CENTRIPETAL ACCELERATION AND FORCE

In the previous subtopics, we established that objects in circular motion experience acceleration toward the centre, and we developed the mathematical tools (angular quantities) to describe rotation. We have not yet answered the deeper question: *exactly how large is this centripetal acceleration? And what force is required to produce it?*

These are not merely academic questions. They are life-and-death questions.

When engineers design curved highways, they must calculate exactly how much friction is needed to keep cars on the road at various speeds. If the required force exceeds what friction can deliver, vehicles will slide off the road, just as our bus did at Kilimani Hill. When satellites are launched into orbit, mission controllers must determine precisely what velocity produces the right centripetal acceleration for a stable orbit. When you whirl a stone on a string, the tension you feel in your hand is the centripetal force, and if that tension exceeds what the string can handle, it snaps, sending the stone flying tangentially just as we saw earlier.

Understanding centripetal acceleration quantitatively transforms circular motion from an interesting concept into a powerful tool for solving real problems. To accomplish this, we need formulas. And those formulas begin with one central derivation: the magnitude of centripetal acceleration.

Deriving the Magnitude of Centripetal Acceleration

Earlier in this chapter, we established that an object in uniform circular motion is always accelerating toward the centre of the circle, even though its speed remains constant. We called this centripetal acceleration and understood its direction. Now we determine its magnitude.

The key question is: *If an object moves at constant speed v around a circle of radius r, how large is the centripetal acceleration?*

Setting up the problem

Consider an object moving at constant speed v along a circular path of radius r. At some instant, the object is at point A, moving with velocity $\mathbf{v_A}$. A short time Δt later, the object has moved to point B, where its velocity is $\mathbf{v_B}$. Both velocities have the same magnitude v (because speed is constant), but they differ in direction.

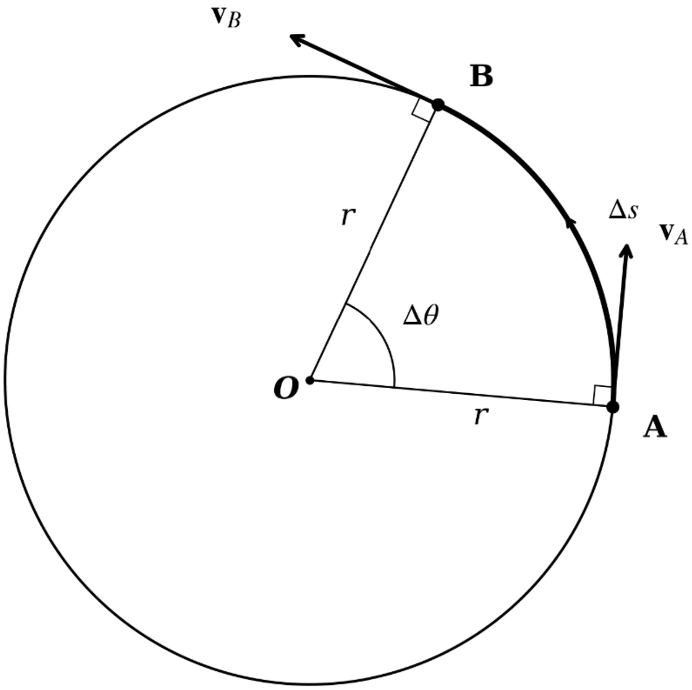


Figure (a): *An object moves at constant speed along a circular path of radius r from point A to point B, sweeping through angle Δθ and covering arc length Δs. The velocity at each point is tangent to the circle and perpendicular to the radius.*

Finding the change in velocity

The change in velocity is:

$$\mathbf{\Delta v} = \mathbf{v_B} - \mathbf{v_A}.$$

This difference $\mathbf{\Delta v}$ points toward the center of the circle.

To find $\mathbf{\Delta v}$, we use a velocity triangle. Place $\mathbf{v_A}$ and $\mathbf{v_B}$ tail-to-tail. Because both have magnitude **v** and the angle between them equals the angular displacement **Δθ** (the same angle the radius sweeps through), the velocity triangle is an isosceles triangle with two sides of length v and an included angle Δθ.

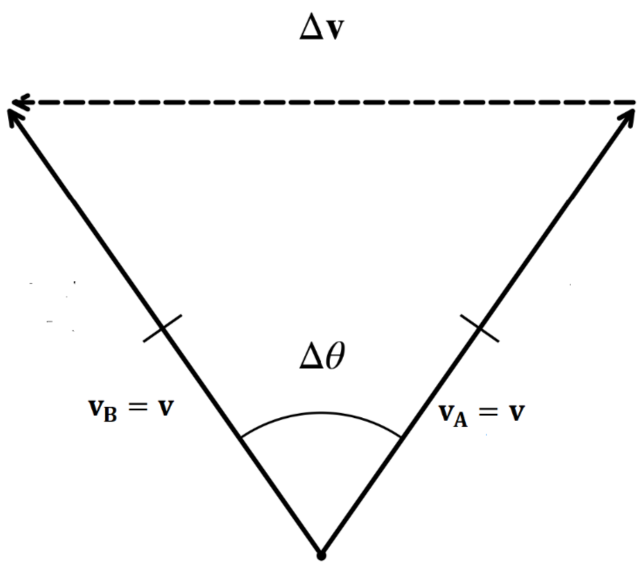


Figure (b): *The velocity vectors from (a) placed tail-to-tail form an isosceles triangle. Both sides have magnitude v, and the angle between them equals* $\boldsymbol{\Delta\theta}$. *The change in velocity,* $\boldsymbol{\Delta v = v_B - v_A}$, *is shown by the dashed arrow.*

For small angles Δθ (measured in radians), the length of the arc connecting the tips of $\mathbf{v_A}$ and $\mathbf{v_B}$ approximately equals the straight-line distance, $\mathbf{\Delta v}$.

But; arc length, s = radius, r × angle, Δθ

Where: s = Δv, r = v.

Thus:

$$\Delta v \approx v\Delta\theta \text{ (for small } \Delta\theta)$$

Now, dividing both sides by Δt:

$$\frac{\Delta v}{\Delta t} \approx v\left(\frac{\Delta\theta}{\Delta t}\right)$$

As Δt approaches zero, $\frac{\Delta v}{\Delta t}$ becomes the instantaneous acceleration, and $\frac{\Delta\theta}{\Delta t}$ becomes angular velocity ω. Therefore:

$$a = v\omega$$

But we know that $v = r\omega$, so we can substitute:

$$a = (\omega r)\omega = \omega^2 r$$

This gives us one formula for centripetal acceleration:

$$\mathbf{a = \omega^2 r}$$

Alternatively, we can eliminate ω using $\omega = \frac{v}{r}$:

$$a = r\left(\frac{v}{r}\right)^2 = r\left(\frac{v^2}{r^2}\right) = \frac{v^2}{r}$$

This gives us the more commonly used formula:

$$\mathbf{a = \frac{v^2}{r}}$$

Let us pause and appreciate what we have discovered. The centripetal acceleration depends on two factors:

1. **Speed squared (v^2):** Doubling your speed quadruples the required centripetal acceleration. This is why *curves that feel comfortable at 30 km/h become terrifying at 60 km/h*; the required acceleration increases by a factor of four.
2. **Inverse of radius:** Tighter curves (smaller r) require greater acceleration. This is why *a gentle highway curve can be navigated at high speed, but a sharp corner demands slowing down.*

These formulas explain why the bus at Kilimani Hill could not make the turn. The curve had a certain radius r, and the bus was traveling at speed v. The required centripetal acceleration was $a = \frac{v^2}{r}$. But friction could only provide a limited acceleration. When $\frac{v^2}{r}$ exceeded what friction could deliver, the tires lost grip and the bus slid off tangentially.

Direction of centripetal acceleration

The magnitude formulas $a = \frac{v^2}{r}$ and $a = \omega^2 r$ tell us how large the acceleration is, but they do not specify direction. We established earlier that centripetal acceleration points toward the centre of the circle, but let us see why this must be true.

The change in velocity $\mathbf{\Delta v} = \mathbf{v_B} - \mathbf{v_A}$ connects the tip of $\mathbf{v_A}$ to the tip of $\mathbf{v_B}$ when both are drawn from a common origin. As the time interval Δt becomes infinitesimally small, the angle $\Delta\theta$ approaches zero, and the chord connecting v_A and v_B becomes perpendicular to both velocity vectors. Since the velocity vectors are themselves tangent to the circle, $\mathbf{\Delta v}$ must be perpendicular to the tangent, which means it points along the radius.

Furthermore, since $\mathbf{\Delta v} = \mathbf{v_B} - \mathbf{v_A}$ represents the change needed to go from $\mathbf{v_A}$ to $\mathbf{v_B}$ as the object moves counterclockwise around the circle, and since v_B has rotated counterclockwise from v_A, the difference vector points inward, toward the centre.

Therefore, *centripetal acceleration always points radially inward, toward the centre of the circle.*

This radial direction is crucial. It means centripetal acceleration is always perpendicular to velocity (which points tangentially). *A force perpendicular to velocity changes the direction of motion without changing speed* which is exactly what we need for uniform circular motion.

Centripetal force

Now we bring in Newton's second law: $\mathbf{F = ma}$.

If an object of mass m experiences centripetal acceleration a toward the centre, there must be a net force F toward the centre producing that acceleration.

The centripetal force is:

$$\mathbf{F_c = ma_c = m\left(\frac{v^2}{r}\right) = m\omega^2 r}$$

Let us be absolutely clear about what "centripetal force" means. *It is not a new type of force like gravity, friction, or tension. Centripetal force is simply* ***whatever*** *net force happens to be directed toward the centre of the circular path.* It is a descriptive term, not a fundamental force.

As mentioned earlier, different situations provide centripetal force in different ways:

Planetary orbits: Gravity provides centripetal force.

Car on curve: Friction between tires and road provides centripetal force.

Stone on string: Tension in the string provides centripetal force.

Satellite in orbit: Earth's gravity provides centripetal force.

Clothes in washing machine: Normal reaction force from drum wall provides centripetal force.

Electron around nucleus (classical model): Electric attraction provides centripetal force.

In each case, we simply identify what physical force acts toward the centre, then set it equal to the required centripetal force $\frac{mv^2}{r}$. This gives us an equation we can solve for unknowns like speed, radius, or the magnitude of the providing force.

What happens when centripetal force is insufficient?

The formula $F_c = \frac{mv^2}{r}$ tells us what force is **required** to maintain circular motion at speed v and radius r. *But what if the available force is less than this required amount?*

Consider our bus at Kilimani Hill. The maximum static friction force available between tires and road might be $F_{max} = \mu_s R$, where μ_s is the coefficient of static friction and R is the normal reaction. For the bus to successfully navigate a curve of radius r, the required centripetal force $\frac{mv^2}{r}$ must not exceed this maximum:

$$\frac{mv^2}{r} \leq F_{max}$$

If v is too large (excessive speed), or r is too small (sharp curve), or μ is too small (wet or icy road), this inequality is violated. When that happens, the available force cannot provide the necessary centripetal acceleration. The tires slip, the inward force decreases, and the vehicle slides outward; though "outward" is slightly misleading! What actually happens is that the vehicle continues along the tangent to the curve, which appears to be "outward" relative to the curve's centre.

This is why speed limits on curves exist. Engineers calculate the curve radius r and the typical friction coefficient μ for various road conditions, then determine the maximum safe speed as follows:

$$F_{max} = \frac{mv_{max}^2}{r}$$

For a car on a level road: $F_{max} = \mu_s mg$

So $\mu_s mg = \frac{mv_{max}^2}{r}$ or $\mu_s g = \frac{v_{max}^2}{r}$

Making v_{max} the subject:

$$\mathbf{v_{max} = \sqrt{\mu_s gr}}$$

If you exceed this speed $\sqrt{\mu_s gr}$, you are demanding more centripetal force than friction can supply. Once this limit is crossed, the tyres can no longer maintain the necessary grip, and the vehicle will begin to slide outward. Eventually, you face the same consequence as the Miono Bus Kilimani Hill accident; a stark reminder that physics does not negotiate!

Connecting to angular velocity

Since we have two expressions for centripetal acceleration ($\frac{v^2}{r}$ and $\omega^2 r$), we similarly have two expressions for centripetal force:

$$F_c = \frac{mv^2}{r} = m\omega^2 r$$

Use whichever is more convenient for the problem at hand. Often, when dealing with rotating rigid objects (wheels, turntables, planets), angular velocity ω is given or easier to work with, making $F_c = m\omega^2 r$ more useful.

For example, every point on a rotating disk has the same ω but different v (since $v = r\omega$ increases with r). The centripetal force required increases with radius: $F_c = m\omega^2 r$. So *points farther from the centre need stronger centripetal force to maintain circular motion at the same angular velocity*. This is why *standing at the edge of a* *spinning merry-go-round is more dangerous than standing near the centre:* ***the required centripetal force is much greater at larger radii.***

Summary of key formulas

Let us consolidate what we have learned:

Centripetal acceleration:

- $a_c = \frac{v^2}{r}$ (when linear velocity v is known).
- $a_c = \omega^2 r$ (when angular velocity ω is known).
- Direction: always toward the centre (radially inward).

Centripetal force:

- $F_c = \frac{mv^2}{r}$ (when linear velocity v is known).
- $F_c = m\omega^2 r$ (when angular velocity ω is known).
- Direction: always toward the centre (radially inward).
- Not a new type of force; it is whatever net force acts toward the centre.

These formulas are the foundation for analyzing all circular motion problems. In the following subtopics, we will apply them to specific situations: objects on strings, conical pendulums, cars on curves, and objects in vertical circles. But the physics remains the same: identify what provides the centripetal force, set it equal to $\frac{mv^2}{r}$ or $m\omega^2 r$, and solve.

With this mathematical framework established, let us practice applying it through worked examples.

BINDER Example 9

A car of mass 1200kg travels around a circular track of radius 50m at constant speed 20m/s. Calculate: (a) The centripetal acceleration. (b) The centripetal force required.

Solution

(a) Centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{(20\text{m/s})^2}{50\text{m}} = 8\text{m/s}^2$$

The centripetal acceleration is 8m/s^2 toward the center of the track.

(b) Centripetal force:

$$F_c = ma_c = 1200\text{kg} \times 8\text{m/s}^2 = 9600\text{N}$$

The required centripetal force is 9600 N toward the centre.

Making Sense of the Answer: *The acceleration 8m/s² is less than g = 9.8m/s², so it is a comfortable turn; not too extreme. The force 9600N must be provided by friction between the tires and track. If friction cannot provide this (wet track, bald tires), the car will* ***slide off tangentially****.*

Think Like a Physicist: *It is better to specify direction when stating acceleration or force. "8m/s²" is incomplete; "8 m/s² toward the centre" is complete. In circular motion, the inward direction is as important as the magnitude.*

BINDER Example 10

A stone of mass 0.5kg is tied to a string and whirled in a horizontal circle of radius 0.8m. The stone completes 4 revolutions per second. Calculate:

(a) The angular velocity.

(b) The centripetal acceleration.

(c) The tension in the string.

Solution

(d) Using:

$$\omega = 2\pi f$$

Where: f = 4 Hz

Substituting:

$$\omega = 2\pi \times 4\mathrm{s}^{-1} = 8\pi = 25.1\mathrm{rad/s}$$

The angular velocity is 25.1 rad/s.

(b) Using:

$$a = \omega^2 r$$

Where: ω = 25.1 rad/s, r = 0.8m

Substituting:

$$a = (25.1\mathrm{rad/s})^2 \times 0.8\mathrm{m} = 504\mathrm{m/s}^2$$

The centripetal acceleration is $504\mathrm{m/s}^2$.

(c) The tension in the string provides the centripetal force. Therefore:

$$T = F_c = ma$$

Where: m = 0.5 kg, a = $504\mathrm{m/s}^2$

Substituting:

$$T = 0.5\mathrm{kg} \times 504\mathrm{m/s}^2 = 252\mathrm{N}$$

The tension in the string is 252N.

Making Sense of the Answer: *The centripetal acceleration is about 51 times the acceleration due to gravity! This shows how rapidly the required acceleration grows with angular velocity. Even a light stone (0.5 kg) demands 252N of tension, which is equivalent to supporting a mass of about 25.7 kg. This is why strings break when you whirl objects too fast.*

Think Like a Physicist: *When frequency or angular velocity is given instead of linear velocity, use* $a = \omega^2 r$ *directly. Converting to linear velocity first and then using* $a = v^2/r$ *gives the same answer but takes more steps.*

BINDER Example 11

The moon orbits Earth at approximately constant distance 3.84×10^8m from Earth's centre, completing one orbit in 27.3 days. Calculate:

(a) The moon's orbital speed (b) The centripetal acceleration of the moon.

Solution

(a) The moon travels a distance equal to the circumference 2πr in time T:

$$v = \frac{2\pi r}{T}$$

Where: $T = 27.3\ \text{days} = 27.3\text{days} \times 24\text{min/day} \times 3600\text{sec/min} = 2.36 \times 10^6\text{s}$

$$r = 3.84 \times 10^8\ \text{m}$$

Substituting values:

$$T = \frac{2\pi \times 3.84 \times 10^8 \text{m}}{2.36 \times 10^6 \text{s}} = 1020 \text{ m/s} \approx 1.02 \text{ km/s}$$

The moon's orbital speed is approximately 1020 m/s or 1.02 km/s.

(b) Centripetal acceleration is given by:

$$a_c = \frac{v^2}{r} = \frac{(1020 \text{m/s})^2}{3.84 \times 10^8 \text{m}} = 0.0027 \text{ m/s}^2$$

The moon's centripetal acceleration is 0.0027m/s² toward Earth's centre.

Making Sense of the Answer: *This tiny acceleration (about 1/3600 of g) is Earth's gravitational pull on the moon at that distance. It is small because the moon is far away, but it's enough to continuously deflect the moon's path into a circle instead of a straight line. Without this acceleration, the moon would fly off tangent to its orbit.*

Think Like a Physicist: *This example shows that circular motion does not require large accelerations; it just requires continuous acceleration toward the centre. Even tiny centripetal acceleration can maintain circular motion if applied persistently. The moon has been held in orbit by this gentle 0.0027 m/s² for billions of years!*

REAL Example 12

During a school trip to the Dar es Salaam amusement park, **Kipanga** sits on a spinning ride that rotates faster and faster. At first he enjoys it. But as the ride speeds up, he feels increasing pressure from the seat pushing against his back. He shouts to Kipute, *"Something is pushing me outward!"*

Kipute, sitting next to him, shouts back, *"Nothing is pushing you outward, Kipanga. Think about what is really happening!"*

Explain, why Kipanga feels pushed outward even though no outward force acts on him.

Solution

When the ride rotates, Kipanga must move in a circular path. This requires a centripetal force directed toward the centre, which is provided by the seat pushing against his back. However, due to inertia (Newton's first law), Kipanga's body tends to continue in a straight line. The seat must continuously push him inward to overcome this tendency. What he feels as being "pushed outward" is actually his inertia resisting the inward push of the seat. As the ride speeds up, the required centripetal force increases (since $F_c = m\omega^2 r$), so the seat pushes harder, making the sensation feel stronger. But no outward force exists; the real force is always directed inward toward the centre.

Making Sense of the Answer: *What we call "feeling pushed outward" is actually feeling the inward force that maintains circular motion. It is the same sensation you feel when a car accelerates forward and the seat pushes your back. You feel "pushed backward," but the real force is forward. In both cases, the sensation is caused by inertia resisting the applied force.*

Think Like a Physicist: *Never write "centrifugal force" as a real force in your solutions. The outward sensation is a consequence of inertia, not a real force. In the reference frame of the ground (an inertial frame), only the inward centripetal force exists.*

REAL Example 13

Kipanga, a Form Six student preparing for his physics practical session, decided to demonstrate circular motion using a simple setup. He tied a 0.5kg stone to a string and began whirling it in a horizontal circle of radius 0.8m in the school playground. However, the string he used could withstand a maximum tension of only 50N before breaking. Curious about the limits of the experiment, Kipanga wondered how fast he could safely whirl the stone without snapping the string.

(a) Explain what provides the centripetal force for the stone's circular motion.
(b) Calculate the maximum speed at which the stone can be whirled before the string breaks.

(c) Explain what happens to the stone if this maximum speed is exceeded.

Solution

(a) In this situation, tension in the string provides the centripetal force. The string pulls the stone inward toward the centre of the circle. This inward pull continuously deflects the stone's velocity, bending its path into a circle.

(b) At maximum speed, tension reaches its maximum value before breaking:

$$F_c = T_{max} = \frac{mv_{max}^2}{r}$$

Making v_{max} the subject:

$$v_{max} = \sqrt{\frac{rT_{max}}{m}}$$

Substituting values:

$$v_{max} = \sqrt{\frac{0.8\text{m} \times 50\text{N}}{0.5\text{kg}}} = 8.94\text{m/s}$$

The maximum safe speed is approximately 8.9m/s.

(c) If the speed exceeds 8.9 m/s, the required centripetal force mv^2/r exceeds the maximum tension the string can provide (50N). The string cannot supply the needed inward force, so it breaks. At the instant of breaking, the centripetal force vanishes. With no force to deflect its path, the stone immediately obeys Newton's first law and continues moving in a straight line along the tangent to the circular path at the breaking point.

Making Sense of the Answer: *The faster you whirl the stone, the greater the required centripetal force (because $F \propto v^2$). Eventually, you demand more force than the string cannot provide. The string does not break because of "centrifugal force throwing the stone outward." It breaks because you are asking for more inward force than it can deliver.*

Think Like a Physicist: *When analyzing circular motion, always identify what provides the centripetal force (tension, friction, gravity, normal reaction, etc.), then set it equal to the required mv^2/r or $m\omega^2r$. This equation can then be solved for any unknown: speed, radius, force, or mass.*

HOT Example 14

A car of mass 1500 kg travels around a level (unbanked) circular curve of radius 80m. The coefficient of static friction between the tyres and the dry road surface is 0.65. Take $g = 9.8$ m/s².

(a) Calculate the maximum speed at which the car can negotiate the curve without sliding.

(b) After rainfall, the coefficient of static friction drops to 0.30. Calculate the new maximum safe speed.

(c) Determine the percentage reduction in maximum safe speed caused by the wet road.

Solution

(a) For a car on a level curve, friction provides the centripetal force. The car slides when the required centripetal force exceeds the maximum static friction.

At maximum speed:

$$\frac{mv_{max}^2}{r} = \mu_s mg \text{ or } \frac{v_{max}^2}{r} = \mu_s g$$

Making v_{max} the subject:

$$v_{max} = \sqrt{\mu_s g r}$$

Where: $\mu_s = 0.65$ $g = 9.8\ m/s^2, r = 80m$

Substituting:

$$v_{max} = \sqrt{0.65 \times 9.8m/s^2 \times 80m} = \mathbf{22.6m/s}$$

The maximum safe speed on the dry road is 22.6 m/s (about 81.3 km/h).

(b) With $\mu_s = 0.30$:

$$v_{max} = \sqrt{\mu_s g r}$$

Substituting:

$$v_{max} = \sqrt{0.30 \times 9.8m/s^2 \times 80m} = \mathbf{15.3m/s}$$

The maximum safe speed on the wet road is 15.3 m/s (about 55.2 km/h).

(c) Percentage reduction:

$$\%\ \text{reduction} = \frac{v_{dry} - v_{wet}}{v_{dry}} \times 100\%$$

$$\%\ \text{reduction} = \frac{(22.6 - 15.3)m/s}{22.6m/s} \times 100\% = 32.3\%$$

The maximum safe speed is reduced by approximately 32%.

Making Sense of the Answer: *Notice that the maximum safe speed depends on the square root of* μ_s. *So even though friction dropped by more than half (from 0.65 to 0.30), the speed dropped by only about a third. However, the required force depends on* v^2*, so even a moderate speed increase beyond the safe limit causes a dramatic increase in the force demanded from the tyres.*

Think Like a Physicist: *The mass cancels in the equation for maximum safe speed:* $v_{max} = \sqrt{\mu_s g r}$

This means the maximum safe speed on a level curve does not depend on the mass of the vehicle. A heavy lorry and a light car have the same maximum safe speed on the same curve, assuming identical tyres and road conditions.

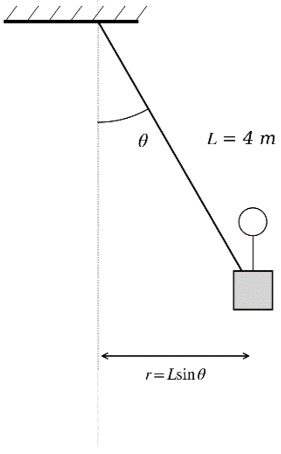
HOT Example 15

A fairground ride consists of chairs hanging from chains 4m long attached to a rotating circular platform. When rotating at constant speed, the chains make an angle of 30° with the vertical.

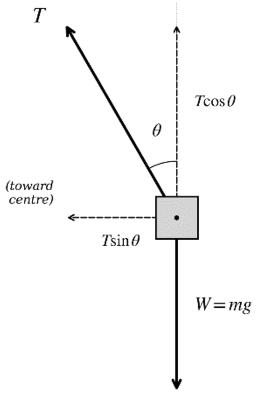
(a) Draw a free-body diagram for a chair with a person sitting in it during rotation.
(b) Explain what force provides the centripetal force for the circular motion.
(c) Calculate the radius of the circular path followed by the chair.
(d) Calculate the speed of rotation.

Solution

(a) Free-body diagram for chair + person (mass m):



(i) Physical setup



(ii) Free-body diagram

(b) The tension T has two components:

Vertical component: Tcos 30° (upward)

Horizontal component: Tsin 30° (toward centre of rotation)

Hence, **the centripetal force is provided by horizontal component of tension, Tsin 30°.**

(c) From the geometry (length L = 4.0 m, angle θ = 30°):

The radius r is the horizontal distance from the rotation axis:

$$\mathrm{r} = \mathrm{Lsin}\,\theta = 4\mathrm{m} \times \sin 30° = 2\mathrm{m}$$

The radius of the circular path is 2m.

(d) Vertical equilibrium (no vertical acceleration):

$$\mathrm{Tcos}\theta = \mathrm{mg}$$

$$\mathrm{T} = \frac{\mathrm{mg}}{\cos\theta}$$

Horizontally (centripetal force equation):

$$\mathrm{Tsin}\theta = \frac{\mathrm{mv}^2}{\mathrm{r}}$$

Substituting $\mathrm{T} = \frac{\mathrm{mg}}{\cos\theta}$:

$$\left(\frac{\mathrm{mg}}{\cos\theta}\right) \times \sin\theta = \frac{\mathrm{mv}^2}{\mathrm{r}}$$

Cancelling m and simplifying $\sin\theta / \cos\theta$:

$$\mathrm{gtan}\,\theta = \frac{\mathrm{v}^2}{\mathrm{r}}$$

From which:

$$\mathrm{v} = \sqrt{\mathrm{rgtan}\,\theta} = \sqrt{2\mathrm{m} \times 9.8\mathrm{ms}^{-2} \times \tan 30°} = 3.36\mathrm{m/s}$$

The speed of rotation is 3.36m/s.

Making Sense of the Answer: *The tilted chain shows that the tension must both* ***support the combined weight of rider and chair****, and* ***pull the chair toward the centre of the circle****. The vertical component balances the weight mg, while the horizontal component provides the centripetal force. Because the chain is inclined at* $30°$*, the chair moves in a circular path of radius 2m. To maintain this angle during rotation, the ride must move at a speed of 3.36 m/s.*

Think Like a Physicist: *This problem shows how circular motion problems often involve both centripetal force equations and equilibrium in other directions. The vertical forces balance (no vertical acceleration), giving one equation. The horizontal force provides centripetal acceleration, giving another equation. Solving these simultaneously yields the motion parameters. This pattern appears in many circular motion scenarios: banked curves, conical pendulums, and vertical circles.*

These examples demonstrate the power and versatility of the centripetal force equation. By identifying what physical force provides the centripetal force and setting it equal to the required mv^2/r, we can solve a vast array of circular motion problems. In the next subtopic, we will apply these principles specifically to horizontal circular motion, including detailed analysis of conical pendulums and banking.