Conservation of Linear Momentum

We have seen that momentum describes the *quantity of motion* an object possesses, while impulse represents the change in that momentum (Δp). From this emerges a powerful and elegant conclusion: ***forces change momentum*.** Whenever a force acts on an object, it can increase its momentum, decrease it, or simply redirect it without changing its magnitude.

But now comes a profound question: ***What happens to the total momentum when two objects interact with each other?***

For example, when two billiard balls collide, each ball experiences forces and its momentum changes. But *what about the **total momentum** of both balls combined? Does it increase, decrease, or stay the same?*

The answer to this question is one of the most beautiful and powerful principles in all of physics: **the law of conservation of linear momentum**.

Why do we study Conservation of Momentum?

Consider these situations:

Situation 1: A gun fires a bullet. The bullet shoots forward while the gun recoils backward. *How are these motions related?*

Situation 2: Two ice skaters push against each other on a frozen lake. Both move in opposite directions. *Why?*

Situation 3: A rocket in space ejects exhaust gases backward and accelerates forward even though there is nothing to "push against." *How is this possible?*

In all these cases, analysing individual forces would be extremely complicated. But conservation of momentum provides an elegant shortcut that makes solutions almost trivial.

To develop a clear and confident understanding of the law of conservation of momentum, let us examine what happens when two objects collide or interact each other. During their interaction:

- Object A exerts force F_{AB} on object B
- Object B exerts force F_{BA} on object A

Now applying the impulse-momentum theorem to each object over the interaction time Δt:

For object A:

$$F_{BA} \times \Delta t = \Delta p_A \text{ ... (equation 1)}$$

For object B:

$$F_{AB} \times \Delta t = \Delta p_B \text{ ... (equation 2)}$$

Adding equations (1) and (2):

$$F_{BA} \times \Delta t + F_{AB} \times \Delta t = \Delta p_A + \Delta p_B$$

But from Newton's third law (*Chapter 3*), F_{AB} and F_{BA} are equal in magnitude but opposite in direction:

$$\mathbf{F_{AB} = -F_{BA}}$$

It follows that:

$$F_{BA} \times \Delta t - F_{BA} \times \Delta t = \Delta p_A + \Delta p_B$$

$$0 = \Delta p_A + \Delta p_B$$

Therefore:

$$\mathbf{\Delta p_A + \Delta p_B = 0}$$

But:

$$\Delta p_A = (p_A)_{final} - (p_A)_{initial}$$

And:

$$\Delta p_B = (p_B)_{final} - (p_B)_{initial}$$

So the equation, $\Delta p_A + \Delta p_B = 0$ becomes:

$$(p_A)_{final} - (p_A)_{initial} + (p_B)_{final} - (p_B)_{initial} = 0$$

Rearranging:

$$\mathbf{(p_A)_{final} + (p_B)_{final} = (p_A)_{initial} + (p_B)_{initial}}$$

The final result is the **law of conservation of linear momentum** which can be stated as:

If no external force acts on a system, the total momentum of the system remains constant.

Or equivalently:

In an isolated system, the total momentum before any interaction equals the total momentum after the interaction.

Mathematically the law can be re-written as follows:

$$\sum p_{initial} = \sum p_{final}$$

If only two objects interact with their respective initial and final velocities: u_1, u_2 and v_1, v_2.

Then:

$$\sum p_{initial} = m_1u_1 + m_2u_2$$

And:

$$\sum p_{final} = m_1v_1 + m_2v_2$$

Hence:

$$\mathbf{m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2}$$

When Is Momentum Really Conserved?

The law of conservation of momentum applies only when there is **no net external force** acting on a system. This immediately raises an important question: *What exactly do we include in the "system"?* The answer determines whether momentum will be conserved or not.

Within a chosen system, we may have **internal forces** which *are forces that objects inside the system exert on each other*. These forces *always occur in equal and opposite pairs* (Newton's third law). Although *they can change the momentum of individual objects, they cannot change the **total** momentum of the system*. For example, during a collision, the forces between the colliding objects are internal; momentum may transfer from one object to another, but the total remains unchanged.

In contrast, **external forces** *come from outside the system* such as friction with the ground, gravity, or air resistance. Unlike internal forces, *external forces can change the total momentum* of the system because they are not balanced by equal and opposite partners within the system.

The key distinction, therefore, is this:

- **Momentum is conserved** when all interacting objects are included in the system and there is no net external force.
- **Momentum is not conserved** when an unbalanced external force acts on the system.

In practice, even when external forces like gravity exist, if they are balanced or act for a very short time during a collision, their effect may be negligible. In such cases, we can often treat momentum as approximately conserved during the interaction.

Examples of isolated systems:

To make these ideas concrete, let us examine a few real-life situations where systems can be treated as isolated and momentum is conserved.

1. **Two ice skaters pushing apart:**

On frictionless ice, no external horizontal forces exist. Momentum is conserved.

2. **Gun and bullet:**

Before firing, total momentum is zero. After firing, bullet goes forward with momentum $+p$, gun recoils backward with momentum $-p$, so total momentum remains zero.

3. **Rocket in space:**

The rocket and ejected gases form an isolated system. Total momentum is conserved.

4. **Colliding billiard balls:**

During the brief collision (milliseconds), external forces like friction and air resistance have negligible effect. Momentum is approximately conserved during the collision.

Momentum Conservation in Different Directions: Signs, Application, and Common Misconceptions

Since momentum is a **vector quantity**, its conservation must be considered separately along each direction. This makes analysis clearer, especially in collisions and interactions involving motion in more than one direction.

In one dimension (along x-axis):

$$\sum (p_x)_{\text{initial}} = \sum (p_x)_{\text{final}}$$

In two dimensions:

$$\sum (p_x)_{\text{initial}} = \sum (p_x)_{\text{final}}$$

$$\sum (p_y)_{\text{initial}} = \sum (p_y)_{\text{final}}$$

This means you can apply the law of conservation of momentum separately in the x-direction and y-direction. To do this effectively:

1. Choose a positive direction (usually **right** or **upward**).
2. Velocities in that direction are positive.
3. Velocities opposite are negative.
4. Apply the conservation: $\Sigma p_{\text{initial}} = \Sigma p_{\text{final}}$ with proper signs.

However, some common misconceptions can interfere with correct application. This include the following:

Misconception 1: "Momentum is always conserved."

Truth: *Momentum is conserved only when no net external force acts on the system.*

Misconception 2: "If two objects collide, each keeps its original momentum."

Truth: *Individual momenta change; only the total momentum is conserved.*

Misconception 3: "Conservation of momentum means objects do not change velocity."

Truth: *Objects can and do change velocity. What stays constant is the total momentum.*

With this solid theoretical foundation established, let us now bring these ideas to life through carefully crafted worked examples.

BINDER Example 6

A boy of mass 40kg stands on a stationary skateboard of mass 2kg. He jumps off the skateboard horizontally with a velocity of 3m/s. Calculate the velocity of the skateboard immediately after he jumps.

Solution

Interpreting the data:

Mass of boy: $m_1 = 40\text{kg}$

Mass of skateboard: $m_2 = 2\text{kg}$

Initial velocity of both (at rest): $u_1 = u_2 = 0\text{m/s}$

Final velocity of boy: $v_1 = 3\text{m/s}$ (take as positive direction)

Final velocity of skateboard: v_2 (to find)

Applying conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Since both start at rest:

$$0 + 0 = m_1v_1 + m_2v_2$$

Rearranging to make v_2 the subject:

$$v_2 = \frac{-m_1v_1}{m_2}$$

Substituting:

$$v_2 = -\frac{40\text{kg} \times 3\text{m/s}}{2\text{m/s}} = -60\text{m/s}$$

The skateboard moves at 60m/s in the opposite direction to the boy's jump.

Making Sense of the Answer: *The negative sign indicates the skateboard moves backward (opposite to the boy's jump direction). The skateboard moves much faster (60m/s) than the boy (3m/s) even though the boy is much heavier. This happens because momentum depends on both mass and velocity: the light skateboard needs high velocity to balance the momentum of the heavy boy moving at low velocity.*

Think Like a Physicist: *Before the jump, total momentum was zero. After the jump, the boy's momentum is +120 kg·m/s and the skateboard's momentum is −120 kg·m/s. They cancel perfectly, keeping total momentum at zero. This demonstrates that* ***when a system starts at rest, objects must move in opposite directions with equal and opposite momenta.***

BINDER Example 7

Two ice skaters stand facing each other on a horizontal ice surface (assume frictionless). Skater A has mass 50kg and skater B has mass 60kg. They push against each other and move apart. If skater A moves backward at 2.4m/s, find the velocity of skater B.

Solution

Interpreting the data:

Mass of A: $m_A = 50\text{kg}$

Mass of B: $m_B = 60\text{kg}$

Initial velocities: $u_A = u_B = 0\text{m/s}$ (both at rest)

Final velocity of A: $v_A = -2.4\text{ m/s}$ (backward, take as negative)

Final velocity of B: v_B (to find)

Again applying conservation of momentum:

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

$$0 = m_A v_A + m_B v_B$$

Making v_B the subject and substituting values:

$$v_B = -\frac{m_A v_A}{m_B} = -\frac{50\text{kg} \times (-2.4\text{m/s})}{60\text{kg}} = +2\text{m/s}$$

The velocity of skater B is 2m/s forward.

Making Sense of the Answer: *The positive sign indicates skater B moves forward (opposite to skater A). Skater B is heavier (60kg vs 50kg) so moves more slowly (2m/s vs 2.4 m/s) to achieve equal and opposite momentum. Check: momentum of A = 50 × (−2.4) = −120 kg·m/s; momentum of B = 60 × 2.0 = +120 kgm/s. They cancel!* ✓

Think Like a Physicist: *The skaters pushed against each other with equal and opposite forces (Newton's third law). These are internal forces that changed individual momenta but not the total. This is why momentum conservation is so powerful; with it we did not need to know how hard they pushed or for how long!*

REAL Example 8

A fisherman of mass 80kg stands at one end of a stationary boat of mass 120kg floating on still water. He walks 4 metres toward the other end of the boat. Assuming the water offers negligible resistance, describe what happens to the boat.

Solution

Since the fisherman and the boat form an isolated system (water resistance is negligible) and both start from rest, the **total momentum remains zero** throughout the motion.

As the fisherman walks forward, he gains momentum in the forward direction. To keep the total momentum zero, the boat must gain an equal momentum in the opposite direction. Hence, **the boat moves backward**.

Applying conservation of momentum:

$$0 = m_f v_f + m_b v_b$$

Where: m_f, m_b are the masses of fisherman and boat, and v_f, v_b are their velocities relative to ground (water).

Making v_b the subject and substituting values:

$$v_b = -\frac{m_f v_f}{m_b} = -\frac{80\text{kg}v_f}{120\text{kg}} = \frac{2}{3}v_f \text{ or } \frac{2}{3} = -\frac{v_b}{v_f}$$

But for constant time (both fisherman and boat moved at the same time):$v \propto \text{displacement}, s$. Thus:

$$\frac{v_b}{v_f} = \frac{s_b}{s_f}$$

It follows that:

$$\frac{2}{3} = -\frac{s_b}{s_f}$$

But the fisherman moves **4m relative to the boat**, not relative to the water (ground). Therefore:

Relative displacement = $s_f - s_b = 4;\ s_f = 4 + s_b$

Substituting:

$$\frac{2}{3} = -\frac{s_b}{4 + s_b} \text{ or } 8 + 2s_b = -3s_b$$

Solving the equation gives:

$$s_b = -1.6\text{m (and } s_f = 4 + s_b = 2.4\text{m)}$$

Hence, the boat moves 1.6m backward, opposite to the fisherman's motion.

Making Sense of the Answer: *The fisherman is lighter than the boat, so he moves a greater distance (2.4m) while the boat moves less (1.6m). But because the boat is heavier, their momenta can still cancel. The fisherman thinks he walked 4m along the boat, but relative to the shore, he only moved 2.4m (4m−1.6m) forward!*

Think Like a Physicist: *This example shows that momentum conservation applies even when objects do not collide or push apart suddenly. Any internal forces (fisherman's feet pushing on boat) cannot change the total momentum of the system. This is why you cannot make a boat move forward by standing inside it and pushing on its walls; you are part of the system!*

REAL Example 9

Kipute and Kipanga are playing with a toy cannon at school during physics club. They load it with a small ball and place it on a smooth table.

Kipanga: *"Watch! When I fire the cannon, the ball will shoot forward very fast!"*

He fires. The ball shoots forward at high speed, but the cannon rolls backward off the table and crashes to the floor.

Kipanga: *"Oh no! I didn't expect that!"*

Kipute (laughing): *"That's momentum conservation, Kipanga! You forgot about the cannon recoiling!"*

Mr. Akilikubwa (walking over): *"Excellent demonstration, though unintentional! Kipanga, can you calculate how fast the cannon was moving when it fell off the table?"*

Information:

- Mass of ball: 0.05kg
- Mass of cannon: 2kg
- Velocity of ball after firing: 40m/s forward
- Both cannon and ball were initially at rest.

Questions:

(a) Calculate the recoil velocity of the cannon.
(b) If the cannon were bolted firmly to the table so it could not recoil, explain what would happen and state whether momentum would still be conserved.

Solution

(a) Initial total momentum = 0 (both at rest)

By conservation of momentum:

$$m_{ball}v_{ball} + m_{cannon}v_{cannon} = 0$$

From which:

$$v_{cannon} = -\frac{m_{ball}v_{ball}}{m_{cannon}} = -\frac{0.05kg \times 40m/s}{2kg} = -1m/s$$

The recoil velocity is $1m/s$ backward.

(b) If the cannon were bolted firmly to the table, the cannon, table, and Earth would recoil together. However, because the Earth's mass is extremely large, the resulting recoil velocity would be extremely small and practically undetectable. Momentum would still be conserved, but the Earth's motion would be negligible and impossible to detect.

Making Sense of the Answer: T*he cannon moved much slower (1m/s) than the ball (40m/s) because it is much heavier (2kg vs 0.05kg; that is 40 times heavier!). Despite the huge difference in speeds, their momenta are equal and opposite: ball's momentum = +2kgm/s, cannon's momentum = −2kgm/s. Total = zero, as required.*

Think Like a Physicist: *This is exactly how real cannons and guns work. The recoil is unavoidable; it is a direct consequence of momentum conservation. Artillery soldiers must brace cannons to prevent them from rolling backward. This also explains rocket propulsion: rockets do not need anything to "push against"; they eject gases backward (like the cannon ball) and recoil forward (like the cannon), all by momentum conservation!*

HOT Example 10

A railway wagon of mass 2000kg moving at 5m/s collides and couples with a stationary wagon of mass 3000kg. Calculate:

(a) The common velocity of the two wagons after coupling.
(b) The loss in kinetic energy during the collision.
(c) Explain clearly what happened to the "lost" kinetic energy.

Solution

Interpreting the data:

Mass of moving wagon: $m_1 = 2000kg$

Mass of stationary wagon: $m_2 = 3000kg$

Initial velocity of moving wagon: $u_1 = 5m/s$

Initial velocity of stationary wagon: $u_2 = 0m/s$

After coupling, both move together: $v_1 = v_2 = v$ (common velocity)

(a) Applying conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v + m_2v$$

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

Making v the subject:

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

Substituting:

$$v = \frac{2000kg \times 5m/s + 3000kg \times 0m/s}{2000kg + 3000kg} = 2m/s$$

The common velocity is $2m/s$.

(b) The loss in kinetic energy is given by:

$$\text{Loss in K.E} = \text{Initial kinetic energy} - \text{Final kinetic energy}$$

Initial kinetic energy $= ½m_1u_1^2 + ½m_2u_2^2 = ½(2000kg)(5m/s)^2 + ½(3000kg)(0m/s)^2$

$= 25000J$

Final kinetic energy $= ½(m_1 + m_2)v^2 = ½(5000kg)(2m/s)^2 = 10000J$

Loss in $K.E = 25000J - 10000J = 15000J$

The loss in kinetic energy is $15000J$.

(c) The lost kinetic energy (15,000 J) was converted into other forms of energy such as:

1. **Heat (thermal) energy:** Friction between the coupling mechanisms and deformation of metal generated heat
2. **Sound energy:** The loud "clang" when wagons collided carried away energy as sound waves.
3. **Deformation energy:** Slight permanent deformation of the coupling devices and wagon structures.
4. **Vibrational energy:** The wagons vibrated after impact, which eventually dissipated as heat.

So the energy was not destroyed (energy is always conserved), but it changed from organised kinetic energy into disorganised forms that cannot be recovered as useful motion.

Making Sense of the Answer: *The final velocity (2m/s) is less than the initial velocity (5m/s) of the moving wagon, which makes sense because the moving wagon had to accelerate the stationary wagon, sharing its momentum. Notice that 60% of the kinetic energy was lost (15000 out of 25000J), yet momentum was perfectly conserved. This demonstrates that* ***momentum conservation and kinetic energy conservation are independent principles!***

Think Like a Physicist: *Momentum is* ***always*** *conserved in isolated systems (it is a fundamental law), but kinetic energy is only conserved in elastic collisions. Most real collisions are inelastic as some kinetic energy always converts to heat and sound. We will explore elastic vs inelastic collisions in depth later in this chapter.*

HOT Example 11

A projectile of mass 10kg moving horizontally at 50m/s explodes into two fragments (**A** and **B**). Fragment **A** has mass 4kg and moves at 80m/s in the original direction. Calculate:

(a) The velocity of fragment **B**.
(b) The energy released by the explosion.

Solution

Interpreting the data:

Initial mass: $m = 10kg$

Initial velocity: $u = 50m/s$ (horizontal, take as positive)

Mass of fragment A: $m_A = 4kg$

Velocity of fragment A: $v_A = 80m/s$ (positive direction)

Mass of fragment B: $m_B = 10kg - 4kg = 6kg$

Velocity of fragment B: v_B (to find)

(a) Applying conservation of momentum:

$$mu = m_Av_A + m_Bv_B$$

Making v_B the subject:

$$v_B = \frac{mu - m_Av_A}{m_B}$$

Substituting:

$$v_B = \frac{10\text{kg} \times 50\text{m/s} - 4\text{kg} \times 80\text{m/s}}{6\text{kg}} = 30\text{m/s}$$

The velocity of fragment B is 30m/s in the original direction.

(b) The energy released is given by:

$\text{Energy released} = \text{Gain in K.E} = \text{Final kinetic energy} - \text{Initial kinetic energy}$

Where:

$\text{Initial kinetic energy} = \text{Kinetic energy before explosion}$

$$= \tfrac{1}{2}mu^2 = \tfrac{1}{2}(10\text{kg})(50\text{m/s})^2 = 12500\text{J}$$

$\text{Final kinetic energy} = \text{Kinetic energy after explosion}$

$$= \tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2 = \tfrac{1}{2}(4\text{kg})(80\text{m/s})^2 + \tfrac{1}{2}(6\text{kg})(30\text{m/s})^2 = 15500\text{J}$$

$\text{Energy released} = 15500\text{J} - 12500\text{J} = 3000\text{J}$

The energy released by the explosion is 3000J.

Making Sense of the Answer: *Fragment A (the lighter piece) was blown forward faster than the original projectile (80m/s vs 50m/s), while fragment B (the heavier piece) slowed down to 30m/s. Both moved forward because the explosion happened while the projectile was already moving. The explosion added 3000J of energy (from chemical energy of the explosive) to the system, increasing the total kinetic energy from 12500 J to 15500J.*

Think Like a Physicist: *This example shows that:*

1. *Momentum is always conserved (even during explosions).*
2. *Energy is conserved if we account for **all** forms (the 3000J came from chemical energy).*
3. *Kinetic energy alone is **not** conserved during explosions; it increases! (This is opposite to inelastic collisions where kinetic energy decreases).*

The worked examples have beautifully demonstrated the power and universality of momentum conservation. Whether objects collide, push apart, explode, or move within a system, the total momentum remains constant as long as no external forces act. This principle will serve as the foundation for our study of collisions in the next subtopic.