HORIZONTAL CIRCULAR MOTION

In the previous subtopic, we developed the formulas for centripetal acceleration and centripetal force, and we applied them to several situations: a car on a track, a stone on a string, the moon orbiting the Earth, and even a fairground ride. Along the way, we derived the maximum safe speed on a level curve, $v_{max} = \sqrt{\mu_s gr}$, and saw clearly why our bus at Kilimani Hill could not hold the road.

But those examples only introduced the ideas. In this subtopic, we examine horizontal circular motion in full detail. We will use free-body diagrams and Newton's second law applied carefully in two directions, just as we learned in Chapter 4. Every situation we meet here involves one central question: *what real force provides the centripetal force?* The answer changes from one situation to the next, but the method remains the same. Identify the forces, resolve them, and set the net inward component equal to $\frac{mv^2}{r}$.

We begin with the simplest case: a car negotiating a level curve.

Car on a Level (Unbanked) Curve

In the previous subtopic, we showed that the maximum safe speed on a level curve is $v_{max} = \sqrt{\mu_s gr}$. We arrived at this result quickly by equating friction to centripetal force. Now we approach the same situation more carefully, using a proper free-body diagram and Newton's second law in both directions. This systematic method will serve us well when we meet more complex situations later.

Setting up the problem

Consider a car of mass **m** moving at constant speed **v** around a horizontal circular curve of radius **r** on a flat road. The road is not banked; it is perfectly level.

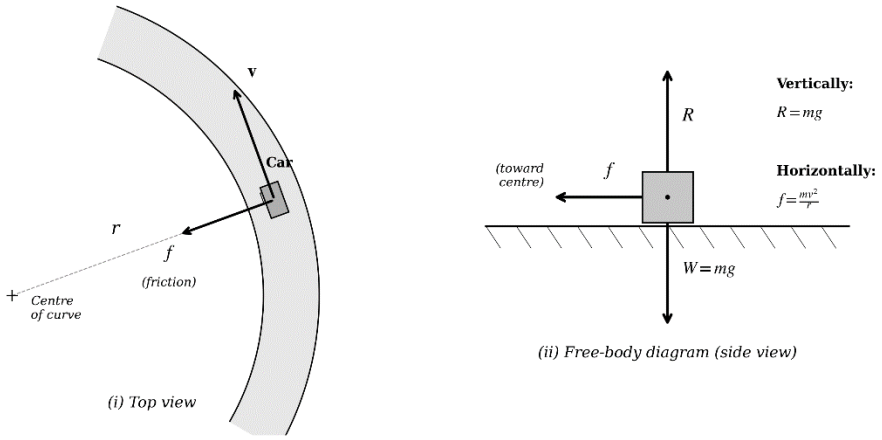

Figure: ***A car negotiating a level (unbanked) circular curve.*** *(i) Top view showing velocity tangent to the path and friction directed toward the centre of the curve. (ii) Free-body diagram: the normal reaction R balances the weight mg vertically, while static friction f provides the centripetal force horizontally.*

Identifying the forces

Three forces act on the car:

1) **Weight** (W = mg), acting vertically downward through the centre of mass.
2) **Normal reaction** (R), acting vertically upward from the road surface, perpendicular to the road.

3) **Static friction** (f), acting horizontally toward the centre of the circular path. This is the force that prevents the tyres from sliding sideways across the road surface.

Notice that *friction here acts sideways (toward the centre of the curve), not forward or backward along the direction of motion. Many students picture friction as always opposing motion along the road. In circular motion on a level surface, it is the sideways friction between tyre and road that bends the car's path into a curve.*

Applying Newton's second law

We apply Newton's second law in two perpendicular directions:

Vertically (no acceleration):

The car does not accelerate up or down, so the net vertical force is zero:

$$R - mg = 0$$

From which:

$$\mathbf{R = mg}$$

Horizontally (toward the centre):

The only horizontal force is friction, and it acts toward the centre, providing the centripetal acceleration:

$$f = \frac{mv^2}{r}$$

This is the force that the road must supply to keep the car on the circular path. If the road cannot supply this force, the car slides.

The maximum safe speed

The maximum friction the road can provide is limited by:

$$f_{max} = \mu_s R = \mu_s mg$$

The car remains on the curve as long as the required centripetal force does not exceed this maximum friction:

$$\frac{mv^2}{r} \leq \mu_s mg$$

At the maximum safe speed, these are equal:

$$\frac{mv_{max}^2}{r} = \mu_s mg$$

Cancelling m and making v_{max} the subject:

$$\mathbf{v_{max} = \sqrt{\mu_s gr}}$$

It is important for you to understand that:

The mass *m* cancels completely from the equation. *This means the maximum safe speed on a level curve does not depend on the mass of the vehicle.* A fully loaded truck and an empty saloon car have exactly the same maximum safe speed, provided their tyres have the same coefficient of friction with the road. This surprises many students, but the physics is clear: *a heavier vehicle needs more centripetal force, but it also has more friction available (because $R = mg$ is larger), and these two effects cancel exactly.*

The result depends on three factors only:

4) The coefficient of static friction μ_s between the tyres and the road (tyre condition, road surface, wetness).
5) The acceleration due to gravity g.
6) The radius r of the curve (sharper curves are more dangerous).

With the ideas now set in place, let us anchor them through worked examples.

BINDER Example 16

A car moves at 15m/s around a horizontal circular curve of radius 60m on a level road. The coefficient of static friction between the tyres and the road is 0.5. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the centripetal acceleration of the car.

(b) Determine whether the car can safely negotiate the curve at this speed.

Solution

(a) Using:

$$a = \frac{v^2}{r}$$

Where: $v = 15\text{m/s}$, $r = 60\text{m}$

Substituting:

$$a = \frac{(15\text{m/s})^2}{60\text{m}} = 3.75\text{m/s}^2$$

The centripetal acceleration is 3.75m/s^2.

(b) The maximum centripetal acceleration that friction can provide is given by:

$$a_{max} = \frac{v_{max}^2}{r}$$

But from:

$$\frac{mv_{max}^2}{r} = \mu_s mg;$$

$$\frac{v_{max}^2}{r} = \mu_s g$$

Hence:

$$a_{max} = \mu_s g = 0.5 \times 9.8\text{m/s}^2 = 4.9\text{m/s}^2 > 3.75\text{m/s}^2(\text{required})$$

Since the required centripetal acceleration (3.75m/s^2) is less than the maximum available (4.9m/s^2), the friction is sufficient. The car can safely negotiate the curve.

Alternative solution

By comparing velocities:

$$v_{max} = \sqrt{\mu_s gr} = \sqrt{0.5 \times 9.8\text{m/s}^2 \times 60\text{m}} = 17.15\text{m/s} > 15\text{m/s (given)}$$

Since the maximum allowable velocity is greater than the given velocity, the car can safely negotiate the curve.

Making Sense of the Answer: *The car needs 3.75 m/s² of centripetal acceleration, but friction can provide up to 4.9 m/s². There is a safety margin. However, if the speed increases or the road becomes wet (reducing μ_s), that margin shrinks. At some critical speed, the required acceleration equals the maximum available, and beyond that, the car slides.*

Think Like a Physicist: *Comparing accelerations or velocities is a quick way to check whether circular motion is possible without calculating forces. If $\frac{v^2}{r} > \mu_s g$ or $v_{max} > v$, the car will slide.*

REAL Example 17

During a road safety lesson, **Kipanga** argues: *"A heavy truck is more dangerous on a curve than a small car, because the truck needs much more centripetal force and the road cannot provide it."*

Kipute disagrees: *"The truck is heavier, but it also presses harder on the road, so friction is stronger too."*

Who is correct? Explain clearly.

Solution

Kipute is correct.

Explanation

A heavier vehicle does require more centripetal force to maintain circular motion, since $F_c = \frac{mv^2}{r}$ increases with mass. However, the maximum friction the road can provide also increases with mass, since $f_{max} = \mu_s mg$.

At maximum speed:

$$\frac{mv_{max}^2}{r} = \mu_s mg$$

The mass m appears on both sides and cancels, giving $v_{max} = \sqrt{\mu_s gr}$, which is independent of mass.

Therefore, the maximum safe speed is the same for both vehicles, provided they have the same tyre-road friction coefficient and travel the same curve. The truck's greater weight demands more force, but the road supplies more friction in exact proportion. The two effects cancel.

Making Sense of the Answer: *This is why speed limit signs on curves do not show different limits for trucks and cars. The safe speed depends on the curve's radius and the road condition, not on the vehicle's mass.*

Think Like a Physicist: *Whenever mass cancels from a physics result, it signals something deep: the phenomenon does not depend on the amount of matter involved. This is the same reason all objects fall with the same acceleration under gravity (mass cancels in $mg = ma$). Here, mass cancels for a similar reason: both the required force and the available force are proportional to mass.*

HOT Example 18

A driver travelling at 25m/s approaches a curve on a level road. The coefficient of static friction between the tyres and the road is 0.4. Take $g = 9.8\text{m/s}^2$.

(a) Calculate the minimum radius of curve the driver can safely negotiate at this speed.

(b) If the road curves with a radius of only 120m, determine whether the driver must slow down, and if so, find the maximum safe speed for that curve.

(c) The driver enters the 120m curve at 25m/s without slowing down. Calculate the centripetal force required and the maximum friction force available for a car of mass 1400kg. Use these values to explain what happens to the car.

Solution

(a) At the maximum safe speed:

$$\frac{mv_{max}^2}{r_{min}} = \mu_s mg$$

Making r_{min} the subject:

$$r_{min} = \frac{v^2}{\mu_s g}$$

Where: $v = 25\text{m/s}$, $\mu_s = 0.40$, $g = 9.8\text{m/s}^2$

Substituting:

$$r_{min} = \frac{(25\text{m/s})^2}{0.40 \times 9.8\text{m/s}^2} = 159.4\text{m}$$

The minimum safe radius is 159.4m.

(b) Since the actual radius (120m) is less than the minimum safe radius (159.4m), the driver **must slow down**.

The maximum safe speed for the 120m curve:

$$v_{max} = \sqrt{\mu_s gr}$$

Substituting:

$$v_{max} = \sqrt{0.40 \times 9.8\text{m/s}^2 \times 120\text{m}} = 21.7\text{m/s}$$

The maximum safe speed is 21.7m/s (about 78km/h).

So the driver must slow from 25m/s to at most 21.7m/s.

(c) Required centripetal force at 25m/s:

$$F_{required} = \frac{mv^2}{r} = \frac{1400\text{kg} \times (25\text{m/s})^2}{120\text{m}} = 7291.7\text{N}$$

Maximum friction available:

$$f_{max} = \mu_s mg = 0.40 \times 1400\text{kg} \times 9.8\text{m/s}^2 = 5488\text{N} < 7291.7\text{N(required)}$$

The required centripetal force (7291.7N) exceeds the maximum friction (5488N) by 1803.7N. Since friction cannot supply the force needed to maintain circular motion at this speed, the tyres lose grip and the car slides outward from the curve, following a tangential path in accordance with Newton's first law.

Making Sense of the Answer: *The numbers tell the story clearly: the car demands 7292N of inward force, but the road can only offer 5488 N. The deficit of about 1804 N means the car cannot follow the curve. It is not a matter of the driver turning the steering wheel harder; the physics simply does not allow it. This is why speed limits on sharp curves are not suggestions; they are calculated from the laws of physics.*

Think Like a Physicist: *Rearranging* $v_{max} = \sqrt{\mu_s gr}$ *to find* $r_{min} = \frac{v^2}{\mu_s g}$ *is a useful trick. It tells you the minimum curve radius you can safely negotiate at a given speed. Highway engineers use exactly this formula when designing roads.*

With the car on the level curve now thoroughly understood, let us see what happens when a cyclist, rather than a car, must negotiate a curved path. The physics is the same, but the human body reveals it in a beautifully visible way.

Cyclist on a Curved Rough Level Road

When a car turns a corner, the physics is hidden inside the vehicle. The tyres grip, friction acts sideways, and the car follows the curve. But when a cyclist takes the same corner, something remarkable happens that the car cannot show: **the cyclist leans inward.**

This leaning is not a style choice. It is a physical necessity. A cyclist who tries to turn a corner while sitting perfectly upright will topple outward. The lean is the body's way of ensuring that the forces acting on the bicycle and rider can produce the required centripetal force while maintaining balance.

Why must the cyclist lean?

A bicycle is a narrow vehicle. Unlike a car, which has four widely spaced wheels, a bicycle makes contact with the ground along a very thin line. If the cyclist sits upright and turns, friction acts horizontally at the contact

point (just as for a car), but the cyclist's weight acts vertically through the centre of mass, which is high above the ground. These two forces would then create a net *turning effect* (**torque**) that tips the cyclist outward.

To prevent this, the cyclist leans the entire bicycle inward. This shifts the line of action of the forces so that they balance not only in terms of net force, but also in terms of torque. The result is stable circular motion without toppling.

The forces acting on the cyclist

Consider the following diagram:

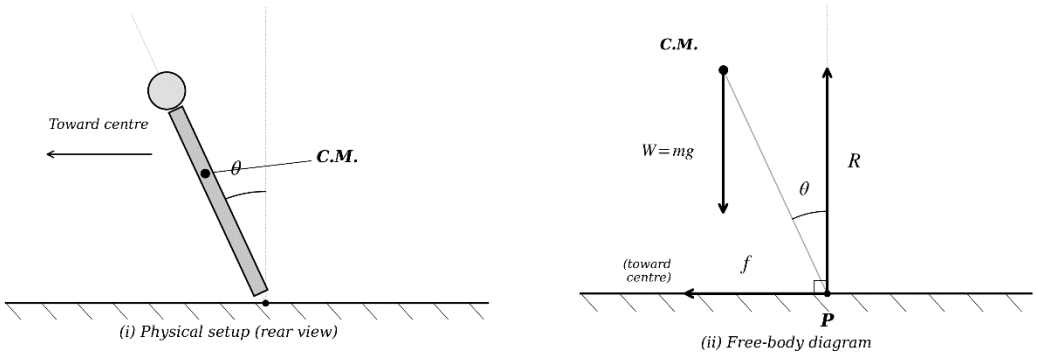

Figure: ***A cyclist negotiating a level circular curve, shown from behind (rear view).*** *(i) The cyclist and bicycle lean inward at angle θ from the vertical. The centre of mass (C.M.) is displaced toward the centre of the curve. (ii) Free-body diagram: weight W = mg acts downward through the centre of mass, while the normal reaction R and friction f act at the contact point P.*

Two forces act on the cyclist-bicycle system:

1. **Weight** (W = mg), acting vertically downward through the centre of mass.
2. **Contact force from the road**, which has two components at the point of contact:

- **Normal reaction** (R), acting vertically upward, perpendicular to the road surface.
- **Static friction** (f), acting horizontally toward the centre of the curve.

These are the same forces as for the car on a level curve. The difference is that for the cyclist, the angle of lean θ connects them geometrically.

Applying Newton's second law

Vertically (no acceleration):

$$R = mg$$

Horizontally (toward the centre):

$$f = \frac{mv^2}{r}$$

These are identical to the equations for a car on a level curve.

Finding the angle of lean

For the cyclist to remain balanced (no toppling), the net force must pass through the contact point. This requires the resultant of weight and friction to act along the line of the bicycle frame. Geometrically, this gives:

$$\tan\theta = \frac{f}{R} = \frac{mv^2/r}{mg}$$

From which:

$$\mathbf{\tan\theta = \frac{v^2}{rg}}$$

Where θ is the angle of lean measured from the vertical.

This equation reveals several important insights:

1. The angle of lean increases with speed. Faster turns require greater lean.
2. The angle of lean increases as the radius decreases. Sharper turns require greater lean.
3. The angle of lean does not depend on mass. A heavy cyclist and a light cyclist lean at the same angle for the same speed and radius.

Notice that this is the same expression, $\tan\theta = \frac{v^2}{rg}$, that will appear again in the conical pendulum and in banking of roads. The physics is universal; only the physical situation changes.

Maximum speed before skidding

The cyclist skids when friction reaches its maximum:

$$f_{max} = \mu_s mg$$

At maximum speed:

$$\frac{mv_{max}^2}{r} = \mu_s mg$$

$$\mathbf{v_{max} = \sqrt{\mu_s gr}}$$

This is exactly the same result as for the car.

Maximum lean angle before skidding

Since $\tan\theta = \frac{v^2}{rg}$, a larger speed requires a larger lean angle. Therefore, the maximum speed and the maximum lean angle, θ_{max} occur at the same instant: just before skidding. The cyclist cannot lean beyond θ_{max} because doing so would require a higher speed that demands more friction than the road can provide.

We showed earlier that the lean angle satisfies:

$$\tan\theta = \frac{f}{R} = \frac{f}{mg}$$

At maximum speed, friction reaches its limit:

$$f_{max} = \mu_s mg$$

Substituting:

$$\tan\theta_{max} = \frac{\mu_s mg}{mg}$$

From which:

$$\tan\theta_{max} = \mu_s$$

Hence, at this maximum speed, the lean angle satisfies:

$$\mathbf{\tan\theta_{max} = \mu_s}$$

This is a remarkable result. The maximum lean angle depends only on the coefficient of static friction between the tyres and the road. It does not depend on the radius of the curve or the mass of the cyclist. A heavier cyclist, a lighter cyclist, a wide curve, a tight curve; none of these matter. Only the grip of the tyres on the road determines how far the cyclist can lean before skidding.

With the ideas laid out clearly, let us practise them through worked examples.

BINDER Example 19

A cyclist rounds a level circular curve of radius 25m at a constant speed of 10m/s. Take $g = 9.8\ m/s^2$.

(a) Calculate the angle at which the cyclist must lean from the vertical.

(b) If the cyclist doubles the speed to 20 m/s on the same curve, find the new lean angle.

Solution

(a) Using:

$$\tan\theta = \frac{v^2}{rg}$$

Where: $v = 10\ m/s$, $r = 25\ m$, $g = 9.8\ m/s^2$

Substituting:

$$\tan\theta = \frac{(10\ m/s)^2}{25\ m \times 9.8\ m/s^2} = 0.408$$

$$\theta = \tan^{-1}(0.408) = 22.2°$$

The cyclist must lean at $22.2°$ from the vertical.

(b) At $v = 20\ m/s$:

$$\tan\theta = \frac{(20\ m/s)^2}{25\ m \times 9.8\ m/s^2} = 1.633$$

$$\theta = \tan^{-1}(1.633) = 58.5°$$

The new lean angle is $58.5°$.

Making Sense of the Answer: *Doubling the speed more than doubled the lean angle, from 22.2° to 58.5°. This is because the lean angle depends on* v^2*, not* v*. A 58.5° lean is extreme and very difficult to maintain in practice, which explains why cyclists slow down significantly before entering sharp curves.*

Think Like a Physicist: *If the calculated lean angle becomes very large (say above 45°), the required friction is enormous and the cyclist is almost certainly going to skid. In practice, lean angles beyond about 40° are dangerous on ordinary road surfaces.*

REAL Example 20

Kipanga rides his bicycle fast around a sharp corner near the school. **Kipute** notices that he leans his body and the bicycle strongly toward the inside of the curve. She asks: *"Why do you lean so much when turning? You don't lean when riding straight."*

Explain why a cyclist must lean inward when negotiating a curve.

Solution

When riding in a straight line, only two forces act on the cyclist: weight downward and the normal reaction upward. These are balanced, and no lean is needed.

When turning, however, a horizontal centripetal force is required toward the centre of the curve. This force is provided by static friction at the contact point between the tyres and the road. Friction acts horizontally at ground level, while the weight acts vertically through the centre of mass, which is high above the ground.

If the cyclist remains upright, friction and weight act along different lines, creating a turning effect that would topple the cyclist outward. By leaning inward, the cyclist shifts the centre of mass so that the combined effect of weight and friction passes through the contact point. This eliminates the toppling tendency and allows stable circular motion. The faster the turn or the sharper the curve, the greater the required friction and the more the cyclist must lean to maintain balance.

Making Sense of the Answer: *Leaning is not about courage or style. It is the cyclist's only way to keep the forces balanced during a turn. A car does not need to lean because it has four widely spaced wheels that prevent toppling. A bicycle, with only two narrow contact points, must lean to achieve the same balance.*

Think Like a Physicist: *The lean angle satisfies* $tan\theta = \frac{v^2}{rg}$. *This same expression appears in the conical pendulum and in road banking, because the underlying physics is the same: balancing a horizontal centripetal requirement against a vertical gravitational force.*

HOT Example 21

A cyclist of mass 70 kg (including the bicycle) rounds a level curve of radius 30m. The coefficient of static friction between the tyres and the rough road surface is 0.55. Take $g = 9.8\ m/s^2$.

(a) Calculate the maximum speed at which the cyclist can negotiate the curve without skidding.

(b) Find the lean angle at maximum speed.

(c) At maximum speed, calculate the friction force and the normal reaction.

(d) If the road becomes wet and the coefficient of friction drops to 0.25, determine the new maximum speed and the new lean angle.

Solution

(a) Using:

$$v_{max} = \sqrt{\mu_s gr}$$

Where: $\mu_s = 0.55$, $g = 9.8\ m/s^2$, $r = 30\ m$

Substituting:

$$v_{max} = \sqrt{0.55 \times 9.8\ m/s^2 \times 30\ m} = 12.7\ m/s$$

The maximum speed is 12.7 m/s (about 45.7 km/h).

(b) At maximum speed:

$$\tan\theta_{max} = \mu_s = 0.55$$

$$\theta_{max} = \tan^{-1}(0.55) = 28.8°$$

The lean angle at maximum speed is 28.8°.

(c) Normal reaction:

$$R = mg = 70\ kg \times 9.8\ m/s^2 = 686N$$

Friction force at maximum speed:

$$f = \mu_s mg = 0.55 \times 686\ N = 377.3N$$

The normal reaction is 686N and the friction force is 377.3N.

(d) With $\mu_s = 0.25$:

$$v_{max} = \sqrt{0.25 \times 9.8\ \text{m/s}^2 \times 30\ \text{m}} = 8.6\ \text{m/s}$$

$$\tan\theta_{max} = 0.25; \quad \theta_{max} = \tan^{-1}(0.25) = 14°$$

The new maximum speed on the wet road is 8.6 m/s (about 30.9 km/h) and the lean angle drops to 14°.

Making Sense of the Answer: *On a wet road, the cyclist must reduce speed from 12.7 m/s to 8.6 m/s, a reduction of about 32%. The lean angle also drops sharply, from 28.8° to 14°. This makes sense: less friction means less centripetal force available, so the cyclist must go slower and lean less. Attempting the original speed on a wet road would cause the tyres to skid sideways.*

Think Like a Physicist: *At maximum speed, $tan\theta_{max} = \mu_s$. This is elegant: the maximum lean angle depends only on friction, regardless of speed, radius, or mass. It provides a quick way to estimate the danger of a curve: if the required lean angle exceeds* $\tan^{-1}(\mu_s)$*, skidding is inevitable.*

Having explored the cyclist, we notice something interesting: the expression $\tan\theta = \frac{v^2}{rg}$ keeps appearing. It appeared for the cyclist's lean angle, and it will appear again in our next subtopic, the conical pendulum, where a mass on a string traces a horizontal circle. The angle is different, the physical setup is different, but the mathematics is remarkably the same. Let us see why.

Conical Pendulum

Imagine tying a small stone to the end of a string, holding the other end fixed above your head, and then setting the stone in motion so that it traces a horizontal circle. The string sweeps out the surface of a cone as the stone revolves, and the arrangement is called a **conical pendulum**.

Unlike the cyclist or the car, there is no friction here. The only forces acting on the stone are its weight and the tension in the string. Yet the stone follows a perfectly circular horizontal path. *How does it manage this without friction?* The answer lies in the geometry: the string is not vertical but inclined at an angle θ to the vertical, and it is the horizontal component of the tension that provides the centripetal force.

This is a beautiful example of how a single force (tension) can serve two roles simultaneously: its vertical component supports the weight, while its horizontal component maintains the circular motion.

Setting up the problem

Consider a small object of mass m attached to a string of length **L**, with the other end fixed to a point directly above the centre of the circular path. The object moves at constant speed in a horizontal circle of radius **r**. The string makes a constant angle **θ** with the vertical.

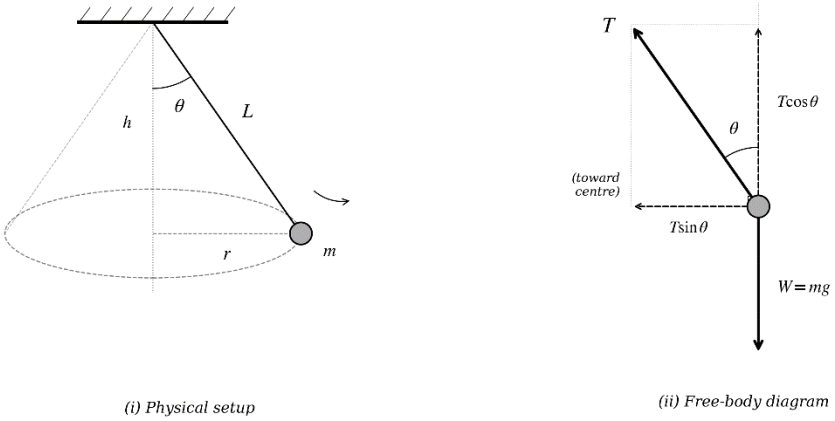

Figure: A conical pendulum. *(i) A mass m attached to a string of length L revolves in a horizontal circle of radius r. The string makes angle θ with the vertical and sweeps out a cone. (ii) Free-body diagram: only tension T and weight W = mg act on the mass. The vertical component Tcos θ balances the weight, while the horizontal component Tsin θ provides the centripetal force.*

Identifying the forces

Only two forces act on the object:

1) **Weight** (W = mg), acting vertically downward.
2) **Tension** (T), acting along the string, directed from the object toward the fixed point.

There is no friction, no normal reaction, and no additional horizontal force. The centripetal force must come entirely from the tension.

Applying Newton's second law

We resolve the tension into vertical and horizontal components:

Vertically (no acceleration):

The object does not move up or down, so the net vertical force is zero:

$$T\cos\theta - mg = 0$$

From which:

$$T\cos\theta = mg \quad \ldots (i)$$

Horizontally (toward the centre):

The horizontal component of tension provides the centripetal force:

$$T\sin\theta = \frac{mv^2}{r} \quad \ldots (ii)$$

Finding the relationship between angle and speed

Dividing equation (ii) by equation (i):

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2/r}{mg}$$

The tension T and mass m cancel:

$$\mathbf{\tan\theta = \frac{v^2}{rg}}$$

This is the same expression we met for the cyclist's lean angle. The physics is different, the forces are different, but the geometry of balancing a horizontal centripetal requirement against a vertical gravitational force produces the same result.

It is important for you to understand that:

The angle θ does not depend on the mass of the object. A heavy stone and a light stone, whirled on the same string at the same speed, will both make the same angle with the vertical. This is because mass appears in both the centripetal force requirement (mv^2/r) and the weight (mg), and cancels when we divide.

What the angle does depend on is the speed v and the radius r. A faster speed or a smaller radius requires a larger angle.

Finding the tension

From equation (i):

$$\mathbf{T = \frac{mg}{\cos\theta}}$$

Since $\cos\theta < 1$ for any angle $\theta > 0°$, the tension is always greater than the weight. This is because the string has two duties: it must support the weight vertically and simultaneously pull the object inward. This double duty means the tension must exceed mg.

Relating radius to string length

From the geometry of the cone:

$$\mathbf{r = L\sin\theta}$$

This connects the radius of the circular path to the string length and the angle. It is the same geometric relationship we used for the fairground chair ride in Example 15.

Deriving the period of revolution

The period T_p (we use T_p instead of the usual T because in this subtopic the symbol T is used for tension; using the same letter for both would cause confusion) can be found by combining our results. Starting from:

$$\tan\theta = \frac{v^2}{rg}$$

We substitute $v = \frac{2\pi r}{T_p}$:

$$\tan\theta = \frac{4\pi^2 r}{T_p^2 g}$$

Making T_p^2 the subject:

$$T_p^2 = \frac{4\pi^2 r}{g\tan\theta}$$

Now substituting $r = L\sin\theta$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$T_p^2 = \frac{4\pi^2 L\sin\theta}{g \times \frac{\sin\theta}{\cos\theta}} = \frac{4\pi^2 L\cos\theta}{g}$$

From which:

$$\mathbf{T_p = 2\pi\sqrt{\frac{L\cos\theta}{g}}}$$

Understanding what this equation reveals

This is a striking result. The period depends on the string length L, the angle θ, and gravity g. It does not depend on the mass of the object.

As the angle θ increases (faster rotation), cosθ decreases, and the period becomes shorter. The object revolves faster. In the limiting case where θ approaches 90° (string nearly horizontal), the period approaches zero and the speed becomes very large, but the tension becomes extremely large because $T = \frac{mg}{\cos\theta}$ increases without bound.

In the opposite limit, as θ approaches 0° (string nearly vertical), the period approaches $2\pi\sqrt{\frac{L}{g}}$, which is the period of a simple pendulum of the same length. This makes physical sense: at very small angles, the conical pendulum barely traces a circle and its motion resembles a simple pendulum.

With the theory complete, let us work through some examples.

BINDER Example 22

A small ball of mass 0.3 kg is attached to a string of length 0.8 m and whirled as a conical pendulum. The string makes an angle of 40° with the vertical. Take $g = 9.8\ m/s^2$.

(a) Calculate the tension in the string.

(b) Calculate the radius of the circular path.

(c) Calculate the speed of the ball.

Solution

(a) Using:

$$T\cos\theta = mg$$

Making T the subject:

$$T = \frac{mg}{\cos\theta}$$

Where: $m = 0.3\ kg$, $g = 9.8\ m/s^2$, $\theta = 40°$

Substituting:

$$T = \frac{0.3\ kg \times 9.8\ m/s^2}{\cos 40°} = 3.84N$$

The tension is 3.84 N.

(b) Using:

$$r = L\sin\theta$$

Where: $L = 0.8m$, $\theta = 40°$

Substituting:

$$r = 0.8\ m \times \sin 40° = 0.514\ m$$

The radius is 0.514 m.

(c) Using:

$$T\sin\theta = \frac{mv^2}{r}$$

Making v the subject:

$$v = \sqrt{\frac{T\sin\theta \cdot r}{m}}$$

Substituting:

$$v = \sqrt{\frac{3.84\ \text{N} \times 0.643 \times 0.514\ \text{m}}{0.3\ \text{kg}}} = \sqrt{4.23} = 2.06\text{m/s}$$

The speed is 2.06 m/s.

Making Sense of the Answer: *The tension (3.84N) is greater than the weight (2.94N), as expected. The string must support the weight vertically and pull the ball inward simultaneously. At a 40° angle, the radius is about 0.51m, which is reasonable for a 0.8m string. The speed of 2.06m/s is modest, consistent with a gentle swing.*

Think Like a Physicist: *Always find the tension first using $T\cos\theta = mg$, because tension appears in both the vertical and horizontal equations. Once you have T, the rest follows easily.*

REAL Example 23

Kipanga ties a small stone to a string and whirls it in a horizontal circle above his head. He notices that when he spins the stone faster, the string rises and the angle with the vertical increases. He asks **Kipute**: *"Why does the string go higher when I spin faster? Shouldn't gravity keep pulling it down the same way?"*

Explain why the angle increases when the speed increases.

Solution

Gravity does pull the stone downward with the same force (mg) regardless of speed. However, as the speed increases, the stone requires a larger centripetal force to maintain its circular path, since the centripetal force is proportional to v^2.

The only horizontal force available is the horizontal component of the tension: $T\sin\theta$. To increase this horizontal component, either the tension must increase, or the angle must increase, or both. In practice, as the speed increases, the string rises to a larger angle θ. A larger angle means a greater proportion of the tension is directed horizontally (larger $\sin\theta$), providing the extra centripetal force needed.

The relationship $\tan\theta = \frac{v^2}{rg}$ confirms this: increasing v increases $\tan\theta$, which means θ increases.

So gravity has not changed, but the balance between vertical support and horizontal pull has shifted. The string rises because the stone's circular motion demands more horizontal force, and the only way to get it is by tilting the string further from the vertical.

Making Sense of the Answer: *Think of it this way: at low speed, the stone barely needs any centripetal force, so the string hangs almost vertically and nearly all the tension goes toward supporting the weight. At high speed, the centripetal demand is large, so the string must tilt outward to redirect more tension horizontally.*

Think Like a Physicist: *The angle is a visible indicator of how much centripetal force the system demands. A large angle means the circular motion is demanding a large fraction of the available tension. This is why conical pendulums are used in some engineering speed governors: the angle directly indicates the rotation speed.*

HOT Example 24

A conical pendulum has a string of length 1.2m. The period of revolution is 1.8s. Take $g = 9.8\ \text{m/s}^2$ and $\pi = 3.14$.

(a) Calculate the angle the string makes with the vertical.

(b) Calculate the radius of the circular path.

(c) Calculate the tension in the string if the mass is 0.5 kg.

(d) Determine the speed of the object.

Solution

(a) Using:

$$T_p = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

Making $\cos\theta$ the subject:

$$T_p^2 = 4\pi^2 \times \frac{L\cos\theta}{g}$$

$$\cos\theta = \frac{T_p^2 g}{4\pi^2 L}$$

Where: $T_p = 1.8$ s, $g = 9.8$ m/s^2, $L = 1.2$ m

Substituting:

$$\cos\theta = \frac{(1.8\text{ s})^2 \times 9.8\text{ m/s}^2}{4 \times (3.14)^2 \times 1.2\text{ m}} = 0.671$$

$$\theta = \cos^{-1}(0.671) = 47.9°$$

The angle is 47.9°.

(b) Using:

$$r = L\sin\theta = 1.2\text{ m} \times \sin 47.9° = 1.2 \times 0.742 = 0.890\text{ m}$$

The radius is 0.890 m.

(c) Using:

$$T = \frac{mg}{\cos\theta} = \frac{0.5\text{ kg} \times 9.8\text{ m/s}^2}{0.671} = 7.30\text{ N}$$

The tension is 7.30 N.

(d) Using:

$$v = \frac{2\pi r}{T_p} = \frac{2 \times 3.14 \times 0.890\text{ m}}{1.8\text{ s}} = \frac{5.59\text{ m}}{1.8\text{ s}} = 3.1\text{m/s}$$

The speed is 3.1 m/s.

Making Sense of the Answer: *An angle of nearly 48° is quite large; the string is closer to horizontal than vertical. This tells us the rotation is fairly fast. The tension (7.3N) is about 1.5 times the weight (4.9N), confirming that the string works hard to provide both vertical support and horizontal centripetal force. The speed of 3.1m/s over a radius of 0.89m gives a reasonable revolution time of 1.8 s.*

Think Like a Physicist: *When the period is given instead of the speed, use the period formula directly to find the angle. Do not try to find the speed first; it creates unnecessary steps. The formula* $\cos\theta = \frac{T_p^2 g}{4\pi^2 L}$ *gives the angle in one calculation.*

With the conical pendulum now understood, we recognise a pattern that runs through all of horizontal circular motion. The leaning cyclist and the conical pendulum both produce the same central relationship: $\tan\theta = \frac{v^2}{rg}$. In each case, the angle θ connects the vertical force (gravity) to the horizontal force (centripetal). This same relationship will appear once more in our next subtopic, where engineers deliberately tilt the road surface to help vehicles navigate curves safely. Welcome to banking of roads.

Banking of Roads

On a level curve, the centripetal force comes entirely from friction. But friction is unreliable. Rain reduces it. Worn tyres reduce it. Oil spills destroy it. And when friction fails, vehicles slide off the road, as we learned at Kilimani Hill.

Engineers asked a simple but powerful question: *What if the road itself could help provide the centripetal force, even without any friction at all?*

The answer is **banking:** *tilting the road surface so that the outer edge is higher than the inner edge.* On a banked curve, the road surface is no longer horizontal. The normal reaction, which is always perpendicular to the surface, now tilts inward. This tilted normal reaction has a horizontal component that points toward the centre of the curve, providing some or all of the centripetal force.

Banking does not eliminate friction; it reduces the dependence on friction. A well-banked curve at the correct speed requires no friction at all. At other speeds, friction assists or opposes the motion, but the banking carries the main load.

Case 1: Ideal banking (no friction)

Consider a car of mass **m** moving at constant speed **v** around a banked curve of radius **r**. The road surface makes an angle θ with the horizontal. We analyse the ideal case where friction is zero (perfectly smooth surface).

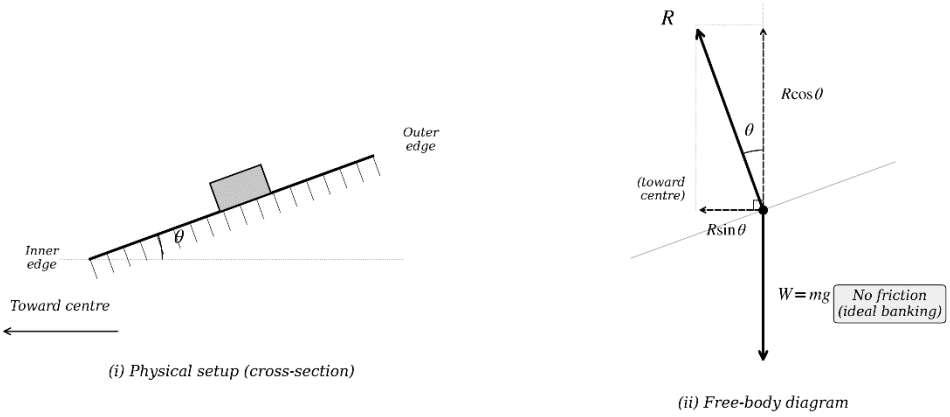

Figure: ***A car on a banked (tilted) curve, shown in cross-section.*** *(i) The road surface makes angle θ with the horizontal, with the outer edge higher than the inner edge. (ii) Free-body diagram for ideal banking (no friction): the normal reaction R is perpendicular to the tilted surface. Its vertical component Rcosθ balances the weight, while its horizontal component Rsinθ provides the centripetal force.*

Identifying the forces

With no friction, only two forces act on the car:

1) **Weight** (W = mg), acting vertically downward.
2) **Normal reaction** (R), acting perpendicular to the banked surface. Because the surface is tilted, R is no longer vertical; it tilts inward.

Resolving the normal reaction

We resolve R into two components:

- **Vertical component:** $R\cos\theta$ (upward)
- **Horizontal component:** $R\sin\theta$ (toward the centre of the curve)

Applying Newton's second law

Vertically (no acceleration):

$$R\cos\theta = mg \quad \text{... (i)}$$

Horizontally (toward the centre):

$$R\sin\theta = \frac{mv^2}{r} \quad \text{... (ii)}$$

Finding the banking angle

Dividing equation (ii) by equation (i):

$$\frac{R\sin\theta}{R\cos\theta} = \frac{mv^2/r}{mg}$$

R and m cancel:

$$\mathbf{\tan\theta = \frac{v^2}{rg}}$$

This is the same expression we derived for the cyclist's lean angle and the conical pendulum. The physics is universal: *whenever gravity provides vertical equilibrium and a horizontal component of some force provides centripetal acceleration, this relationship emerges.*

The design speed

The equation $\tan\theta = \frac{v^2}{rg}$ can be rearranged to give the speed for which the banking angle is perfectly matched:

$$\mathbf{v_{design} = \sqrt{rg\tan\theta}}$$

At this speed, the car negotiates the curve without any friction at all. The entire centripetal force comes from the horizontal component of the normal reaction. This is called the **design speed** of the banked curve.

It is important for you to understand that:

The design speed does not depend on the mass of the vehicle. A motorcycle, a car, and a fully loaded bus all have the same design speed on the same banked curve, because mass cancels from the equation. This is the same mass-independence we observed for the maximum safe speed on a level curve, the cyclist's lean angle, and the conical pendulum angle.

What happens at speeds other than the design speed?

If the car travels at exactly the design speed, no friction is needed.

If the car travels **faster** than the design speed, the required centripetal force exceeds what the banking alone provides. Friction must act *inward* (down the slope) to supply the extra centripetal force. Without sufficient friction, the car slides outward (up the slope).

If the car travels **slower** than the design speed, the banking provides more inward force than needed. Friction must act *outward* (up the slope) to prevent the car from sliding inward (down the slope).

At the design speed, the forces are perfectly balanced and friction is zero. This is why highway engineers choose the banking angle based on the expected speed of traffic on that section of road.

Case 2: Banked curve with friction (maximum speed)

In real life, roads are not frictionless. When a car travels faster than the design speed on a banked curve, the horizontal component of the normal reaction alone cannot supply the centripetal force required. The car tends to slide outward (up the slope). Friction therefore acts to prevent this sliding, which means friction acts **down the slope** (toward the inner edge), helping to push the car inward.

Why does friction act down the slope?

This is the point that confuses many students. On a level road, friction acts horizontally toward the centre of the curve. On a banked road, however, if the speed is too high, the vehicle tends to slide up the slope toward the outside of the curve. Friction therefore acts **down the slope** toward the centre to oppose this motion. In general, friction always acts in the direction that opposes the motion that would occur if friction were absent.

Identifying the forces

Three forces now act on the car:

3) **Weight** ($W = mg$), acting vertically downward.
4) **Normal reaction** (R), acting perpendicular to the banked surface (tilted inward from the vertical).
5) **Friction** (f), acting along the banked surface, directed down the slope (toward the inner edge).

Resolving the forces

Each force must be resolved into vertical and horizontal components. This is where careful thought is needed, because both R and f contribute to both directions.

Normal reaction R (perpendicular to road, tilted at angle **θ from the vertical**):

- Vertical component: $R\cos\theta$ (upward)
- Horizontal component: $R\sin\theta$ (toward centre)

Friction f (along the road surface, directed down the slope, at angle **θ from the horizontal**):

- Vertical component: $f\sin\theta$ (downward, because friction points down the slope)
- Horizontal component: $f\cos\theta$ (toward centre, because the down-slope direction has a horizontal inward component)

This is the key insight: friction has a **vertical component that acts downward**, reducing the vertical support, and a **horizontal component that acts toward the centre**, adding to the centripetal force. **Both R and f contribute to the centripetal force**, but friction slightly undermines the vertical balance.

Applying Newton's second law

Vertically (no acceleration):

The upward forces must balance the downward forces:

$$R\cos\theta = mg + f\sin\theta$$

Rearranging:

$$R\cos\theta - f\sin\theta = mg \quad \ldots (i)$$

Notice that friction's vertical component ($f\sin\theta$) appears on the same side as weight. Friction acting down the slope pulls the car slightly downward, so the normal reaction must support both the weight and this downward component of friction.

Horizontally (toward the centre):

Both the horizontal component of R and the horizontal component of f point toward the centre:

$$R\sin\theta + f\cos\theta = \frac{mv^2}{r} \quad \ldots (ii)$$

Banking and friction work together to provide the centripetal force.

Finding the maximum speed

The maximum speed occurs when friction reaches its limit:

$$f = \mu_s R$$

Substituting $f = \mu_s R$ into equation (i):

$$R\cos\theta - \mu_s R\sin\theta = mg$$

$$R(\cos\theta - \mu_s\sin\theta) = mg \quad \ldots (iii)$$

Substituting $f = \mu_s R$ into equation (ii):

$$R\sin\theta + \mu_s R\cos\theta = \frac{mv_{max}^2}{r}$$

$$R(\sin\theta + \mu_s\cos\theta) = \frac{mv_{max}^2}{r} \quad \ldots (iv)$$

Dividing equation (iv) by equation (iii) to eliminate both R and m:

$$\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta} = \frac{v_{max}^2}{rg}$$

Dividing numerator and denominator of the left side by $\cos\theta$:

$$\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta} = \frac{v_{max}^2}{rg}$$

Making v_{max} the subject:

$$\mathbf{v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)}}$$

Checking the formula against known results

A good formula should reduce to simpler cases when we set certain quantities to zero. Let us verify:

- If $\theta = 0$ (level road, no banking): $\tan\theta = 0$, so the formula gives $v_{max} = \sqrt{rg \times \mu_s} = \sqrt{\mu_s gr}$, which is our result we established earlier.
- If $\mu_s = 0$ (no friction): the formula gives $v_{max} = \sqrt{rg\tan\theta}$, which is the design speed from Case 1.

Both limiting cases check out, giving us confidence that the derivation is correct.

Case 3: Banked curve with friction (minimum speed)

So far we have found the maximum speed. But on a banked curve, there is also a **minimum** safe speed, and this surprises many students.

When the car travels slower than the design speed, the horizontal component of the normal reaction provides more inward force than the circular motion requires. The car tends to slide **down the slope** (toward the inner edge). Friction must now act **up the slope** (toward the outer edge) to prevent this inward sliding.

Notice the reversal: *at high speed, friction acts down the slope to help the car turn; at low speed, friction acts up the slope to stop the car from sliding inward. At exactly the design speed, friction is zero.*

Identifying the forces

The same three forces act on the car: weight W = mg (downward), normal reaction R (perpendicular to road, tilted inward), and friction f (along the road surface). But now friction acts **up the slope** instead of down.

Resolving the forces

Because friction is reversed, its components change sign compared to Case 2:

Friction f (up the slope):

- Vertical component: $f\sin\theta$ (**upward**, helping to support the weight)
- Horizontal component: $f\cos\theta$ (**away from centre**, opposing the centripetal force)

Applying Newton's second law

Vertically (no acceleration):

$$R\cos\theta + f\sin\theta = mg \quad \ldots (v)$$

Compare this with equation (i) from Case 2: the friction term has changed from $-f\sin\theta$ to $+f\sin\theta$ because friction now has an upward vertical component.

Horizontally (toward the centre):

$$R\sin\theta - f\cos\theta = \frac{mv^2}{r} \quad \ldots (vi)$$

Compare with equation (ii) from Case 2: the friction term has changed from $+f\cos\theta$ to $-f\cos\theta$ because friction now opposes the centripetal direction.

Finding the minimum speed

At the minimum speed, friction reaches its limit: $f = \mu_s R$.

Substituting into equations (v) and (vi):

$$R(\cos\theta + \mu_s\sin\theta) = mg \quad \ldots (vii)$$

$$R(\sin\theta - \mu_s\cos\theta) = \frac{mv_{min}^2}{r} \quad \ldots (viii)$$

Dividing (viii) by (vii):

$$\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta} = \frac{v_{min}^2}{rg}$$

Dividing numerator and denominator by $\cos\theta$:

$$\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta} = \frac{v_{min}^2}{rg}$$

Making v_{min} the subject:

$$\mathbf{v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)}}$$

Understanding what this equation reveals

Compare the minimum and maximum speed formulas side by side:

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)} \qquad v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)}$$

The pattern is elegant: wherever the maximum speed formula has a $+$, the minimum speed formula has a $-$, and vice versa. This reflects the reversal of friction direction.

An important special case arises when $\tan\theta \leq \mu_s$. In this case, the numerator $(\tan\theta - \mu_s)$ becomes zero or negative, which means $v_{min} = 0$. Physically, this means friction is strong enough to hold the car stationary on the banked surface without sliding. The car can safely travel at any speed from zero up to v_{max}.

However, if the banking is steep (large θ) and friction is weak (small μ_s), then $\tan\theta > \mu_s$, and there is a real minimum speed. A car travelling too slowly on such a curve will slide down the slope toward the inner edge.

The complete picture

Every banked curve with friction has a safe speed range:

$$v_{min} \leq v \leq v_{max}$$

Below v_{min}, the car slides inward (down the slope). Above v_{max}, the car slides outward (up the slope). At the design speed $v_{design} = \sqrt{rg\tan\theta}$, friction is zero and the ride is smoothest.

The worked examples that follow will sharpen all of these ideas.

BINDER Example 25

A highway curve has a radius of 200m. The road is banked at an angle of 12° to the horizontal. Take $g = 9.8\ m/s^2$.

(a) Calculate the design speed of the curve.

(b) A car of mass 1200 kg travels at the design speed. Calculate the normal reaction acting on the car.

Solution

(a) Using:

$$v_{design} = \sqrt{rg\tan\theta}$$

Where: $r = 200$ m, $g = 9.8\ m/s^2$, $\theta = 12°$

Substituting:

$$v_{design} = \sqrt{200\ m \times 9.8\ m/s^2 \times \tan 12°} = 20.4\ m/s$$

The design speed is 20.4 m/s (about 73.4 km/h).

(b) Using:

$$R\cos\theta = mg$$

Making R the subject:

$$R = \frac{mg}{\cos\theta}$$

Where: $m = 1200$ kg, $g = 9.8\ m/s^2$, $\theta = 12°$

Substituting:

$$R = \frac{1200\ \text{kg} \times 9.8\ \text{m/s}^2}{\cos 12^\circ} = \frac{11760\text{N}}{0.978} = 12024\text{N}$$

The normal reaction is 12024N.

Making Sense of the Answer: *The design speed of about 73 km/h is typical for a gently banked highway curve. The normal reaction (12024N) is slightly larger than the weight (11760N) because the normal force must support the weight vertically while also providing centripetal force horizontally. This is the same effect we saw in the conical pendulum, where tension exceeded the weight.*

Think Like a Physicist: *At the design speed, friction is zero, so the only contact force is the normal reaction. The car feels exactly as if it were driving on a perfectly smooth surface. Passengers experience no sideways sensation because the banking angle matches the speed perfectly.*

REAL Example 26

On a school trip to Iringa, the bus passes through a mountain road with several sharp curves. At one particular curve, **Kipanga** looks out the window and sees that the road is visibly tilted, with the outer edge much higher than the inner edge.

Kipanga (alarmed): *"Mr. Akilikubwa! Look at this road! It is slanting badly. I think the road has been damaged by heavy rains. Should we report it to TANROADS?"*

Mr. Akilikubwa (smiling): *"Kipanga, if you report that road, the engineers at TANROADS will laugh. They are the ones who built it that way on purpose."*

Kipanga (confused): *"On purpose? Why would anyone build a slanting road?"*

Kipute (grinning): *"Because physics said so, Kipanga."*

Explain why the road is deliberately tilted (banked) at curves, and what advantage this provides over a flat road.

Solution

On a level (flat) curve, the centripetal force required to keep the bus on its circular path must come entirely from friction between the tyres and the road. If friction is insufficient (due to rain, worn tyres, or excessive speed), the bus slides off the curve.

A banked road is deliberately tilted so that the outer edge is higher than the inner edge. This causes the normal reaction from the road to tilt inward. The horizontal component of this tilted normal reaction points toward the centre of the curve and provides centripetal force.

At the design speed, the banking alone supplies all the centripetal force needed, and friction is not required at all. This means the bus can safely negotiate the curve even on a wet or icy surface, provided it travels near the design speed. Thus banking greatly reduces dependence on friction and makes the curve safer under all road conditions.

The engineers at TANROADS calculated the banking angle using $\tan\theta = \frac{v^2}{rg}$, based on the expected traffic speed and the radius of the curve.

Making Sense of the Answer: *Kipanga's instinct was understandable. A tilted road looks wrong if you do not know why it is tilted. But the tilt is deliberate engineering, not damage. Every banked curve on a highway represents a careful calculation: the angle is chosen so that normal vehicles at normal speeds require little or no friction to stay on the road.*

Think Like a Physicist: *Banking is nature's way of saying: "If you cannot trust friction, tilt the surface." The same principle appears in athletics (track bicycle racing), railway curves, and aircraft turns. Whenever something must follow a curved path reliably, banking is the engineer's first tool.*

HOT Example 27

(a) A curved section of road has a radius of 150m and is banked at an angle of 15°. The coefficient of static friction between the tyres and the road surface is 0.4. Take $g = 9.8\ m/s^2$.

(b) Compare this maximum speed with: the design speed of the curve (no friction), and the maximum speed if the same curve were level (unbanked) with the same friction. What does this comparison reveal about the advantage of combining banking with friction?

Solution

(a) Using the result derived for the maximum speed on a banked curve with friction:

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)}$$

Where: $r = 150\ m$, $g = 9.8\ m/s^2$, $\theta = 15°$, $\mu_s = 0.4$

Substituting:

$$v_{max} = \sqrt{150m/s \times 9.8\ m/s^2 \times \left(\frac{\tan 15° + 0.4}{1 - 0.4\tan 15°}\right)} = 33.2m/s$$

The maximum speed on the banked curve is 33.2 m/s (about 119.4 km/h).

(b) Comparisons:

On a level road (no banking, $\theta = 0$), the maximum speed depends on friction alone:

$$v_{max,\ level} = \sqrt{\mu_s gr} = \sqrt{0.40 \times 9.8 \times 150} = \sqrt{588} = 24.2\ m/s \text{ (about 87.3 km/h)}$$

The design speed of the banked curve (no friction):

$$v_{design} = \sqrt{rg\tan\theta} = \sqrt{150 \times 9.8 \times 0.2679} = \sqrt{393.8} = 19.8\ m/s \text{ (about 71.4 km/h)}$$

So the three scenarios give:

- Banking alone (no friction): 19.8 m/s
- Friction alone (no banking): 24.2 m/s
- Banking and friction together: 33.2 m/s

The combination of banking and friction supports a maximum speed that is significantly higher than either banking or friction could achieve alone.

Making Sense of the Answer: *Banking alone gives 71 km/h. Friction alone gives 87 km/h. But together they give 119 km/h, which is far more than either alone. This is not simply addition; the two effects reinforce each other through the geometry of the tilted surface. This is why highway engineers never rely on friction alone for high-speed curves; they always bank the road as well.*

Think Like a Physicist: *Notice that the formula has $1 - \mu_s\tan\theta$ in the denominator. If $\mu_s\tan\theta$ approaches 1, the denominator approaches zero and the maximum speed becomes very large. Physically, this means a steeply banked road with high friction can support extremely high speeds.*

With banking of roads now understood, we have seen how engineers use geometry to assist circular motion. The final application in horizontal circular motion is simpler but equally instructive: an object placed on a rotating platform. Let us see what keeps it from sliding off, and what happens when the rotation becomes too fast.

Object on a Rotating Platform

Place a coin on a spinning turntable. At low speed, the coin stays in place. Increase the speed gradually, and at some critical point the coin suddenly slides outward and flies off the edge. This simple observation contains a complete physics lesson.

The coin is not glued to the turntable. The only horizontal force available to keep it moving in a circle is **static friction** between the coin and the surface. As the turntable spins faster, the required centripetal force increases. The moment this required force exceeds the maximum static friction, the coin can no longer maintain circular motion and it slides outward.

Setting up the problem

Consider an object of mass m placed at a distance **r** from the centre of a horizontal turntable that rotates at angular velocity $\boldsymbol{\omega}$.

Identifying the forces

Three forces act on the object:

1) **Weight** (W = mg), acting vertically downward.
2) **Normal reaction** (R), acting vertically upward. Since the surface is horizontal, R is vertical (unlike the banked road).
3) **Static friction** (f), acting horizontally toward the centre of the turntable. This is the only force available to provide centripetal acceleration.

Applying Newton's second law

Vertically (no acceleration):

$$R = mg$$

Horizontally (toward the centre):

$$f = m\omega^2 r$$

Be careful: We use $m\omega^2 r$ rather than $\frac{mv^2}{r}$ here because angular velocity ω is the natural quantity for a rotating platform: every point on the turntable shares the same ω, but points at different radii have different linear speeds.

The critical condition for sliding

The object remains in place as long as friction can supply the required centripetal force:

$$m\omega^2 r \leq \mu_s mg$$

Sliding occurs when the required centripetal force exceeds the maximum friction. At the critical point:

$$m\omega_{max}^2 r = \mu_s mg$$

Making ω_{max} the subject:

$$\boldsymbol{\omega_{max} = \sqrt{\frac{\mu_s g}{r}}}$$

This can also be expressed as a maximum linear speed at radius r (from $v = \omega r$):

$$\boldsymbol{v_{max} = \sqrt{\mu_s g r}}$$

Or as a maximum frequency (from $\omega = 2\pi f$):

$$\boldsymbol{f_{max} = \frac{1}{2\pi}\sqrt{\frac{\mu_s g}{r}}}$$

It is important for you to understand that:

The formula $\omega_{max} = \sqrt{\frac{\mu_s g}{r}}$ reveals something that many students find surprising: *objects farther from the centre slide off first. Why?*

At a larger radius r, the maximum angular velocity ω_{max} is smaller. So when the turntable gradually speeds up, objects near the edge reach their sliding limit before objects near the centre. The outer objects fly off while the inner objects remain safely in place.

This makes physical sense: at the same angular velocity, an object at larger radius has greater linear speed ($v = \omega r$) and therefore needs greater centripetal force ($m\omega^2 r$). The required force is proportional to r, but the available friction ($\mu_s mg$) is the same regardless of position. So objects farther out are always the first to lose the battle against inertia.

With the theory in hand, let us explore through worked examples.

BINDER Example 28

A small coin of mass 20 g is placed 15cm from the centre of a horizontal turntable. The coefficient of static friction between the coin and the surface is 0.35. Take $g = 9.8\ m/s^2$.

Calculate the maximum angular velocity at which the turntable can rotate before the coin slides off.

Solution

Using:

$$\omega_{max} = \sqrt{\frac{\mu_s g}{r}}$$

Where: $\mu_s = 0.35$, $g = 9.8\ m/s^2$, $r = 0.15\ m$
Substituting:

$$\omega_{max} = \sqrt{\frac{0.35 \times 9.8\ m/s^2}{0.15\ m}} = 4.78\ rad/s$$

The maximum angular velocity is 4.78 rad/s.

Making Sense of the Answer: *At 4.78 rad/s, the turntable completes about* $\frac{4.78}{2\pi} = 0.76$ *revolutions per second. This is a gentle spin. The coin slides at a modest speed because friction is relatively weak* ($\mu_s = 0.35$) *and the coin is fairly far from the centre (15cm). Moving the coin closer to the centre would allow faster rotation before sliding.*

Think Like a Physicist: *Notice that mass does not appear in the formula. A heavy coin and a light coin at the same radius, with the same friction coefficient, slide off at exactly the same angular velocity. Mass cancels for the same reason as in every other horizontal circular motion problem: both the required force and the available friction are proportional to mass.*

REAL Example 29

During a physics demonstration, **Mr. Akilikubwa** places several coins at different distances from the centre of a spinning turntable. He slowly increases the speed. **Kipute** observes that the coin nearest the edge slides off first, then the next one, and the coin closest to the centre is the last to slide.

Kipanga is puzzled: *"All the coins are on the same turntable spinning at the same speed. Why don't they all slide off at the same time?"*

Explain why the outermost coin slides first.

Solution

All the coins share the same angular velocity ω because they are on the same rigid turntable. However, the centripetal force required to keep each coin moving in a circle is $f = m\omega^2 r$, which is proportional to the radius r.

A coin at a larger radius needs a greater centripetal force. The maximum friction available is $\mu_s mg$, which is the same for all coins (assuming equal mass and the same surface). So the coin at the largest radius demands the most force while having the same friction limit as the others. So it reaches the sliding threshold first.

As the turntable speeds up further, the sliding threshold is reached at progressively smaller radii, so the coins slide off one by one, starting from the outermost.

Making Sense of the Answer: *Think of it as a competition between demand and supply. Every coin has the same friction supply ($\mu_s mg$). But the outer coins have greater centripetal demand ($m\omega^2 r$). The coin with the greatest demand loses the competition first.*

Think Like a Physicist: *The critical angular velocity is* $\omega_{max} = \sqrt{\frac{\mu_s g}{r}}$. *Since* ω_{max} *decreases as* r *increases, outer objects always have a lower threshold. This principle applies to any rotating system: centrifuges, spin dryers, and even planetary rings all obey the same rule.*

HOT Example 29

Two coins are placed on a horizontal turntable. Coin A (mass 10g) is placed 10cm from the centre, and coin B (mass 25g) is placed 20cm from the centre. The coefficient of static friction is 0.4 for both coins. The turntable starts from rest and its angular velocity increases gradually. Take $g = 9.8\ m/s^2$.

Which coin slides first, and at what angular velocity does it happen? At that instant, what fraction of the maximum friction is being used by the other coin?

Solution

The maximum angular velocity before sliding is:

$$\omega_{max} = \sqrt{\frac{\mu_s g}{r}}$$

For coin A ($r = 0.1m$):

$$\omega_{max,\,A} = \sqrt{\frac{0.4 \times 9.8}{0.10}} = 6.26\ rad/s$$

For Coin B ($r = 0.20\ m$):

$$\omega_{max,\,B} = \sqrt{\frac{0.4 \times 9.8}{0.20}} = 4.43\ rad/s$$

Since $\omega_{max,\,B} < \omega_{max,\,A}$, **coin B slides first** at $\omega = 4.43$ rad/s.

At $\omega = 4.43$ rad/s, the friction force on coin A is:

$$f_A = m_A \omega^2 r_A = 0.010\ kg \times (4.43 rad/s)^2 \times 0.10\ m = 0.0196\ N$$

The maximum friction on coin A is:

$$f_{max,\,A} = \mu_s m_A g = 0.4 \times 0.01 m \times 9.8\ m/s^2 = 0.0392 N$$

Fraction used:

$$\frac{f_A}{f_{max,\,A}} = \frac{0.0196}{0.0392} = 0.5$$

At the instant coin B slides off, coin A is using exactly **half** (50%) of its available friction. It has a comfortable safety margin.

Making Sense of the Answer: *Coin B slides first despite being heavier. Mass is irrelevant; only radius matters. Coin B is twice as far from the centre, so it needs twice the centripetal force per unit mass. At the moment coin B reaches its limit, coin A is only halfway to its own limit because it sits at half the radius. The factor of exactly one-half is not coincidence: since* $r_A = \frac{1}{2} r_B$, *and friction used is proportional to* $\omega^2 r$, *the ratio is* $\frac{r_A}{r_B} = \frac{1}{2}$.

Think Like a Physicist: *This problem tests whether you truly understand that radius, not mass, determines which object slides first. Many students instinctively expect the heavier coin to slide first because it "has more*

inertia." But more mass also means more friction, and the two cancel. Only radius remains as the deciding factor.

That brings horizontal circular motion to a close. From the simplest case of a car on a flat road, through the leaning cyclist, the conical pendulum, the banked curve, and finally the rotating platform, one principle has guided every analysis: identify the real forces, resolve them, and set the net inward component equal to $\frac{mv^2}{r}$ or $m\omega^2 r$.

In the next subtopic, we leave the horizontal plane and enter a world where gravity no longer sits quietly in the background. In vertical circular motion, gravity actively participates: it helps the centripetal force at the top and opposes it at the bottom. The speed changes, the tension changes, and new critical conditions emerge. The physics becomes richer, but the method remains the same.