MODULE 20

CONNECTED BODIES IN VERTICAL MOTION

Introduction

Imagine two students connected by a rope during a tug-of-war revision session. One pulls confidently, the other resists bravely, and suddenly both start moving together, not because either planned it, but because the forces refused to stay balanced. Physics describes such situations as **motion of connected bodies**.

In the previous sections, you mastered equilibrium for single objects: books on tables, blocks on inclines, and signboards hanging from cables. But the real world rarely presents isolated objects. Instead, systems work together: elevators carrying people, cranes lifting loads, cars towing trailers, and even your school bag hanging from a desk hook while pulling on the strap.

When two or more objects are physically connected by strings, cables, ropes, chains, or rigid bars, they form what we call a **connected bodies system**. The fascinating aspect of such systems is this: although the objects may have different masses and experience different forces, they often share the same motion. A hanging mass pulls on a string that accelerates a block on a table. The string transmits force between them, and both objects accelerate together (though in different directions).

What makes this different from what we have studied?

Unlike equilibrium situations where forces cancel perfectly, connected bodies often reveal what happens when that balance is slightly disturbed.

Previously, we analysed equilibrium where acceleration was zero. Now we extend our understanding to connected systems where acceleration may occur, but the key principle remains: we apply Newton's laws systematically to each body, paying special attention to how the connection (string or cable) transmits force through tension.

Instead of the equilibrium condition: $\Sigma F = 0$; we now apply Newton's second law: $\Sigma F = ma$.

Thus, this topic does not replace equilibrium principles; it builds directly on them.

Three common configurations:

Connected bodies can move in three main ways:

1. **Vertical motion:** Bodies connected by a string over a pulley, where one hangs and pulls the other vertically, as in an elevator and its counterweight system.
2. **Horizontal motion:** Bodies connected by a string or tow-bar on a horizontal surface, where one pulls the other along the same plane, similar to a car towing a trailer.
3. **Inclined plane motion:** Bodies connected by a string over a pulley at the top of an inclined plane, where one moves up or down the slope while the other hangs vertically, as seen in some mountain rescue lifting systems.

Let us explore each configuration systematically, building your understanding step by step.

Connected Bodies in Vertical Motion

One of the simplest connected-body systems consists of two masses joined by a light string passing over a smooth pulley. Such a system is often called an **Atwood-type machine**.

Consider two masses m_1 and m_2 connected by a light, inextensible string that passes over a smooth (frictionless) pulley fixed to the ceiling. The masses hang freely on either side of the pulley.

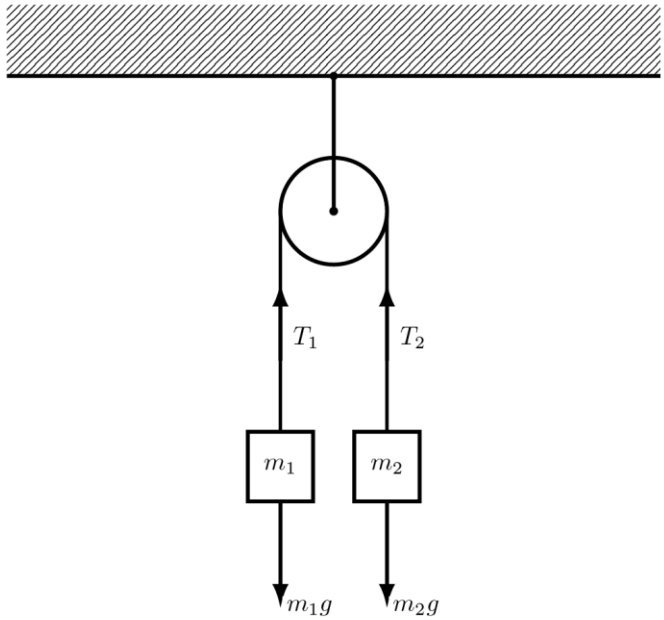


Figure: *Two masses m_1 and m_2 connected by a light inextensible string passing over a smooth pulley. Tensions T_1 and T_2 act along the string, while the weights m_1g and m_2g act vertically downward.*

Three possible scenarios:

1) **If $\mathbf{m_1 = m_2}$**: The system is in equilibrium. There is no acceleration; the masses remain stationary (or move with constant velocity if already in motion).
2) **If $\mathbf{m_1 < m_2}$**: The heavier mass m_2 pulls the system downward. Mass m_2 accelerates downward while m_1 accelerates upward. Both have the **same** magnitude of acceleration **a**.
3) **If $\mathbf{m_1 > m_2}$**: The heavier mass m_1 pulls the system downward. Mass m_1 accelerates downward while m_2 accelerates upward, again with the same acceleration magnitude a.

Deriving the acceleration and tension formulas:

Let us analyse the case where m_2 is heavier than m_1 ($m_2 > m_1$)

Acceleration formula:

Forces on mass m_1 (moving upward):

- Weight: $W_1 = m_1g$ (downward)
- Tension: T (upward)

Forces on mass m_2 (moving downward):

- Weight: $W_2 = m_2g$ (downward)
- Tension: T (upward)

Applying Newton's second law to each body

For m_1 (taking upward as positive): $\Sigma F_1 = m_1 a;\ T_1 - m_1 g = m_1 a$ ... (equation 1)

For m_2 (taking downward as positive): $\Sigma F_2 = m_2 a;\ m_2 g - T_2 = m_2 a$ ... (equation 2)

Solving the simultaneous equations:

Since the pulley is smooth, $T_1 = T_2 = T$

Adding equation 1 and equation 2:

$$(T - m_1 g) + (m_2 g - T) = m_1 a + m_2 a$$

$$m_2 g - m_1 g = m_1 a + m_2 a$$

$$(m_2 - m_1)g = (m_1 + m_2)a$$

Hence, acceleration is:

$$\mathbf{a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g}$$

The final result is our key formula for acceleration in vertical connected bodies.

The formula shows:

- Greater mass difference leads to greater acceleration. If $m_2 >> m_1$ (much heavier), then a approaches g, meaning the system accelerates nearly in free fall.
- Equal masses ($m_2 - m_1 = 0$) means zero acceleration (equilibrium).
- The acceleration cannot exceed the acceleration due to gravity, g.

$$\left(\text{Since } (m_2 - m_1) < (m_1 + m_2), \left(\frac{m_2 - m_1}{m_1 + m_2}\right) < 1;\ \text{hence } a < g\right).$$

Tension formula:

Also from equation 1:

$T_1 = T = m_1 g + m_1 a$

$T = m_1 g + m_1 \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g$

$T = m_1 g \left(1 + \frac{m_2 - m_1}{m_1 + m_2}\right)$

$T = m_1 g \left(\frac{m_1 + m_2 + m_2 - m_1}{m_1 + m_2}\right)$

$T = m_1 g \left(\frac{2m_2}{m_1 + m_2}\right)$

Hence, tension is:

$$\mathbf{T = \frac{2m_1 m_2 g}{m_1 + m_2}}$$

The same result can be obtained by substituting the expression for **a** into equation 2.

It is important to note that *the tension is always* ***greater than the lighter weight*** *and* ***less than the heavier weight****:* $m_1g < T < m_2g$.

Two key insights regarding a vertical connected-body system:

Insight 1: ***Why same acceleration magnitude?***

Because the string is inextensible (does not stretch), any distance m_1 moves up equals the distance m_2 moves down in the same time interval. Therefore, their speeds and accelerations must have the same magnitude, though opposite directions.

Insight 2: ***The smooth pulley assumption***

A smooth pulley means frictionless. This ensures the tension in the string is the same on both sides of the pulley. If friction existed, tension would differ on each side, complicating the analysis significantly. For A-level problems, we almost always assume smooth pulleys unless stated otherwise.

For now, let us pause the theory and bring the ideas down to earth with a few carefully chosen worked examples.

BINDER Example 17

Show that, in a vertical connected-body system, if one body is much heavier than the other, the system accelerates at nearly the acceleration due to gravity, g.

Solution

Consider two masses m_1 and m_2 connected by a light inextensible string over a smooth pulley, with $m_2 > m_1$ so that m_2 moves downward.

The acceleration of the vertical connected-body system is given by the following formula:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g$$

If $m_2 >> m_1$ (much heavier), then m_2 is very large compared to m_1. Thus:

$$m_2 - m_1 \approx m_2 \text{ and } m_2 + m_1 \approx m_2$$

Substituting into the acceleration expression:

$$a \approx \left(\frac{m_2}{m_2}\right) g \approx g$$

Hence, when one body is much heavier than the other, the system accelerates at almost g, (almost in free fall).

Making Sense of the Answer: *If the heavier mass is far larger than the lighter one, the lighter mass becomes almost negligible. The heavy mass then behaves nearly like an object falling under gravity with very little "load" to lift, so its acceleration gets very close to g.*

Thinking Like a Physicist: *This is a limiting-case test. If m_1 were zero (an "almost nothing" mass), the heavy mass would essentially be falling freely and the formula would give $a = g$ exactly. Real systems never reach g because the other mass is never truly zero, but as the mass difference becomes extreme, the system approaches free-fall behaviour.*

BINDER Example 18

Show that, in a vertical connected-body system, the tension, T is always less than the weight of the heavier mass and greater than the weight of the lighter mass.

Solution

Assume a light inextensible string over a smooth pulley with $m_2 > m_1$, so the system accelerates with magnitude $a > 0$, with m_2 downward and m_1 upward.

Applying Newton's second law on each mass:

For m_1 (accelerating upward):

$$T - m_1 g = m_1 a$$

So: $\mathbf{T = m_1 g + m_1 a}$

Since $a > 0, \rightarrow \mathbf{T > m_1 g}$ … …(i)

For m_2 (accelerating downward):

$$m_2 g - T = m_2 a$$

So: $\mathbf{T = m_2 g - m_2 a}$

Since $a > 0, \rightarrow \mathbf{T < m_2 g}$ … …(ii)

Combining (i) and (ii) gives:

$$\mathbf{m_1 g < T < m_2 g}$$

Hence, the tension is always less than the weight of the heavier mass and greater than the weight of the lighter mass.

Making Sense of the Answer: *The heavier mass pulls downward, but part of its weight accelerates the lighter mass upward, so the tension is less than the heavier weight. At the same time, it (tension) must exceed the lighter weight to lift it.*

Thinking Like a Physicist: *If tension equalled either weight, one mass would not accelerate. Since both move, the tension must lie between the two weights.*

BINDER Example 19

Two masses of 4kg and 6kg are connected by a light inextensible string passing over a smooth pulley. The system is released from rest. Take $g = 9.8 \text{ m/s}^2$.

Calculate: (a) the acceleration of each mass (b) the tension in the string

Solution

Using:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g$$

Where:

$$m_2 = 6\text{kg (heavier)}$$

$$m_1 = 4\text{kg (lighter)}$$

Substituting:

$$a = \left(\frac{6kg - 4kg}{4kg + 6kg}\right) \times 9.8m/s^2 = 1.96m/s^2$$

Using:

$$T = \frac{2m_1m_2g}{m_1 + m_2}$$

Substituting:

$$T = \frac{2 \times 4kg \times 6kg \times 9.8m/s^2}{4kg + 6kg} = 47.04N$$

(a) The acceleration is $1.96m/s^2$.
(b) The tension is $47.04N$.

Making Sense of the Answer: *The heavier mass ($6kg$) pulls the system downward, so both masses accelerate together at $1.96m/s^2$. The tension ($47.04N$) lies between the two weights: greater than 4g ($39.2N$) but less than $6g$ ($58.8N$), which confirms the result is physically reasonable.*

Thinking Like a Physicist: *In the vertical connected-body system, always identify the heavier mass first to determine the direction of motion. Also as a quick check, the tension must satisfy $m_1g < T < m_2g$; if it does, the analysis is likely correct.*

BINDER Example 20

Two masses are connected by a string over a smooth pulley. When released from rest, the system accelerates at $1.5\ m/s^2$. If one of the masses is 4kg and moves upward, determine the mass of the other body. Take $g = 9.8\ m/s^2$.

Solution

Since the given mass was moving upward, it is the lighter mass. Thus the given mass is m_1 and the unknown mass is m_2 (heavier mass).

Using:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$$

Substituting:

$$1.5\ m/s^2 = \left(\frac{m_2 - 4kg}{4kg + m_2}\right) \times 9.8\ m/s^2$$

Solving for m_2 gives: $m_2 = 5.45kg$

The mass of the other body is 5.45kg.

Making Sense of the Answer: *The unknown mass ($5.45kg$) is moderately heavier than the known mass ($4kg$), producing a modest acceleration ($1.5m/s^2$). If they were equal, acceleration would be zero. If the difference were larger, acceleration would be greater. The result fits the physical expectation perfectly.*

Thinking Like a Physicist: *Always identify the direction of motion first; the mass moving upward must be the lighter one. Also check whether the result makes physical sense: a slightly heavier opposing mass should produce a moderate acceleration, not zero and not close to free fall (g).*

REAL Example 20

Kipanga watches workers installing a new elevator at his uncle's building. He notices a large concrete block hanging on the other side of the pulley.

Kipanga: *"Why is there a heavy block hanging there? Isn't the elevator motor strong enough to lift people by itself?"*

Kipute: *"I think it's a counterweight. It helps balance the elevator so the motor doesn't have to work as hard."*

Mr. Akilikubwa: (joining them) *"Excellent observation, Kipute! Without that counterweight, the motor would need to lift the full weight of the elevator cabin plus passengers. Imagine lifting a 1000kg load straight up! But with a properly sized counterweight, the motor only needs to overcome the difference. If the cabin with passengers weighs 1000kg and the counterweight is 800kg, the motor effectively only lifts 200kg. Much easier, much more energy efficient."*

Kipanga: *"So it's like those connected masses we studied in class?"*

Mr. Akilikubwa: *"Exactly! The elevator and counterweight are connected bodies in vertical motion. When the elevator goes up, the counterweight comes down, just like our pulley system. The tension in the cable does most of the work, and the motor just provides the extra push or pull to accelerate or decelerate the system smoothly. Without the counterweight, your electricity bill would be shocking, and not in a good way!"*

Kipute: (smiling) *"So Physics saves money too?"*

Mr. Akilikubwa: *"Always! Good engineering is just applied physics with a practical goal. Every elevator in the world uses this principle. Next time you ride one, thank Newton's laws and the counterweight quietly working behind the scenes."*

Question: Based on this conversation:

(a) Using the given masses, show quantitatively how the counterweight reduces the force required from the elevator motor compared with lifting the cabin alone.

(b) If an elevator cabin with passengers has a total mass of 900kg and the counterweight has a mass of 700 kg, calculate:

(i) the acceleration if the motor provides an additional upward force of 2450N to the cabin,

(ii) the tension in the cable connecting them.

(Take g = 9.8 m/s²)

Solution

(a) Minimum motor force required without counterweight:

Without a counterweight, the motor must provide the entire upward force to lift the cabin and overcome its full weight. This requires at least: $F = W = mg$

For a 900kg cabin: $F_{motor} = 900\text{kg} \times 9.8\,\text{m/s}^2 = 8820\text{N}$

Minimum motor force required with counterweight:

With a counterweight of 700kg, the downward pull of the counterweight partially balances the cabin's weight. The net force the motor must provide becomes much smaller because it only needs to overcome the difference (200kg):

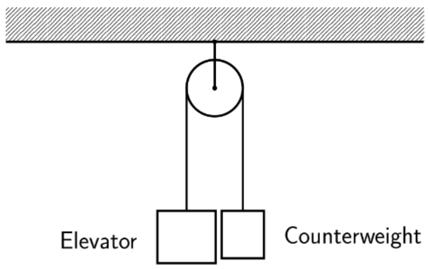
$$F_{motor} = (m_{cabin} - m_{counterweight})g = (900 - 700)kg \times 9.8m/s^2 = 1960N$$

This is only 22% of the original force! The counterweight does most of the "heavy lifting" through gravity, and the motor provides just the extra force needed for acceleration and control.

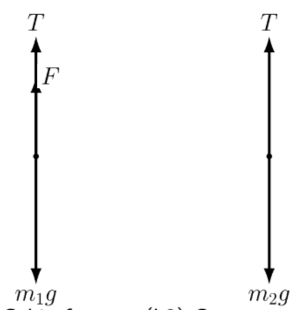
Additionally, when the cabin descends, the counterweight assists by pulling upward, again reducing motor effort.

This bidirectional efficiency explains why nearly all elevators use counterweights as they dramatically reduce energy consumption and allow smaller, cheaper motors.

(b) To fully understand this, consider the following situation diagram and its corresponding free-body diagram.



(a) Elevator cabin and counterweight



(b1) Cabin forces (b2) Counterweight forces

Forces acting on the elevator cabin (moving upward):

- Weight of cabin (downward): $W_{cabin} = m_1g = 900kg \times 9.8\,m/s^2 = 8820N$
- Tension (upward): T
- Motor force (upward): $F_{motor} = 2450N$

Total upward forces $= 2450N + T$

Total downward forces $= 8820N$

Applying Newton's second law (upward positive): $\Sigma F = m_1a$

$(2450 + T) - 8820 = 900a$ or

$T - 900a = 6370$ … …(i)

Forces acting on the counterweight (moving downward):

- Weight of counterweight (downward):$W_{counter} = m_2g = 700kg \times 9.8\,m/s^2 = 6860N$
- Tension (upward): T

Again applying Newton's second law (downward positive for counterweight): $\Sigma F = m_2a$

$6860 - T = 700a$ or

$T + 700a = 6860 \ldots \ldots (ii)$

Solving (i) and (ii) simultaneously gives: $a = 0.31 m/s^2, T = 6646N$

(i) The acceleration is $0.306 m/s^2$.
(ii) The tension is $6646N$.

Making Sense of the Answer: *The counterweight principle dramatically reduces the force requirement. Instead of the motor lifting 900kg, it effectively lifts only the difference (200kg), plus any additional force for acceleration. This is why elevators are energy-efficient despite moving heavy loads many times per day. The tension in the cable is substantial (6646N) but shared between supporting the cabin and the counterweight; the motor just provides the marginal difference.*

Thinking Like a Physicist: *Real engineering applications like elevators demonstrate why understanding connected bodies matters beyond textbooks. Engineers must calculate safe cable tensions, motor power requirements, and energy efficiency. A poorly designed counterweight (too light or too heavy) wastes energy or creates unsafe accelerations. Physics isn't just formulas, it's the foundation of technologies we use every day without thinking about the Newton's laws quietly working behind the scenes.*

HOT Example 21

A container of mass 8kg hangs from one end of a light inextensible string that passes over a smooth pulley. The other end of the string is attached to a 12kg block of ice resting on a platform. The system is initially at rest. After the ice block is released and starts moving, it melts at a constant rate of 0.5kg every 10 seconds due to the pulley mechanism warming the string.

(a) Calculate the initial acceleration of the system immediately after release.
(b) Determine the acceleration of the system after 20 seconds.
(c) Explain clearly what happens to the system's motion as the ice continues to melt.

Take $g = 9.8 m/s^2$

Solution

(a) Initial masses:
$m_1 = 8kg$ (container, moving upward)
$m_2 = 12kg$ (ice block, moving downward initially)
Using:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g$$

Substituting:

$$a = \left(\frac{12kg - 8kg}{8kg + 12kg}\right) \times 9.8 m/s^2 = 1.96 m/s^2$$

The initial acceleration is $1.96 m/s^2$.

(b) Calculating mass of ice after melting:
Melting rate = 0.5kg per 10 seconds

Time elapsed = 20 seconds

Mass melted = $\frac{0.5\text{ kg}}{10\text{s}} \times 20\text{s} = 1\text{kg}$

New mass of ice block: $m_2' = 12 - 1 = 11\text{kg}$

Again using the acceleration formula:

$$a' = \left(\frac{m_2' - m_1}{m_1 + m_2'}\right)g$$

Substituting:

$$a' = \left(\frac{11\text{kg} - 8\text{kg}}{8\text{kg} + 11\text{kg}}\right) \times 9.8\text{m/s}^2 = 1.55\text{m/s}^2$$

The acceleration after 20 seconds is 1.55m/s^2.

(c) As ice melts, the mass difference decreases, causing acceleration to decrease. At 80 seconds, masses become equal (both 8kg) and acceleration becomes zero, making the system to move at constant velocity. If melting continues beyond this point, the container becomes heavier, causing acceleration to reverses direction. Eventually, if all the ice melts, the container is left unsupported (the system is no longer connected bodies) and will undergo free fall with acceleration 9.8m/s^2 until it hits the ground or another obstacle.

The worked examples have filled the plate nicely; now it is time to enjoy the next subtopic and see what new ideas it brings to the table.