MODULE 17

THE TWO EQUILIBRIUM EQUATIONS

Up to now, we have talked about equilibrium in a simple way: if the resultant force on a body is zero, the body either remains at rest or continues moving with constant velocity. That idea works perfectly when all forces act along one straight line. But real life is rarely that cooperative. Forces often act in different directions: a hanging sign pulls downward while its cables pull sideways and upward, a ladder leans against a wall, a crane cable pulls at an angle, or even your school bag straps pull in directions you did not plan when you overloaded the bag.

To deal with such situations, Physics does not panic. Instead, it separates forces into two perpendicular components (directions): horizontal component (x-direction) and vertical component (y-direction). This process is called **resolution of forces**. To have better understanding on this consider the following two cases.

Case 1: Angle measured from the horizontal

In most cases, angles are measured from the horizontal, as shown in the figure below.

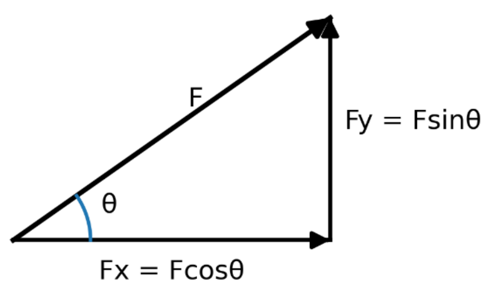


Figure: *A force resolved into horizontal and vertical components when the angle is measured from the horizontal.*

From the diagram, a force **F** act at angle **θ** above the horizontal (**measured from positive x-axis**).

From right-triangle geometry:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{F_x}{F}$$

From which:

$$\mathbf{F_x = F\cos\theta = Horizontal\ component}$$

Analogously:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{F_y}{F}$$

From which:

$$\mathbf{F_y = F\sin\theta \ = Vertical\ component}$$

Case 2: Angle Measured from the Vertical

In some situations, angles are measured from the vertical, as shown in the figure below.

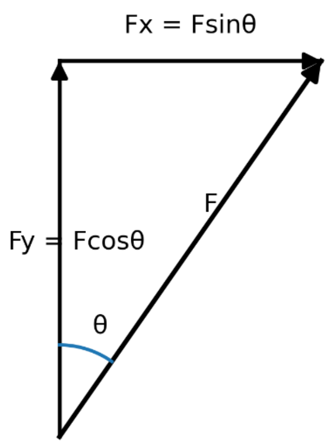


Figure: *Resolution of a force when the angle is measured from the vertical; note that the component expressions reverse.*

In this case:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{F_x}{F}$$

From which:

$$\mathbf{F_x = F\sin\theta \ = Horizontal\ component}$$

Similarly:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{F_y}{F}$$

From which:

$$\mathbf{F_y = F\cos\theta \ = Vertical\ component}$$

Important Alert!

In the second case, *the components reverse.*

This happens because the horizontal component is now opposite to the angle, while the vertical component is adjacent to it. So, always ***remember to check which axis the angle is measured from.***

It is also worth noting that, from geometry (Pythagoras' theorem), the magnitude of a force is related to its components in both cases by:

$$F^2 = (F_x)^2 + (F_y)^2 = (F\cos\theta)^2 + (F\sin\theta)^2$$

Hence:

$$\mathbf{F} = \sqrt{(\mathbf{F_x})^2 + (\mathbf{F_y})^2} \text{ or } \mathbf{F} = \sqrt{(\mathbf{F\cos\theta})^2 + (\mathbf{F\sin\theta})^2}$$

This confirms that the resultant magnitude is independent of whether the angle is measured from the horizontal or the vertical.

Before the ideas start colliding in our heads, let us calm them down with a few simple worked examples.

BINDER Example 3

A force of 50N acts on a body at an angle of 30° above the horizontal. Calculate the horizontal and vertical components of this force.

Solution

Since the angle was measured from horizontal:

Horizontal component, $F_x = F\cos\theta = 50\cos 30° = \mathbf{43.3N}$

Vertical component, $F_y = F\sin\theta = 50\sin 30° = \mathbf{25N}$

Making Sense of the Answer: *The horizontal component is larger because the force leans more toward the horizontal. If the angle were 60°, the vertical component would dominate.*

Thinking Like a Physicist: *Component analysis allows us to treat two-dimensional force problems as two separate one-dimensional problems. This simplification is one of Physics' most elegant problem-solving techniques.*

BINDER Example 4

A hanging sign is supported by a cable. The cable pulls on the sign with a force of 120N and makes an angle of 40° with the vertical. Without worrying yet about the detailed theory of tension, resolve this force into:

(a) the horizontal component, and

(b) the vertical component.

Solution

Since the angle was measured from horizontal:

Horizontal component, $F_x = F\sin\theta = 120\cos 40° = \mathbf{77.1N}$

Vertical component, $F_y = F\cos\theta = 120\sin 40° = \mathbf{91.9N}$

Making Sense of the Answer: *Notice how the component equations reversed compared to Example 3. This happens because now the vertical component is adjacent to the angle (cosine) while the horizontal is opposite (sine).*

Thinking Like a Physicist: *The physics doesn't change with how we measure angles, but the mathematics adjusts accordingly. Staying alert to reference axes prevents sign errors and incorrect component calculations.*

The worked examples have done their job nicely; now let us invite the next concept to step forward and show us its charm.

Fundamental Condition for Equilibrium in Two Dimensions

From Newton's second law:

Resultant force $= \Sigma F = ma$; where Σ stands for the *summation of*.

When a resultant force (ΣF) acts diagonally, it has components in both horizontal and vertical directions. If these components are unbalanced, the body will experience acceleration in both directions, leading to **motion in two dimensions**.

For horizontal motion:

$$\Sigma F_x = ma_x$$

For vertical motion:

$$\Sigma F_y = ma_y$$

But for equilibrium, acceleration is zero. Thus:

$$a_x = 0 \text{ and } a_y = 0$$

Hence:

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

The final result is the set of **two equilibrium equations for forces acting in two dimensions**.

The equations mean that for equilibrium to be maintained:

- The algebraic sum of horizontal components of all forces must be zero.
- The algebraic sum of vertical components of all forces must be zero.

When either of these conditions fails, equilibrium is broken. If:

- $\Sigma F_x \neq 0$; horizontal acceleration occurs resulting to sideway motion.

- $\Sigma F_y \neq 0$; vertical acceleration occurs resulting to vertical (upward or downward) motion.

Equilibrium is therefore a multidirectional condition.

You have to remember that:

A body could move with constant velocity and still satisfy: $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

To truly understand and enjoy these ideas, let us serve them in the form of worked example.

BINDER Example 5

Three coplanar forces act on a particle in equilibrium: 10N eastward, 6N northward, and a third force F at an angle. If the particle is in equilibrium, find the magnitude and direction of the third force F.

Solution

For equilibrium: $\Sigma F_x = 0$ and $\Sigma F_y = 0$

Taking east as positive x-direction and north as positive y-direction.

Applying horizontal equilibrium equation: $\Sigma F_x = 0$

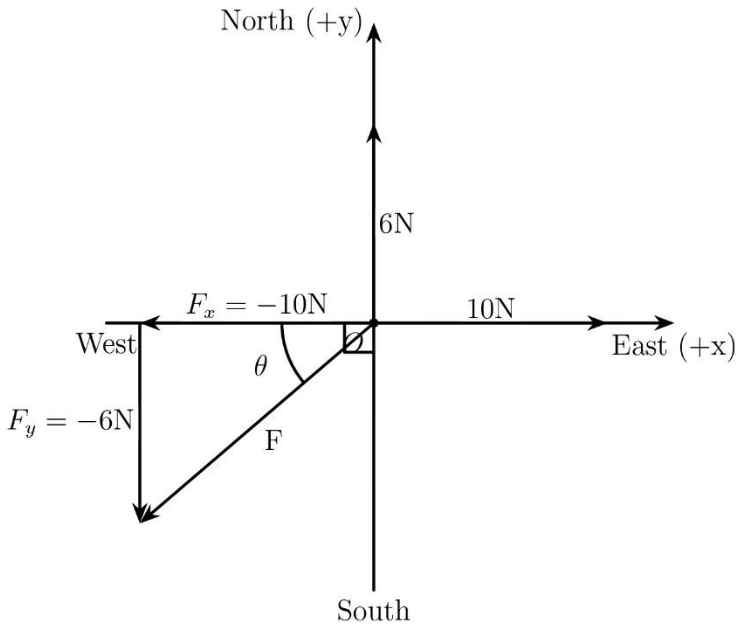
$$F_x + 10 = 0;\ F_x = -10\text{N} = \text{Horizontal component of the third force}$$

The negative sign means the horizontal component acts ***westward****.*

Applying vertical component equilibrium equation: $\Sigma F_y = 0$

$$F_y + 6 = 0;\ F_y = -6\text{N} = \text{Vertical component of the third force}$$

Here, *the negative sign means the vertical component acts* ***southward****.*



From the diagram:

$$\tan\theta = \frac{F_y}{F_x} = \frac{-6}{-10} = 0.6;\ \theta = \tan^{-1} 0.6 = 31°$$

Also, using $F = \sqrt{(F_x)^2 + \left(F_y\right)^2} = \sqrt{(-10)^2 + (-6)^2} = 11.7\text{N}$

The magnitude is 11.7N.

The direction is 31° south of west.

Making Sense of the Answer: *The third force must "cancel" the combined effect of the other two. Since we have eastward and northward forces, the equilibrant must pull westward (to balance eastward force) and southward (to balance northward force).*

Thinking Like a Physicist: *Equilibrium means the vector sum is zero. Treating x and y directions separately transforms a vector problem into two simple algebraic equations.*

With this worked example now neatly packed away, let us move on and meet the next subtopic; it has been waiting patiently to join the conversation.