VARIABLE MASS SYSTEMS

Throughout this chapter, we have analysed collisions and momentum conservation for systems where mass remains constant. We used Newton's second law in the form F = ma, and later learned the more general form $F = dp/dt$. *But what happens when the mass of a moving object changes continuously?*

Consider these fascinating situations:

Situation 1: A rocket in space burns fuel and ejects exhaust gases backward. Its mass decreases continuously while it accelerates forward, even though nothing external pushes it!

Situation 2: Rain falls vertically onto a toy car rolling horizontally on a smooth surface. The car's mass increases as it collects rainwater. What happens to its velocity?

Situation 3: A conveyor belt moves at constant velocity while sand falls onto it from a stationary hopper above. What force must the motor provide to maintain constant velocity?

These are **variable mass systems** (objects whose mass changes with time). They cannot be analyzed using F = ma because that equation assumes constant mass. We must return to the fundamental form: $\mathbf{F} = \frac{\mathbf{dp}}{\mathbf{dt}}$.

Why F = ma Fails for Variable Mass

Newton's second law in the form F=ma is derived from $F = dp/dt$ by assuming mass is constant:

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\left(\frac{dv}{dt}\right) + v\left(\frac{dm}{dt}\right)$$

If mass is constant $\left(\frac{dm}{dt} = 0\right)$, this simplifies to:

$$F = m\left(\frac{dv}{dt}\right) = ma$$

But when mass changes $\left(\frac{dm}{dt} \neq 0\right)$, we cannot ignore the second term. The complete equation is:

$$\mathbf{F} = \mathbf{m}\left(\frac{\mathbf{dv}}{\mathbf{dt}}\right) + \mathbf{v}\left(\frac{\mathbf{dm}}{\mathbf{dt}}\right)$$

Or equivalently:

$$\mathbf{F} = \mathbf{ma} + \mathbf{v}\left(\frac{\mathbf{dm}}{\mathbf{dt}}\right)$$

Where:

F = external force on the system,

m = instantaneous mass at time t,

$a = \frac{dv}{dt}$ = acceleration,

v = velocity of the system,

$\frac{dm}{dt}$ = rate of mass change.

It is important to understand that the term $v\left(\frac{dm}{dt}\right)$ represents the force associated with the mass flow; either mass leaving the system (e.g. a rocket) or mass joining the system (e.g. rain on cart). Therefore, we have two categories of variable mass systems:

Category 1: Mass decreasing systems

- Examples: Rockets, jet engines, leaking containers.
- $\frac{dm}{dt} < 0$ (mass decreases).

- Typically: external force $F \approx 0$ (rocket in space).
- Result: Object accelerates as it loses mass.

Category 2: Mass increasing systems

- Examples: Rain falling on cart, sand on conveyor, snow on train.
- $\frac{dm}{dt} > 0$ (mass increases).
- Often: maintain constant velocity ($v =$ constant, $a = 0$).
- Result: Force needed to prevent slowing down (to maintain constant velocity).

Let us explore each category in depth.

Mass Decreasing Systems: Rockets and Jets

Rocket Propulsion:

A common question students ask is: *How can a rocket move in empty space if there is nothing to push against?*

The answer is that a rocket does not rely on air, the ground, or any external surface. Its motion is entirely explained by the **conservation of momentum**.

A rocket moves by expelling exhaust gases at very high velocity in the backward direction. When the gases are pushed backward, they carry momentum with them. To conserve total momentum, the rocket gains an equal amount of forward momentum. This interaction occurs entirely within the rocket–exhaust system, which is why a rocket functions perfectly even in a vacuum.

As the fuel burns:

- Rocket's mass decreases with time: $\frac{dm}{dt} < 0$.
- Exhaust gases are expelled with velocity u relative to rocket.
- Rocket accelerates forward (by conservation of momentum).
- The backward momentum of the expelled gases produces a forward thrust on the rocket.

Thus, the rocket accelerates forward as a direct consequence of momentum conservation in a variable-mass system.

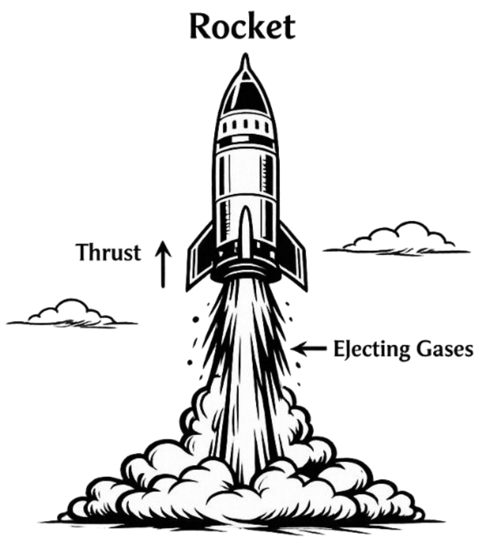

Figure:
Rocket propulsion mechanism. Expelled gases carry momentum downward, generating upward thrust on the rocket by conservation of momentum.

Deriving Rocket Equations:

Consider a rocket of instantaneous mass m moving with velocity v in free space (no external forces).

In a small time interval dt:

- Rocket ejects a small mass dm of exhaust.
- The exhaust leaves with velocity u_{rel} relative to the rocket (directed backward).
- Rocket mass decreases to $(m - dm)$.
- Rocket velocity increases to $(v + dv)$.

Taking the rocket as our reference frame initially:

Before ejection (in rocket's initial rest frame, where $v = 0$):

Total momentum $= 0$

After ejection (in the same initial rest frame):

- Rocket momentum $= (m - dm)(v + dv) = (m - dm)(0 + dv) =) = (\mathbf{m - dm})\mathbf{dv}$.
- Exhaust momentum = $dm(-u_{rel})$ (negative because backward).

Conservation of momentum:

$0 = (m - dm)(dv) + dm(-u_{rel})$

$0 = m(dv) - dm(dv) - u_{rel}(dm)$

Since dm and dv are very small quantities, $dm(dv) \approx 0$ (negligible). So:

$0 = m(dv) - u_{rel}(dm)$

$m(dv) = u_{rel}(dm)$

Rearranging:

$$\mathbf{dv = u_{rel}\left(\frac{dm}{m}\right)}$$

Important note on sign convention: In this derivation, u_{rel} is taken as a **positive** exhaust velocity relative to the rocket, and dm represents the **positive** mass expelled, so the rocket's mass decreases by this amount; that is $m_{rocket} = m - dm$.

Tsiolkovsky Rocket Equation:

Integrating the differential equation obtained from momentum conservation leads to the Tsiolkovsky rocket equation, which relates the rocket's change in velocity to the exhaust velocity and the ratio of its initial mass to its final mass.

From the differential equation:

$$dv = u_{rel}\left(\frac{dm}{m}\right)$$

Integrating from initial mass m_i to final mass m_f, with their respective change in velocity, v_i to v_f where total mass expelled = m_i(upper limit) $-m_f$ (lower limit) and total increase in rocket's velocity = v_f(upper limit)) $-$ v_i(lower limit).

$$\int_{v_i}^{v_f} dv = u_{rel}\int_{m_f}^{m_i}\frac{dm}{m}$$

$$v_f - v_i = u_{rel}[\ln m]_{m_f}^{m_i} = u_{rel}(\ln m_i - \ln m_f)$$

$$\mathbf{v_f = v_i + u_{rel}\ln\left(\frac{m_i}{m_f}\right)}$$

The final equation is known as **Tsiolkovsky Rocket Equation**. In this form, the equation applies if and only if **no external forces act on the rocket**.

Alternatively, the equation can be written as:

$$v_f - v_i = \Delta\mathbf{v} = \mathbf{u_{rel}\ln\left(\frac{m_i}{m_f}\right)}$$

The Tsiolkovsky Rocket Equation reveals several important practical insights:

1. Exhaust velocity and mass ratio:

A higher exhaust velocity (u_{rel}) or a larger mass ratio m_i/m_f produces a greater final velocity. The exhaust velocity depends on engine design and fuel quality, while the mass ratio depends on how much fuel the rocket can carry relative to its structure and payload.

2. Logarithmic limitation:

The velocity depends logarithmically on the mass ratio. This means that achieving large increases in velocity requires disproportionately large increases in fuel mass. For example, to double the velocity increase, the mass ratio must be squared. This logarithmic behaviour is one reason space travel is extremely expensive.

3. Staged rockets:

Instead of building a single massive rocket with an extremely high mass ratio, engineers use multiple stages. Each stage burns its fuel and is then discarded, reducing the remaining mass. This greatly improves efficiency and makes high velocities achievable in practice.

The Thrust Force:

The thrust produced by a rocket arises from the rate of change of momentum of the expelled gases.

$$\mathbf{F_{thrust} = u_{rel}\left(\frac{dm}{dt}\right)}$$

Where $\frac{dm}{dt}$ is the rate at which mass is ejected (positive value).

Thus, thrust is directly proportional to both the exhaust velocity and the rate of fuel consumption.

The relationship between thrust and resultant force of rocket in absence of external forces can be derived as follows:

From Rocket Differential Equation:

$$dv = u_{rel}\left(\frac{dm}{m}\right)$$

Multiplying by m both sides:

$$m(dv) = u_{rel}(dm)$$

Dividing by dt throughout:

$$m\left(\frac{dv}{dt}\right) = u_{rel}\left(\frac{dm}{dt}\right)$$

Where $m\left(\frac{dv}{dt}\right)$ is the rocket's resultant (net) force $= ma$

If external forces (e.g. gravity, drag) exist:

$$\text{Resultant force} = \text{Thrust} + \text{External forces}$$

$$\mathbf{m\left(\frac{dv}{dt}\right) = u_{rel}\left(\frac{dm}{dt}\right) + F_{external}}$$

For vertical launch against gravity where the rocket must first overcome gravity (mg) before accelerating upward, $F_{external} = -mg$; thus:

$$\mathbf{m\left(\frac{dv}{dt}\right) = u_{rel}\left(\frac{dm}{dt}\right) - mg}$$

From this equation we can derive modified Tsiolkovsky Rocket Equation under presence of force of gravity as follows:

Multiplying by dt and dividing by m throughout the above equation:

$$dv = u_{rel}\left(\frac{dm}{m}\right) - gdt$$

Integrating:

$$\int_{v_i}^{v_f} dv = u_{rel}\int_{m_f}^{m_i}\frac{dm}{m} - g\int_0^t dt$$

$$v_f - v_i = u_{rel}[\ln m]_{m_f}^{m_i} - g(t-0) = u_{rel}(\ln m_i - \ln m_f) - gt$$

$$\mathbf{v_f = v_i + u_{rel}\ln\left(\frac{m_i}{m_f}\right) - gt}$$

The final equation is the modified Tsiolkovsky rocket equation for vertical flight under gravity (neglecting drag).

Jet Propulsion:

Jet engines operate on the same fundamental principle as rockets: thrust is produced by the rate of change of momentum of expelled gases. However, there is one crucial difference in how combustion is sustained.

Rockets: Carry both fuel and oxidiser, allowing them to operate independently of the surrounding environment. As a result, rockets function in space as well as in the atmosphere.

Jet engines: Draw in atmospheric air to supply oxygen for combustion. Therefore, they can operate only within the atmosphere.

This difference leads to important practical consequences.

Advantage: Since jet engines do not carry oxidiser, they are much more efficient in the atmosphere. The absence of heavy oxidiser significantly reduces the aircraft's mass.

Disadvantage: Because they depend on atmospheric oxygen, jet engines cannot operate in space, where no air is available.

Operation of a Jet Engine

The basic working process of a jet engine can be summarised as follows:

1. **Air intake:**

Air enters the front of the engine with velocity v_{in} (relative to the ground).

2. **Compression and combustion:**

The incoming air is compressed, mixed with fuel, and ignited in the combustion chamber.

3. **Exhaust:**

The high-temperature, high-pressure gases expand and exit the rear of the engine with velocity, v_{out}(relative to the ground).

4. **Thrust generation:**

The difference between the incoming and outgoing momentum of the air–gas mixture produces a net forward thrust.

Thrust Equation for a Jet Engine

A jet engine produces thrust by changing the momentum of the air–fuel mixture passing through it. Let:

$\dot{m}_{air}$ = mass flow rate of incoming air,

$\dot{m}_{fuel}$ = mass flow rate of injected fuel,

v_{in} = velocity of incoming air (relative to ground),

v_{out} = velocity of exhaust gases (relative to ground).

Applying momentum principle:

Thrust is equal to the rate of change of momentum:

$$F_{thrust} = \frac{d(\text{momentum})}{dt}$$

For steady flow:

$$\mathbf{F_{thrust}} = (\textbf{momentum out per second}) - (\textbf{momentum in per second})$$

Momentum in:

Only air enters the engine from outside. So:

$$\text{momentum out per second} = \dot{m}_{air} v_{in}$$

You should understand that fuel is injected inside the engine, so it does not contribute to incoming external momentum.

Momentum out:

Both air and fuel leave as exhaust gases.

So total mass leaving per second $= \dot{m}_{air} + \dot{m}_{fuel}$

And momentum out per second $= (\dot{m}_{air} + \dot{m}_{fuel})v_{out}$

Net thrust:

$$\mathbf{F_{thrust} = (\dot{m}_{air} + \dot{m}_{fuel})v_{out} - \dot{m}_{air}v_{in}}$$

Rearranging:

$$F_{thrust} = \dot{m}_{air}(v_{out} - v_{in}) + \dot{m}_{fuel}v_{out}$$

Hence, the final thrust equation for a jet engine is:

$$\mathbf{F_{thrust} = \dot{m}_{air}(v_{out} - v_{in}) + \dot{m}_{fuel}v_{out}}$$

Where $\dot{m} = \frac{dm}{dt}$ (mass flow rate)

Simplified form:

If the aircraft velocity equals the incoming air velocity:

$$v_{in} = v_{jet}$$

And if fuel mass flow is small compared to air flow, then the thrust equation for jet engine can be simplified as follows:

From one the form of the thrust equation above:

$$F_{thrust} = (\dot{m}_{air} + \dot{m}_{fuel})v_{out} - \dot{m}_{air}v_{in}$$

Subtracting and adding $\dot{m}_{fuel}v_{in}$ (this is just adding zero in a useful form):

$$F_{thrust} = (\dot{m}_{air} + \dot{m}_{fuel})v_{out} - \dot{m}_{air}v_{in} - \dot{m}_{fuel}v_{in} + \dot{m}_{fuel}v_{in}$$

Grouping the first three terms:

$$F_{thrust} = (\dot{m}_{air} + \dot{m}_{fuel})v_{out} - v_{in}(\dot{m}_{air} + \dot{m}_{fuel}) + \dot{m}_{fuel}v_{in}$$

$$F_{thrust} = (\dot{m}_{air} + \dot{m}_{fuel})(v_{out} - v_{in}) + \dot{m}_{fuel}v_{in}$$

Applying the common jet assumption $v_{in} = v_{jet}$

$$F_{thrust} = (\dot{m}_{air} + \dot{m}_{fuel})(v_{out} - v_{jet}) + \dot{m}_{fuel}v_{jet}$$

In most aircraft engines, $\dot{m}_{fuel} \ll \dot{m}_{air}$, and thus the extra term $\dot{m}_{fuel}v_{jet}$ is relatively small compared with the main momentum-change term and hence it can be neglected. Therefore, the simplified thrust equation for jet engine becomes:

$$F_{thrust} = (\dot{m}_{air} + \dot{m}_{fuel})(v_{out} - v_{jet})$$

This form emphasizes that thrust depends on:

- The total mass flow rate,
- The difference between exhaust velocity and aircraft velocity.

So we can conclude that:

- Thrust increases if exhaust velocity increases.
- Thrust increases if mass flow rate increases.
- If $v_{out} = v_{in} \approx v_{jet}$, thrust is zero.

- If exhaust velocity is only slightly greater than aircraft velocity, thrust is small.

Warning!
The simplified form

$$F_{thrust} = (\dot{m}_{air} + \dot{m}_{fuel})(v_{out} - v_{jet})$$

is an **approximation**. It neglects the additional momentum term associated with the fuel entering the engine at aircraft velocity. **In solving problems, always use the full momentum expression unless explicitly told otherwise.**

Mass Increasing Systems: Constant Velocity

So far, we have considered systems that lose mass, such as rockets. Let us now examine the opposite situation: systems whose mass increases with time.

Examples include:

- Rain falling onto a moving cart
- Sand pouring onto a moving conveyor belt
- Water flowing from a pipe onto a moving platform
- Grain being loaded into a moving truck

In this case, the key question is this: *if mass is being added to a moving system, what force is required to maintain constant velocity?*

When mass is added to a moving system, the added mass must be accelerated from rest (or its initial velocity) to match the system's velocity. This requires a force, even if the system's velocity does not change! If no force is applied, the system slows down.

Deriving the required force:

Consider a system of mass m moving at constant velocity v. In time dt, additional mass dm is added (initially at rest in the ground frame).

Conservation of momentum approach:

Before addition:

- System momentum: $p_1 = mv$
- Added mass momentum: $p_2 = 0$ (at rest)
- Total: $p_{initial} = mv$

After addition (system maintains velocity v):

- Combined momentum: $p_{final} = (m + dm)v$

Change in momentum:

$\Delta p = p_{final} - p_{initial}$

$\Delta p = (m + dm)v - mv = v(dm)$

This momentum change must be provided by an external force, F.

$\Delta p = F \times dt = v(dm)$

Hence:

$$\mathbf{F = v\left(\frac{dm}{dt}\right)}$$

The equation tells that to maintain constant velocity while mass increases, you must apply a force equal to the velocity times the rate of mass increase.

Alternative derivation using F $= \frac{dp}{dt}$:

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\left(\frac{dv}{dt}\right) + v\left(\frac{dm}{dt}\right)$$

If velocity is constant $\left(\frac{dv}{dt} = 0\right)$:

$$F = v\left(\frac{dm}{dt}\right)$$

The equation reveals several important practical insights:

1. **Force is needed even though velocity is constant** (a = 0 but F ≠ 0!)
2. **F is proportional to v:** Faster systems need more force to maintain velocity while collecting mass.
3. **F is proportional to dm/dt:** Heavier mass flow requires more force.
4. **Direction:** Force must be in the direction of motion.

With this comprehensive theoretical foundation established, we are now ready to bring these concepts to life through carefully crafted worked examples.

BINDER Example 21

A toy rocket has a total mass of 500g (including 300g of fuel). The rocket engine ejects exhaust gases at a speed of 800m/s relative to the rocket. Calculate:

(a) The final velocity of the rocket after all fuel is burned (assume it starts from rest in space).
(b) The thrust force if fuel burns at a rate of 20g/s.

Solution

Interpreting the data:

Initial mass: $m_0 = 500g = 0.5kg$

Fuel mass: $300g = 0.3kg$

Final mass (after fuel burned): $m_f = (500 - 300)g = 200g = 0.2kg$

Exhaust velocity: $u_{rel} = 800m/s$

Initial velocity: $v_i = 0$ (starts from rest)

Fuel burn rate: $\frac{dm}{dt} = 20g/s = 0.02kg/s$

(a) Using the integral rocket equation:

$$v_f = v_i + u_{rel} \ln\left(\frac{m_i}{m_f}\right)$$

With $v_i = 0$, the above equation becomes:

$$v_f = u_{rel} \ln\left(\frac{m_i}{m_f}\right)$$

Substituting values:

$$v_f = 800m/s \times \ln\left(\frac{0.5kg}{0.2kg}\right) = 733m/s$$

The velocity is $733m/s$.

(b) The thrust force is given by the following equation:

$$F_{thrust} = u_{rel}\left(\frac{dm}{dt}\right)$$

Substituting values:

$$F_{thrust} = 800m/s \times 0.02kg/s = 16N$$

The thrust force is 16N.

Making Sense of the Answer: *The rocket reached 733m/s (about 2600km/h) by burning 60% of its initial mass as fuel. The thrust of 16N is modest but sufficient for a 500g rocket as it provides initial acceleration of* $a = F/m = 16N/0.5kg = 32m/s^2$*, which is impressive! Notice that the thrust remains constant as long as* u_{rel} *and* dm/dt *stay constant, but the acceleration increases as the mass of rocket decreases.*

Think Like a Physicist: *The rocket equation* $v_f = u_{rel} \times ln(m_i/m_f)$ *shows a logarithmic relationship. To double the final velocity, you would need to square the mass ratio,* $m_i : m_f$ *(from* 2.5 *to* 6.25*), meaning even more fuel. This is why* ***multi-stage rockets are used for space missions because they drop empty fuel tanks to improve the mass ratio.*** *Also notice that thrust is independent of the rocket's mass - it depends only on exhaust velocity and mass flow rate.*

BINDER Example 22

A small jet aircraft is flying horizontally at 200m/s. The jet engine takes in 50kg of air per second and ejects exhaust gases at 500m/s relative to the ground. The fuel consumption rate is 2kg/s. Calculate:

(b) The thrust produced by the engine.
(c) The net force on the aircraft if air resistance is 8000N

Solution

Interpreting the data:

Aircraft velocity: $v_{jet} = 200m/s$

Air intake rate: $\dot{m}_{air} = 50kg/s$

Exhaust velocity (relative to ground): $v_{out} = 500m/s$

Fuel burn rate: $\dot{m}_{fuel} = 2kg/s$

Air resistance: $F_{drag} = 8000N$

(a) Air enters the engine at the aircraft's velocity: $v_{in} = v_{jet} = 200m/s$ (relative to ground).

Using the jet thrust equation:

$$F_{thrust} = \dot{m}_{air}(v_{out} - v_{in}) + \dot{m}_{fuel}v_{out}$$

Substituting values:

$$F_{thrust} = 50kg/s(500 - 200)m/s + 2kg/s \times 500m/s = 16000N$$

The thrust is 16000N (or 16kN).

(b) Net force = Thrust − Air resistance

$$F_{net} = F_{thrust} - F_{drag} = (16000 - 8000)N = 8000N$$

The net force is 8000N (or 8kN).

Making Sense of the Answer: *The engine produces 16kN of thrust by increasing the momentum of the flow through it. Each second, a 52kg of mass (air + fuel) leaves as exhaust at 500m/s, while 50kg of incoming air enters at 200m/s, giving a momentum increase equivalent to a velocity change of 300m/s for the main airflow.*

The net forward force of 8kN means the aircraft is accelerating. The large air intake (50kg/s) is typical for jet engines; they use atmospheric oxygen rather than carrying it, making them much more efficient than rockets in atmosphere.

Think Like a Physicist: *Jets are fundamentally different from rockets: jets take in air (free mass + oxygen), while rockets carry everything. This is why jets are efficient in atmosphere but useless in space, while rockets work everywhere but are less efficient in atmosphere (must carry oxidiser). The thrust depends on the velocity difference ($v_{out} - v_{in}$), which is why jets are less efficient at higher speeds because as the aircraft speeds up, the relative exhaust velocity decreases.*

BINDER Example 23

A jet aircraft is flying horizontally at 250m/s. The engines take in 80kg of air per second and eject it at 600m/s relative to the ground. The fuel burn rate is 3kg/s. Calculate:

(a) The thrust produced by the engines.
(b) The mass of fuel consumed to travel 100km at this velocity.

Solution

Interpreting the data:

Aircraft velocity: $\mathrm{v_{jet} = v_{in} = 250m/s}$

Air intake rate: $\mathrm{\dot{m}_{air} = 80kg/s}$

Exhaust velocity (relative to ground): $\mathrm{v_{out} = 600m/s}$

Fuel burn rate: $\mathrm{\dot{m}_{fuel} = 3kg/s}$

Distance: $\mathrm{d = 100\ km = 100000m}$

(a) Using the jet thrust equation:

$$\mathrm{F_{thrust} = \dot{m}_{air}(v_{out} - v_{in}) + \dot{m}_{fuel}v_{out}}$$

Substituting values:

$$\mathrm{F_{thrust} = 80kg/s(600 - 250)m/s + 3kg/s \times 600m/s = 29800N}$$

The thrust is 29800N.

(b) Time to travel 100km:

$$\mathrm{t = \frac{d}{v} = \frac{100000m}{250m/s} = 400s}$$

Mass of fuel consumed = $\mathrm{\dot{m}_{fuel} \times t = 3kg/s \times 400 = 1200kg}$

The mass consumed is 1200kg.

Making Sense of the Answer: *The thrust ($29800N$) is substantial, typical for a small jet engine. The exhaust exits at only $350m/s$ faster than it entered ($600 - 250 = 350m/s$), but the large mass flow ($83kg/s$) produces significant thrust. In 400 seconds (6.7 minutes), the aircraft burned 1,200 kg (1.2 tonnes) of fuel! This illustrates how rapidly fuel is used in jet propulsion and why fuel efficiency is a critical factor in aircraft operation and design.*

Think Like a Physicist: *Jet engines are more efficient than rockets in atmosphere because they do not carry oxidiser as they get oxygen "free" from the air. The $80:3$ ratio (air:fuel) is typical, showing that most mass flow is air, not fuel. This is why jets are practical for aircraft but rockets are needed for space.*

BINDER Example 24

A flatbed railway wagon moves at constant velocity of 10m/s along a horizontal track. Coal is being dropped vertically onto the wagon from a stationary hopper at a rate of 40kg/s. Calculate the horizontal force required to maintain the wagon's constant velocity.

Solution

Interpreting the data:

Wagon velocity: $v = 10\text{m/s}$ (constant)

Coal loading rate: $\frac{dm}{dt} = 40\text{kg/s}$

Coal initially has zero horizontal velocity (drops vertically)

For mass increasing system at constant velocity:

$$F = v\left(\frac{dm}{dt}\right)$$

Substituting values:

$$F = 10\text{m/s} \times 40\text{kg/s} = 400\text{N}$$

The force is 400N acting in the direction of motion.

Making Sense of the Answer: *The force requirement (400N) seems modest, but it must be applied continuously as long as coal is being loaded.*

Think Like a Physicist: *This problem demonstrates an important principle: maintaining constant velocity while mass increases requires force, even though acceleration is zero (a = 0 but F ≠ 0). This seems to contradict F = ma, but remember the complete equation is F = m(dv/dt) + v(dm/dt). When dv/dt = 0, we still have F = v(dm/dt). The falling coal must be given horizontal momentum, and that requires a horizontal force on the system. This is why conveyor belt motors must work harder when material is being loaded, even if belt speed stays constant.*

BINDER Example 25

Water flows horizontally from a stationary fire hose at a rate of 20kg/s with velocity 25m/s. The water strikes a vertical wall and falls vertically downward (assume no bouncing). Calculate:

(a) The force exerted by water on the wall.
(b) If the wall is mounted on wheels and can move freely, at what rate will it accelerate if the wall's mass is 500kg?

Solution

Interpreting the data:

Water flow rate: $\frac{dm}{dt} = 20\text{kg/s}$

Water velocity: $v = 25\text{m/s}$ (horizontal)

Wall mass: $m_{wall} = 500\text{kg}$

Water falls vertically after hitting (final horizontal velocity = 0)

(a) The horizontal velocity of the water changes from **25m/s** to **0**.

Rate of change of momentum = $F = \dot{m}(v_{final} - v_{initial}) = 20\text{kg/s}\,(0 - 25)\text{m/s} = -500\text{N}$

The negative indicates that momentum of water decreases (water slows horizontally).

The force exerted is 500N (pushing the wall horizontally).

(b) If the wall is free to move, this force (500N) acts on it.

Using Newton's second law:

$$a = \frac{F}{m} = \frac{500N}{500kg} = 1m/s^2$$

The acceleration would be $1m/s^2$.

Making Sense of the Answer: *The force of 500N arises because every second, 20kg of water moving at 25m/s has its horizontal velocity reduced to zero. Stopping this steady stream of horizontal momentum requires a continuous force. Even though the water simply falls downward after impact, the key effect is that its horizontal momentum is destroyed, and the wall must provide the force to make that happen. If the wall is free to move, this 500N force produces an acceleration of 1m/s², which is quite noticeable for a 500 kg mass.*

Thinking Like a Physicist: *Whenever a moving fluid strikes a surface, do not focus on velocity but on* ***momentum change per second****. The force comes from how much momentum is being removed (or redirected) each second. Even if the velocity of the fluid remains constant before impact, a change in direction or a reduction to zero velocity still represents a change in momentum. Always ask:*

- *What is the initial velocity?*
- *What is the final velocity?*
- *What is the rate at which mass is flowing?*

Force in fluid problems is almost always a ***momentum-rate problem****, not an energy problem.*

REAL Example 26

A space agency is designing a launch vehicle. One engineer suggests building one very large rocket with enough fuel to reach orbit. Another proposes a multi-stage rocket where empty fuel tanks are discarded during flight.

By using the rocket equation, explain which design is more suitable.

Solution

Suitable choice: Multi-stage rocket.

Explanation

A rocket gains velocity by ejecting mass backward at high speed. The total gain in velocity (Δv) is given by the integral (Tsiolkovsky) rocket equation:

$$\Delta \mathbf{v} = \mathbf{u_{rel}} \mathbf{ln}\left(\frac{\mathbf{m_i}}{\mathbf{m_f}}\right)$$

This shows that velocity gain increases as the mass ratio, m_i/m_f increases. For a single massive rocket, the initial mass (m_i) includes: fuel, structure and payload. After fuel is burned, the final mass (m_f) still includes the entire empty structure. That empty structure contributes no further thrust but must still be accelerated. This reduces the effective mass ratio.

In a multi-stage rocket, once fuel in one stage is exhausted, that empty structure is discarded. The next stage then accelerates a smaller mass. This increases the effective mass ratio for each stage and allows a greater total velocity increase. Therefore, multi-stage rockets are preferred because they increase achievable velocity without requiring unrealistically large fuel masses.

Making Sense of the Answer: *The rocket equation contains a logarithm. This means that to double the final velocity, the mass ratio must increase exponentially. Carrying empty tanks is inefficient because they not only increase final mass (which makes the mass ratio smaller) but also do not contribute to thrust. So discarding them improves efficiency dramatically.*

Thinking Like a Physicist: *Focus on the mass ratio, not just the fuel amount. The rocket equation shows that useless mass in the final mass reduces the achievable velocity. Multi-staging works because it removes dead weight, improving the mass ratio at each stage. Better mass ratio means greater velocity gain.*

REAL Example 27

During a safety drill, Kipute uses a CO_2 fire extinguisher. When she presses the handle, the extinguisher jerks backward in her hands. Explain why this recoil occurs, even though the extinguisher is not "pushing against the air."

Solution

When the extinguisher is activated, carbon dioxide is expelled at high velocity in one direction. The escaping gas carries momentum. According to conservation of momentum, if the gas gains momentum in one direction, the extinguisher must gain equal momentum in the opposite direction.

This produces a backward thrust on the extinguisher. The force experienced equals the rate at which momentum is expelled:

$$F_{thrust} = u_{rel}\left(\frac{dm}{dt}\right)$$

Thus, the recoil is not due to pushing against air but due to the continuous ejection of mass. The faster and greater the mass flow rate, the stronger the recoil.

Making Sense of the Answer: *The extinguisher behaves exactly like a small rocket. It does not need air to push against; it moves because it throws mass forward. The same principle explains rocket propulsion in space.*

Thinking Like a Physicist *Ask: **what momentum leaves the system each second?** If mass is expelled forward, momentum is carried away. By conservation of momentum, the source must recoil backward. Thrust is simply the rate at which momentum is expelled.*

HOT Example 28

A rocket in deep space has an initial mass of 3000kg (including 2400kg of fuel). The rocket engines eject exhaust gases at a speed of 2500m/s relative to the rocket. Calculate:

(a) The final velocity of the rocket after all fuel is burned (starting from rest).
(b) The thrust force when the fuel is being burned at a rate of 15kg/s.
(c) The time taken to burn all the fuel at this rate.

Solution

Interpreting the data:

Initial mass: $m_i = 3000\text{kg}$

Fuel mass: 2400kg

Final mass (after fuel burned): $m_f = (3000 - 2400)\text{kg} = 600\text{kg}$

Exhaust velocity: $u_{rel} = 2500\text{m/s}$

Initial velocity: $v_i = 0$ (starting from rest)

Burn rate: $\frac{dm}{dt} = 15\text{kg/s}$

(a) Using the integral rocket equation with $v_i = 0$:

$$v_f = u_{rel}\ln\left(\frac{m_i}{m_f}\right)$$

Substituting values:

$$v_f = 2500\text{m/s} \times \ln\left(\frac{3000\text{kg}}{600\text{kg}}\right) = 4023\ \text{m/s}$$

The velocity is 4023m/s.

(b) The thrust force is given by the following equation:

$$F_{thrust} = u_{rel}\left(\frac{dm}{dt}\right)$$

Substituting values:

$$F_{thrust} = 2500m/s \times 15kg/s = 37500N$$

The thrust force is 37500N.

(c) Using basic rate equation:

$$\text{Burn rate} = \frac{\text{Total fuel burned}}{\text{Time taken}}$$

From which:

$$\text{Time taken} = \frac{\text{Total fuel burned}}{\text{Burn rate}} = \frac{2400kg}{15kg/s} = 160s$$

The time is 160s.

Making Sense of the Answer: *The rocket's final velocity (4km/s) is impressive but still less than what is needed for Earth orbit (~8 km/s). The mass ratio of 5:1 ($m_i/m_f = 5$) is typical for single-stage rockets. To achieve orbital velocity, rockets need multiple stages. The thrust (37500N) would give initial acceleration $a = F/m = 37500/3000 = 12.5\ m/s^2$, which increases as mass decreases.*

Think Like a Physicist: *Notice the logarithmic relationship: to double the velocity, you would need to square the mass ratio (from 5 to 25). This is why 90% of a rocket's initial mass is fuel! Also notice that thrust is constant (if dm/dt and u_{rel} are constant), but acceleration increases during the burn because mass decreases.*

HOT Example 29

A rocket of mass 800kg (including 600kg fuel) is launched vertically from Earth's surface. The engines produce a thrust of 12000N. Taking g = 9.8 m/s², calculate:

(a) The initial acceleration of the rocket.
(b) The acceleration when half the fuel has been burned.
(c) Compare the performance of a rocket operating in space with that of a rocket launching from Earth. Hence, explain why rockets are generally more efficient in space than during launch from the Earth's surface.

Solution

(a) Forces acting on rocket at launch:
- Thrust: $F_{thrust} = 12000N$ (upward)
- Weight: $W = m_i g = 800kg \times 9.8m/s^2 = 7840N$ (downward)

By Newton's second law:

$$\text{Resultant force} = F_{thrust} - W = m_i a$$

From which:

$$a = \frac{F_{thrust} - W}{m_i} = \frac{(12000 - 7860)N}{800kg} = 5.2\ m/s^2$$

The initial acceleration is $5.2\ m/s^2$ upward.

(b) When half fuel $\left(\frac{1}{2}\times 600kg = 300kg\right)$burned:
- Remaining mass: $m_f = (800 - 300)kg = 500kg$
- Thrust: still 12000N (unchanged)
- Weight: $W = m_f g = 500kg \times 9.8m/s^2 = 4900N$

Again:

$$a = \frac{F_{thrust} - W}{m_f} = \frac{(12000 - 4900)N}{500kg} = 14.2\ m/s^2$$

The acceleration is $14.2\ m/s^2$ upward.

(c) Comparison:

On Earth (launching):

- The rocket must overcome Earth's gravitational force; a significant portion of the thrust is used to overcome its weight (mg).
- It experiences air resistance: drag force opposes the motion and waste energy.
- At lower altitudes, the air is denser, resulting in greater drag forces.
- Each kilogram of fuel must not only propel the rocket forward but also lift its own mass against gravity.

In space:

- Gravitational effects are negligible compared to those at Earth's surface.
- There is no air resistance, since space is essentially a vacuum.
- The entire thrust contributes to changing the rocket's momentum (no energy is lost to drag).
- For the same thrust and mass, the rocket achieves greater acceleration because opposing forces are minimal.

Quantitative comparison:

In space:

$$a = \frac{\mathbf{F_{thrust}}}{\mathbf{m}}$$

On Earth:

$$a = \frac{F_{thrust} - mg}{m} = \frac{\mathbf{F_{thrust}}}{\mathbf{m}} - g$$

Hence, even when drag is ignored, the rocket on Earth loses exactly g ($9.8\ m/s^2$) of acceleration simply to counteract its weight.

Conclusive explanation:

In space, rockets do not waste thrust in overcoming force of gravity or air resistance. Every newton of thrust goes directly into accelerating the rocket, making them far more efficient in space than on Earth.

Making Sense of the Answer: *The acceleration nearly tripled (from 5.2 to 14.2m/s²) when half the fuel burned. This happens for two reasons: (1) mass decreased, and (2) weight decreased, so more of the thrust goes into acceleration rather than overcoming gravity. This is why rockets seem to accelerate faster and faster as they rise; they are getting lighter and escaping Earth's gravity!*

Think Like a Physicist: *The rocket equation $v = u_{rel} \times ln(m_i/m_f$ applies perfectly in space but must be modified for Earth launch to account for gravity losses. A rocket launching from Earth loses velocity $\Delta v \approx g \times t_{burn}$ due to gravity, where t_{burn} is burn time. This is called "gravity loss" and is why Earth launches need higher mass ratios than theoretical predictions suggest.*

HOT Example 30

A toy cart of mass 2kg rolls freely at 4m/s on a smooth horizontal surface. Rain begins falling vertically at a rate such that the cart collects water at 0.5kg/s. Calculate:

(a) The velocity of the cart after 3 seconds if no horizontal force is applied.
(b) The horizontal force needed to maintain the cart's velocity at 4m/s.

(c) The distance travelled in the first 3 seconds with no applied force.

Solution

(a) Mass after 3 seconds: m_3

$$m_3 = m_0 + \left(\frac{dm}{dt}\right) \times t = 2kg + 0.5kg/s \times 3s = 3.5kg$$

Conservation of horizontal momentum:

Since the rain was falling vertically, it had zero horizontal velocity initially (and therefore zero horizontal momentum). So:

$$m_0 v_0 = m_3 v_3$$

From which:

$$v_3 = \frac{m_0 v_0}{m_3} = \frac{2kg \times 4m/s}{3.5kg} = 2.29m/s$$

The velocity is $2.29m/s$.

(b) For mass increasing system at constant velocity:

$$F = v\left(\frac{dm}{dt}\right)$$

Substituting values:

$$F = 4m/s \times 0.5kg/s = 2N$$

The required force is 2N in the direction of motion.

(c) This requires integration because velocity decreases as mass increases.

From momentum conservation at any time t:

$$m_0 v_0 = \left(m_0 + \left(\frac{dm}{dt}\right) \times t\right) \times v_t$$

$$v_t = \frac{m_0 v_0}{\left(m_0 + \left(\frac{dm}{dt}\right) \times t\right)} = \frac{2 \times 4}{2 + 0.5 \times t} = \frac{8}{2 + 0.5t}$$

Distance is given by:

$$s = v_t dt = \left(\frac{8}{2 + 0.5t}\right) dt$$

After 3 seconds:

$$s_3 = \int_0^3 \left(\frac{8}{2 + 0.5t}\right) dt = 8 \times 2[\ln(2 + 0.5t)]_0^3 = 16(\ln 3.5 - \ln 2) = 16\ln\left(\frac{3.5}{2}\right) = 8.96m$$

The distance travelled is 8.96m.

Making Sense of the Answer: *The cart slowed from 4m/s to 2.29 m/s as it collected rain, losing 43% of its velocity. This happened because horizontal momentum was conserved whereby the cart shared its original momentum with the collected rainwater (which had zero horizontal momentum initially). To prevent slowing, a modest 2N force is needed to continuously accelerate the incoming rain from zero to 4m/s horizontally.*

Think Like a Physicist: *This demonstrates an important principle: when a moving object collects stationary mass, it slows down (if no force applied) because momentum is conserved but mass increases. Industrial* *conveyor systems must account for this when calculating motor requirements as the motor must provide extra force to accelerate incoming material to the belt's speed.*

HOT Example 31

A flatbed truck of mass 2000kg moves at constant velocity 15m/s along a horizontal road. Sand falls vertically onto the truck bed from a stationary hopper at a rate of 50kg/s. The truck maintains its velocity by increasing engine power. Taking coefficient of friction between truck and road as $\mu = 0.4$ and $g = 9.8\text{m/s}^2$, calculate:

(a) The force needed to accelerate the falling sand horizontally.
(b) The friction force between truck and road after 10 seconds.
(c) The total forward force needed from the engine after 10 seconds.

Solution

(a) For mass increasing system at constant velocity:

$$F = v\left(\frac{dm}{dt}\right)$$

Substituting values:

$$F = 15\text{m/s} \times 50\text{kg/s} = 750\text{N}$$

The force needed is 750N in the direction of motion.

(b) After 10 seconds:

$$m_{total} = m_{truck} + m_{sand} = m_{truck} + \left(\frac{dm}{dt}\right)t = 2000\text{kg} + 50\text{kg/s} \times 10\text{s} = 2500\text{kg}$$

Friction force: $f = \mu R = \mu mg = 0.4 \times 2500\text{kg} \times 9.8\text{m/s}^2 = 9800\text{N}$

The friction force is 9800N.

(c) The engine must overcome:
- Friction force: $f = 9800\text{N}$ (opposes motion)
- Force to accelerate incoming sand: $F_{sand} = 750\text{N}$ (continuous)

Thus, the total force needed will be:

$$F_{total} = f + F_{sand} = 9800\text{N} + 750\text{N} = 10550\text{N}$$

The total forward force needed is 10550N.

Making Sense of the Answer: *The engine must continuously provide 750N just to accelerate the incoming sand (50kg/s from 0 to 15m/s). Additionally, as sand accumulates, the truck gets heavier, increasing friction. After 10 seconds, friction dominates (9800N) compared sand acceleration force (750N).*

Think Like a Physicist: *Notice that the sand acceleration force (750N) is constant because dm/dt and v are constant. But friction increases linearly with time as mass accumulates. Eventually, friction becomes the dominant force requirement. This is why trucks can start moving easily when empty but need much more power when fully loaded; friction increases with weight.*

Important note: *We assumed the truck maintains constant velocity throughout. In reality, the engine would need to increase power continuously as friction increases. Maximum engine power limits how much sand can be loaded while maintaining velocity.*

That brings our subtopic-by-subtopic worked examples to a satisfying close. The plates are cleared! Now let us enjoy the full buffet, where all the ideas of this topic come together in miscellaneous worked examples.