Connected Bodies on an Inclined Plane

You have now mastered two connected-body configurations: masses hanging vertically over a pulley, and masses moving horizontally on surfaces. Both involved motion in a single direction: pure vertical or pure horizontal. Now we combine these ideas: one mass moves along an incline while another hangs vertically. This is perhaps the most common connected-body scenario in real engineering such as mountain railway systems, mine hoists, cable cars, and even simple construction pulleys lifting materials up slopes.

The general setup:

Consider a mass m_1 rests on an inclined plane making angle θ with the horizontal. It is connected by a light inextensible string passing over a smooth pulley fixed at the top of the incline to a mass m_2 hanging freely on the vertical side.

When released, the system moves in a direction determined by the relative magnitudes of m_1, m_2, and angle θ. Because the string is inextensible, if m_1 moves distance d up the slope, then m_2 descends distance d vertically. Therefore, both masses share the same acceleration magnitude a, though they move in different directions (one along the incline, another vertically).

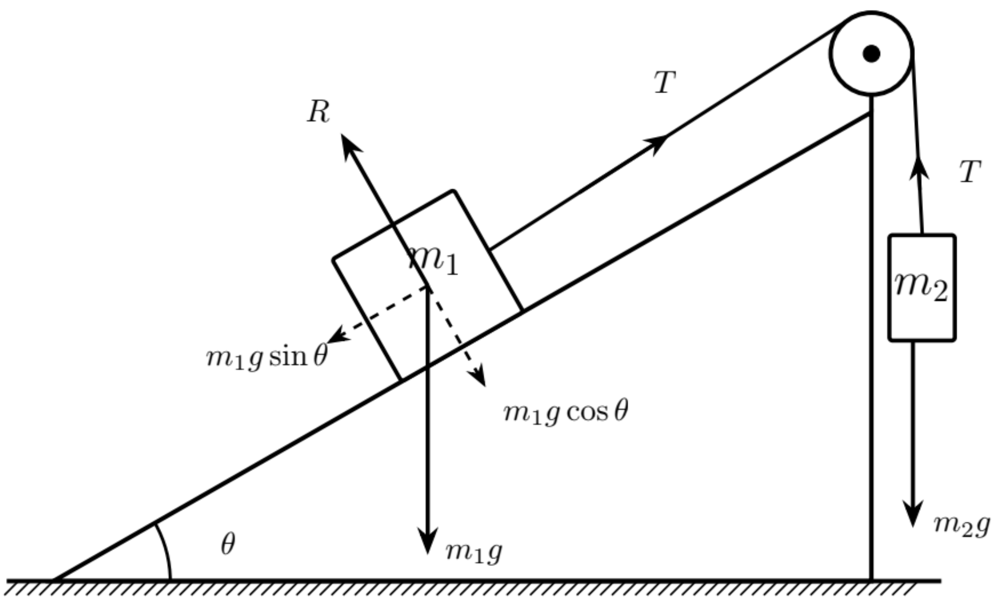

Figure: *Connected-body system on an inclined plane.*

First case: Smooth incline (no friction)

Forces acting on m_1:

- Weight: m_1g (vertically downward)
  - ✓ Component parallel to incline: $m_1g\sin\theta$ (down the slope)
  - ✓ Component perpendicular to incline: $m_1g\cos\theta$ (into the incline)
- Tension: T (up the slope, parallel to incline)
- Normal reaction: $R = m_1g\cos\theta$ (perpendicular to slope, outward)

Forces acting on m_2 (being pulled by string):

- Weight: m_2g (downward)
- Tension: T (upward)

Identifying forces trying to move the system:

- Weight of hanging mass, m_2: $W = m_2g$ (try to move m_1 up the slope while m_2 is moving vertically downward).
- Weight component of m_1 parallel to incline: $m_1g\sin\theta$ (try to move m_2 vertically upward while m_1 is moving down the slope).

So two possibilities exist on direction of motion:

1) If $\mathbf{m_2g > m_1g\sin\theta}$: Mass m_2 is heavier (or the incline is gentle as θ is small), so m_2 descends and m_1 moves up the slope.
2) If $\mathbf{m_2g < m_1g\sin\theta}$: Weight component of m_1 dominates, so m_1 slides down and m_2 ascends.

For analysis, assume case (1): m_2 descends, m_1 ascends.

Applying Newton's second law:

For m_1:

$T - m_1g\sin\theta = m_1a$ ... (equation 1)

For m_2:

$m_2g - T = m_2a$ ... (equation 2)

Finding acceleration:

Adding equations (1) and (2):

$T - m_1g\sin\theta + m_2g - T = m_1a + m_2a$

$m_2g - m_1g\sin\theta = (m_1 + m_2)a$

Making a the subject:

$$a = \frac{m_2g - m_1g\sin\theta}{m_1 + m_2}$$

Hence:

$$\mathbf{a = \frac{(m_2 - m_1\sin\theta)g}{m_1 + m_2}}$$

Alternatively; the pulling force (m_2g) must overcome weight component of m_1 parallel to incline ($m_1g\sin\theta$) before accelerating both m_1 and m_2 with acceleration a. Thus:

$$m_2g - m_1g\sin\theta = (m_1 + m_2)a$$

From which:

$$a = \frac{m_2g - m_1g\sin\theta}{m_1 + m_2} = \mathbf{\frac{(m_2 - m_1\sin\theta)g}{m_1 + m_2}}$$

Finding tension:

From equation (1):

$T = m_1a + m_1g\sin\theta = m_1(a + g\sin\theta)$

Substituting expression for a:

$$T = m_1\left(\frac{(m_2 - m_1\sin\theta)g}{m_1 + m_2} + g\sin\theta\right)$$

$$T = m_1g\left(\frac{m_2 - m_1\sin\theta + m_1\sin\theta + m_2\sin\theta}{m_1 + m_2}\right)$$

$$T = m_1g\left(\frac{m_2(1 + \sin\theta)}{m_1 + m_2}\right)$$

Hence:

$$\mathbf{T = \frac{m_1m_2g(1 + \sin\theta)}{m_1 + m_2}}$$

Interesting physical interpretation

From the acceleration and tension equations, we can deduce that:

- If $\theta = 0°$ (horizontal surface): $\sin\theta = 0$, so $a = m_2g/(m_1 + m_2)$, $T = m_1m_2g/(m_1 + m_2)$ which matches the horizontal case we studied earlier.
- If $\theta = 90°$ (vertical pulley system): $\sin\theta = 1$, therefore $a = (m_2 - m_1)g/(m_1 + m_2)$, and $T = 2m_1m_2g/(m_1 + m_2)$ which matches the vertical case (Atwood machine).

These limiting cases confirm that our general results are consistent with the familiar horizontal and vertical configurations. Let us now extend the analysis further by introducing friction, leading us to the more realistic case of motion on a rough incline.

Second case: Rough incline (With friction)

When the incline surface is rough with coefficient of kinetic friction μ, friction opposes the motion of m_1 along the slope.

Friction force: $f = \mu R = \mu m_1 g\cos\theta$

The direction of friction depends on the direction of motion of m_1.

- If m_1 moves up the slope: Friction acts down the slope (opposing upward motion).
- If m_1 moves down the slope: Friction acts up the slope (opposing downward motion).

Now, let us return to our assumed scenario where m_2 descends, m_1 ascends.

For this case, the pulling force (m_2g) must overcome both weight component of m_1 parallel to incline ($m_1g\sin\theta$) and frictional force ($\mu m_1g\cos\theta$) before accelerating both m_1 and m_2 with acceleration a. Thus:

$$m_2g - m_1g\sin\theta - \mu m_1g\cos\theta = (m_1 + m_2)a$$

From which:

$$a = \frac{m_2g - m_1g\sin\theta - \mu m_1g\cos\theta}{m_1 + m_2} = \frac{m_2g - (m_1g\sin\theta + \mu m_1g\cos\theta)}{m_1 + m_2}$$

$$a = \frac{m_2g - m_1g(\sin\theta + \mu\cos\theta)}{m_1 + m_2}$$

Hence:

$$\mathbf{a = \frac{\big(m_2 - m_1(\sin\theta + \mu\cos\theta)\big)g}{m_1 + m_2}}$$

Once the acceleration has been determined, the tension can be found by considering forces acting on the hanging mass.

That is:

$$m_2g - T = m_2a$$

From which:

$$\mathbf{T = m_2g - m_2a}$$

Never overlook this important condition!

For motion to begin from rest, the driving force must first overcome static friction. In this system, the weight of the hanging mass must exceed the combined resisting forces along the incline:

$$m_2g > m_1g\sin\theta + \mu_s m_1 g\cos\theta$$

Where μ_s is the coefficient of static friction. If this condition is not met, the system remains stationary.

The ideas have finished cooking; now let us sit down and enjoy them through a few worked examples.

BINDER Example 27

A block of mass 6kg rests on a smooth plane inclined at 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a hanging mass of 4kg. The system is released from rest. Take $g = 9.8\text{m/s}^2$.

(a) Determine the acceleration of the system.
(b) Determine the tension in the string.

Solution

Identifying the system:

$m_1 = 6\text{kg}$ (on incline)

$m_2 = 4\text{kg}$ (hanging)

$\theta = 30°$

Surface: smooth (no friction)

Comparing m_2g and $m_1g\sin\theta$ to determine the direction of motion:

$m_2g = 4\text{kg} \times 9.8\text{m/s}^2 = 39.2\text{N}$ (pulls m_1 up)

$m_1g\sin\theta = 6\text{kg} \times 9.8\text{m/s}^2 \times \sin 30° = 29.4\text{N}$ (pulls m_1 down)

Since m_2g (39.2N) $>$ $m_1g\sin\theta$ (29.4N), the hanging mass moves downward and pulls the block up the plane.

(a) The weight of hanging mass (m_2g) must overcome weight component of m_1 parallel to incline ($m_1g\sin\theta$) before accelerating both m_1 and m_2 with acceleration a. Thus:

$$m_2g - m_1g\sin\theta = (m_1 + m_2)a$$

From which:

$$a = \frac{m_2g - m_1g\sin\theta}{m_1 + m_2} = \frac{9.8\text{m/s}^2(4\text{kg} - 6\text{kg}\sin 30°)}{6\text{kg} + 4\text{kg}} = 0.98\text{m/s}^2$$

The acceleration is 0.98m/s^2.

(b) Considering forces acting on the hanging mass, m_2.

$$m_2g - T = m_2a \text{ or } T = m_2g - m_2a$$

Substituting:

$$T = (4\text{kg} \times 9.8\text{m/s}^2) - (4\text{kg} \times 0.98\text{m/s}^2) = 33.32\text{N}$$

The tension is 35.28N.

Making Sense of the Answer: *The hanging side is stronger ($39.2N$) than the block's down-slope pull ($29.4N$), so the system accelerates but only gently because both masses must be moved together. The tension is slightly less than $39.2N$ (since the hanging mass is accelerating downward) and greater than $29.4N$ (since the block is being pulled up the plane), which fits $T = 35.28N$.*

Thinking Like a Physicist: *On inclined connected-body problems, always compare m_2g with $m_1gsin\theta$ first to predict the direction. That single comparison prevents wrong friction directions later and saves you from sign confusion. A quick check is that the tension should lie between the two opposing pulls along the string.*

BINDER Example 28

A block of mass 9kg rests on a rough plane inclined at 40° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a hanging mass of 4kg. The coefficient of friction between the block and the plane is $\mu = 0.20$. The system is released from rest. Take $g = 9.8\text{m/s}^2$.

Calculate: (a) the acceleration of the system, (b) the tension in the string.

Solution

Identifying the system:

$m_1 = 9$kg (on incline)

$m_2 = 4$kg (hanging)

$\theta = 40°$

Surface: rough with $\mu = 0.20$

Determining the direction motion:

In this case, the driving force must overcome both the opposing pull from the other mass and the frictional force. Therefore, the comparison is either between m_2g and ($m_1g\sin\theta + f$), or between $m_1g\sin\theta$ and ($m_2g + f$), where:

- $m_2g = 4\text{kg} \times 9.8\text{m/s}^2 = 39.2\text{N}$
- $m_1g\sin\theta = 9\text{kg} \times 9.8\text{m/s}^2 \times \sin 40° = 56.7\text{N}$
- $f = \mu m_1 g\cos\theta = 0.2 \times 9\text{kg} \times 9.8\text{m/s}^2 \times \cos 40° = 13.5\text{N}$

So $m_1g\sin\theta$ (56.7N) is greater than $m_1g\sin\theta + f$ (52.7N). Hence, the block moves down the plane and the hanging mass moves upward.

(a) The driving force ($m_1g\sin\theta$) must overcome all resistive forces so as to accelerate both m_1 and m_2 with acceleration a. Thus:

$$m_1g\sin\theta - m_2g - f = (m_1 + m_2)a \text{ or } a = \frac{m_1g\sin\theta - m_2g - f}{m_1 + m_2}$$

Substituting:

$$a = \frac{(56.7 - 39.2 - 13.5)\text{N}}{(9 + 4)\text{kg}} = 0.31\text{m/s}^2$$

The acceleration is 0.31m/s^2.

(b) Considering forces acting on the hanging mass, m_2.

$$T - m_2g = m_2a \text{ or } T = m_2g + m_2a$$

Substituting:

$$T = 39.2\text{N} + (4\text{kg} \times 0.31\text{m/s}^2) = 40.44\text{N}$$

The tension is 40.44N.

Making Sense of the Answer: *The block's down-slope pull (56.7N) is slightly larger than the combined resistance from the hanging weight and friction (52.7N), so motion occurs but slowly. The tension lies between the opposing pulls, which confirms the result is physically reasonable.*

Thinking Like a Physicist: *Never assume the hanging mass must move downward. Always compare the possible driving force with its corresponding resistive forces. That quick comparison tells you the direction before any algebra and saves many sign mistakes.*

REAL Example 29

During road construction, a small machine is lowered down a sloping surface using a rope connected over a pulley to a counterweight. Workers observe that when the slope is made steeper, the machine begins to move downward even though the counterweight remains unchanged. Account for this observation.

Solution

As the slope becomes steeper, the component of the machine's weight acting down the slope increases. This increases the driving pull downward along the rope. If this down-slope pull becomes greater than the upward pull provided by the counterweight (together with friction), the machine will begin to move downward. Therefore, increasing the slope angle changed the balance of forces and altered the direction of motion even though the masses remained the same.

Making Sense of the Answer: *A steeper slope (greater value of $\sin\theta$) means a stronger downward pull along the ramp. Even without changing the counterweight, the balance can shift simply because geometry ($\sin\theta$) changes the effective force along the slope.*

Thinking Like a Physicist: *Always remember: on an inclined plane, the angle controls how much of the weight acts along the slope. Changing the angle changes the driving force even if the mass stays constant.*

HOT Example 30

A block of mass 10kg rests on a rough plane inclined at 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a hanging mass of 4kg. The coefficient of static friction between the block and the plane is $\mu_s = 0.25$, and the coefficient of kinetic friction is $\mu_k = 0.2$. The system is released from rest. Take $g = 9.8\text{m/s}^2$.

(a) Determine whether the system moves. If it moves, state the direction of motion.
(b) If it moves, calculate the acceleration. If it does not move, determine the frictional force and the tension in the string.

Solution

(a) *Calculating key forces in the system:*

For the 10kg block:

Weight component parallel to the plane: $m_1 g\sin30° = 10 \times 9.8 \times 0.5 = 49\text{N}$

Limiting friction: $f_{max} = \mu_s R = \mu m_1 g\cos30° = 0.25 \times 10\text{kg} \times 9.8\text{m/s}^2 \times \sin30° = 21.22\text{N}$

For the hanging mass:

Weight: $m_2 g = 4\text{kg} \times 9.8\ \text{m/s}^2 = 39.2\text{N}$

Testing possible directions:

First possibility: Suppose the hanging mass goes down (block moves up the plane).

The weight of hanging mass must overcome: $m_1 g\sin30° + f_{max} = (49 + 21.22)\text{N} = 70.22\text{N}$

But the weight, $m_2 g = 39.2\text{N} < 70.22\text{N}(m_1 g\sin30° + f_{max})$. So motion in this direction is impossible.

Second possibility: Suppose the block goes down the plane (hanging mass moves upward).

The weight acting along the plane must overcome: $m_2g + f_{max} = (39.2 + 21.22)N = 60.42N$

But the weight acting along the plane, $m_1gsin30° = 49N < 60.42N(m_2g + f_{max})$. So motion in this direction is impossible too.

Conclusion: The system does **not** move from rest. Static friction holds it in equilibrium.

(b) Finding frictional force and tension (since $a = 0$).

Since $m_1gsin30°(49N) > m_2g$ (39.2N), the block on the plane attempts to move down the plane.

To move in this direction, the weight component has to overcome the weight of hanging mass and static friction. Thus for the block along the plane in **equilibrium:**

$$m_1gsin30° - m_2g - f = 0 \text{ or } f = m_1gsin30° - m_2g$$

Substituting:

$$f = (49 - 39.2)N = 9.8N$$

The frictional force is $9.8N$.

Considering forces acting on the hanging mass in equilibrium, m_2.

$$T - m_2g = 0 \text{ or } T = m_2g = 39.2N$$

The tension is $39.2N$.

Making Sense of the Answer: *Even though the block has a strong down-slope pull ($49N$), the hanging mass provides a $39.2N$ counter-pull through the tension. Only $9.8N$ of friction is needed to stop motion, and since the surface can provide up to $21.22N$, the system stays comfortably at rest. Technically, the system is "locked" by friction.*

Thinking Like a Physicist: *In real engineering, this "locking" effect is both useful and dangerous. Useful: prevents cable cars from sliding backward when motors are off. Dangerous: can trap loads on slopes, requiring extra force to start motion. Engineers must calculate whether static friction will hold loads safely OR whether it will prevent necessary motion. Sometimes "stuck" is good; sometimes it is catastrophic.*

As the worked examples quietly leave the table, the next subtopic arrives; not to overwhelm us, but to be understood and enjoyed.