KEY PARAMETERS OF THE TRAJECTORY

Having developed the fundamental equations governing projectile motion in the previous subtopic, we now turn our attention to deriving expressions for specific quantities that characterize the trajectory. When Kipanga asks "How far will my shot put travel?" or Kipute wonders "How high will the orange I toss reach at its peak?", they are asking about what we call the parameters of projectile motion. These parameters: time of flight, maximum height, and horizontal range, provide complete practical information about any projectile's trajectory. In this subtopic, we shall derive mathematical expressions for each parameter and discover some surprising relationships between them.

Time of flight, T

The **time of flight** *is the total time interval from the moment a projectile is launched until the moment it returns to the same vertical level from which it was projected.* This parameter is crucial because it determines how long the projectile remains in the air, which in turn affects how far it can travel horizontally.

Consider a projectile launched from ground level with initial velocity u at angle θ above the horizontal. We established in the previous subtopic that the vertical displacement at time t is given by:

$$s_y = (u\sin\theta)t - \tfrac{1}{2}gt^2$$

The projectile returns to ground level when $s_y = 0$. This occurs at two instants: $t = 0$ (the moment of launch) and at some later time T (the time of flight). Setting $s_y = 0$:

$$0 = (u\sin\theta)t - \tfrac{1}{2}gt^2$$

Factorising the expression:

$$0 = t(u\sin\theta - \tfrac{1}{2}gt)$$

This equation has two solutions:

$$t = 0 \text{ and } u\sin\theta - \tfrac{1}{2}gt = 0.$$

For the second solution, $t = T$, thus:

$$u\sin\theta = \tfrac{1}{2}gT$$

Solving for T:

$$\mathbf{T = \frac{2u\sin\theta}{g}}$$

This is the time of flight for a projectile launched and landing at the same level.

Notice the physical insight embedded in this formula: time of flight depends only on the vertical component of initial velocity (usinθ) and gravity. The horizontal component plays no role whatsoever in determining how long the projectile stays in the air. This makes perfect sense from our principle of independence: *the time spent in the air is determined entirely by vertical motion, exactly as if we had thrown the object straight upward with initial velocity usinθ, as we studied in* ***Chapter 2***.

Connecting to Chapter 2: *Do you remember when we derived the time for an object thrown vertically upward to return to its starting point?* We found $t = \frac{2u}{g}$, where u was the upward initial velocity. Here, the vertical component usinθ plays exactly that role, giving us $T = \frac{2(u\sin\theta)}{g}$ The physics is identical; we have simply replaced u with usinθ to account for the angled launch.

Alternative derivation:

We can verify this result using another approach. The projectile reaches maximum height when $v_y = 0$. From the previous subtopic, we know that $v_y = u\sin\theta - gt$. Setting this to zero:

$$0 = u\sin\theta - gt_{max}$$

$$t_{max} = \frac{u\sin\theta}{g}$$

This is the time to reach maximum height (the upward journey). By symmetry, the downward journey takes equal time. Therefore, total time of flight is:

$$T = 2t_{max} = \frac{2(u\sin\theta)}{g}$$

This confirms our result and demonstrates the symmetric nature of projectile motion when launch and landing occur at the same level.

Maximum height, H

The **maximum height** *is the greatest vertical displacement achieved by the projectile above its launch point.* This occurs at the instant when the vertical component of velocity becomes zero, when the projectile stops rising and is about to begin falling.

We can derive the maximum height using the third equation of motion from Chapter 2:

$$v^2 = u^2 + 2as.$$

Applying this to vertical motion:

$$v_y^2 = u_y^2 + 2a_y s_y$$

At maximum height:

$$v_y = 0, u_y = u\sin\theta, a_y = -g, \text{and } s_y = H.$$

Substituting:

$$0 = (u\sin\theta)^2 - 2gH$$

Solving for H:

$$\mathbf{H = \frac{u^2\sin^2\theta}{2g}}$$

This elegant formula reveals that maximum height depends on the square of the vertical component of initial velocity. Doubling the vertical component quadruples the maximum height.

Alternative derivation:

We can also derive this using the time to maximum height and the displacement equation.

We found $t_{max} = \frac{u\sin\theta}{g}$.

Substituting into $s_y = (u\sin\theta)t - ½gt^2$:

$$H = u\sin\theta\left(\frac{u\sin\theta}{g}\right) - ½g\left(\frac{u\sin\theta}{g}\right)^2$$

$$H = \frac{u^2\sin^2\theta}{g} - ½\frac{g(u^2\sin^2\theta)}{g^2}$$

$$H = \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g}$$

$$\mathbf{H = \frac{u^2\sin^2\theta}{2g}}$$

Both methods yield the same result, providing mathematical confirmation of our formula.

An important observation: If we launch the projectile vertically upward ($\theta = 90°$), then $\sin^2 90° = 1$ (*at this angle, the maximum height becomes greatest for a given initial velocity*), and we get $H_{max} = \frac{u^2}{2g}$, which is exactly the formula we derived in Chapter 2 for maximum height of vertical projection. This serves as another verification that our projectile motion analysis correctly reduces to simpler cases.

Horizontal range, R

The **horizontal range** *is the total horizontal distance travelled by the projectile from launch point to landing point (when both are at the same level).* This is perhaps the most practically important parameter, as it determines how far the projectile will travel.

From the previous subtopic, horizontal displacement is given by:

$$s_x = (u\cos\theta)t$$

At landing ($t = T$), the horizontal displacement is equal to the range R:

$$R = (u\cos\theta)T$$

We already derived $T = \frac{2u\sin\theta}{g}$.

Substituting:

$$R = u\cos\theta\left(\frac{2u\sin\theta}{g}\right)$$

From which:

$$R = \frac{2u^2\cos\theta\sin\theta}{g}$$

This can be simplified by using the trigonometric identity: $2\sin\theta\cos\theta = \sin2\theta$:

$$\mathbf{R = \frac{u^2\sin2\theta}{g}}$$

This remarkably simple formula captures everything about horizontal range. Notice that range depends on the initial velocity squared, inversely on gravity, and on the sine of twice the angle of projection.

The factor $\sin2\theta$ deserves special attention. Since the maximum value of sine function is 1 (occurring when the angle is 90°), the range is maximized when:

$$2\theta = 90°$$

Which gives:

$$\theta = 45°$$

And the range formula becomes:

$$R_{max} = \frac{u^2}{g}$$

This is the famous result: *for a given initial velocity, maximum range occurs at 45° launch angle*. This explains why Kipute's 45° throw in our introduction travelled farther than both of Kipanga's attempts; she intuitively chose the optimal angle!

Physical interpretation: *Why does 45° give maximum range?* At angles below 45°, the projectile has greater horizontal velocity but insufficient vertical velocity, so its time of flight is small. At angles above 45°, the projectile has larger time of flight but moves too slow horizontally. The 45° angle perfectly balances these competing requirements: enough vertical component to provide adequate flight time, and enough horizontal velocity to cover large horizontal distance during that time.

Moreover, the factor sin2θ has an important property. The sine function satisfies the identity:

$$\sin x = \sin(180 - x)$$

If we let $x = 2\theta$. Then:

$$\sin 2\theta = \sin(180 - 2\theta)$$

Since the horizontal range depends on sin2θ, the same range will be obtained when:

$$2\theta' = 180 - 2\theta$$

Solving for θ':

$$\boldsymbol{\theta' = 90 - \theta}$$

Hence, *a projectile launched at angle* $\boldsymbol{\theta}$ *will have the same horizontal range as one launched at an angle* $(\boldsymbol{90 - \theta})$**.** In other words: *two projectiles launched at complementary angles (angles that sum to 90°), have the same horizontal range*. For example, launches at 30° and 60° produce identical horizontal range, and the same is true for 20° and 70°.

Relationship between parameters

The three parameters we have derived: time of flight, maximum height, and range are not independent. They are intimately connected through the initial conditions u (velocity of projection) and θ (angle of projection). Let us explore some of these relationships.

From our formulas:

- $T = \frac{2u\sin\theta}{g}$
- $H = \frac{u^2\sin^2\theta}{2g}$
- $R = \frac{u^2\sin 2\theta}{g}$

We can express **maximum height in terms of time of flight:**

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{(u\sin\theta)^2}{2g}$$

But from $T = \frac{2u\sin\theta}{g}$:

$$u\sin\theta = \frac{gT}{2}$$

So:

$$H = \frac{\left(\frac{gT}{2}\right)^2}{2g} = \frac{g^2T^2}{4 \times 2g} = \frac{gT^2}{8}$$

Therefore:

$$\mathbf{H = \frac{gT^2}{8}}$$

This shows that for any projectile, maximum height is proportional to the square of time of flight. Thus, a projectile that stays in the air four times as long reaches sixteen times the height!

We can also relate **range to maximum height.**

Starting from:

$$R = \frac{u^2\sin2\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g}$$

And:

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin\theta\sin\theta}{2g}$$

Taking $\frac{R}{H}$:

$$\frac{R}{H} = \frac{2u^2\sin\theta\cos\theta}{g} \times \frac{2g}{u^2\sin\theta\sin\theta} = \frac{4\cos\theta}{\sin\theta} = 4\cot\theta$$

Hence:

$$\mathbf{R = 4H\cot\theta}$$

This relationship tells us that for a given maximum height, the range depends on the launch angle. Shallow angles (small θ) give large cotθ and thus large range for the same height. Steep angles give smaller range for the same height. It also shows that for a given angle of projection, if the maximum height increases, the range increases proportionally.

For the special case of 45° launch:

$$\cot 45° = 1$$

So:

$$\mathbf{R = 4H}$$

Thus, *at 45° launch angle, the range is exactly four times the maximum height*. This is a useful rule of thumb: ***if a projectile launched at 45° reaches height H, it will land at distance 4H from its starting point.***

After all these derivations, the mathematics has revealed the structure of projectile motion. But equations alone can feel abstract until we see them at work. Before they start arguing among themselves, let us invite a few worked examples to settle the matter.

BINDER Example 5

A cricket ball is struck at 30m/s at an angle of 60° above horizontal. Calculate:

(a) the time of flight, (b) the maximum height reached, (c) the horizontal range. Take $g = 9.8\ m/s^2$.

Solution

(a) Time of flight is given by:

$$T = \frac{2u\sin\theta}{g}$$

Substituting values:

$$T = \frac{2 \times 30\text{m/s} \times \sin 60°}{9.8\text{m/s}^2} = 5.3\text{s}$$

The time of flight is 5.3s.

(b) Maximum height is given by:

$$H = \frac{u^2 \sin^2\theta}{2g}$$

Substituting values:

$$H = \frac{(30\text{m/s})^2 \times \sin^2 60°}{2 \times 9.8\text{m/s}^2} = 34.44\text{m}$$

The maximum height is 34.44m.

(c) Horizontal range is given by:

$$R = \frac{u^2 \sin 2\theta}{g}$$

Substituting values:

$$R = \frac{(30\text{m/s})^2 \times \sin 120°}{9.8\text{m/s}^2} = 79.55\text{m}$$

The horizontal range is 79.55m.

Making Sense of the Answer: *The ball stays in the air for over 5 seconds, reaches a height of 34 metres, and travels nearly 80 metres horizontally. These are realistic values for a powerfully struck cricket ball. Notice that 60° is quite steep, so it gives impressive height but not maximum range. If the same ball were struck at 45° instead, it would travel farther* ($R_{max} = \frac{900}{9.8} = 91.8m$) *but not as high* ($H = \frac{900}{39.2} = 23m$)*.*

Think Like a Physicist: *Observe the complementary angle relationship: if we launched at 30° (the complement of 60°), we would get the same range (79.5m) but different flight time and height. We can verify: sin 2(30°) = sin 60° = 0.866, same as sin 2(60°) = sin 120° = 0.866. This symmetry is not coincidental; it is built into the trigonometry of projectile motion. In cricket, fielders must position themselves based on the likely range, which depends on both the batsman's power (determining u) and shot selection (determining θ).*

BINDER Example 6

A stone is thrown from ground level and reaches a maximum height of 20m. If the stone was thrown at 45°, calculate:

(a) the initial velocity of the stone, (b) the range of the stone, (c) the time of flight.

Take g = 9.8 m/s².

Solution

(a) From the maximum height formula:

$$H = \frac{u^2 \sin^2\theta}{2g}$$

Making u the subject:

$$u = \sqrt{\frac{2gH}{\sin^2\theta}}$$

Substituting values:

$$u = \sqrt{\frac{2 \times 9.8\ m/s^2 \times 20m}{\sin^2 45}} = 28m/s$$

The initial velocity is $28m/s$.

(b) If the angle of projection is 45°:

$$R = 4H = 4 \times 20m = 80\ m$$

The range is 80m.

(c) Using:

$$T = \frac{2u\sin\theta}{g} = \frac{2 \times 28m/s \times \sin 45°}{9.8\ m/s^2} = 4.04s$$

The time of flight is $4.04s$.

Making Sense of the Answer: *The special angle of 45° creates the elegant relationship R = 4H, making part (b) trivial once we know the height. The initial velocity of 28m/s is enough fast to reach 20 metres height. The 4-second flight time is substantial, giving plenty of time for the stone to travel the 80-metre range. This example demonstrates how knowing one parameter (height) and the launch angle allows us to determine all other parameters.*

Think Like a Physicist: *This example reverses the usual problem: instead of starting with initial conditions (u and θ) and finding the trajectory parameters, we start with a parameter (H) and work backward to find initial velocity. This "inverse problem" approach is common in real applications. For instance, if you observe a projectile reaching a certain height and range, you can deduce its initial velocity and launch angle. This is how military analysts estimate the capabilities of observed missile launches; they measure the trajectory and work backward to the launch conditions.*

BINDER Example 7

A javelin thrower achieves a throw of 60m when the javelin leaves her hand at 25m/s. Calculate:

(a) the two possible angles at which the javelin could have been launched,
(b) explain which angle is more realistic for javelin throwing. Take $g = 9.8\ m/s^2$.

Solution

(a) Javelin is thrown for maximum horizontal distance, so we are going to apply the range formula:

$$R = \frac{u^2 \sin 2\theta}{g}$$

Substituting values:

$$60m = \frac{(25m/s)^2 \times \sin 2\theta}{9.8m/s^2}$$

From which:

$$\sin 2\theta = 0.941 \text{ or } 2\theta = \sin^{-1}(0.941)$$

Thus:

$$2\theta = 70.2° \text{ or } 2\theta = 180° - 70.2° = 109.8°$$

And:

$$\theta = \frac{70.2°}{2} = 35.1° \text{ or } \frac{109.8°}{2} = 54.9°$$

two possible angles are 35.1° and 54.9°.

(b) For javelin throwing, the smaller angle (35.1°) is more realistic.

Explanation:

In real situations, air resistance reduces the horizontal range, and this effect becomes more significant for steeper launch angles because the javelin spends more time in the air. Consequently, the optimum angle for maximum range becomes less than 45° (typically around 35°–40°).

Making Sense of the Answer: *The two angles (*35.1° *and* 54.9°*) are complementary (they sum to 90°), which is exactly what we expect from the theory: complementary angles give the same range. Both are physically possible, but aerodynamics favour the shallower trajectory. Notice that neither angle is exactly 45° because the range (60m) is less than the maximum possible range:* $R_{max} = \frac{u^2}{g} = \frac{625}{9.8} = 63.8m$*. The maximum range would require 45°.*

Think Like a Physicist: *This inverse problem requires finding angle from range has two solutions because of the symmetry in the sine function:* $\sin 2\theta_1 + \sin 2\theta_2$ *when* $2\theta_1 + 2\theta_2 = 180°$*. In sports, military applications, and firefighting, this symmetry creates interesting choices. In practice, other factors (obstacles, wind, equipment limitations) often make one choice clearly superior.*

BINDER Example 8

Two identical stones are thrown from ground level with the same initial velocity of 20m/s, but at different angles: stone A at 30° and stone B at 60°. Taking g = 9.8 m/s². Calculate:

(a) the horizontal range for each stone and verify they are equal,
(b) the maximum height reached by each stone,
(c) the time of flight for each stone.

Solution

(a) Range for stone A:

$$R_A = \frac{u^2 \sin 2\theta_A}{g} = \frac{(20\text{m/s})^2 \times \sin 60°}{9.8\text{m/s}^2} = 35.3\text{m}$$

Range for stone B:

$$R_B = \frac{u^2 \sin 2\theta_B}{g} = \frac{(20\text{m/s})^2 \times \sin 120°}{9.8\text{m/s}^2} = 35.3\text{m}$$

The horizontal range for each stone is 35.3m.

(b) Maximum height for stone A:

$$H_A = \frac{u^2 \sin^2\theta_A}{2g} = \frac{(20\text{m/s})^2 \times \sin^2 30°}{2 \times 9.8\text{m/s}^2} = 5.1\text{m}$$

Maximum height for stone B:

$$H_B = \frac{u^2 \sin^2\theta_B}{2g} = \frac{(20\text{m/s})^2 \times \sin^2 60°}{2 \times 9.8\text{m/s}^2} = 15.3\text{m}$$

The maximum height of stone A is 5.1m.

The maximum height of stone B is 15.3m. (Stone B goes 3 times higher!)

(c) Time of flight for stone A:

$$T_A = \frac{2u\sin\theta_A}{g} = \frac{2 \times 20\text{m/s} \times \sin 30°}{9.8\text{m/s}^2} = 2.04\text{s}$$

Time of flight for stone B:

$$T_B = \frac{2u\sin\theta_B}{g} = \frac{2 \times 20\text{m/s} \times \sin 60°}{9.8} = 3.54\text{s}$$

The time of flight for stone A is 2.04s.

The time of flight for stone B is 3.54s.

Making Sense of the Answer: *Although both stones travel the same horizontal distance, stone B accomplishes this by staying in the air longer while moving more slowly horizontally, while stone A covers the same distance by moving faster horizontally in less time. So the complementary angle relationship creates fascinating trade-offs: same range, but completely different flight characteristics. Stone A (30°): low, fast trajectory which is good for minimizing flight time. Stone B (60°): high, slow trajectory which is good for clearing obstacles. If there were a 10-metre wall in the path, only stone B would clear it (15.3m > 10m), even though both eventually land at the same spot. This explains why artillery can hit the same target with two different firing solutions: one "flat" trajectory for quick strikes, one "high" trajectory for clearing obstacles.*

Think Like a Physicist: *This example reveals a profound principle: there are often multiple ways to achieve the same outcome in physics, each with different trade-offs. Both angles reach 35.3m, but the "how" differs dramatically. In engineering and warfare, these choices matter: flat trajectories are faster but cannot clear obstacles; high trajectories clear obstacles but take longer and are more affected by wind. Similarly, in life, different paths can lead to the same destination: some fast and risky, some slow and safe! Physics teaches us to quantify these trade-offs rather than just acknowledge them qualitatively.*

REAL Example 9

During the annual school sports day, the long jump competition takes place on a sand pit. After the competition ends, Kipanga sits dejectedly on the bench; he managed only 4.8 metres while his classmate Musa won with 6.5 metres.

Kipanga: (frustrated) *"It's not fair, sir! I ran down that track just as fast as Musa, maybe even faster! But he jumped way farther. He must have longer legs or something."*

Mr. Akilikubwa: (sitting down beside him) *"Kipanga, I watched both your jumps carefully. Your running speed looked similar to Musa's, that's true. But tell me, what happened at the moment you left the takeoff board?"*

Kipanga: *"I jumped as hard as I could, sir! I remember going quite high. I could see the whole sand pit below me for a moment. It felt like flying!"*

Kipute: (joining them) *"I noticed that too! Kipanga jumped very high. Musa's jump looked different; he barely went upward. His jump was much flatter."*

Mr. Akilikubwa: (smiling) *"Interesting observations. But neither of you used the best possible angle for your running speed. Long jump actually hides an important secret of* ***projectile motion****."*

Kipanga: (puzzled) *"Projectile motion? Like the shot put we studied?"*

Mr. Akilikubwa: *"Exactly. The moment you leave the board; your body behaves like a projectile moving through the air. Two things determine how far you travel horizontally: how long you stay in the air, and how fast you move forward while airborne."*

Kipute: *"But sir, those seem to contradict each other! To stay in the air longer, you need to jump upward more. But to move fast horizontally, you need to jump forward more. You can't do both!"*

Mr. Akilikubwa: (enthusiastically) *"Exactly, Kipute! That's the heart of it! Your running speed gives you a certain amount of initial velocity. At takeoff, you must decide how to share this velocity between going up and going forward. Jump too steep like Kipanga did, and yes, you'll be airborne for a long time, but you're crawling forward so slowly that you waste all that airtime. Jump too flat like Musa, and yes, you're rocketing forward, but you hit the sand so quickly that you don't have time to travel far."*

Kipanga: *"So... there's a perfect angle in the middle somewhere?"*

Mr. Akilikubwa: *"Now you're thinking like a physicist! In ideal projectile motion, when the launch point and landing point are at the same height and air resistance is neglected, the mathematics shows that the maximum horizontal range occurs when the launch angle is 45°. At that angle, the initial velocity is shared equally between the horizontal and vertical directions. The vertical component keeps the projectile in the air long enough, while the horizontal component carries it forward efficiently. This balance produces the greatest possible range"*

Kipute: (excited) *"It's like that time we learned about the see-saw! The balance point is in the middle!"*

Mr. Akilikubwa: *"Beautiful analogy, Kipute! And here's the magical part: if you could somehow jump at exactly 45 degrees, the distance you travel horizontally will always be exactly four times the maximum height you reach. Always! Whether you're a child jumping 2 metres, or an Olympic athlete jumping 8 metres; 45 degrees makes the range exactly four times the height."*

Kipanga: (thoughtful) *"But sir, I've watched Olympic long jumpers on TV. They don't seem to go that high..."*

Mr. Akilikubwa: *"Ah! Sharp observation, Kipanga. You're right! Do you know why?"*

Kipanga: *"Um... they're making a mistake?"*

Mr. Akilikubwa: (laughing) *" "Not at all! They are actually being clever. Real long-jump motion is more complicated than the simple projectile model. The athlete does not land at exactly the same height as the take-off point, and air resistance also affects the motion. Because of these factors, the best take-off angle in real long jumping is usually slightly less than 45°."*

Kipute: *"So Kipanga wasn't entirely wrong to jump upward, and Musa wasn't entirely wrong to jump flat; they were both just too extreme?"*

Mr. Akilikubwa: *"Precisely! And here's your homework: both of you, practice finding that middle ground. Kipanga, try to drive your leading knee forward more, not just up. Musa.... well, he needs the opposite advice: lift that knee higher at takeoff. You're both capable of jumping over 6 metres if you find your 45-degree sweet spot."*

Kipanga: (grinning) *"Next year, sir, I'm going to beat Musa! Now that I know the secret physics formula!"*

Mr. Akilikubwa: *"That's the spirit! But remember, knowing the physics is one thing, training your body to execute it is another. Even knowing the optimal angle, it takes years of practice to actually achieve it consistently. Physics tells you the target; dedication gets you there!"*

Based on this dialogue, explain:

(a) Explain clearly why both Kipanga's high jump (steep angle) and Musa's flat jump (shallow angle) failed to achieve maximum distance, even though they had similar running speeds.

(b) Explain why the optimal take-off angle of 45° gives the maximum range, and explain the meaning of Mr. Akilikubwa's statement that "range equals four times the height" for 45-degree launches.

(c) Explain why Olympic long jumpers actually use angles slightly below 45° rather than exactly 45°, as Mr. Akilikubwa explained.

Solution

(a) Both jumpers failed because they did not balance the vertical and horizontal components of their initial velocity optimally.

Kipanga (too steep, $\theta > 45°$): By jumping at a steep angle, too much of his initial velocity went into the vertical component ($u\sin\theta$) and too little into the horizontal component ($u\cos\theta$).

Result: He rose very high and stayed in the air for a long time (large $T = 2u\sin\theta/g$), but his horizontal velocity was so small that he barely traveled forward during all that airtime. High time of flight but low horizontal velocity leads to mediocre range.

Musa (too flat, θ < 45°): By jumping at a shallow angle, too much velocity went into horizontal component and too little into vertical.

Result: He moved forward very quickly (large $u\cos\theta$), but he did not stay in the air for long enough (small T because small $u\sin\theta$) to take full advantage of that velocity. High horizontal velocity but low time of flight leads to mediocre range again.

(b) Explanation for maximum range at 45°:

Mathematically, $R = \frac{u^2\sin 2\theta}{g}$. Since the maximum value of sine function is 1, range is maximized when $\sin 2\theta = 1$, which means $2\theta = 90°$, giving $\theta = 45°$.

At this angle, the initial velocity is distributed equally between the horizontal and vertical directions, giving the best balance between: sufficient **time of flight**, and sufficient **horizontal velocity** and the range becomes maximum.

Explanation on the range-height relationship:

For 45° launch:

- Maximum height: $H = \frac{u^2\sin^2 45°}{2g} = \frac{u^2 \times 0.5}{2g} = \frac{\mathbf{u^2}}{\mathbf{4g}}$
- Range: $R = \frac{u^2 \sin 90°}{g} = \frac{u^2}{g}$ (since $\sin 2 \times 45° = \sin 90° = 1$)

Therefore: $\frac{R}{H} = \frac{\frac{u^2}{g}}{\frac{u^2}{4g}} = 4$

So: $R = 4H$

This means that for a projectile launched at **45°**, the horizontal range is always **four times the maximum height reached**. For example, if the projectile rises to a maximum height of **2m**, then its range is **8m**.

(c) Real long-jump athletes like Olympic long jumpers usually use take-off angles slightly less than 45° because actual long-jump motion is not exactly the same as ideal projectile motion.

In the ideal model, the projectile is launched and lands at the same height and air resistance is neglected. Under those conditions, 45° gives maximum range. In real long jump, however:

- the athlete's landing position is effectively **lower than the take-off position**, and
- **air resistance** also affects the motion.

When the landing point is lower than the launch point, the projectile can stay in the air long enough even with a slightly smaller launch angle. Therefore, the angle for practical maximum range becomes slightly less than 45°.

Making Sense of the Answer: *The dialogue beautifully captures the central insight of projectile optimization: extreme strategies fail, balanced approaches succeed. Kipanga and Musa represent the two failure modes: too much vertical emphasis versus too much horizontal emphasis. Mr. Akilikubwa guides them to the middle ground where physics achieves the optimal result. The conversation also reveals that real-world applications (Olympic long jump) must account for factors beyond the idealized physics (like landing height differences), showing how theory guides practice even when perfect application is not possible.*

Think Like a Physicist: *This example demonstrates how physics provides both explanation and optimization strategy. Without physics, an athlete might spend years trying random angles, hoping to improve. With physics, we know immediately that 45° (or slightly below for long jump) is the target, and training can focus on achieving that specific goal. The dialogue format makes the abstract mathematics tangible: students see Kipanga's* *frustration, understand his confusion, and follow his journey from "it's not fair" to "now I know the secret formula." This is how physics transforms from formulas on paper into actionable knowledge that changes outcomes.*

HOT Example 10

A projectile is fired from ground level with initial velocity u at angle θ. At the highest point of its trajectory, the projectile explodes into two equal fragments. One fragment falls straight down from the explosion point and lands directly below.

(a) Show that the second fragment lands at a distance $\mathbf{\frac{3R}{2}}$ from the launch point, where R is the range the projectile would have achieved without exploding.
(b) If the projectile was fired at 40m/s at 45°, calculate:
  (i) the range the projectile would have travelled if no explosion occurred,
  (ii) the position where each fragment lands,
  (iii) the horizontal velocity of the second fragment immediately after explosion.

Take g = 9.8 m/s².

Solution

(a) Before explosion:

At the highest point:

- Horizontal position $= \frac{R}{2}$ (halfway through the trajectory)
- Vertical velocity $= 0$
- Horizontal velocity $=$ ucosθ (unchanged throughout flight)

Total horizontal momentum before explosion:

$p_{\text{before}} = m(u\cos\theta)$ (where m is the projectile mass)

After explosion:

Fragment 1 (mass $m/2$) falls straight down; thus, horizontal velocity $= 0$.

Fragment 2 (mass $m/2$) has horizontal velocity v_2.

Conservation of horizontal momentum:

$$m(u\cos\theta) = \left(\frac{m}{2}\right)(0) + \left(\frac{m}{2}\right)v_2$$

Solving for v_2:

$\mathbf{v_2 = 2u\cos\theta}$ *(Fragment 2 has twice the original horizontal velocity!)*

Landing position:

Fragment 2 travels from horizontal position $R/2$ with velocity 2ucosθ for time t (time to fall from maximum height H).

But the time to reach the maximum height is equal to the falling time. Thus:

$$t = t_{\max} = \frac{u\sin\theta}{g}$$

Additional horizontal distance traveled by fragment 2:

$$\Delta x = \mathbf{v_2} \times t = (\mathbf{2u\cos\theta}) \times \left(\frac{u\sin\theta}{g}\right) = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{u^2\sin 2\theta}{g} = \text{original range R}$$

Therefore: $\mathbf{\Delta x = R}$

Fragment 2 starts at horizontal position, $\mathbf{R/2}$ and travels additional distance **R**:

Landing position of fragment 2 $= \frac{R}{2} + R = \frac{\mathbf{3R}}{\mathbf{2}}$

Hence, the second fragment lands at a distance $\frac{\mathbf{3R}}{\mathbf{2}}$ from the launch point.

(b) The solution of each part is as follows:

(i) Using:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(40\mathrm{m/s})^2 \times \sin 90°}{9.8\mathrm{m/s^2}} = 163.3\mathrm{m}$$

The range the projectile would be $163.3\mathrm{m}$.

(ii) Fragment 1 falls vertically from height H at horizontal position $R/2$. So it lands at:

$$\frac{R}{2} = \frac{163.3\mathrm{m}}{2} = 81.7\mathrm{m}$$

Fragment 2 lands at:

$$\frac{3R}{2} = 3 \times \frac{163.3\mathrm{m}}{2} = 245\mathrm{m}$$

Fragment 1 lands at $81.7\mathrm{m}$, while fragment 2 lands at $245\mathrm{m}$.

(iii) Horizontal velocity of fragment 2:

$$v_2 = 2u\cos\theta = 2 \times 40\mathrm{m/s} \times \cos 45° = 56.6\mathrm{m/s}$$

Making Sense of the Answer: *Fragment 2 lands at 245m which is equivalent to a 50% farther than the original projectile would have traveled (163m)! This seems counterintuitive until we realize that the explosion doubles its horizontal velocity from 28.3 m/s to 56.6 m/s. Even though it only has half the time left to travel, the doubled horizontal velocity allows the fragment to travel a further horizontal distance of 163.3m (R) before reaching the ground. Fragment 1 lands exactly halfway along the original trajectory, which makes sense since it falls straight down from the peak.*

Think Like a Physicist: *This problem beautifully demonstrates momentum conservation in two dimensions. The explosion provides internal forces between fragments, but no external horizontal force acts on the system. Therefore, total horizontal momentum must be conserved. Since one fragment stops horizontally (falls straight down), the other must carry the entire horizontal momentum, and having half the mass, it must travel at twice the original horizontal velocity.*

HOT Example 11

A ball is thrown from ground level at 20m/s at an angle of 40° toward a wall. The wall is 25m away and 3.5m high. Taking $g = 9.8\ m/s^2$ and using sin 40° = 0.643, cos 40° = 0.766, sin 80° = 0.985:

(a) Show that the ball clears the wall.
(b) Calculate by how much the ball clears the top of the wall.
(c) Calculate where the ball lands beyond the wall.
(d) If the wall height were increased, determine the maximum wall height that the ball could just clear.

Solution

(a) We need to find the height of the ball when it is at horizontal position of $s_x = 25\text{m}$.

Using the trajectory equation:

$$s_y = s_x\tan\theta - \left(\frac{g}{2u^2\cos^2\theta}\right)s_x^2$$

Where: $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$s_y = s_x\left(\frac{\sin40°}{\cos40°}\right) - \left(\frac{g}{2u^2\cos^2 40°}\right)s_x^2$$

Substituting values:

$$s_y = 25\text{m}\left(\frac{0.643}{0.766}\right) - \left(\frac{9.8\ \text{m/s}^2}{2(20\text{m/s})^2(0.766)^2}\right)(25\text{m})^2 = 7.91\text{m}$$

Since $s_y = 7.91\text{m} > 3.5\text{m}$ (wall height), **the ball clears the wall**.

(b) $\text{Clearance} = \text{height of ball} - \text{height of wall}$

$$\text{Clearance} = (7.91 - 3.5)\text{m} = 4.41\text{m}$$

The ball clears the wall by 4.41m.

(c) Total horizontal distance travelled by ball:

$$R = \frac{u^2\sin 2\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{2(20\text{m/s})^2(0.643)(0.766)}{9.8\ \text{m/s}^2} = 40.2\text{m}$$

Distance beyond wall = $40.2\text{m} - 25\text{m} = 15.2\text{m}$

The ball lands 15.2m beyond the wall.

(d) For the ball to just clear the wall: $s_y = h$ at $s_x = 25\text{m}$.

But from (a):

When $s_x = 25\text{m}$, $s_y = 7.91\text{m} = h$

The maximum wall height is 7.91m (as long as the wall stays at 25m distance).

Making Sense of the Answer: *The ball reaches 7.91m height at the wall location, clearing the 3.5m wall by a comfortable margin of 4.41m which is more than the wall height itself! This is not surprising since the ball reaches maximum height of 8.43m (by applying maximum height formula) at its peak (which occurs before the wall because R/2 is smaller than 25m). If the wall were higher than 7.91m, the ball would hit it. The ball's trajectory is already descending when it reaches the wall position, which is why it lands only 15.1 m beyond the wall rather than much farther.*

Think Like a Physicist: *In real applications (sports, military, construction), clearance calculations are critical: will the basketball clear the defender's outstretched hand? Will the artillery shell clear the hill? Will the water from the hose reach over the fence? The mathematics gives definitive answers. Notice how the maximum* *clearable wall height depends on where the wall is positioned; obstacles at different distances pose different challenges, even for the same projectile. A wall placed at the peak position ($R/2$ which is about 20m for our example) is the easiest to clear because the projectile is at its maximum height there.*

Summary of key formulas

For a projectile launched at angle θ with initial velocity u from and returning to the same level:

Time of flight: $T = \frac{2u \sin \theta}{g}$

Maximum height: $H = \frac{u^2 \sin^2 \theta}{2g}$

Horizontal range: $R = \frac{u^2 \sin 2\theta}{g}$

Special relationships:

Maximum range occurs at $\theta = 45°$: $R_{max} = \frac{u^2}{g}$

At 45°, the range is four times the height: $R = 4H$

Complementary angles (θ and $90° - \theta$) give the same range

Maximum height relates to time of flight: $H = \frac{gT^2}{8}$

These formulas, combined with those from previous subtopic, provide complete tools for analysing any projectile motion problem where launch and landing occur at the same level. In the next subtopic, we shall extend our analysis to more complex cases where these conditions do not hold.