ANGULAR MOTION

In the previous topic, we established the conceptual foundation of circular motion: velocity is tangential, acceleration is radial, and continuous inward force is required. Now we develop the mathematical tools to describe and calculate circular motion quantitatively. To do this, we introduce angular quantities which is a powerful alternative way of describing circular motion that often proves more convenient than linear quantities like distance and speed.

Radian Measure

Before discussing angular motion, we must understand the proper unit for measuring angles in physics: the radian.

You are already familiar with degrees: a full circle contains 360°, a right angle is 90°, and so on. Degrees are arbitrary; there is no fundamental reason why a circle should be divided into 360 parts rather than 100 or 1000. The radian, by contrast, emerges naturally from the geometry of the circle itself. By definition:

One radian is the angle subtended at the center of a circle by an arc length equal to the radius of the circle.

To understand it properly, imagine a circle of radius r. If you measure an arc along the circumference with length exactly equal to r, the angle at the center corresponding to that arc is one radian. This definition creates a natural relationship between arc length s, radius r, and angle θ (measured in radians):

$$\boldsymbol{\theta = \frac{s}{r}}$$

From this, we can derive the conversion between radians and degrees. The full circumference of a circle is 2πr. This circumference subtends a complete revolution, which is 360° or, in radians:

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians}$$

Therefore: $\boldsymbol{2\pi \text{ radians} = 360°}$

In physics, we almost always use radians rather than degrees when working with circular motion. The mathematics becomes cleaner, the formulas simpler, and the relationships more transparent. Unless explicitly stated otherwise, assume **angles in circular motion are measured in radians.**

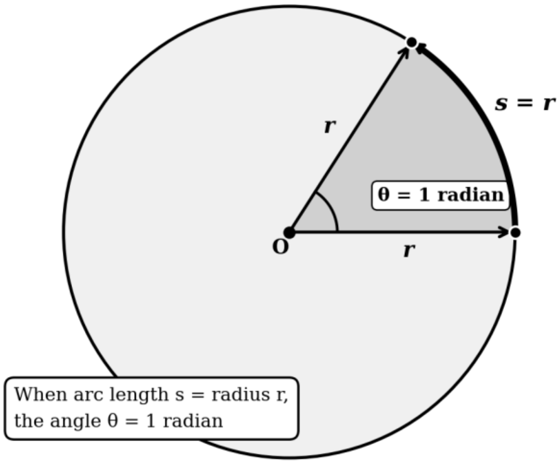

Figure: *A circle of radius r with an arc of length s = r. The angle θ subtended at the centre O is exactly 1 radian. Since the full circumference equals 2πr, a complete revolution equals 2π radians.*

Angular displacement

When an object moves along a circular path from one position to another, we can describe its motion using either linear displacement (arc length travelled) or angular displacement (angle swept out).

Angular displacement (θ) *is the angle through which an object has rotated about the centre of the circle.* It is measured in radians.

If an object moves along an arc of length s on a circle of radius r, the angular displacement is:

$$\theta = \frac{s}{r}$$

Angular displacement is the same for all points on a rotating rigid object, regardless of how far they are from the centre. A point near the rim and a point near the hub of a wheel rotate through the same angle θ, even though the point on the rim travels a much longer arc length. This is one advantage of angular quantities as they describe the rotation of the entire object with a single number.

Angular velocity

Just as linear velocity describes how quickly linear position changes with time, angular velocity describes how quickly angular position changes with time. **Angular velocity (ω)** *is the rate of change of angular displacement:*

$$\boldsymbol{\omega} = \frac{\boldsymbol{\theta}}{\mathbf{t}} \text{ (for motion starting from rest)}$$

Or more generally:

$$\boldsymbol{\omega} = \frac{\boldsymbol{\Delta\theta}}{\boldsymbol{\Delta t}}$$

Angular velocity is measured in radians per second (rad/s). It tells us how many radians the object rotates through per unit time. A wheel rotating at ω = 10 rad/s sweeps out 10 radians every second. Since **2π radians equals one complete revolution**, this wheel completes $10/(2\pi) \approx 1.59$ revolutions every second.

Angular velocity is a vector quantity, with direction along the axis of rotation, but for our purposes in uniform circular motion, we focus on its magnitude, which tells us the rate of rotation.

Relationship between linear and angular velocity

Here is the crucial connection between the linear and angular velocity of circular motion:

$$\mathbf{v = r\omega}$$

This beautifully simple equation relates:

- v: linear velocity (in m/s)
- r: radius of the circular path (in m)
- ω: angular velocity (in rad/s)

Derivation:

If an object moves along an arc of length s on a circle of radius r, the angular displacement is:

$$\theta = \frac{s}{r}$$

From which:

$$s = r\theta$$

The linear velocity (v) is displacement divided by time:

$$v = \frac{s}{t} = \frac{r\theta}{t}$$

But $\frac{\theta}{t} = \omega$

Hence:

$$\mathbf{v = r\omega}$$

This relationship reveals a profound truth: *all points on a rotating rigid object have the same angular velocity ω, but points farther from the centre (with greater value of r) have greater linear speed v.* A point on the outer edge of a spinning disk moves faster (covers more distance per second) than a point near the center, but both rotate through the same angle per second.

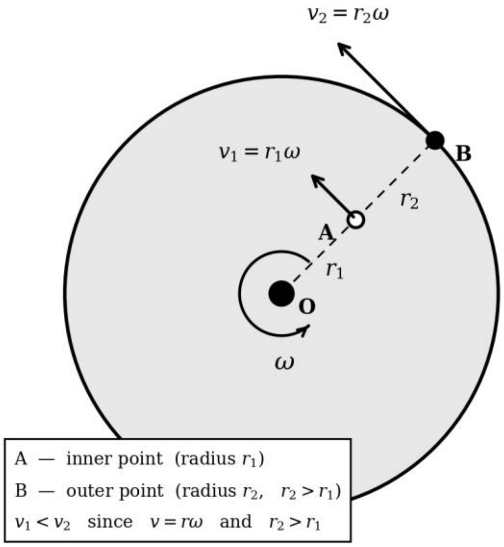

Figure 7.2: *Two points A and B on a rotating rigid disk. Both rotate with the same angular velocity ω, but since v = rω, the outer point B (radius r_2) moves with greater linear velocity than the inner point A (radius r_1). The length of each velocity arrow reflects this difference.*

Period and Frequency

For objects in uniform circular motion that complete regular revolutions, we can describe their motion using period and frequency.

Period (T) *is the time required for one complete revolution.* It is measured in seconds.

$$T = \frac{\text{Total time taken, t}}{\text{Number of revolutions, n}}$$

Frequency (f) *is the number of complete revolutions per unit time.* It is measured in hertz (Hz), where 1 Hz = 1 revolution per second = 1 s^{-1}.

$$f = \frac{\text{Number of revolutions, n}}{\text{Total time taken, t}}$$

Period and frequency are reciprocals of each other:

$$\mathbf{T} = \frac{1}{f} \text{ and } \mathbf{f} = \frac{1}{T}$$

Relating angular velocity to period and frequency

In one complete revolution, an object rotates through 2π radians. This takes time T (the period), and the angular velocity is:

$$\boldsymbol{\omega} = \frac{2\pi}{T}$$

Alternatively, since $f = \frac{1}{T}$:

$$\boldsymbol{\omega} = 2\pi f$$

These relationships allow us to convert easily between angular velocity, period, and frequency:

- *Know the period?* Calculate $\omega = \frac{2\pi}{T}$
- *Know the frequency?* Calculate $\omega = 2\pi f$
- *Know the angular velocity?* Calculate $T = \frac{2\pi}{\omega}$ or $f = \frac{\omega}{2\pi}$

Combined with $v = r\omega$, we can now express linear velocity in terms of period or frequency:

$$\mathbf{v} = \omega r = r\left(\frac{2\pi}{T}\right) = \frac{\mathbf{2\pi r}}{\mathbf{T}}$$

Or $$\mathbf{v} = r\omega = r(2\pi f) = \mathbf{2\pi rf}$$

These formulas are powerful tools for analyzing circular motion problems.

Summary of key relationships

Let us consolidate the mathematical framework we have developed:

Arc length and angle:

$s = r\theta$ (θ in radians)

Angular velocity:

- $\omega = \frac{\Delta\theta}{\Delta t}$ (rad/s)
- $\omega = \frac{2\pi}{T}$
- $\omega = 2\pi f$

Linear and angular velocity:

$v = \omega r$

Linear velocity from period/frequency:

- $v = \frac{2\pi r}{T}$
- $v = 2\pi rf$

Period and frequency:

- $T = \frac{1}{f}$
- $f = \frac{1}{T}$

With these tools, we can solve a wide variety of circular motion problems. Let us practice applying them through worked examples.

BINDER Example 5

A wheel of radius 0.4m rotates at 5 revolutions per second. Calculate:

(a) The period of rotation. (b) The angular velocity. (c) The linear velocity of a point on the rim.

Solution

$f = 5\ \text{rev/s} = 5\text{Hz}$

(a) Period:

$$T = \frac{1}{f} = \frac{1}{5s^{-1}} = 0.2s$$

The period is 0.2s.

(b) Angular velocity:

$\omega = 2\pi f = 2\pi \,\text{rad} \times 5\text{s}^{-1} = 10\pi = 31.4 \,\text{rads}^{-1}$

The angular velocity is 31.4rad/s

(c) Linear velocity:

$v = \omega r = 0.4\text{m} \times 31.4\text{rads}^{-1} = 12.6\text{m/s}$

The linear velocity at the rim is 12.6 m/s.

Making Sense of the Answer: *The wheel spins 5 times per second, so each revolution takes* $0.2s$. *A point on the rim travels the circumference (*$2\pi r = 2.51m$*) in 0.2s, giving speed* $v = \frac{2.51}{0.2} = 12.6ms^{-1}$, *which matches our calculation.*

Think Like a Physicist: *Always check that your units work out correctly. Angular velocity must be in rad/s (not rev/s), and when you multiply ω (rad/s) by r (m), you get v (m/s). The "rad" unit effectively cancels because* ***radians are dimensionless*** *(they are a ratio of lengths).*

BINDER Example 6

A point on the equator of the Earth moves in a circle as the Earth rotates. Taking the Earth's radius as 6400km and the period of rotation as 24 hours, calculate the linear velocity of a point on the equator.

Solution

Using $v = \frac{2\pi r}{T}$;

Where: $r = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}$, $T = 24 \text{ hours} = 24 \times 3600\text{s} = 86400\text{s}$

Substituting values:

$$v = \frac{2\pi \times 6.4 \times 10^6\text{m}}{86400\text{s}} = 465\text{m/s}$$

The linear velocity at the equator is 465m/s.

Making Sense of the Answer: *This is about 1670 km/h, quite fast! Yet we do not feel this motion because we are moving with the Earth. This velocity is why launching rockets eastward from equatorial locations provides a "boost"; the rocket already has this velocity before its engines fire.*

Think Like a Physicist: *Always convert to standard SI units (metres and seconds) before calculating. Period in hours must become seconds:* $24\,h \times 3600\,s/h = 86{,}400\,s$. *Failing to convert units properly is a common source of errors in circular motion problems.*

REAL Example 7

Two children sit on a merry-go-round: Amina sits 1m from the center, while Baraka sits 2m from the centre (at the outer edge). The merry-go-round rotates at constant angular velocity.

(a) Compare the angular velocities of the two children. Explain.
(b) Compare their linear velocities.

Solution

(a) **Both children have the same angular velocity ω.** They are on the same rigid platform, so they both rotate through the same angle in the same time. When the platform completes one revolution, both Amina and Baraka complete one revolution together, even though they travel different distances.

(b) Using $v = r\omega$:
Amina ($r = 1\text{m}$): $v_A = 1\omega$
Baraka ($r = 2.0 \text{ m}$): $v_B = 2\omega$

Hence, **Baraka's linear velocity is twice Amina's linear velocity** (Baraka moves twice as fast as Amina in linear terms).

Making Sense of the Answer: *Think of a spinning disk with dots painted on it. All dots complete one rotation together (same ω), but outer dots trace larger circles and therefore move faster (larger v = rω).*

Think Like a Physicist: *When analyzing rotating rigid objects, angular quantities (ω, θ) are the same for all points, but linear quantities (v, s) increase with radius. This is why ω is often more convenient than v for describing rotation; one number describes the entire object.*

HOT Example 8

A DVD rotates in a player at 1800 revolutions per minute. The DVD has inner radius 2cm (where data begins) and outer radius 6cm (edge of data region). Take π = 3.14.

(a) Calculate the linear velocity at the inner radius.
(b) Calculate the linear velocity at the outer radius.
(c) Calculate how much faster (in percentage) the outer edge moves compared to the inner radius.
(d) Explain why DVD players must vary their rotation speed when reading from different parts of the disk to maintain constant linear read speed.

Solution

(a) $f = 1800\,\frac{\text{rev}}{\text{min}} \times \frac{1\text{min}}{60\text{s}} = 30\text{rev/s} = 30\text{Hz}$

Angular velocity:

$$\omega = 2\pi f = 2 \times 3.14 \times 30 = 188.4\ \text{rad/s}$$

Linear velocity at inner radius:

$$v_{\text{inner}} = r_{\text{inner}} \times \omega = 0.02\text{m} \times 188.4\text{rad/s} = 3.77\text{m/s}$$

The linear velocity at the inner radius is 3.77m/s.

(b) Linear velocity at outer radius:

$$v_{\text{outer}} = r_{\text{outer}} \times \omega = 0.06\text{m} \times 188.4\text{rad/s} = 11.3\text{m/s}$$

The linear velocity at the outer radius is 11.3m/s.

(c) Speed difference $= v_{\text{outer}} - v_{\text{inner}} = (11.3 - 3.77)\text{m/s} = 7.53\text{m/s}$

$$\text{Percentage increase} = \frac{\text{Speed difference}}{v_{\text{inner}}} \times 100\% = \frac{7.53}{3.77} \times 100\% = 200\%$$

The outer edge moves **200% faster** than the inner radius.

(d) If the angular velocity ω were constant, the linear speed $v = r\omega$ would vary dramatically from inner to outer radius. To maintain constant linear read speed:

- When reading inner tracks (small r), the player must spin the disk faster (larger ω) to achieve $v = r\omega =$ constant.
- When reading outer tracks (large r), the player must spin slower (smaller ω).

Therefore, since $v = r\omega$ must stay constant, and r increases as we read outward, ω must decrease proportionally: $\omega \propto \frac{1}{r}$ for constant v.

Making Sense of the Answer: *Old vinyl records played at constant angular velocity, which meant outer grooves moved faster past the needle than inner grooves. This limited sound quality. CDs and DVDs solved this by varying angular velocity to maintain constant linear velocity, improving performance and allowing more efficient use of disk space.*

Think Like a Physicist: *The relationship* $v = r\omega$ *creates an inverse relationship between r and ω when v must remain constant:* $v = \frac{v}{r}$*. Doubling the radius requires halving the angular velocity to maintain the same linear velocity. This principle applies anywhere constant linear speed is needed despite varying radius: car transmissions, conveyor systems, and optical disk drives all use this concept.*

These angular quantities and relationships provide powerful tools for analyzing circular motion. In the next subtopic, we will use these tools to develop formulas for centripetal acceleration and centripetal force, allowing us to calculate the forces required to maintain circular motion and understand why certain speeds or radii make circular motion impossible without sufficient inward force.