PROJECTILES FROM HEIGHT AND ON INCLINED PLANES

In the previous two subtopics, we developed a complete mathematical framework for projectile motion under the assumption that the projectile is launched and lands at the same horizontal level. While this idealized case is fundamental for understanding the basic principles, it represents only a limited range of real-world situations. In practice, projectiles are often launched and land at different heights. For example, a basketball player releases the ball about 2m above the ground toward a hoop 3m high. A stone may be thrown from the edge of a cliff toward the ocean below. A skier may launch off a slope and land farther down the same incline. In each of these situations, the projectile does not return to its original launch height, and therefore the formulas developed earlier must be modified to account for the difference in vertical levels.

This subtopic extends our analysis to two important special cases that frequently arise in practice: **projectiles launched from a height above the landing level**, and **projectiles moving on inclined planes**. In both situations, the projectile does not return to the same vertical level from which it was launched.

We shall see that the fundamental principles remain unchanged. The horizontal and vertical components of velocity still act independently, and the trajectory of the projectile remains parabolic. However, the specific parameters such as the time of flight, horizontal range, and landing velocity must be recalculated to account for the asymmetry between the launch and landing conditions.

Horizontal Projection from a Height

Consider the simplest extension of our previous analysis: a projectile launched horizontally from height **h** above the ground. This scenario occurs when you roll a ball off a table, kick a football from a cliff edge, or drop supplies from a moving aircraft, and other many familiar circumstances. Since the initial velocity is horizontal, there is no upward component. The projectile therefore begins falling immediately while simultaneously moving forward.

To analyse the motion clearly, let us first define a suitable coordinate system and specify the initial conditions.

Let the origin be at the point of projection, with the x-axis horizontal (direction of initial velocity) and the y-axis vertical (upward positive). The initial conditions are:

- $\mathbf{u_x = u}$ (horizontal initial velocity).
- $\mathbf{u_y = 0}$ (no vertical component).
- **Initial position:** $(0, 0)$.
- **Landing position:** $(R, -h)$ where R is the range and h is the launch height.

Time of flight (time to hit the ground)

The vertical displacement equation gives:

$$s_y = u_y t - \tfrac{1}{2}gt^2$$

At landing: $s_y = -h$ (negative because downward from origin) and $u_y = 0$:

$$-h = 0 - \tfrac{1}{2}gt^2$$

Making **t** the subject:

$$\mathbf{t = \sqrt{\frac{2h}{g}}}$$

This is exactly the same result we obtained in **Chapter 2**: *fall time depends only on height, not on horizontal velocity*. Whether the ball rolls off the table slowly or is strongly kicked horizontally, **the time to hit the ground** remains identical, determined solely by **h** and **g**.

Horizontal range

During time t, the projectile travels horizontally at constant velocity $u_x = u$:

$$\mathbf{R} = \mathrm{ut} = \mathbf{u} \times \sqrt{\frac{\mathbf{2h}}{\mathbf{g}}}$$

This elegant formula reveals that range is proportional to both initial velocity and the square root of height. Doubling the launch height increases range by factor √2 ≈ 1.41. Doubling the initial velocity doubles the range directly.

Landing velocity

At time $t = \sqrt{\frac{2h}{g}}$, the velocity components are:

$v_x = u$ (unchanged horizontal velocity)

$v_y = u_y - gt$

$v_y = 0 - g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$ (The negative sign means the velocity is **downward**).

The magnitude of landing velocity:

$$\mathbf{v} = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{\mathbf{u^2 + 2gh}}$$

The landing angle α satisfies:

$$\tan\alpha = \frac{v_y}{v_x}$$

However, since v_y is negative while v_x is positive, the value of $\tan\alpha$ becomes negative. This indicates that the angle α lies **below the horizontal direction**. To determine the magnitude of this angle directly, we take the absolute value of v_y. Tus:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right)$$

Where α represents the **angle below the horizontal**.

Notice that landing velocity exceeds initial velocity (v > u) because gravity has added a vertical component of velocity during the fall.

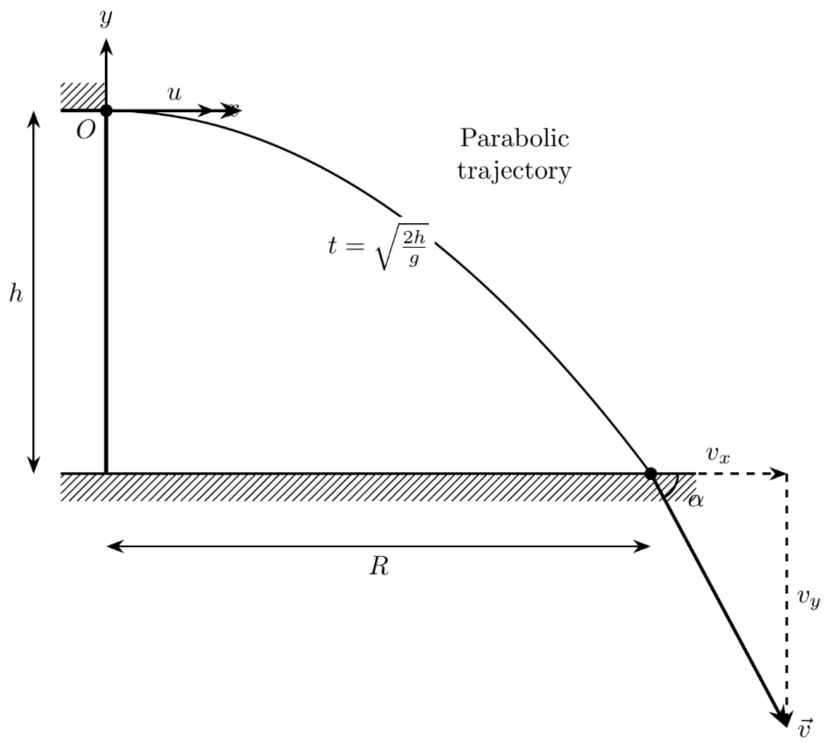

Figure: *Horizontal projection from a height h. The projectile is launched with horizontal velocity u and follows a parabolic trajectory, reaching the ground after time* $t = \sqrt{\frac{2h}{g}}$ *at horizontal range R. The landing velocity has horizontal and vertical components* v_x *and* v_y.

Projection at an Angle from a Height

The more general case occurs when a projectile is launched at angle **θ** above horizontal from height **h**. This describes situations like a basketball shot, a water fountain on elevated ground, or an archer shooting from a castle wall. The analysis combines our previous work on angled projection with the modifications introduced by the height difference.

Now, let us specify the initial conditions for this case:

- $\mathbf{u_x = u\cos\theta}$ (horizontal component)
- $\mathbf{u_y = u\sin\theta}$ (vertical component, upward)
- **Initial position:** $(0, 0)$
- **Landing position:** $(R, -h)$

Time of flight

Again, this requires solving the vertical displacement equation:

$$s_y = (u\sin\theta)t - \tfrac{1}{2}gt^2$$

At landing, $s_y = -h$:

$$-h = (u\sin\theta)t - \tfrac{1}{2}gt^2$$

Rearranging to standard quadratic equation in t:

$$\tfrac{1}{2}gt^2 - (u\sin\theta)t - h = 0$$

Using the quadratic formula with: $a = \tfrac{1}{2}g, b = -(u\sin\theta), c = -h$:

$$t = \frac{\left(u\sin\theta \pm \sqrt{(u\sin\theta)^2 + 2gh}\right)}{g}$$

Since $\sqrt{(u\sin\theta)^2 + 2gh} > u\sin\theta$, the negative root make the numerator negative and therefore give a **negative value of t**. Because negative time has no physical meaning, that root must be rejected. Hence, only the **positive root** is physically meaningful, and therefore:

$$\mathbf{t} = \frac{\mathbf{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}}{\mathbf{g}}$$

Comparing with level-ground formula: When $h = 0$, this reduces to:

$$t = \frac{\left(u\sin\theta + \sqrt{(u\sin\theta)^2}\right)}{g} = \frac{(u\sin\theta + u\sin\theta)}{g} = \frac{2u\sin\theta}{g}$$

This matches our result we derived earlier, confirming that our new formula generalizes the previous one.

Notice the following important physical insight: The term $\sqrt{(u\sin\theta)^2 + 2gh}$ is always greater than $u\sin\theta$ when $h > 0$. So time of flight from a height is always longer than for the same launch on level ground. The projectile gets "bonus time" in the air from the additional height.

Range

Substituting the time of flight into $R = (u\cos\theta)t$:

$$\mathbf{R = u\cos\theta\left(\frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g}\right)}$$

This formula can be simplified but is often left in this form for calculation purposes.

Maximum height above launch point

The projectile reaches maximum height when $v_y = 0$:

$$v_y = u\sin\theta - gt_{max} = 0$$

$$t_{max} = \frac{u\sin\theta}{g}$$

The height above launch point is:

$$\mathbf{H} = (u\sin\theta)t_{max} - \tfrac{1}{2}gt_{max}^2 = \frac{\mathbf{u^2\sin^2\theta}}{\mathbf{2g}}$$

This is the same as the formula we derived earlier. *The maximum height above the launch point depends only on the vertical component of initial velocity, regardless of the absolute elevation of the launch point.*

Keep in mind that the maximum height above the ground is $H + h$ (the height above launch plus the launch elevation).

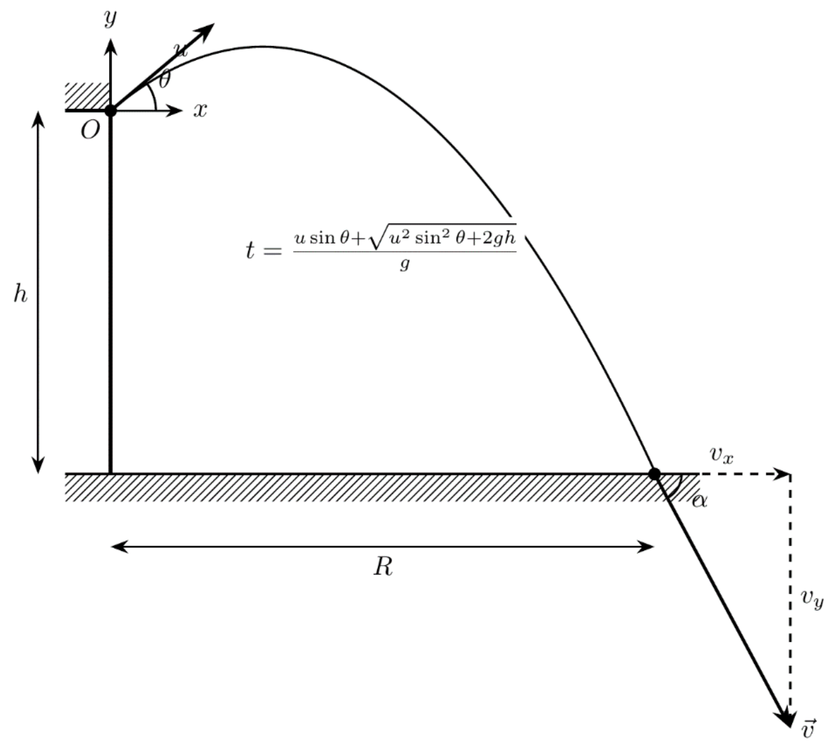

Figure: *Projectile launched from a height h above the ground with initial velocity u at angle θ. The projectile follows a parabolic trajectory and lands at horizontal range R. The landing velocity has horizontal and vertical components v_x and v_y, making an angle α below the horizontal.*

For now, give theory a short holiday and let us sharpen our understanding with some useful worked examples.

BINDER Example 12

A ball is rolled off a table 1.2m high with horizontal velocity 3m/s. Taking g = 9.8 m/s²; Calculate:

(a) the time taken for the ball to reach the floor,
(b) the horizontal distance from the table edge where the ball lands
(c) the velocity of the ball just before it hits the floor.

Solution

(a) Time for the ball to reach the floor is given by:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1.2\text{m}}{9.8\text{m/s}^2}} = 0.495\text{s}$$

The time is 0.495s.

(b) Horizontal distance is given by:

$$R = ut = 3\text{m/s} \times 0.495\text{s} = 1.49\text{m}$$

The horizontal distance is 1.49m.

(c) Velocity components at landing:

$v_x = u = 3m/s$ (unchanged)

$v_y = \sqrt{2gh} = \sqrt{2 \times 9.8m/s^2 \times 1.2m} = 4.85m/s$ (downward)

Magnitude: $v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(3m/s)^2 + (4.85m/s)^2} = 5.7m/s$

Angle below horizontal:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{4.85}{3}\right) = 58.3°$$

The velocity is $5.7m/s$ at $58.3°$ below horizontal.

Making Sense of the Answer: *The ball takes half a second to fall 1.2 m, reasonable for free fall. The landing velocity (5.7m/s) exceeds initial velocity (3m/s) because gravity added vertical component of velocity.*

Think Like a Physicist: *In projectile motion, the horizontal and vertical components of motion are independent. Gravity affects only the vertical motion, which is why the fall time depends solely on the height (and of course the vertical component of velocity).*

BINDER Example 13

A stone is thrown horizontally from the top of a cliff 45 m high with velocity 15m/s. Taking the value of g = 9.8 m/s^2; calculate:

(a) how long the stone is in the air,
(b) how far from the base of the cliff the stone lands,
(c) the angle at which the stone strikes the ground.

Solution

(a) Time of flight:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 45m}{9.8m/s^2}} = 3.03s$$

The stone stays in the air for 3.03s.

(b) Range:

$R = ut = 15m/s \times 3.03s = 45.45m$

The stone lands 45.5m from the base of the cliff.

(c) Landing velocity components:

$v_x = 15m/s$

$v_y = \sqrt{2gh} = \sqrt{2 \times 9.8m/s^2 \times 45.45} = \sqrt{882} = 29.85m/s$

Landing angle:

$$\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{29.85}{15}\right) = 63.4°$$

The stone strikes at 63.4° below horizontal.

Making Sense of the Answer: *The 3s fall time and 45.45m range are both substantial, reflecting the considerable cliff height. The steep landing angle (63°) shows the stone is falling quite rapidly by the time it reaches the ground (the vertical velocity (29.7m/s) is nearly double the horizontal velocity (15m/s).*

Think Like a Physicist: *Compare this to dropping the stone straight down (u = 0). The fall time would be identical (3.03s), but the landing would occur at the cliff base (R = 0) instead of 45.45m away. The horizontal motion is "free"; it adds range without costing any flight time. This principle explains why supply drops from aircraft are so effective: the aircraft's forward velocity gives range without reducing the time available for parachutes to deploy.*

BINDER Example 14

A projectile is launched at 25m/s at 37° above horizontal from a platform 5m above ground level. Taking g = 9.8 m/s², calculate:

(a) the time of flight,
(b) the range,
(c) the maximum height above the ground.

Solution

(a) Time of flight from height is given by:

$$t = \frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g} = \frac{25\sin37° + \sqrt{(25\sin37°)^2 + 2 \times 9.8 \times 5}}{9.8} = 3.37\text{s}$$

The time of flight is 3.37s.

(b) Range is given by:

$$R = (u\cos\theta)t = 25\cos37° \times 3.37 = 67.3\text{m}$$

The range is 67.3m.

(c) Maximum height above the ground

$$= H + h = \frac{u^2\sin^2\theta}{2g} + h = \frac{(25\sin37°)^2}{2 \times 9.8} + 5 = 16.55\text{m}$$

The height is 16.55m.

Making Sense of the Answer: *The time of flight (3.37s) is longer than it would be for level ground launch (T = 3.06s) because the 5m platform height provides extra airtime. The range (67.3m) reflects this longer flight time combined with the horizontal velocity of about 20 m/s.*

Think Like a Physicist: *The maximum height above the launch point (11.55m) depends only on the vertical velocity component, exactly as in level-ground projection. The 5m platform height simply adds to this, giving 16.55m total above ground. This clear separation of effects shows that the vertical motion is independent of the initial elevation.*

REAL Example 15

A construction worker accidentally kicks a hammer off the roof of a building. At the same moment, another worker standing on the roof simply drops a similar hammer without giving it any horizontal push. Explain why both hammers reach the ground at the same time, even though one travels forward while the other falls straight down.

Solution

Both hammers reach the ground at the same time because the time taken to fall depends only on the vertical motion. This is due to the fact that in both cases, the vertical motion begins with zero vertical velocity and the only force acting vertically is gravity. Therefore, each hammer falls with the same acceleration and from the same height leading to equal time of fall. The horizontal velocity only changes the landing position, not the landing time.

Making Sense of the Answer: *The kicked hammer moves forward while falling, but gravity acts only vertically. Therefore, both hammers fall for the same duration, even though one lands farther from the building.*

Think Like a Physicist: *Projectile motion can be separated into independent horizontal and vertical motions. Horizontal motion determines **where** the object lands, while vertical motion determines **when** it reaches the ground.*

REAL Example 16

During a rescue operation, supplies are dropped from a moving helicopter flying horizontally over a disaster area. Explain why the supplies land some distance ahead of the point directly below the helicopter, even though they are simply released and not thrown forward.

Solution

When the supplies are released from the helicopter, they already possess the same horizontal velocity as the helicopter at that instant. After release, there is no horizontal force acting on the supplies (neglecting air resistance), so they continue to move forward with this horizontal velocity while falling under gravity. Consequently, they land some distance ahead of the point directly below the helicopter at the moment of release.

Making Sense of the Answer: *Although the supplies are simply dropped, they already have the same forward velocity as the helicopter. During the fall they keep moving forward while gravity pulls them downward.*

Think Like a Physicist: *An object released from a moving body retains the horizontal velocity it had at the moment of release. Without a horizontal force to change it, that velocity remains constant while gravity controls the vertical motion.*

HOT Example 17

A basketball player shoots from 2m above the ground, releasing the ball at 7.5m/s at 50° above horizontal. The basket is 3.05 m above the ground and 5m away horizontally. Taking $g = 9.8\ m/s^2$, sin 50° = 0.766, cos 50° = 0.643:

(a) Calculate the time for the ball to reach the basket's horizontal position.
(b) Determine whether the ball passes through the basket or misses (too high or too low).
(c) Calculate the velocity of the ball when it reaches the basket location.
(d) If the shot misses the basket, determine by how much it misses and, without performing further calculations, suggest possible corrections.

Solution

(a) The horizontal displacement after any time, t is given by: $x = (x\cos\theta)t$

Thus, the time to reach horizontal distance of 5m:

$$t = \frac{x}{u\cos\theta} = \frac{5m}{7.5m/s \times 0.643} = 1.04s$$

The time is 1.04s.

(b) Height of ball at $t = 1.04s$ (measured from release point at 2m):

$$s_y = (u\sin\theta)t - ½gt^2$$

$$s_y = (7.5m/s \times 0.766)(1.04s) - ½(9.8m/s^2)(1.04s)^2 = 0.67m$$

The vertical displacement is 0.67m above release point.

Height above ground $= s_y + h_o = 0.67m + 2m = 2.67m$

The ball vertical position above the ground $= 2.67m < 3.05m$ (basket position above the ground)

Hence, the ball **misses too low** (passes under the basket).

(c) Velocity at $t = 1.04s$:

$$v_x = u\cos\theta = 7.5m/s \times 0.643 = 4.82m/s$$

$v_y = u\sin\theta - gt = 7.5\text{m/s} \times 0.766 - 9.8\text{m/s}^2 \times 1.04 = -4.45\text{ m/s}$ (downward)

Magnitude: $v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(4.82\text{m/s})^2 + (-4.45\text{m/s})^2} = 6.56\text{m/s}$

Direction: $\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{4.45}{4.82}\right) = 42.7°$ below horizontal.

The velocity is 6.56m/s at $42.7°$ below horizontal.

(d) But basket height is 3.05m.

Difference $= 3.05\text{m} - 2.67\text{m} = 0.38\text{m}$

The shot misses by $\mathbf{0.38m}$ **low**.

Possible corrections:

1. Increasing angle of projection.
2. Increasing velocity of projection (initial velocity).
3. Launching the ball from higher position (increasing arm extension or jumping higher)

Making Sense of the Answer: The ball reaches correct horizontal position but is 38cm too low; a clear miss! By the time it arrives (1.04 s), it is already descending (v_y negative), having peaked earlier.

Think Like a Physicist: *Basketball shooting is applied projectile motion. The 2-metre release height helps by reducing vertical distance to the 3.05m basket (only 1.05m to climb), making the trajectory less extreme. Professional players develop intuition through thousands of shots, but physics is unforgiving: wrong velocity or angle means a miss, regardless of skill.*

HOT Example 18

A stone is thrown from the top of a cliff at 15m/s at 30° below the horizontal. The cliff is 40m high. Taking g = 9.8 m/s², calculate:

(a) Calculate the time taken for the stone to reach the ground.
(b) Calculate the horizontal distance from the cliff base where the stone lands.
(c) Calculate the magnitude and direction of stone's velocity when it hits the ground.

Solution

Since the angle of projection is **below the horizontal**, the ball is thrown **downward**. Therefore, in the calculations the angle is taken as **negative**, according to the usual sign convention.

Components of initial velocity:

$u_x = u\cos(-30°) = 15\text{m/s} \times \cos(-30°) = 13\text{m/s}$

$u_y = u\sin(-30°) = 15\text{m/s} \times \sin(-30°) = -7.5\text{m/s}$ (downward)

(a) Using vertical displacement equation:

$s_y = u_y t - ½gt^2$

At $s_y = -40\text{m}$ (downward displacement is taken as negative)

$-40 = -7.5t - ½(9.8)t^2$

$4.9t^2 + 7.5t - 40 = 0$

Solving the quadratic equation by using calculator gives the practical (positive) value of $t = 2.19\text{s}$

The time taken is 2.19s.

Alternative solution

By using formula (where $u\sin\theta = -7.5\text{m/s}$):

$$t = \frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g} = \frac{-7.5 + \sqrt{(-7.5)^2 + 2 \times 9.8 \times 40}}{9.8} = 2.19\text{s}$$

The time is 2.19s.

(b) The horizontal distance from the cliff base:

$R = u_x t = 13\text{m/s} \times 2.19\text{s} = 28.47\text{m}$

The horizontal distance is 28.47m.

(c) Final velocity components:

$v_x = u_x = 13\text{m/s}$

$v_y = u_y - gt = -7.5\text{m/s} - 9.8\text{m/s}^2(2.19\text{s}) = -29\ \text{m/s}$

Magnitude: $v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(13\text{m/s})^2 + (-29\ \text{m/s})^2} = \mathbf{31.8m/s}$

Direction: $\alpha = \tan^{-1}\left(\frac{|v_y|}{v_x}\right) = \tan^{-1}\left(\frac{29}{13}\right) = \mathbf{65.9°}$ **below horizontal.**

Making Sense of the Answer: *Starting with downward velocity (−7.5m/s) and falling for 2.19s adds significant vertical velocity, reaching −29m/s vertically. The steep landing angle (66°) shows the stone is falling rapidly, with vertical velocity more than double the horizontal velocity.*

Think Like a Physicist: *Throwing downward at 30° gives the stone an initial downward "head start" of 7.5m/s, reducing fall time compared to throwing horizontally (which would be about 2.86s). This trade-off between faster impact versus more horizontal distance, appears in many applications, from cliff diving to aerial delivery of supplies.*

Projectiles on Inclined Planes

An entirely different special case arises when a projectile is launched from and lands on an inclined plane. This situation occurs in many practical contexts, for example: a skier launching from a jump on a slope, a ball thrown uphill along a mountain path, or supplies released onto an inclined conveyor belt. In such cases, the landing point does not lie on a horizontal surface. Instead, the projectile returns to the **sloping surface**, which introduces a natural asymmetry. Depending on the launch direction, the projectile may land "below" its launch point (**downhill projection**) or "above" it (**uphill projection**), measured along the incline.

To analyse this situation, consider an inclined plane that makes an angle $\boldsymbol{\beta}$ with the horizontal. A projectile is launched from a point on the plane with initial velocity **u** at an angle $\boldsymbol{\theta}$ measured from the horizontal (not from the incline).

Two coordinate systems may be used for this analysis:

1) The horizontal and vertical axes (the standard x–y system).
2) Axes parallel and perpendicular to the incline.

Although both approaches are valid, we use standard x-y, which connects directly to our previous work and makes the mathematics more straightforward.

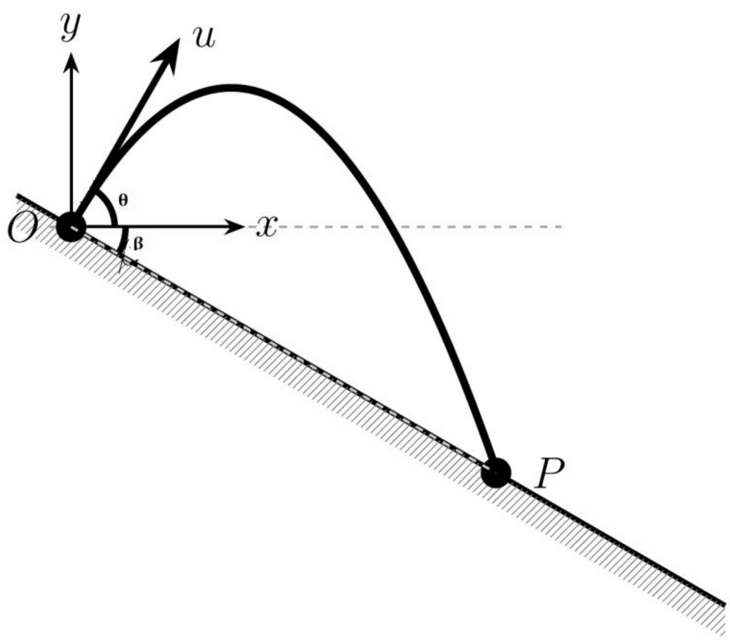

Figure: ***Downhill projection on an inclined plane.*** *A projectile is launched with initial velocity u at an angle θ above the horizontal from a point on an incline that makes an angle β below the horizontal. The projectile follows a parabolic path and lands at point P on the slope.*

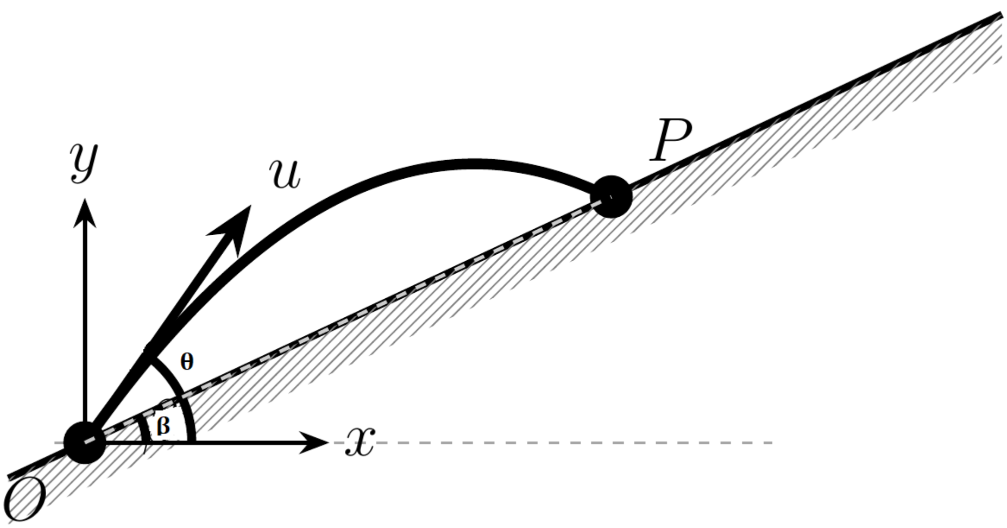

Figure: ***Uphill projection on an inclined plane.*** *A projectile is launched with initial velocity u at an angle θ above the horizontal from a point on an incline that makes an angle β above the horizontal. The projectile follows a parabolic path and lands at point P on the slope.*

From diagrams:

- θ is the **angle of projection** (between u and the horizontal axis).
- β is the **angle of the incline** relative to the horizontal.
- OP is the **range along the incline** (the distance measured along the sloping surface from the point of projection O to the landing point P). This distance will be denoted by **L**.

Important: Even though the projectile is launched from an inclined plane, the angle of projection is still defined relative to the horizontal, not relative to the slope. If the angle θ were measured from the incline itself, the corresponding angle of projection with respect to the horizontal would be $\theta - \beta$ for downhill projection and $\theta + \beta$ for uphill projection.

Range along the incline, L

The derivation and resulting formula are the same for both **uphill** and **downhill** projections. The only difference arises when applying the formula: for **uphill projection**, the angle β is taken as **positive** (above the horizontal), whereas for **downhill projection**, β is taken as **negative** (below the horizontal). Therefore, for the purpose of this derivation, we shall consider the **uphill case.**

Writing the incline equation

From the diagram:

$$\tan\beta = \frac{y}{x}$$

From which, the equation of the inclined plane is:

$$y = x\tan\beta \ldots \text{(i)}$$

(It is equivalent to general equation of a straight line through the origin, $y = mx$, where m is the slope. Here the slope of the incline is $\tan\beta$).

Comparing with trajectory equation:

$$y = x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2 \ldots \text{(ii)}$$

The projectile lands where it intersects the incline. At this point, the coordinates (x, y) must satisfy **both** equations simultaneously.

Thus, by equating (i) and (ii):

$$x\tan\beta = x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$$

Dividing by x throughout (if $x \neq 0$)

$$\tan\beta = \tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)x$$

Making x the subject:

$$\mathbf{x} = \left(\frac{2u^2\cos^2\theta}{g}\right)(\tan\theta - \tan\beta)$$

Simplifying ($\tan\theta - \tan\beta$)

$$\tan\theta - \tan\beta = \frac{\sin\theta}{\cos\theta} - \frac{\sin\beta}{\cos\beta} = \frac{\sin\theta\cos\beta - \cos\theta\sin\beta}{\cos\theta\cos\beta}$$

Using the sine difference formula: $\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\sin\theta\cos\beta - \sin\beta\cos\theta = \sin(\theta - \beta)$

Therefore:

$$\mathbf{\tan\theta - \tan\beta = \frac{\sin(\theta - \beta)}{\cos\theta\cos\beta}}$$

Substituting back into expression for x

$$\mathbf{x} = \left(\frac{2u^2\cos^2\theta}{g}\right)\left(\frac{\sin(\theta - \beta)}{\cos\theta\cos\beta}\right) = \mathbf{\frac{2u^2\cos\theta\sin(\theta - \beta)}{g\cos\beta}}$$

This is the **horizontal distance to the landing point**.

Finding the range L along the incline

Also from the diagram:

$\cos\beta = \frac{x}{OP} = \frac{x}{L}$ or $\mathbf{x} = L\cos\beta$

Equating:

$$L\cos\beta = \frac{2u^2\cos\theta\sin(\theta - \beta)}{g\cos\beta}$$

$$\mathbf{L = \frac{2u^2\cos\theta\sin(\theta - \beta)}{g\cos^2\beta}}$$

This is the **range along the inclined plane.**

Simplifying the formula

$2\cos\theta\sin(\theta - \beta)$ can be simplified by using the product-to-sum formula:

$2\cos A\sin B = \sin(A + B) - \sin(A - B)$

With $A = \theta$ and $B = \theta - \beta$:

$$\mathbf{2\cos\theta\sin(\theta - \beta)} = \sin(\theta + (\theta - \beta)) - \sin(\theta - (\theta - \beta)) = \mathbf{\sin(2\theta - \beta) - \sin\beta}$$

Substituting into our expression for L:

$$L = \frac{u^2(\sin(2\theta - \beta) - \sin\beta)}{g\cos^2\beta}$$

Hence, the simplified formula for the **range along the inclined plane** is:

$$\mathbf{L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)}$$

This compact result is remarkably general. It applies equally well to both **uphill** and **downhill** projections; the only difference lies in the sign of β. For an uphill incline β is positive, while for a downhill incline β is taken as negative as mentioned earlier.

A useful check on the correctness of this formula is obtained by considering the special case where the plane becomes horizontal. If $\beta = 0$, the incline disappears and the situation reduces to ordinary ground-level projection. Substituting $\beta = 0$ gives:

$$\mathbf{L} = \frac{u^2}{g\cos^2 0}(\sin(2\theta - 0) - \sin 0) = \frac{\mathbf{u^2 \sin 2\theta}}{\mathbf{g}},$$

which is exactly the familiar **range formula for level ground**. This confirmation shows that the inclined-plane result is simply a natural extension of the standard projectile theory developed earlier.

Having obtained the expression for the range, we can now ask an important question: ***for what angle of projection will the range along the incline be greatest?*** Since the θ containing term sin(2θ−β) determines the variable part of the expression (β is constant for given incline, so other factors with β are constant, they do not affect the maximum), the range is maximized when:

$$\sin(2\theta - \beta) = 1$$

This occurs when:

$$2\theta - \beta = 90°$$

Solving for θ gives:

$$\boldsymbol{\theta} = \frac{90° + \beta}{2} = \mathbf{45°} + \frac{\boldsymbol{\beta}}{\mathbf{2}}$$

Hence, the **optimal angle** for maximum range along the incline is given by:

$$\boldsymbol{\theta}_{\mathbf{optimal}} = \mathbf{45°} + \frac{\boldsymbol{\beta}}{\mathbf{2}}$$

This result also connects beautifully with the familiar case of projection on level ground. When the incline disappears ($\beta = 0$), the expression reduces immediately to $\theta = 45°$, the well-known angle that produces **maximum horizontal range** on a flat surface.

Time of flight

We can derive this by using the vertical displacement equation.

The vertical position at any time t is given by: $y = (u\sin\theta)t - ½gt^2$

At landing (time $t = T$), the projectile is on the incline at horizontal position x: $y = x\tan\beta$

And also: $x = (u\cos\theta)T$

So: $y = (u\cos\theta)T \times \tan\beta$

The vertical displacement equation after time $t = T$ (at landing) is:

$$y = (u\sin\theta)T - ½gT^2$$

Where: $y = (u\cos\theta)T \times \tan\beta$

Substituting:

$$(u\cos\theta)T \times \tan\beta = (u\sin\theta)T - ½gT^2$$

Divide by T (if $T \neq 0$):

$$(u\cos\theta)\tan\beta = u\sin\theta - ½gT$$

$$½gT = u\sin\theta - (u\cos\theta)\tan\beta = u\sin\theta - (u\cos\theta)\frac{\sin\beta}{\cos\beta} = u\sin\theta - \frac{u\cos\theta\sin\beta}{\cos\beta}$$

$$½gT = \frac{u(\sin\theta\cos\beta - \cos\theta\sin\beta)}{\cos\beta} = \frac{u(\sin(\theta - \beta))}{\cos\beta}$$

Hence, by making T the subject:

$$\mathbf{T = \frac{2u(\sin(\theta - \beta))}{g\cos\beta}}$$

After a rather lengthy chain of algebra and trigonometric manipulation, it is easy for the main physical ideas to become lost in the mathematics. Before those ideas begin to drift away, let us pause for a moment and anchor them with a few carefully chosen worked examples. These examples will translate the formulas we have just derived into concrete situations, showing how the theory predicts the motion of projectiles on inclined surfaces and revealing the practical meaning behind the equations.

BINDER Example 19

A skier launches off a jump on a slope inclined at 30° below horizontal. The skier leaves the jump with velocity 20m/s at 25° above horizontal. Use $g = 9.8\ m/s^2$ to calculate:

(a) how far down the slope (measured along the slope) the skier lands,
(b) the time the skier spends in the air before landing.

Solution

Interpreting the data:

$\theta = 25°$ (above horizontal).

$\beta = -30°$ (negative for downhill).

(a) Range along the incline is given by:

$$L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)$$

Substituting values:

$$L = \frac{(20m/)^2}{9.8\ m/s^2 \times \cos^2(-30)}(\sin(2 \times 25° + 30°) - \sin(-30°)) = 80.8m$$

The skier lands at distance of 80.8m down the slope.

(b) From $x = (u\cos\theta)t$

Just before landing:

- $t = T$
- $x = L\cos\beta$

Thus:$(u\cos\theta)T = L\cos\beta$

From which:

$$T = \frac{L\cos\beta}{u\cos\theta} = \frac{80.8m \times \cos(-30°)}{20m/s \times \cos 25°} = 3.86s$$

The time the skier spends in the air is 3.86s.

Alternative solution

By using the time of flight formula for an incline projection:

$$T = \frac{2u(\sin(\theta - \beta))}{g\cos\beta}$$

Substituting values:

$$T = \frac{2 \times 20\text{m/s} \times \sin(25° + 30°)}{9.8\ \text{m/s}^2 \times \cos(-30°)} = \mathbf{3.86s}$$

Making Sense of the Answer: *The skier travels 80.8m along the slope in just 3.86s. The 25° launch angle is close to the optimal 30° for maximum range in 30° downslope, explaining the large range achieved.*

Think Like a Physicist: *The optimal angle for maximum range on a −30° inclined plane is:* $45° + \frac{-30°}{2} = 30°$. *The skier's 25° is close, that is why good range was achieved. Interestingly, experienced ski jumpers naturally learn to use angles below 45° when jumping downhill. The mathematics simply confirms what skilled athletes discover through practice.*

BINDER Example 20

A ball is thrown with velocity 18m/s up a hill inclined at 25° to the horizontal. Take $g = 9.8\ \text{m/s}^2$:

(a) Calculate the optimal angle of projection for maximum range up the slope,
(b) Calculate the maximum range achieved at this optimal angle,
(c) Compare this with the maximum range the ball would achieve on horizontal ground.

Solution

(a) The optimal angle is given by:

$$\theta_{\text{optimal}} = 45° + \frac{\beta}{2}$$

Where, $\beta = 25°$ (uphill, positive)

$$\theta_{\text{optimal}} = 45° + \frac{25°}{2} = 57.5°$$

The optimal angle is 57.5° above the horizontal.

(b) Range along the incline is given by:

$$L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)$$

For optimal angle, $(\theta = 57.5°)$, L becomes maximum.

Substituting values:

$$L_{\max} = \frac{(18\text{m/s})^2}{9.8\ \text{m/s}^2 \times \cos^2(25°)}(\sin(2 \times 57.5° - 25°) - \sin 25°) = 23.2\text{m}$$

The maximum range is 23.2m.

(c) For horizontal ground, range is given by:

$$R = \frac{u^2 \sin 2\theta}{g}$$

The optimal angle of projection for maximum range is 45°.

$$R_{\max} = \frac{(18\text{m/s})^2 \sin 90°}{9.8\ \text{m/s}^2} = 33.1\text{m} > L_{\max}(23.2\text{m})$$

The maximum range on horizontal ground is greater than that on an uphill inclined plane because, when moving uphill, the ball must overcome both **gravity** and the **rising slope**.

Making Sense of the Answer: *The optimal 57.5° angle is steeper than the familiar 45° because the ball must climb the 25° slope. The uphill range (23.2m) is therefore smaller than the range on horizontal ground (33.1m), as expected; with the same initial velocity, a projectile cannot travel farther uphill than it can on level ground.*

Think Like a Physicist: *This illustrates how changing the reference frame (level ground versus inclined plane) changes the optimal strategy. On level ground, 45° maximizes horizontal distance. On a 25° upslope, the optimal angle shifts to 57.5° to maximize distance along that slope but the actual range achieved is still less than what is possible on flat ground. Physics respects the fundamental constraint: going uphill is harder than going on level ground.*

REAL Example 21

At a ski training ground, Kipute notices that ski jumpers launching from a downward slope rarely jump at the familiar 45° angle taught in physics for maximum range on level ground. Instead, their launch angles appear noticeably smaller.

(a) Explain why the optimal launch angle on a downhill slope is less than 45°.
(b) Explain why ski jumpers can travel much farther (measured along the slope) than they could on flat ground with the same launch velocity.

Solution

(a) On a downhill slope, the ground falls away beneath the jumper, giving extra flight time. A flatter angle (less than 45°) maximizes horizontal distance while the falling ground provides the vertical component naturally. The optimal angle is $45° + \frac{\beta}{2}$; for downhill β is negative, so optimal angle < 45°.
(b) There two reasons that combine to give much greater range:

1) **Falling time:** The falling ground gives the jumper extra time in the air compared to flat ground, allowing more horizontal distance to accumulate.
2) **Measurement along slope:** Distance is measured along the slope, not horizontally. Since jumper lands at a much lower elevation, the sloped path from launch to landing is geometrically longer than just the horizontal distance.

Making Sense of the Answer: *The launch angle for downslope may appear quite shallow, but this is reasonable because the ground drops away beneath the projectile as it moves forward. This increases the time before it meets the surface again, allowing the projectile to travel a much greater distance along the slope.*

Think Like a Physicist: *This illustrates optimization in physics: what is "optimal" depends on constraints. On flat ground, 45° balances horizontal and vertical motion equally. On a downslope, the ground falling away changes the optimization; you do not need as much vertical velocity because the slope provides vertical descent. Nature finds the angle that maximizes the goal (range along slope) given the constraints (gravity + sloped landing). Engineers designing ski jumps use this physics to set ramp angles for maximum safety and distance.*

REAL Example 22

A student throws a ball with the same velocity both uphill and downhill on identical slopes (same angle magnitude, opposite directions). The ball travels much farther down the slope than up the slope.

(a) Explain why the downhill throw travels farther even though the slope angles have the same steepness.
(b) Explain why the optimal throwing angle is different for uphill versus downhill throws.

Solution

(a) The difference comes from flight time:

For downhill: The ground falls away, so the ball has more flight time before landing. The projectile gets extra time from the descending slope, and gravity helps the descent.

For uphill: The ground rises to meet the ball, cutting flight time short. The ball must fight both gravity and the rising slope.

Consequently, the same initial velocity and slope steepness, but downhill range is significantly larger than that of uphill because of dramatically different flight times.

(b) The optimal angle must balance vertical velocity against the slope direction:

For downhill: Ground falls away, so the aim is flatter angle which is **less than 45°** to maximize horizontal velocity. This is because, gravity and the falling slope together provide sufficient vertical descent time. This is justified by optimal angle formula, $\theta_{optimal} = 45° + \frac{\beta}{2}$; for downhill β is negative, so optimal angle < 45°.

For uphill: Ground rises, so the aim is steeper angle which is **greater than 45°** to give more vertical velocity to overcome both gravity and the rising slope. Again, this is justified by optimal angle formula, $\theta_{optimal} = 45° + \frac{\beta}{2}$; for uphill β is positive, so optimal angle $> 45°$.

Making Sense of the Answer: *To make sense the difference, think in this way: downhill, you are working* ***with*** *gravity and the falling ground. Uphill, you are working* ***against*** *both. It is like the difference between running downhill versus uphill at the same speed; going down is much easier and you cover more ground. The physics quantifies this intuition precisely.*

Think Like a Physicist: *This problem demonstrates broken symmetry. The setup looks symmetric (same slope angle, just opposite directions), but gravity breaks the symmetry. In physics, apparent symmetries do not always produce symmetric results; you must identify what breaks the symmetry. Here, gravity's downward direction means "downhill" and "uphill" are fundamentally different, even if the slope angles match.*

HOT Example 23

An agricultural irrigation system uses a water pump to spray water onto a sloped field. The nozzle is positioned at the bottom of the field. The field slopes upward at 15° to the horizontal. The water leaves the nozzle at 12m/s at an adjustable angle θ above horizontal. Taking g = 9.8 m/s²:

(a) Calculate the nozzle angle that maximizes the distance up the slope where water lands.
(b) Calculate the maximum distance up the slope that can be irrigated.
(c) If the field extends 25m up the slope, determine whether this irrigation system can adequately water the entire field.
(d) Without performing further calculations, suggest possible modifications to reach the full 25m if needed.

Solution

(a) The optimal angle is given by:

$$\theta_{optimal} = 45° + \frac{\beta}{2}$$

Where, $\beta = 15°$ (uphill, positive)

$$\theta_{optimal} = 45° + \frac{15°}{2} = 52.5°$$

The optimal nozzle angle is 52.5° above the horizontal.

(b) Maximum distance up the slope is found by range formula along the incline:

$$L = \frac{u^2}{g\cos^2\beta}(\sin(2\theta - \beta) - \sin\beta)$$

For optimal angle, $(\theta = 52.5°)$, L becomes maximum.

Substituting values:

$$L_{max} = \frac{(12\mathrm{m/s})^2}{9.8\ \mathrm{m/s^2} \times \cos^2(15°)}(\sin(2 \times 52.5° - 15°) - \sin 15°) = 11.7\mathrm{m}$$

The maximum distance up slope is 11.7m.

(c) Field extends 25m, but system reaches only 11.7m:

Shortfall $= 25\text{m} - 11.7\text{m} = 13.3\text{m}$

Hence, the irrigation system **cannot** adequately water the entire field. It falls short by 13.3m.

(d) Possible modifications are:

1. **Increase the water velocity** by using higher-power pressure pump.
2. **Reposition the nozzle** by moving it partway up the slope.
3. **Use multiple nozzles** placed at different positions along the slope.

Making Sense of the Answer: The 52.5° angle is steeper than 45° because of the upward slope. The 11.7m range is limited because we are throwing uphill, so the water must overcome both gravity and the 15° rising ground.

Think Like a Physicist: *This demonstrates practical engineering constraints. The physics clearly shows the system is inadequate for a 25m field because no angle adjustment can extend 12m/s water beyond 11.7m on a 15° upslope. Physics sets fundamental limits; engineering works within or around them.*

APPLICATIONS OF PROJECTILE MOTION

The parabolic paths we have studied mathematically appear throughout the natural and engineered world. From ancient battlefields to modern sports arenas, from water fountains to spacecraft trajectories, projectile motion shapes our physical reality. Here are some of the countless ways these principles manifest in real life:

1. Basketball and free throws

Professional players intuitively understand the 45° rule and how height affects range. Shot tracking technology now analyzes launch angle, velocity, and arc height in real-time, using projectile equations to predict whether a shot will score before the ball reaches the hoop. The optimal arc for a free throw is typically 52° due to the elevated hoop.

2. Artillery and military ballistics

For centuries, armies have relied on projectile motion to aim cannons and mortars. The maximum range at 45° on level ground was discovered empirically by gunners long before Newton formalized the mathematics. Modern artillery computers instantly solve the projectile equations, accounting for wind, air resistance, and terrain elevation to hit targets kilometers away.

3. Irrigation and Water Distribution

Agricultural sprinklers use projectile motion principles to distribute water evenly across fields. Engineers design nozzle angles and pressures to achieve specific ranges, often placing systems on slopes where the inclined plane formulas determine optimal configurations. The parabolic water arcs you see from fountains and garden sprinklers are pure projectile motion.

4. Long jump and athletic performance

Olympic long jumpers launch at angles between 20−25° (not 45°!) because they carry substantial horizontal velocity from their run-up. Sports scientists use projectile analysis to optimize takeoff angles for maximum distance. The same principles apply to triple jump, javelin throw, shot put, and discus, each requiring different optimal angles based on initial conditions.

5. Ski jumping and winter sports

Ski jump designers use downhill projectile formulas to create safe landing zones. Jumpers launch at angles around 30° (much less than 45°) because the downward-sloping landing hill provides the vertical component naturally. The world's longest ski jumps exceed 250 metres, with flight times over 7 seconds; all predictable using our inclined plane equations.

6. Firefighting and emergency response

Fire hoses project water in parabolic arcs to reach elevated windows and distant flames. Firefighters adjust nozzle angles to maximize horizontal range or achieve specific heights, applying projectile principles under pressure. Aerial firefighting aircraft drop water or retardant, calculating release points to account for the aircraft's velocity and the target's location below.

7. Space mission trajectories

Spacecraft launches are sophisticated projectile problems. Rockets follow parabolic paths during initial ascent (before achieving orbital velocity), and mission controllers must calculate precise launch angles and velocities. Even orbital mechanics reduces to projectile motion when analyzed in small segments (satellites in low orbit are continuously "falling" in parabolic arcs around Earth's curvature).

8. Video games and animation

Every video game with jumping, throwing, or shooting implements projectile motion equations in its physics engine. Game developers use the same formulas we have studied to create realistic ball trajectories, bullet paths, and character jumps. Angry Birds, basketball simulations, and first-person shooters all run projectile calculations thousands of times per second.

9. Goalkeeper strategies in football

When a goalkeeper kicks a football from their hands, they are solving a projectile problem: what angle gives maximum distance downfield? Professional keepers typically launch at $30-40°$, balancing range against "hang time" that allows teammates to reach the landing zone. Goalkeepers intuitively account for wind and other factors, applying projectile principles.

10. Cliff diving and extreme sports

Professional cliff divers leap from heights exceeding 25 metres, with horizontal launch velocities that determine how far from the cliff they land. Divers must account for projection from height to ensure they clear dangerous rocks below and land safely in deep water. A miscalculation of just one degree can mean the difference between safety and catastrophe.

11. Fireworks displays

Pyrotechnic shells launched from mortars trace parabolic paths skyward before exploding at their maximum height. Display designers calculate launch angles and velocities to create patterns at specific altitudes and locations. Large displays coordinate hundreds of projectiles, each following calculated trajectories to burst at precisely timed moments.

12. Volcanic eruptions and natural phenomena

Nature's most spectacular demonstrations of projectile motion occur during volcanic eruptions, where molten rock follows parabolic trajectories reaching hundreds of metres. Geologists use these paths to estimate eruption velocities and predict hazard zones. Even simpler phenomena like water spraying from a broken pipe, obeys the same mathematical laws we have mastered.

We have explored the theory, derived the formulas, and seen how projectile motion shapes the world around us. Now it is time to put these principles to work through a diverse collection of problems that will test your understanding and sharpen your problem-solving skills. Ready? Let us tackle some projectiles in miscellaneous worked examples!