = \frac{r_E}{2}$:

$$g_p = \frac{GM_E}{\left(\frac{r_E}{2}\right)^2} = \frac{GM_E}{\frac{r_E^2}{4}} = 4 \times \frac{GM_E}{r_E^2} = 4 \times g$$

The surface gravity is 4g.

(b) For a planet with the same density ρ but radius $r_p = 2r_E$:

The mass of the planet is:

$$M_p = \rho \times \frac{4}{3}\pi r_p^3 = \rho \times \frac{4}{3}\pi(2r_E)^3 = \rho \times \frac{4}{3}\pi \times 8r_E^3 = 8\left(\rho \times \frac{4}{3}\pi r_E^3\right) = 8M_E$$

The surface gravity is:

$$g_p = \frac{GM_p}{r_p^2} = \frac{G \times 8M_E}{(2r_E)^2} = \frac{8GM_E}{4r_E^2} = 2 \times \frac{GM_E}{r_E^2} = 2g$$

The surface gravity is 2g.

Making Sense of the Answer: *Part (a): same mass squeezed into a smaller sphere means the surface is closer to the centre, dramatically increasing g. Part (b): same density but larger radius means more mass, but the surface is also farther from the centre. The mass grows as r^3 while g divides by r^2, leaving a net factor of r. Double the radius, double the g.*

Think Like a Physicist: *When comparing planets, always check what is held constant. Same mass, different radius: $g \propto \frac{1}{r^2}$. Same density, different radius: $g \propto r$. Same radius, different mass: $g \propto M$. Each gives a completely different scaling.*

HOT Example 12

Calculate the percentage decrease in the weight of a body when it is taken to a depth of 64km below the Earth's surface. Take $r_E = 6400\text{km}$.

Solution

Using the depth formula (assuming uniform density):

$$g'' = g\left(1 - \frac{d}{r_E}\right)$$

Substituting $d = 64\text{km}$ and $r_E = 6400\text{km}$:

$$g'' = g\left(1 - \frac{64\text{km}}{6400\text{km}}\right) = g(1 - 0.01) = 0.99g$$

The fractional decrease in g (and therefore in weight) is:

$$\frac{g - g''}{g} = \frac{g - 0.99g}{g} = 0.01 = 1\%$$

The weight decreases by 1%.

Making Sense of the Answer: *$64km$ is 1% of the Earth's radius, and the weight decreases by 1%. This is the beauty of the linear relationship: the percentage decrease in g at depth equals the percentage of the radius descended.*

Think Like a Physicist: *Compare altitude and depth at the same distance of $64km$. At $64km$ altitude: $g' \approx g\left(1 - \frac{2\times64}{6400}\right) = g(1 - 0.02) = 0.98g$, a 2% decrease. At $64km$ depth: only a 1% decrease. Gravity decreases* ***twice as fast*** *with altitude as with depth (for small distances), because the altitude formula has a factor of* **2** *that the depth formula does not.*

HOT Example 13

Calculate the effective value of the acceleration due to gravity at the equator, taking into account the Earth's rotation. Take $\mathrm{g} = 9.8\mathrm{m/s^2}$, $\mathrm{r_E} = 6.4 \times 10^6\mathrm{m}$, and the Earth completes one rotation in 24 hours.

Solution

Calculating the angular velocity of the Earth:

$$\omega = \frac{2\pi}{\mathrm{T}} = \frac{2 \times 3.14}{24 \times 3600\mathrm{s}} = 7.27 \times 10^{-5}\mathrm{rad/s}$$

At the equator ($\theta = 0°$, so $\cos\theta = 1$), the effective gravity is:

$$\mathrm{g}' = \mathrm{g} - \omega^2\mathrm{r_E}\cos^2\theta = \mathrm{g} - \omega^2\mathrm{r_E}$$

Calculating $\omega^2 r_E$:

$$\omega^2\mathrm{r_E} = (7.27 \times 10^{-5}\mathrm{rad/s})^2 \times 6.4 \times 10^6\mathrm{m} = 0.0338\mathrm{m/s^2}$$

Therefore:

$$\mathrm{g}' = 9.8\mathrm{m/s^2} - 0.034\mathrm{m/s^2} = 9.766\mathrm{m/s^2}$$

The effective g at the equator is $9.766\mathrm{m/s^2}$, (a reduction of $0.034\mathrm{m/s^2}$ (about 0.35%) due to the Earth's rotation).

Making Sense of the Answer: *The effect is tiny (about a third of one percent). A $70kg$ person would weigh about $0.24N$ less at the equator than at the pole. This is negligible for everyday purposes, but it matters for precision instruments and for satellite launches from equatorial sites.*

Think Like a Physicist: *If the Earth rotated fast enough that $\omega^2\mathrm{r_E} = \mathrm{g}$, objects at the equator would be weightless. This would require a rotation period of about 84 minutes, which is the same as the orbital period of a low-Earth satellite. This is not a coincidence.*

The variation of g with altitude, depth, and latitude reveals that gravity is richer and more subtle than the simple constant we used in Chapter 2. In the next subtopic, we formalise this by introducing the concept of a gravitational field.