NEWTON'S LAW OF UNIVERSAL GRAVITATION

Kepler showed us the pattern. Three laws, elegant and precise, describing exactly how planets orbit the Sun. But Kepler could not answer the obvious question: *what force makes them do this?*

Isaac Newton answered that question with one of the boldest claims in the history of science. He proposed that the force pulling an apple to the ground and the force holding the Moon in orbit are **the same force**. Not similar forces, not analogous forces, but the *same* force, obeying the *same* law, differing only in the masses involved and the distances between them. He called it **gravitation**, and he declared it to be **universal**: every object in the universe attracts every other object. The law states that:

Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres.

Mathematically:

$$F = \frac{Gm_1m_2}{r^2}$$

Where:

F is the gravitational force of attraction between the two bodies (in N),

m_1 and m_2 are the masses of the two bodies (in kg),

r is the distance between their centres (in m),

G is the **universal gravitational constant**, with value:

$$G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$$

Understanding the law

Several features of this law deserve careful attention.

1. The force is mutual

If the Earth pulls you downward with a gravitational force, you pull the Earth upward with the same force. This is Newton's third law in action. The forces are equal in magnitude and opposite in direction. The reason you accelerate more than the Earth is that your mass is vastly smaller, not that the force is different.

2. The force depends on the product of the masses

Doubling either mass doubles the force. Doubling both masses quadruples the force. If one mass is zero, the force is zero.

3. The force obeys an inverse square law

Doubling the distance reduces the force to one-quarter. Tripling the distance reduces it to one-ninth. This rapid decrease with distance is why gravitational effects from distant objects are usually negligible, even though the force technically extends to infinity.

4. The force acts along the line joining the centres

Gravity is always attractive and always acts along the straight line connecting the centres of the two masses. It never pushes; it only pulls.

The Gravitational Constant G

The constant G is extraordinarily small: $6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. This is why you do not feel a gravitational pull toward the person sitting next to you in class, even though Newton's law says the force exists. For two people of mass 60kg each, sitting 1m apart, the gravitational force between them is about 2.4×10^{-7}N (roughly the weight of a bacterium!).

Gravity only becomes significant when at least one of the masses is enormous, like a planet or a star. The Earth's mass (6.0×10^{24}kg) compensates for the tiny value of G and produces the familiar gravitational pull we experience every day.

G was first measured experimentally by Henry Cavendish in 1798, more than a century after Newton proposed the law. Cavendish used a sensitive torsion balance to measure the weak gravitational attraction between lead spheres. His experiment is often described as "weighing the Earth," because knowing G allows the mass of the Earth to be calculated.

The Shell Theorem

Newton's law of gravitation is stated for **point particles**. But planets and stars are not points, they are extended spheres. *How do we apply the law to them?* Newton himself answered this question by proving what is now called the **Shell Theorem**, which has two parts:

Part 1: *A uniform spherical shell attracts a particle **outside** it as if the entire mass of the shell were concentrated at its centre.*

Part 2: *A uniform spherical shell exerts **zero** gravitational force on a particle **inside** it.*

Since a solid sphere can be thought of as a collection of concentric shells, Part 1 means that a uniform solid sphere (like a planet) attracts any external object as if all the sphere's mass were at its centre. This is the reason we are justified in using $F = \frac{Gm_1m_2}{r^2}$ with r measured from **centre to centre**, even though the masses are spread out over large volumes.

When you stand on the Earth's surface, the relevant distance in $F = \frac{Gm_Em}{r^2}$ is the distance from you to the **centre** of the Earth ($r_E = 6.4 \times 10^6$m), not the distance to the ground beneath your feet.

Part 2 will become important later when we study how gravity varies with depth below the Earth's surface.

Deriving Kepler's Third Law from Newton's Law

This is where the connection between Chapters 7 and 8 becomes explicit. Consider a planet of mass **m** orbiting the Sun of mass **M** in a circular orbit of radius **r**.

From Chapter 7, the planet in circular motion requires a centripetal force:

$$F_{centripetal} = \frac{mv^2}{r}$$

This centripetal force is provided by gravity:

$$F_{gravity} = \frac{GMm}{r^2}$$

Setting them equal:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

The mass of the planet m cancels:

$$v^2 = \frac{GM}{r}$$

Now, the orbital speed is related to the period by $v = \frac{2\pi r}{T}$. Substituting:

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

$$\mathbf{T^2 = \frac{4\pi^2}{GM} r^3}$$

This is Kepler's Third Law, with the constant identified:

$$\mathbf{k = \frac{4\pi^2}{GM}}$$

Three observations are profound:

First: $T^2 \propto r^3$ emerges naturally from Newton's law of gravitation combined with circular motion. Kepler discovered this empirically; Newton explained *why*.

Second: The mass of the orbiting body (m) does not appear. The period depends only on the orbital radius and the mass of the central body (M). This is why *all satellites at the same altitude orbit the Earth with the same period, regardless of their mass.*

Third: the constant k depends only on G and M. If you know k (from observing any orbit), you can calculate M (the mass of the central body). This is how astronomers "weigh" the Sun, the Earth, and other planets.

Let us now put these ideas to work.

BINDER Example 4

(a) Two identical spheres, each of mass 50kg, are placed with their centres 0.5m apart. Calculate the gravitational force between them and explain why this force is not noticeable in everyday life. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

(b) Calculate the gravitational force between the Earth and a 70kg person standing on the surface. Take $M_E = 6.0 \times 10^{24}\text{kg}$, $r_E = 6.4 \times 10^6\text{m}$.

Solution

(a) Using $F = \frac{Gm_1m_2}{r^2}$:

$$F = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 50\text{kg} \times 50\text{kg}}{(0.5\text{m})^2} = 6.67 \times 10^{-7}\text{N}$$

This force is about 0.00000067N. It is far too small to feel because the gravitational constant G is extremely small, and neither mass is large enough to compensate.

(b) Using:

$$F = \frac{GM_Em}{r_E^2} = \frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg} \times 70\text{kg}}{(6.4 \times 10^6\text{m})^2} = 683\text{N}$$

The gravitational force between the Earth and a 70kg person is 683N. (This is the person's weight, close to $mg = 70 \times 9.8 = 686\text{N}$; the small difference comes from rounding).

Making Sense of the Answer: *The gravitational force between everyday objects is negligible. But when one mass is as enormous as the Earth (*$6 \times 10^{24}\text{kg}$*), the tiny constant G is overwhelmed and the force becomes the very weight we feel every moment of our lives.*

Think Like a Physicist: *Newton's law of gravitation and the familiar formula W = mg are not two different things. W = mg is simply the result of applying* $F = \frac{GMm}{r^2}$ *at the Earth's surface, where* $g = \frac{GM}{r_E^2}$*. We will derive this connection explicitly in the next section.*

REAL Example 5

The Earth pulls the Moon with a gravitational force, and the Moon pulls the Earth with the same force. Yet the Moon orbits the Earth, not the other way around. Explain why both bodies experience the same force but respond so differently.

Solution

The gravitational force on the Moon due to the Earth is equal in magnitude to the force on the Earth due to the Moon. However, by Newton's second law ($a = \frac{F}{m}$), the acceleration produced depends on the mass of the body. The Moon is much less massive than the Earth, so the Moon's acceleration is much greater than the Earth's. The Moon therefore moves in a large, visible orbit, while the Earth's motion is barely noticeable.

Making Sense of the Answer: *Same force, different masses, different accelerations. A mosquito and a truck experience the same collision force during impact, but only the mosquito changes its motion dramatically. The same principle applies to the Moon and the Earth.*

Think Like a Physicist: *The phrase "the Moon orbits the Earth" is a convenient simplification. More precisely, both orbit their common centre of mass. But since the Earth is so much more massive, the centre of mass is very close to the Earth's centre, making the simplification excellent.*

HOT Example 6

(a) The Earth orbits the Sun at a mean distance of 1.5×10^{11}m with a period of 3.156×10^{7}s. Take $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$. Calculate the mass of the Sun.

(b) A satellite orbits the Earth at a height of 300km above the surface. Calculate its orbital speed and period in minutes. Take $M_E = 6.0 \times 10^{24}$kg, $r_E = 6.4 \times 10^{6}$m.

Solution

(a) From $T^2 = \frac{4\pi^2}{GM}r^3$, making M the subject:

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{4 \times (3.14)^2 \times (1.5 \times 10^{11}\text{m})^3}{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times (3.156 \times 10^{7}\text{s})^2} = 2.01 \times 10^{30}\text{kg}$$

The mass of the Sun is approximately 2.0×10^{30}kg.

(b) Orbital radius: $R = r_E + h = 6.4 \times 10^6\text{m} + 300 \times 10^3\text{m} = 6.7 \times 10^6\text{m}$

$$F_{centripetal} = \frac{mv^2}{R}$$

This centripetal force is provided by gravity:

$$F_{gravity} = \frac{GM_E m}{R^2}$$

Equating:

$$\frac{mv^2}{R} = \frac{GM_E m}{R^2}$$

From which:

$$v^2 = \frac{GM_E}{R}$$

$$v = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2} \times 6.0 \times 10^{24}\text{kg}}{6.7 \times 10^{6}\text{m}}} = 7726\text{m/s}$$

The orbital speed is 7726m/s.

Period: $T = \frac{2\pi R}{v} = \frac{2 \times 3.14 \times 6.7 \times 10^6\text{m}}{7726\text{m/s}} = \frac{4.21 \times 10^7}{7726} = 5449\text{s}$

$$T = \frac{5449}{60}\text{min} = 90.8\text{min}$$

The period is $\mathbf{90.8}$min.

Making Sense of the Answer: *A satellite at 300km altitude circles the Earth in about 91 minutes which is roughly the period of the International Space Station. Its speed of 7726m/s means it travels faster than a bullet. At lower altitudes, the speed would be even higher and the period shorter.*

Think Like a Physicist: *Notice that the satellite's mass never appeared in any calculation. Orbital speed and period depend only on the mass of the central body and the orbital radius. A feather and a truck at the same altitude would orbit at exactly the same speed, just as Galileo's insight predicts.*

With Newton's law of gravitation and its connection to circular motion now established, we have the tools to explore how gravity behaves at different locations: above, on, and below the surface of the Earth. That is the subject of the next section.