MISCELLANEOUS WORKED EXAMPLES ON UNIFORM MOTION

Example 44

(a) Explain why a satellite in a circular orbit around the Earth is accelerating even though its speed is constant.

(b) A satellite orbits the Earth at a height of 200km above the surface. At this altitude, the acceleration due to gravity is 9.6m/s^2. The radius of the Earth is 6400km. Calculate the orbital speed and the period of revolution of the satellite.

Solution

(a) Acceleration is defined as the rate of change of velocity. Velocity is a vector quantity that has both magnitude (speed) and direction. Although the satellite's speed is constant, its direction of motion changes continuously as it follows the circular path. This continuous change in the direction of the velocity vector constitutes a centripetal acceleration directed toward the centre of the Earth. Therefore, the satellite is accelerating even at constant speed.

(b) The orbital radius is:

$$r = R_E + h = 6400\text{km} + 200\text{km} = 6600\text{km} = 6.6 \times 10^6\text{m}$$

At this altitude, gravity provides the centripetal force. For the orbital speed:

$$\frac{mv^2}{r} = mg'$$

Where $g' = 9.6\text{m/s}^2$ is the gravitational acceleration at that height.

Making v the subject:

$$v = \sqrt{g'r} = \sqrt{9.6\text{m/s}^2 \times 6.6 \times 10^6\text{m}} = \sqrt{6.336 \times 10^7\text{m}^2/\text{s}^2} = 7960\text{m/s}$$

The orbital speed is 7960m/s.

For the period:

$$T = \frac{2\pi r}{v} = \frac{2 \times 3.14 \times 6.6 \times 10^6\text{m}}{7960\text{m/s}} = \frac{4.145 \times 10^7\text{m}}{7960\text{m/s}} = 5207\text{s}$$

$$T = \frac{5207}{60}\text{min} = 86.8\text{min}$$

The period of revolution is 5207s (about 86.8 minutes).

Example 45

(a) A student claims: *"If the net force on a body is always perpendicular to its velocity, the body cannot accelerate because the force does no work."* Explain why the student's reasoning is only partially correct.

(b) A particle of mass 0.05kg moves in a horizontal circle of radius 0.4m at constant speed. The centripetal force acting on the particle is 2N. Calculate the speed and the number of revolutions the particle completes per second.

Solution

(a) The student is correct that a force perpendicular to the velocity does no work and therefore does not change the kinetic energy or the speed of the body. However, the student is wrong to conclude that the body cannot accelerate. A body can accelerate without any change in speed, provided its direction changes. This is because, acceleration is the rate of change of velocity, not the rate of change of speed. A perpendicular force changes the *direction* of the velocity without changing its magnitude.

(b) Using:

$$F = \frac{mv^2}{r}$$

Making v the subject:

$$v = \sqrt{\frac{Fr}{m}} = \sqrt{\frac{2N \times 0.4m}{0.05kg}} = \sqrt{16m^2/s^2} = 4m/s$$

The speed is 4m/s.

The number of revolutions per second (frequency):

$$f = \frac{v}{2\pi r} = \frac{4m/s}{2 \times 3.14 \times 0.4m} = \frac{4}{2.513} s^{-1} = 1.59 rev/s$$

The particle completes 1.59 revolutions per second.

Example 46

(a) Explain why road surfaces at sharp bends are made rough rather than smooth.

(b) A curved section of road has a radius of 120m and is banked at 18° to the horizontal. The coefficient of static friction between the tyres and the road is 0.35. Take $g = 9.8m/s^2$.

Determine the range of speeds at which a car can safely negotiate this curve without sliding.

Solution

(a) When a vehicle negotiates a curve on a flat road, the centripetal force required to maintain the circular path is provided entirely by static friction between the tyres and the road surface. A rough surface provides a higher coefficient of static friction than a smooth surface, allowing a greater maximum centripetal force and therefore a higher safe speed on the curve. A smooth surface would provide very little friction, and vehicles would slide outward even at low speeds.

(b) **Maximum speed** (friction acts down the slope, supplementing centripetal force):

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta}\right)}$$

Where: $r = 120m$, $g = 9.8m/s^2$, $\theta = 18°$, $\mu_s = 0.35$, $\tan 18° = 0.3249$

$$v_{max} = \sqrt{120m \times 9.8m/s^2 \times \frac{0.3249 + 0.35}{1 - 0.35 \times 0.3249}} = 29.9m/s$$

The maximum safe speed is 29.9m/s.

Minimum speed (friction acts up the slope, preventing inward sliding):

$$v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta}\right)}$$

$$v_{min} = \sqrt{120m \times 9.8m/s^2 \times \frac{0.3249 - 0.35}{1 + 0.35 \times 0.3249}} = \sqrt{1176 \times \frac{-0.0251}{1.1137}}$$

The numerator $(\tan\theta - \mu_s) = 0.3249 - 0.35 = -0.0251$ is negative.

Since $\tan\theta < \mu_s$, the minimum speed is **zero**. Friction is strong enough to prevent the car from sliding inward down the banked surface at any speed, including rest. The car can safely travel at any speed from 0 to 29.9m/s on this curve.

The safe speed range is $0 \leq v \leq 29.9m/s$.

Example 47

(a) Kipanga claims that a heavier person is more likely to fall off a spinning ride than a lighter person because *"more mass means more centrifugal force pulling them outward."* Is Kipanga correct? Explain.

(b) A spinning ride has a radius of 4m and rotates at 1.5rev/s. A child of mass 35kg and an adult of mass 80kg sit at the same radius. Calculate the centripetal force required for each person, and show that both require the same coefficient of friction to remain seated.

Solution

(a) Kipanga is wrong in two accounts:

First, there is no outward "centrifugal force" acting on the rider. When the ride spins, friction between the rider and the seat pushes the rider inward, providing the centripetal force needed for circular motion. If friction is insufficient, the rider does not get "pulled outward" by any force. Instead, the rider's body, obeying Newton's first law, continues in a straight line (tangentially), which from the ride's perspective appears as sliding outward.

Second, a a heavier person is not more likely to slide off. A heavier person requires more centripetal force ($F = m\omega^2 r$), but a heavier person also has more weight, which to the same proportion, increases the normal reaction and therefore the maximum friction available ($f_{max} = \mu mg$), providing more centripetal acceleration. Mathematically, when we set the required centripetal force equal to the maximum friction, the mass cancels:

$$m\omega^2 r = \mu mg \Longrightarrow \mu = \frac{\omega^2 r}{g}$$

Hence, the coefficient of friction needed is independent of mass. Both the child and the adult slide off at the same rotation speed.

(b) Angular velocity:

$$\omega = 2\pi f = 2 \times 3.14 \times 1.5 s^{-1} = 9.42 rad/s$$

For the child ($m = 35kg$):

$$F_c = m\omega^2 r = 35kg \times (9.42 rad/s)^2 \times 4m = \mathbf{12424N}$$

$$\mu = \frac{F_c}{mg} = \frac{12424N}{35kg \times 9.8m/s^2} = \mathbf{36.22}$$

For the adult ($m = 80kg$):

$$F_c = 80kg \times (9.42 rad/s)^2 \times 4m = \mathbf{28397N}$$

$$\mu = \frac{F_c}{mg} = \frac{28397N}{80kg \times 9.8m/s^2} = \mathbf{36.22}$$

Both require $\mu = 36.22$, confirming that the required friction coefficient is independent of mass.

Example 48

(a) A stone is whirled in a vertical circle on a string. State the position in the circle where the string is most likely to break, and explain why.

(b) A stone of mass 0.2kg is attached to a string of length 0.8m and whirled in a vertical circle. The speed at the top of the circle is 5m/s. The string can withstand a maximum tension of 14N. The centre of the circle is 4m above the ground. Take $g = 9.8m/s^2$.

(i) Find the speed at the bottom of the circle.

(ii) Calculate the tension at the bottom and determine whether the string breaks.

(iii) If it breaks, find the time taken by stone to reach the ground and the horizontal distance covered.

Solution

(a) The string is most likely to break at the **bottom** of the circle. At this position, the tension must support the weight and provide the centripetal force simultaneously ($T_{bottom} = mg + \frac{mv^2}{r}$), making the tension maximum.

(b) The solution of each part is as follows:

(i) Using conservation of energy between the top (height 2r above the bottom) and the bottom:

$$\frac{1}{2}mv_{bottom}^2 = \frac{1}{2}mv_{top}^2 + mg(2r)$$

$$v_{bottom}^2 = v_{top}^2 + 4gr = (5m/s)^2 + 4 \times 9.8m/s^2 \times 0.8m = 25 + 31.36 = 56.36m^2/s^2$$

$v_{bottom} = 7.51m/s$

(ii) Tension at the bottom:

$$T_{bottom} = mg + \frac{mv_{bottom}^2}{r} = 0.2kg \times 9.8m/s^2 + \frac{0.2kg \times 56.36m^2/s^2}{0.8m} = 16.05N$$

Since $16.05N > 14N$, the string breaks at the bottom.

(iii) At the bottom, the velocity is horizontal (7.51m/s). The height of the bottom above the ground:

$$h = 4m - 0.8m = 3.2m$$

The stone becomes a projectile with $u_x = 7.51m/s$ and $u_y = 0$.

Time to reach the ground:

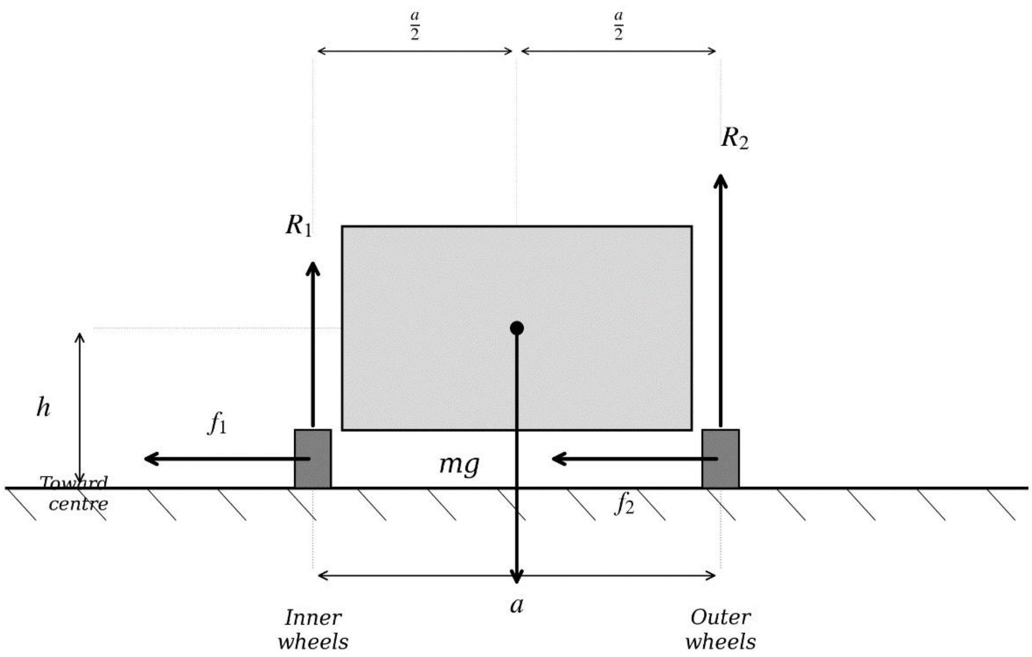
$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 3.2m}{9.8m/s^2}} = \sqrt{0.653s^2} = \mathbf{0.808s}$$

Horizontal distance:

$$x = u_x \times t = 7.51m/s \times 0.808s = \mathbf{6.07m}$$

Example 49

The figure below shows a rear view of a car of mass **m** rounding a level curved road of radius **r** at speed **v.** The lateral distance between the inner and outer wheels is **a**, and the centre of gravity **G** is at height **h** above the ground. $\mathbf{R_1}$ and $\mathbf{R_2}$ are the normal reactions on the inner and outer wheels, and $\mathbf{f_1}$ and $\mathbf{f_2}$ are the friction forces on the inner and outer wheels.



(a) By considering the forces and taking moments about G, derive expressions for:

(i) the normal reaction $\mathbf{R_1}$ on the inner wheels,
(ii) the normal reaction $\mathbf{R_2}$ on the outer wheels.

(b) Hence show that:

(i) the maximum speed before the car overturns is $v_{overturn} = \sqrt{\frac{arg}{2h}}$,
(ii) the maximum speed before the car skids is $v_{skid} = \sqrt{\mu_s gr}$,
(iii) the car skids before overturning if $\mu_s < \frac{a}{2h}$, and overturns before skidding if $\mu_s > \frac{a}{2h}$.

Solution

(a) Three equations govern the system:

Vertical equilibrium:

$$R_1 + R_2 = mg \quad \dots(1)$$

Horizontal (centripetal force from friction):

$$f_1 + f_2 = \frac{mv^2}{r} \quad \dots(2)$$

Taking moments about G:

Each wheel is at horizontal distance $\frac{a}{2}$ from G. The friction forces act at the ground, a vertical distance h below G. Taking anticlockwise as positive:

R_2 acts upward at distance $\frac{a}{2}$ to the right of G → anticlockwise (positive).

R_1 acts upward at distance $\frac{a}{2}$ to the left of G → clockwise (negative).

$(f_1 + f_2)$ acts horizontally at distance h below G → clockwise (negative).

$$R_2 \times \frac{a}{2} - R_1 \times \frac{a}{2} - (f_1 + f_2) \times h = 0$$

$$\frac{a}{2}(R_2 - R_1) = (f_1 + f_2) \times h$$

Substituting (2):

$$\frac{a}{2}(R_2 - R_1) = \frac{mv^2h}{r} \quad \text{... (3)}$$

(i) Finding R_1:

From (1): $R_2 = mg - R_1$. Substituting into (3):

$$\frac{a}{2}(mg - 2R_1) = \frac{mv^2h}{r}$$

$$mg - 2R_1 = \frac{2mv^2h}{ar}$$

$$\boxed{\mathbf{R_1 = \frac{m}{2}\left(g - \frac{2v^2h}{ar}\right)}}$$

As v increases, R_1 decreases. The inner wheels progressively lose contact with the ground.

(ii) Finding R_2:

Substituting R_1 into (1):

$$R_2 = mg - \frac{m}{2}\left(g - \frac{2v^2h}{ar}\right) = \frac{mg}{2} + \frac{mv^2h}{ar}$$

$$\boxed{\mathbf{R_2 = \frac{m}{2}\left(g + \frac{2v^2h}{ar}\right)}}$$

As v increases, R_2 increases. The outer wheels carry progressively more load.

(c) **(i) Maximum speed before overturning:**

The car overturns when $R_1 = 0$:

$$0 = g - \frac{2v_{\text{overturn}}^2 h}{ar}$$

$$\boxed{\mathbf{v_{overturn} = \sqrt{\frac{arg}{2h}}}}$$

(ii) Maximum speed before skidding:

The car skids when total friction reaches its limit: $f_1 + f_2 = \mu_s(R_1 + R_2) = \mu_s mg$

$$\mu_s mg = \frac{mv_{\text{skid}}^2}{r}$$

$$\boxed{\mathbf{v_{skid} = \sqrt{\mu_s gr}}}$$

(iii) Condition for skidding vs overturning:

Skidding occurs first if $v_{\text{skid}} < v_{\text{overturn}}$:

$$\mu_s gr < \frac{arg}{2h}$$

$$\boxed{\boldsymbol{\mu_s < \frac{a}{2h}} \Rightarrow \textit{car skids first (safer)}}$$

$$\boxed{\boldsymbol{\mu_s > \frac{a}{2h}} \Rightarrow \textit{car overturns first (dangerous)}}$$

Example 50

(a) State two design features that make a vehicle less likely to overturn on a curved road.

(b) A bus has its wheels 2m apart laterally and its centre of gravity is 1.2m above the ground. It rounds a level curve of radius 60m. The coefficient of static friction between the tyres and the road is 0.45. Take $g = 9.8\text{m/s}^2$.

(i) Calculate the maximum speed before the bus overturns.
(ii) Calculate the maximum speed before the bus skids.
(iii) Determine which happens first, and comment on the safety of this vehicle.
(iv) A sports car has wheels 1.8m apart and centre of gravity 0.4m above the ground. On the same road with the same friction, determine which happens first for the sports car.

Solution

(a) (1) A low centre of gravity. (2) A wide wheel base.

(b) The solution of each part is as follow:

(i) For the bus ($a = 2\text{m}$, $h = 1.2\text{m}$):

$$v_{overturn} = \sqrt{\frac{arg}{2h}} = \sqrt{\frac{2\text{m} \times 60\text{m} \times 9.8\text{m/s}^2}{2 \times 1.2\text{m}}} = \mathbf{22.1m/s}$$

(ii) Using:

$$v_{skid} = \sqrt{\mu_s gr} = \sqrt{0.45 \times 9.8\text{m/s}^2 \times 60\text{m}} = \mathbf{16.3m/s}$$

(iii) Using:

$$\frac{a}{2h} = \frac{2\text{m}}{2.4\text{m}} = 0.833$$

Since $\mu_s(0.45) < \frac{a}{2h}(0.833)$, the bus **skids first** at 16.3m/s. This is the safer outcome. However, if the road surface improves (higher μ_s approaching 0.833), the bus could overturn instead which is a serious danger for tall vehicles.

(iv) For the sports car ($a = 1.8\text{m}$, $h = 0.4\text{m}$):

$$\frac{a}{2h} = \frac{1.8\text{m}}{0.8\text{m}} = 2.25$$

Since $\mu_s(0.45) \ll \frac{a}{2h}(2.25)$, the sports car **skids first** with a very large safety margin against overturning. Its low centre of gravity and wide track make overturning virtually impossible under normal conditions.

Example 51

(a) Distinguish between the apparent weight of a driver at the top of a hill and at the bottom of a valley, in terms of the direction of centripetal acceleration.

(b) A car travels at 25m/s along a road that passes over a hill of radius 60m followed by a valley of radius 80m. Calculate the apparent weight of the driver (mass 70kg) at the top of the hill and at the bottom of the valley. Take $g = 9.8\text{m/s}^2$.

Solution

(a) At the top of a hill, the centripetal acceleration is directed **downward** (toward the centre of curvature below the road). The normal reaction is therefore less than the weight, so the driver feels lighter.

At the bottom of a valley, the centripetal acceleration is directed **upward** (toward the centre of curvature above the road). The normal reaction is therefore greater than the weight, so the driver feels heavier.

(b) **At the top of the hill ($r = 60m$):**

$$R_{hill} = mg - \frac{mv^2}{r} = 70kg \times 9.8m/s^2 - \frac{70kg \times (25m/s)^2}{60m} = 686N - 729.2N = -43.2N$$

The result is negative. This means the car has **lost contact with the road** at the top of the hill at this speed. The driver is momentarily airborne and experiences weightlessness ($R = 0$).

Hence, the apparent weight at the top of the hill is $0N$.

Checking the answer

You may check the answer by using the fact that, *if $R = 0$, the speed exceeds the critical speed for this hill.*

$$v_{max} = \sqrt{gr} = \sqrt{9.8 \times 60} = 24.2m/s$$

Since $25m/s > 24.2m/s$, the car leaves the road.

At the bottom of the valley ($r = 80m$):

$$R_{valley} = mg + \frac{mv^2}{r} = 70kg \times 9.8m/s^2 + \frac{70kg \times (25m/s)^2}{80m} = 686N + 546.9N = 1232.9N$$

The apparent weight at the bottom of the valley is $1232.9N$, (which is 1.80 times the actual weight (686N). The driver feels almost twice as heavy).

Example 52

(a) Explain why a pendulum hanging from the ceiling of a car swings outward when the car rounds a curve at constant speed.

(b) A small ball of mass 0.1kg is suspended by a light string from the ceiling of a car. The car rounds a horizontal curve of radius 15m at constant speed 12m/s. Find the angle the string makes with the vertical and the tension in the string. Take $g = 9.8m/s^2$.

Solution

(a) When the car rounds a curve, the ball must accelerate toward the centre of the curve to follow the car's circular path. The only horizontal force available to provide this centripetal acceleration is the horizontal component of the tension in the string. For the tension to have a horizontal component, the string must swing outward from the vertical. The ball does not experience an outward force; rather, the string tilts outward to provide the inward (centripetal) force needed for circular motion.

(b) Let θ be the angle the string makes with the vertical.

Vertically: $T\cos\theta = mg$

Horizontally (toward centre): $T\sin\theta = \frac{mv^2}{r}$

Dividing:

$$\tan\theta = \frac{v^2}{rg} = \frac{(12m/s)^2}{15m \times 9.8m/s^2} = \frac{144}{147} = 0.9796$$

$$\theta = \tan^{-1}(0.9796) = 44.4°$$

Tension:

$$T = \frac{mg}{\cos\theta} = \frac{0.1\text{kg} \times 9.8\text{m/s}^2}{\cos 44.4°} = \frac{0.98\text{N}}{0.7147} = 1.37\text{N}$$

The string makes $44.4°$ with the vertical and the tension is 1.37N.

Example 53

(a) Explain why, for an object moving in a vertical circle on a string, the minimum speed at the top required to maintain a complete circle does not depend on the mass of the object.

(b) Two balls A (mass 0.3kg) and B (mass 0.5kg) are connected by a light rigid rod and move inside a smooth vertical circular track of radius 1m. At a certain instant, A is at the top and B is at the bottom. The speed of each ball at this instant is 4m/s. Take $g = 9.8\text{m/s}^2$. Calculate the normal reaction on each ball from the track at this instant.

Solution

(a) At the top of the vertical circle, only gravity provides the centripetal force. A heavier object requires a larger centripetal force $\left(\frac{mv^2}{r}\right)$, but it also experiences a greater gravitational force (mg). Since both the required centripetal force and the gravitational force increase in direct proportion to mass, the effect of mass change cancels. Mathematically, setting the tension to zero at the top gives: $mg = \frac{mv_{min}^2}{r}$. The mass m appears on both sides and cancels, giving $v_{min} = \sqrt{gr}$, which depends only on g and r. Hence, the minimum speed at the top required to maintain the circular motion is the same for all objects, regardless of their mass.

(b) Since both balls are on a smooth track (not a string), the track pushes them. At each position, Newton's second law is applied toward the centre.

Ball A at the top (centre is below):

Both R_A and $m_A g$ act downward (toward centre):

$$R_A + m_A g = \frac{m_A v^2}{r}$$

$$R_A = \frac{m_A v^2}{r} - m_A g = \frac{0.3\text{kg} \times (4\text{m/s})^2}{1.0\text{m}} - 0.3\text{kg} \times 9.8\text{m/s}^2 = 4.8\text{N} - 2.94\text{N} = 1.86\text{N}$$

Ball B at the bottom (centre is above):

R_B acts upward (toward centre), $m_B g$ acts downward (away from centre):

$$R_B - m_B g = \frac{m_B v^2}{r}$$

$$R_B = \frac{m_B v^2}{r} + m_B g = \frac{0.5\text{kg} \times (4\text{m/s})^2}{1.0\text{m}} + 0.5\text{kg} \times 9.8\text{m/s}^2 = 8.0\text{N} + 4.9\text{N} = 12.9\text{N}$$

The normal reaction on A is 1.86N and on B is 12.9N.

Example 54

(a) Explain why a small block placed inside a smooth rotating hemispherical bowl rises up the inner surface as the angular speed of rotation increases.

(b) A small block of mass 0.1kg sits inside a smooth hemispherical bowl of internal radius 0.3m. The bowl rotates about its vertical axis of symmetry. The block is in equilibrium at a position where the line from the centre of the bowl to the block makes an angle of $40°$ with the vertical. Take $g = 9.8\text{m/s}^2$.

(iv) Show that the angular speed of the bowl is given by $\omega = \sqrt{\frac{g}{R\cos\theta}}$, where R is the radius of the bowl and θ is the angle from the vertical.

(v) Calculate the angular speed and the normal reaction on the block.

Solution

(a) As the bowl spins faster, the block requires a greater centripetal force to maintain circular motion at its current radius. The only horizontal force available is the horizontal component of the normal reaction from the bowl surface. To increase this horizontal component, the block must move to a position where the bowl surface is more steeply inclined; that is, higher up the bowl where θ is larger. This increases $R\sin\theta$ (the horizontal component), providing the greater centripetal force needed.

(b) The solution of each part is as follows:

(i) The block moves in a horizontal circle of radius $r = R\sin\theta$. The normal reaction N acts perpendicular to the bowl surface, which at this position is directed along the line from the block toward the centre of the bowl. Resolving N:

Vertically: $N\cos\theta = mg$

Horizontally: $N\sin\theta = m\omega^2 r = m\omega^2 R\sin\theta$

Dividing:

$$\frac{\sin\theta}{\cos\theta} = \frac{\omega^2 R\sin\theta}{g}$$

The $\sin\theta$ cancels:

$$\frac{1}{\cos\theta} = \frac{\omega^2 R}{g}$$

$$\boxed{\boldsymbol{\omega = \sqrt{\frac{g}{R\cos\theta}}}}$$

(vi) Angular speed:

$$\omega = \sqrt{\frac{9.8\text{m/s}^2}{0.3\text{m} \times \cos 40°}} = 6.53\text{rad/s}$$

Normal reaction:

$$N = \frac{mg}{\cos\theta} = \frac{0.1\text{kg} \times 9.8\text{m/s}^2}{\cos 40°} = \frac{0.98\text{N}}{0.766} = 1.28\text{N}$$

The angular speed is 6.53rad/s and the normal reaction is 1.28N.

Example 55

(a) A student says: *"At the top of a vertical circle, the ball is weightless because the velocity is horizontal."* Identify and correct two errors in this statement.

(b) A ball of mass 0.25kg moves in a vertical circle of radius 0.5m on a string. The speed at the bottom is 5.5m/s. Take $g = 9.8\text{m/s}^2$. Find the angle from the bottom at which the tension in the string equals the weight of the ball.

Solution

(a) **Error 1:** The ball is not weightless at the top. Weight (mg) acts on the ball at every point in the circle, including the top. The ball may have zero *apparent* weight (zero tension) at the critical speed, but its actual weight never changes.

Error 2: The direction of velocity does not determine whether the ball is weightless. A ball moving horizontally at the bottom of the circle is certainly not weightless. Weightlessness (zero contact force) depends on the balance between gravity and centripetal requirements, not on the direction of velocity.

(b) Using the general expression:

$$T = \frac{mv_{bottom}^2}{r} - 2mg + 3mg\cos\theta$$

Setting $T = mg$:

$$mg = \frac{mv_{bottom}^2}{r} - 2mg + 3mg\cos\theta$$

$$3mg\cos\theta = 3mg - \frac{mv_{bottom}^2}{r}$$

$$\cos\theta = 1 - \frac{v_{bottom}^2}{3gr}$$

Substituting:

$$\cos\theta = 1 - \frac{(5.5\text{m/s})^2}{3 \times 9.8\text{m/s}^2 \times 0.5\text{m}} = 1 - \frac{30.25}{14.7} = 1 - 2.059 = -1.059$$

Since $\cos\theta = -1.059 < -1$, which is impossible, the tension **never equals the weight** at any point in the circle. The tension is always greater than mg throughout the motion.

Checking of the answer

This can be verified: at the bottom, $T_{bottom} = mg + \frac{mv^2}{r} = 0.25 \times 9.8 + \frac{0.25 \times 30.25}{0.5} = 17.58\text{N}$, which is far greater than $mg = 2.45\text{N}$.

At the top, using energy conservation: $v_{top}^2 = v_{bottom}^2 - 4gr = 30.25 - 19.6 = 10.65\text{m}^2/\text{s}^2$

$$T_{top} = \frac{mv_{top}^2}{r} - mg = \frac{0.25 \times 10.65}{0.5} - 2.45 = 2.88\text{N}$$

Even at the top, the tension (2.88N) exceeds the weight (2.45N). The speed is too high for the tension to ever drop to mg.

Example 56

(a) Explain why a ball released from the top of a smooth hemispherical dome will eventually leave the surface before reaching the bottom.

(b) A small ball of mass 0.15kg sits on top of a smooth hemispherical dome of radius 2m. It is given a tiny push and begins to slide down the surface. Take $g = 9.8\text{m/s}^2$.

(i) By considering forces along the radius and using energy conservation, show that the ball leaves the surface when $\cos\theta = \frac{2}{3}$, where θ is measured from the vertical.

(ii) Calculate the angle, the height above the ground, and the speed at the point where the ball leaves the dome.

Solution

(a) As the ball slides down the dome, it speeds up (gaining kinetic energy from lost potential energy). At any point on the dome, the component of weight along the radius toward the centre ($mg\cos\theta$) must provide both the centripetal force and the normal reaction. As the ball speeds up, the centripetal force requirement ($\frac{mv^2}{r}$) increases while $mg\cos\theta$ decreases (because θ increases as the ball descends). At some angle, the required centripetal force exceeds $mg\cos\theta$, the normal reaction drops to zero, and the ball leaves the surface.

(b) The solution of each part is as follows:

(i) At angle θ from the vertical, applying Newton's second law toward the centre (along the radius, inward):

$$mg\cos\theta - R = \frac{mv^2}{r}$$

The ball leaves the surface when $R = 0$:

$$mg\cos\theta = \frac{mv^2}{r} \quad \ldots(1)$$

Using energy conservation from the top ($\theta = 0$, $v = 0$) to angle θ:

The ball has descended a vertical height $h = r - r\cos\theta = r(1 - \cos\theta)$

$$\frac{1}{2}mv^2 = mgr(1 - \cos\theta)$$

$$v^2 = 2gr(1 - \cos\theta) \quad \ldots(2)$$

Substituting (2) into (1):

$$mg\cos\theta = \frac{m \times 2gr(1 - \cos\theta)}{r}$$

$$g\cos\theta = 2g(1 - \cos\theta)$$

$$\cos\theta = 2 - 2\cos\theta$$

$$3\cos\theta = 2$$

$$\boxed{\mathbf{\cos\theta = \frac{2}{3}}}$$

(ii) Angle:

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2°$$

Height above the ground at the point of leaving:

The ball is on the *outside* of the dome. The centre of the hemisphere is at the base (ground level), and the top of the dome is at height r above the ground. At angle θ from the vertical, the ball is at height:

$$h = r\cos\theta = 2\text{m} \times \frac{2}{3} = 1.33\text{m above the ground}$$

Speed at the point of leaving, from equation (2):

$$v^2 = 2gr(1 - \cos\theta)$$

$$v = \sqrt{2gr(1 - \cos\theta)} = \sqrt{2 \times 9.8\text{m/s}^2 \times 2\text{m} \times \left(1 - \frac{2}{3}\right)} = 3.61\text{m/s}$$

The ball leaves the dome at $\theta = 48.2°$ from the vertical, at a height of 1.33m above the ground, with a speed of 3.61m/s.

And with that, the miscellaneous worked examples take their final bow. Every concept in this chapter: from the gentle lean of a cyclist to the terrifying physics of overturning, from satellites silently orbiting overhead to a bucket of water that defied gravity (and soaked Kipanga), has been tested, twisted, combined, and calculated. If you survived all of that, congratulations! But do not relax just yet. The Digging Deeper Exercise is waiting on the next page, and it has no solutions to hold your hand. Just you, the physics, and a calculator. Good luck! You are ready!